

Motion of a drop in a vertical temperature gradient at small Marangoni number – the critical role of inertia

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When a drop moves in a uniform vertical temperature gradient under the combined action of gravity and thermocapillarity at small values of the thermal Péclet number, it is shown that inclusion of inertia is crucial in the development of an asymptotic solution for the temperature field. If inertia is completely ignored, use of the method of matched asymptotic expansions, employing the Péclet number (known as the Marangoni number) as the small parameter, leads to singular behaviour of the outer temperature field. The origin of this behaviour can be traced to the interaction of the slowly decaying Stokeslet, arising from the gravitational contribution to the motion of the drop, with the temperature gradient field far from the drop. When inertia is included, and the method of matched asymptotic expansions is used, employing the Reynolds number as a small parameter, the singular behaviour of the temperature field is eliminated. A result is obtained for the migration velocity of the drop that is correct to $O(Re^2 \log Re)$.

1. Introduction

In the formation of alloys, it is possible for drops rich in one component to precipitate out of a liquid mixture of two components as the liquid is cooled. An example is the system Al–Si–Bi on which experiments were performed by Prinz & Romero (1993). The drops, rich in bismuth, are subjected to the simultaneous action of gravity and a vertical temperature gradient field in a strip-casting process. Prinz & Romero showed that it is possible to use the temperature gradient to reduce the gravitational settling velocity so that a more homogeneous suspension of bismuth-rich drops in an aluminium-rich matrix could be achieved in the central regions of the cast material. This is because of the action of thermocapillarity, which causes the motion of drops in the direction of the temperature gradient when the interfacial tension decreases with increasing temperature. The authors go on to suggest that alloys such as Al–Si–Bi and Al–Si–Pb with a homogeneous distribution of the droplets can serve as improved bearing materials in the future. Although the practical system involves a collection of droplets of different sizes, it is instructive to examine the behaviour of an

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isolated drop under idealized conditions as a prelude to considering more complicated interaction problems.

For the motion of an isolated drop subjected simultaneously to the action of gravity and a vertical temperature gradient, Young, Goldstein & Block (1959) obtained results for the field variables and the quasi-steady velocity of the drop when convective transport of both momentum and energy are neglected. Several investigators have analysed this problem including convective transport effects, but only when motion is driven purely by thermocapillarity in the absence of a body force, with one exception that will be identified shortly. In these analyses, an isolated drop is assumed to move at a steady velocity in an unbounded continuous phase fluid with which it is immiscible, and in which a uniform temperature gradient is imposed in the undisturbed state. Bratukhin (1975) included convective transport of both momentum and energy, and presented an asymptotic analysis of the problem for small values of the Reynolds number for fixed Prandtl number, using a regular perturbation expansion. He obtained the corrections for the velocity and temperature fields at $O(Re)$, and calculated the resulting small inertial deformation of the shape of the drop. He found the correction to the migration velocity to be zero at $O(Re)$. Thompson, DeWitt & Labus (1980) extended Bratukhin's analysis to the next higher order.

Subramanian (1981) pointed out that the solution obtained by regular perturbation fails to meet the boundary condition at infinity because convective transport effects are not uniformly small everywhere; far from the drop, convection and conduction play equally important roles. He used the method of matched asymptotic expansions to analyse the thermocapillary motion of a bubble when inertial effects are completely negligible, but small amounts of convective transport of energy are included. In this case, the solutions for the fields and the migration velocity of the bubble are expanded in asymptotic series in the thermal Péclet number, termed the Marangoni number, Ma . Subramanian found that the first non-zero correction to the steady migration velocity of the bubble occurs at $O(Ma^2)$. Subsequently, Subramanian (1983) extended the analysis to the case of a drop, calculating the fields both within and outside the drop, and obtained the first non-zero correction to the migration velocity, again at $O(Ma^2)$. Crespo, Migoya & Manuel (1998) performed a similar analysis using the velocity profile corresponding to $Re \rightarrow \infty$.

Crespo & Manuel (1983) found that the Stokes solution for the velocity field in the continuous phase for the steady thermocapillary motion of a bubble satisfies the full Navier–Stokes equation and therefore is an exact solution at all values of the Reynolds number, so long as the temperature field satisfies the Laplace equation. Balasubramaniam & Chai (1987) independently discovered that this result holds for the solutions both within and outside a drop, and used that solution to calculate small inertial deformations of a drop from the spherical shape. Haj-Hariri, Nadim & Borhan (1990) independently discovered the same solution, and went on to calculate the correction to the migration velocity of the drop and the correction to the temperature field due to small inertial deformations of the shape using the Lorentz reciprocal theorem.

Asymptotic analyses have been performed in the thermocapillary migration problem for the case when the Marangoni number is large by Crespo & Jimenez-Fernandez (1992*a, b*), and by Balasubramaniam & Subramanian (1996, 2000). These authors have considered both the limiting cases of $Re \rightarrow 0$ and $Re \rightarrow \infty$. In the only available asymptotic analysis that includes a body force, Balasubramaniam (1998) has examined the motion of a gas bubble in a vertical temperature gradient in the limiting case $Re \rightarrow \infty$ and $Ma \rightarrow \infty$, including the buoyant contribution to the motion as well as

the influence of temperature on viscosity. Balasubramaniam used the potential flow velocity profile for the motion of a bubble in the limit $Re \rightarrow \infty$. This profile applies at leading order whether the motion is driven by gravity or by thermocapillarity. Therefore, the gravitational force only enters the problem via an additional term in the expression for the viscous dissipation that is used to determine the velocity of the bubble. The result given by Balasubramaniam (1998) for the steady migration velocity of the bubble at leading order is a linear combination of the result for purely thermocapillary motion obtained by Balasubramaniam & Subramanian (1996) and the purely gravitational rise velocity of a bubble at large Reynolds number. Inclusion of a body force in the present analysis, which applies in the limit $Re \rightarrow 0$, also leads to a simple superposition of results at leading order because of the linearity of the problem at that order, but introduces significant complications at higher orders. A more detailed discussion of the literature, including contributions in which a numerical technique was used to obtain the solution, can be found in a recent monograph by Subramanian & Balasubramaniam (2001).

The purpose of the present article is to highlight a curious feature that emerges in an asymptotic analysis of the motion of a drop under the combined influence of gravity and thermocapillarity when convective transport effects are included. When inertial effects are small, it is customary to neglect them altogether and assume Stokes flow. Because the Prandtl number in many liquids is larger than unity, and can vary over a few orders of magnitude, one might consider analysing the problem for small values of the thermal Péclet number using a suitable asymptotic expansion. As noted above, convective transport effects are not uniformly small over the domain, however. Convection and conduction play equally important roles far from the drop, and the method of matched asymptotic expansions must be used to solve the problem. Such a procedure works well for the problem of heat transfer between a rigid sphere maintained at a constant temperature and a continuous phase that is at a different but uniform temperature in the undisturbed state, as shown by Acrivos & Taylor (1962). It also continues to work when the analysis is extended to the analogous problem for a fluid drop (Brunn 1982), or to the motion of a drop purely due to thermocapillarity in a continuous phase in which a uniform temperature gradient is imposed in the undisturbed state (Subramanian, 1981, 1983). But, when a similar analysis is attempted while accommodating a body force, we demonstrate that it leads to a singular solution. This is a consequence of the interaction of the slowly decaying Stokeslet flow far from the drop with the temperature gradient field. The remedy is to use the proper velocity distribution obtained by including a slight amount of inertia. The resulting Oseen flow decays correctly far from the drop, and avoids the appearance of the singularity in the heat transfer problem.

2. Analysis

Consider an isolated drop placed in a continuous-phase fluid of unbounded extent in which a vertical temperature gradient $\nabla T_\infty = |\nabla T_\infty| \mathbf{i}_z$ is imposed, where \mathbf{i}_z is a unit vector in the z^* -direction. The acceleration due to gravity points in the direction of the vector $-\mathbf{i}_z$. Figure 1 depicts the system and the spherical polar coordinates used in the analysis, which is performed in a reference frame attached to the moving drop. The density d , dynamic viscosity μ , and the thermal diffusivity κ , of the continuous phase are assumed constant; $\nu = \mu/d$ is used to designate the constant kinematic viscosity of the continuous phase. Similar properties in the drop phase, identified by a caret, are assumed constant as well. Also σ_T , which stands for the rate of change of

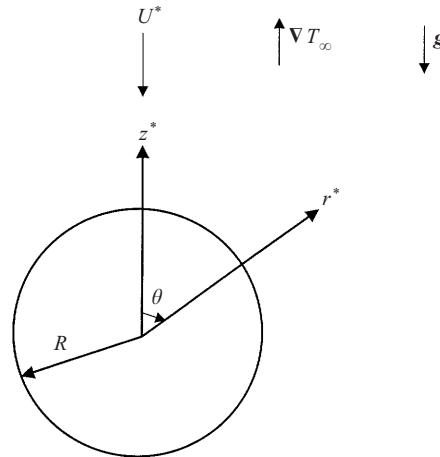


FIGURE 1. Schematic of the system, with a uniform stream $-U^* \mathbf{i}_z$ approaching the drop in a reference frame attached to the drop.

the interfacial tension between the drop and the continuous phase with temperature, is assumed to be a negative constant. For definiteness we assume the drop to move upward at a velocity $U^* \mathbf{i}_z$. Note that downward motion is possible only when the drop is more dense than the continuous phase.

Scaled variables are used in the analysis. The radial coordinate is scaled using the radius R of the drop, and velocity is scaled using a thermocapillary reference velocity $v_0 = -(\sigma_T |\nabla T_\infty| R) / \mu$. Pressure and viscous stresses are scaled using $\mu v_0 / R = -\sigma_T |\nabla T_\infty|$. The temperature is scaled by subtracting the value in the undisturbed continuous phase in the horizontal plane that passes through the instantaneous location of the centre of mass of the drop, and dividing by $|\nabla T_\infty| R$. The symbols \mathbf{v} , p and T are used to designate the scaled velocity, pressure, and temperature fields, and similar symbols with a caret are used for referring to variables within the drop.

In the laboratory reference frame, the temperature field in the undisturbed continuous phase fluid is steady, but spatially non-uniform. In a reference frame moving with the drop, which we use in the analysis, the temperature in the undisturbed continuous-phase fluid will change with time and position. But the corresponding scaled temperature, as defined above, will be independent of time. This permits a steady problem to be posed for the scaled temperature and velocity fields. The fields satisfy the following continuity, Navier–Stokes, and energy equations (Subramanian & Balasubramaniam 2001):

$$\nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$Re[(\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p + \nabla^2 \mathbf{v}, \quad (2)$$

$$Ma[U + \mathbf{v} \cdot \nabla T] = \nabla^2 T, \quad (3)$$

$$\nabla \cdot \hat{\mathbf{v}} = 0, \quad (4)$$

$$\gamma Re[(\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}}] = -\nabla \hat{p} + \alpha \nabla^2 \hat{\mathbf{v}}, \quad (5)$$

$$Ma[U + \hat{\mathbf{v}} \cdot \nabla \hat{T}] = \lambda \nabla^2 \hat{T}. \quad (6)$$

In the above equations, $Re = (Rv_0)/\nu$ is the Reynolds number, and $Ma = (Rv_0)/\kappa$ is a Péclet number, termed the Marangoni number. The scaled velocity of the drop is $\mathbf{U} = U \mathbf{i}_z$; and $\gamma = \hat{d}/d$, $\alpha = \hat{\mu}/\mu$, and $\lambda = \hat{\kappa}/\kappa$, are the ratios of the density, dynamic

viscosity, and the thermal diffusivity of the drop phase to the corresponding properties of the continuous phase, respectively. We assume the capillary number $Ca = \mu v_0 / \sigma$, where σ is a characteristic value of the interfacial tension, to be negligibly small, and therefore neglect deformation from the spherical shape. The boundary conditions satisfied by the velocity field are as follows.

The velocity approaches a uniform stream far from the drop:

$$\mathbf{v} \rightarrow -U \mathbf{i}_z \quad \text{as } r \rightarrow \infty. \tag{7}$$

At the drop surface, the velocity must be continuous, the normal component of the velocity must be zero, and the discontinuity in the tangential stress is balanced by the surface gradient of the interfacial tension:

$$v_\theta(1, s) = \hat{v}_\theta(1, s), \tag{8}$$

$$v_r(1, s) = \hat{v}_r(1, s) = 0, \tag{9}$$

$$\tau_{r\theta}(1, s) - \hat{\tau}_{r\theta}(1, s) = -\sqrt{1-s^2} \frac{\partial T}{\partial s}(1, s). \tag{10}$$

In (8)–(10), the subscripts r and θ represent the corresponding spherical polar components, $\tau_{r\theta}$ and $\hat{\tau}_{r\theta}$ are the scaled tangential stress components in the continuous phase and the drop phase, respectively, and $s = \cos \theta$. In addition to these boundary conditions, we require that the velocity be bounded everywhere within the drop. Because the shape has been assumed to be a sphere, we do not attempt to satisfy the normal stress balance. An expansion for small capillary number can be used in conjunction with that balance to infer the nature of slight deformations from the spherical shape, if desired.

In a like manner, we require that the temperature approach the undisturbed field far from the drop:

$$T \rightarrow rs \quad \text{as } r \rightarrow \infty. \tag{11}$$

At the surface of the drop, the temperature and the heat flux must be continuous. In writing the latter condition, we neglect the relatively small discontinuity arising from the creation and destruction of the interface:

$$T(1, s) = \hat{T}(1, s), \tag{12}$$

$$\frac{\partial T(1, s)}{\partial r} = \beta \frac{\partial \hat{T}(1, s)}{\partial r}. \tag{13}$$

Here, $\beta = \hat{k}/k$ is a thermal conductivity ratio. We must also require the temperature field to be bounded everywhere within the drop. Setting the net force on the drop to zero completes the set of conditions:

$$\int_{-1}^1 \left[sp - \frac{2s}{r} \frac{\partial}{\partial s} (\sqrt{1-s^2} v_\theta) + r \sqrt{1-s^2} \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) \right]_{r=1} ds + 2G = 0. \tag{14}$$

Here $G = gR(\hat{d} - d)/(3\sigma_T |\nabla T_\infty|)$ is a dimensionless group that represents the relative importance of the effect of gravity to that of thermocapillarity. When the drop is more dense than the continuous phase, G assumes negative values because σ_T is a negative constant.

For convenience in the subsequent analysis, we define a Stokes streamfunction ψ via $\mathbf{v} = \nabla \phi \times \nabla \psi$ where ϕ is the azimuthal angular coordinate. A similar definition for

$\hat{\psi}$ applies in the drop phase. The streamfunction fields satisfy the following equations:

$$Re \left[\frac{1}{r^2} \frac{\partial(\psi, E_r^2 \psi)}{\partial(r, s)} + \frac{2}{r^2} E_r^2 \psi L_r \psi \right] = -E_r^4 \psi, \quad (15)$$

$$Re \left[\frac{1}{r^2} \frac{\partial(\hat{\psi}, E_r^2 \hat{\psi})}{\partial(r, s)} + \frac{2}{r^2} E_r^2 \hat{\psi} L_r \hat{\psi} \right] = -\frac{\alpha}{\gamma} E_r^4 \hat{\psi}, \quad (16)$$

where

$$E_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1-s^2}{r^2} \frac{\partial^2}{\partial s^2} \quad (17)$$

and

$$L_r = \frac{s}{1-s^2} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial s}. \quad (18)$$

2.1. Solution when the Reynolds number is set equal to zero

We first proceed to solve for the field variables and the migration velocity for small values of Ma by completely neglecting inertia, setting $Re = 0$. Asymptotic expansions for the fields and the migration velocity with Ma as the perturbation parameter are

$$\hat{\psi} = \hat{\psi}_0(r, s) + o(1), \quad (19)$$

$$\psi = \psi_0(r, s) + o(1), \quad (20)$$

$$\hat{t} = \hat{t}_0(r, s) + o(1), \quad (21)$$

$$t = t_0(r, s) + o(1), \quad (22)$$

$$U = U_0 + o(1). \quad (23)$$

We have used lower-case symbols (\hat{t}, t) to designate the temperature fields in (21) and (22). The order of the coefficient of t_0 should be established by matching, but we have anticipated the result of that process and selected it as unity. As noted earlier, the convective transport terms become comparable to conduction far from the drop (when $r \sim O(1/Ma)$), so that it is necessary to define an outer radial coordinate $\rho^* = Mar$, and rewrite the energy equation in the continuous phase in the outer variables. The outer temperature field in the continuous phase $T(\rho^*, s)$ satisfies

$$\mathbf{V} \cdot \nabla_{\rho^*} T = \nabla_{\rho^*}^2 T - \frac{1}{Ma} U \quad (24)$$

where \mathbf{V} is the velocity field written as a function of (ρ^*, s) , and the subscript ρ^* in the operator ∇_{ρ^*} signifies the use of outer variables along with the condition that $T \rightarrow (\rho^* s)/Ma$ as $\rho^* \rightarrow \infty$ and suitable matching conditions as $\rho^* \rightarrow 0$. The field $T(\rho^*, s)$ can be expanded as

$$T = \frac{\rho^* s}{Ma} + T_0(\rho^*, s) + o(1). \quad (25)$$

The coefficient of T_0 should be permitted to depend upon Ma , but it will be evident from the governing equation for T_0 that it has to be $O(1)$, and we have anticipated this in writing (25). Because the small parameter Ma enters the fluid-mechanical problem only through the tangential stress balance, solutions of the Stokes equation for the complete streamfunction fields within and outside the drop can be written immediately

and specialized to the boundary conditions of this problem. These solutions are

$$\hat{\psi} = \frac{1}{2(1 + \alpha)}(r^2 - r^4)[I_2 - U]\mathcal{C}_2(s) + \frac{1}{4(1 + \alpha)} \sum_{n=3}^{\infty} n(n - 1)I_n(r^n - r^{n+2})\mathcal{C}_n(s), \quad (26)$$

$$\begin{aligned} \psi = & \frac{I_2}{2(1 + \alpha)} \left(\frac{1}{r} - r \right) \mathcal{C}_2(s) + \frac{U}{2} \left[2r^2 + \frac{\alpha}{1 + \alpha} \frac{1}{r} - \frac{2 + 3\alpha}{1 + \alpha} r \right] \mathcal{C}_2(s) \\ & + \frac{1}{4(1 + \alpha)} \sum_{n=3}^{\infty} n(n - 1)I_n \left(\frac{1}{r^{n-1}} - \frac{1}{r^{n-3}} \right) \mathcal{C}_n(s). \end{aligned} \quad (27)$$

Here, $\mathcal{C}_n(s)$ is the Gegenbauer polynomial of order n and degree $-\frac{1}{2}$, and we have omitted the degree designation to avoid clutter. The constants I_n are defined as

$$I_n = - \int_{-1}^1 \frac{\partial t}{\partial s}(1, s)\mathcal{C}_n(s)ds = - \int_{-1}^1 t(1, s)P_{n-1}(s)ds \quad (n \geq 2). \quad (28)$$

The symbol $P_n(s)$ stands for the Legendre polynomial. Because the temperature field is expanded in an asymptotic series in (22), a similar expansion is obtained for I_n :

$$I_n = I_{n0} + o(1). \quad (29)$$

The force balance on the drop yields

$$U = -\frac{I_2}{2 + 3\alpha} + \frac{2(1 + \alpha)}{2 + 3\alpha}G. \quad (30)$$

It is evident that the streamfunction fields and the migration velocity to any order can be written once the temperature fields are known to that order. Substitution of the expansions for the temperature fields into the governing equations and boundary conditions, followed by taking the limit as $Ma \rightarrow 0$, leads to the governing equations and boundary conditions for the leading-order fields:

$$\nabla^2 \hat{t}_0 = 0, \quad (31)$$

$$\nabla^2 t_0 = 0, \quad (32)$$

$$\begin{aligned} \nabla_{\rho^*}^2 T_0 - \left(\frac{I_{20}}{2 + 3\alpha} - \frac{2(1 + \alpha)}{2 + 3\alpha}G \right) \left(s \frac{\partial T_0}{\partial \rho^*} + \frac{1 - s^2}{\rho^*} \frac{\partial T_0}{\partial s} \right) \\ = - \lim_{Ma \rightarrow 0} \left[\frac{1}{Ma} (U + \mathbf{V} \cdot \nabla_{\rho^*}(\rho^* s)) \right] = G \frac{2 + P_2(s)}{3\rho^*}. \end{aligned} \quad (33)$$

We note that the inhomogeneity appearing in (33) for the outer temperature field T_0 arises from the Stokeslet term in the velocity field (proportional to $1/\rho^*$) interacting with the applied temperature gradient field far from the drop. The boundary conditions on the leading-order temperature field within the drop \hat{t}_0 and the corresponding leading-order inner field in the continuous phase t_0 are the same as those given in (12) and (13), along with a boundedness condition on $\hat{t}_0(0, s)$ and a matching requirement on the field t_0 as $r \rightarrow \infty$. The outer field T_0 must vanish as $\rho^* \rightarrow \infty$ and match the inner field as $\rho^* \rightarrow 0$. The procedure for obtaining the solution is straightforward, and is well-described by Proudman & Pearson (1957) and Acrivos & Taylor (1962) in analogous fluid mechanical and heat transport problems. For matching, we use the asymptotic matching principle given by Van Dyke (1975). The details of the analysis

can be found in the doctoral thesis of Zhang (2000). The leading-order streamfunction fields are the same as those obtained from Young *et al.* (1959):

$$\hat{\psi}_0 = \hat{A}(r^2 - r^4)(1 - s^2), \quad (34)$$

$$\psi_0 = \left(U_0 r^2 + \frac{B}{r} - Gr \right) \frac{1 - s^2}{2}. \quad (35)$$

The constants \hat{A} and B can be written as

$$\hat{A} = -\frac{G(2 + \beta) + 3}{2(2 + 3\alpha)(2 + \beta)}, \quad (36)$$

$$B = \frac{G\alpha(2 + \beta) - 2}{(2 + 3\alpha)(2 + \beta)}. \quad (37)$$

The leading-order temperature field within the drop is given by

$$\hat{t}_0(r, s) = \frac{G}{2U_0} - \frac{G}{U_0} \left[\gamma_E + \log \left(\frac{U_0}{2} \right) \right] + \frac{3}{2 + \beta} r P_1(s) \quad (38)$$

and the leading-order inner temperature field in the continuous phase and the contribution to the outer field at $O(1)$ are written as

$$t_0(r, s) = \frac{G}{2U_0} - \frac{G}{U_0} \left[\gamma_E + \log \left(\frac{U_0}{2} \right) \right] + \left(r + \frac{1 - \beta}{2 + \beta} \frac{1}{r^2} \right) P_1(s), \quad (39)$$

$$T_0(\rho^*, s) = \frac{G}{U_0} \left[E_1 \left(\frac{U_0}{2} \rho^* \{1 + s\} \right) - \frac{1}{2} P_1(s) + \log(\rho^* \{1 + s\}) \right] + \frac{G}{\rho^* U_0^2} \left[1 - \exp \left\{ -\frac{U_0 \rho^*}{2} (1 + s) \right\} \right]. \quad (40)$$

Here, $E_1(\eta) = \int_{\eta}^{\infty} (e^{-x}/x) dx$ is the exponential integral, and $\gamma_E = 0.577215 \dots$ is Euler's constant. The term involving the exponential integral in (40) is a homogeneous solution of (33). It is important to write the homogeneous solution in this form to match correctly with the inner field. Merritt (1988), who first analysed the present problem for a gas bubble, was unable to match the inner and outer solutions at leading order because the homogeneous solution was written as an infinite series in the usual specialized form given in Acrivos & Taylor (1962).

It is possible now to obtain the leading-order migration velocity U_0 as

$$U_0 = \frac{2}{(2 + 3\alpha)(2 + \beta)} + \frac{2(1 + \alpha)}{2 + 3\alpha} G. \quad (41)$$

This is a well-known result first obtained by Young *et al.* (1959). Also, one can proceed to obtain a first-order correction to it as shown by Zhang (2000), because the inner fields at the next order can be obtained by solving the governing equations and applying the boundary conditions, followed by matching with the leading-order outer field. Note, however, that the outer field does not satisfy the boundary condition imposed on it as $\rho^* \rightarrow \infty$. Instead $T \rightarrow \rho^* s / Ma + (G/U_0) \log \rho^*$. While the logarithmic growth term is overshadowed by the linear growth term as $\rho^* \rightarrow \infty$, its appearance is nevertheless troublesome, and physically unacceptable. It arises due to the inhomogeneity in the Oseen equation for the outer temperature field that occurs due to the presence of the slowly decaying Stokeslet in the velocity field.

This singular behaviour in the outer solution needs to be relieved by correcting the Stokes flow solution for inertial effects. When inertia is included, the velocity field far from the drop decays more rapidly than the Stokeslet, and we find that there is no singularity in the outer solution. We now proceed to provide results from the analysis including small amounts of inertia.

2.2. Solution including small inertial effects

When the Reynolds number is not set equal to zero, it can serve as a logical perturbation parameter. All the field variables and the migration velocity are expanded in asymptotic series in the limit $Re \rightarrow 0$. The Marangoni number $Ma = Re Pr$, where $Pr = \nu/\kappa$ is the Prandtl number. The Prandtl number is treated as an $O(1)$ constant. The outer variable is defined as $\rho = Re r$. The analysis of the fluid-mechanical problem follows the lines established by Proudman & Pearson (1957), and that of the energy equation is similar to that performed by Gupalo & Ryazantsev (1972) who considered heat or mass transfer from a moving rigid sphere to a fluid, and accounted for small amounts of inertia in their analysis. One difference from these problems is that the uniform stream itself is expanded in an asymptotic series as $U = U_0 + Re U_1 + o(Re)$. Therefore, while higher-order contributions to the outer streamfunction in the analysis of Proudman & Pearson approach zero as $\rho \rightarrow \infty$, here we encounter non-zero uniform stream contributions at every order, each involving an unknown correction to the scaled velocity. On the other hand, in the present problem, the hydrodynamic force is a specified quantity, being balanced by the hydrostatic force on the drop. The latter is a constant independent of the Reynolds number, so that there are no contributions to the hydrodynamic force on the drop beyond leading order. The details of the analysis are lengthy, and can be found in Zhang (2000). Here, we only provide the principal results. In the following governing equations, $\Phi(\rho, s) = Re^2 \psi(r, s)$ represents the outer streamfunction in the continuous phase, and $T(\rho, s)$ stands for the outer temperature field:

$$\frac{1}{\rho^2} \frac{\partial(\Phi, E_\rho^2 \Phi)}{\partial(\rho, s)} + \frac{2}{\rho^2} E_\rho^2 \Phi L_\rho \Phi = -E_\rho^4 \Phi, \tag{42}$$

$$V \cdot \nabla_\rho T = \nabla_\rho^2 T - \frac{1}{Re} U. \tag{43}$$

V represents the velocity field expressed in the outer variables (ρ, s) and the subscript ρ in the operators E^2, E^4, L , and ∇ signifies the use of outer variables. The field Φ must satisfy the boundary condition

$$\Phi \rightarrow \frac{1}{2} U \rho^2 (1 - s^2) \quad \text{as } \rho \rightarrow \infty \tag{44}$$

and matching requirements as $\rho \rightarrow 0$. The field T must satisfy the boundary condition $T \rightarrow (\rho s)/Re$ as $\rho \rightarrow \infty$ and matching conditions with the inner field t as $\rho \rightarrow 0$.

The inner velocity field in the continuous phase v and the velocity field within the drop \hat{v} will satisfy the boundary conditions stated in (8) to (10) and the requirement that the velocity within the drop remain bounded. In addition, the inner velocity field must satisfy matching requirements with the outer velocity field as $r \rightarrow \infty$. We can use the formal expansions in (19) to (23) with the understanding that the small parameter now is Re . The expansion for the outer temperature field $T(\rho, s)$ is slightly modified:

$$T = \frac{\rho s}{Re} + T_0(\rho, s) + o(1). \tag{45}$$

It can be seen that the leading-order inner streamfunction in the continuous phase

ψ_0 as well as the leading-order streamfunction within the drop $\hat{\psi}_0$ satisfy Stokes's equation. General solutions for these fields can be written, but the arbitrary constants appearing in them cannot be evaluated without knowledge of the solution for the temperature fields at leading order. Upon writing $\Phi = \Phi_0 + G_1(Re)\Phi_1 + o(G_1(Re))$ where $\Phi_0 = (U_0/2)\rho^2(1-s^2)$, examination of the coupled problem reveals that we need to obtain Φ_1 to determine the values of all the constants in the leading-order solutions, and to calculate the first corrections in the inner streamfunction fields. The following governing equation can be written for $\Phi_1(\rho, s)$:

$$-E_\rho^4 \Phi_1 = U_0 \left[\frac{1-s^2}{\rho} \frac{\partial}{\partial s} + s \frac{\partial}{\partial \rho} \right] (E_\rho^2 \Phi_1). \quad (46)$$

Upon matching the outer solution $\Phi_0 + G_1(Re)\Phi_1$ at the level of second derivatives in ρ with the inner solution ψ_0 , some of the arbitrary constants in the solutions as well as the dependence of G_1 on Re can be established. We find that $G_1 = Re$.

The leading-order temperature fields satisfy the following governing equations:

$$\nabla^2 \hat{t}_0 = 0, \quad (47)$$

$$\nabla^2 t_0 = 0, \quad (48)$$

$$\nabla_\rho^2 T_0 = Pr \left[V_{r0} \frac{\partial T_0}{\partial \rho} + \frac{V_{\theta 0}}{\rho} (1-s^2)^{1/2} \frac{\partial T_0}{\partial s} \right] + Pr[U_1 + sV_{r1} + (1-s^2)^{1/2}V_{\theta 1}]. \quad (49)$$

The boundary conditions on \hat{t}_0 and t_0 are the coupling conditions given in (12) and (13) and the requirements that \hat{t}_0 remain bounded within the drop and t_0 be matched to the outer solution. The field T_0 must vanish as $\rho \rightarrow \infty$ and satisfy suitable matching conditions with the inner field. We report the solutions next. The details can be found in Zhang (2000).

The streamfunction fields ψ_0 and $\hat{\psi}_0$ are given by the results in (34) and (35) that were obtained in the analysis neglecting inertia. The outer field Φ_1 is

$$\Phi_1 = -\frac{G}{U_0}(1-s) \left[1 - \exp \left\{ -\frac{U_0 \rho}{2}(1+s) \right\} \right] + U_1 \rho^2 \frac{1-s^2}{2}. \quad (50)$$

The leading-order temperature field within the drop, the leading-order inner temperature field in the continuous phase, and the correction to the outer temperature field at $O(1)$ are

$$\hat{t}_0 = -\frac{Pr}{Pr-1} \frac{G}{U_0} \log Pr + \frac{3}{2+\beta} r P_1(s), \quad (51)$$

$$t_0 = -\frac{Pr}{Pr-1} \frac{G}{U_0} \log Pr + \left[r + \frac{1-\beta}{2+\beta} \frac{1}{r^2} \right] P_1(s), \quad (52)$$

$$T_0 = -\frac{G}{U_0^2 \rho} \left[1 + \frac{1}{Pr-1} \left\{ \exp \left(-\frac{PrU_0}{2} \rho \{1+s\} \right) - Pr \exp \left(-\frac{U_0}{2} \rho \{1+s\} \right) \right\} \right] - \frac{G}{U_0} \frac{Pr}{Pr-1} \left[E_1 \left(\frac{U_0}{2} \rho \{1+s\} \right) - E_1 \left(\frac{PrU_0}{2} \rho \{1+s\} \right) \right]. \quad (53)$$

The important aspect of this solution is that T_0 behaves properly as $\rho \rightarrow \infty$. From these solutions, one can proceed to obtain the constants U_0 and U_1 in the asymptotic expansion of the scaled migration velocity, after obtaining the corrections to the streamfunction fields at $O(Re)$. It can be shown that the expansion for the

streamfunction fields must proceed as $\hat{\psi} = \hat{\psi}_0 + Re\hat{\psi}_1 + o(Re)$ and $\psi = \psi_0 + Re\psi_1 + o(Re)$. The governing equations for $\hat{\psi}_1$ and ψ_1 are

$$E_r^4 \hat{\psi}_1 = 0, \quad (54)$$

$$E_r^4 \psi_1 = 6G \left[\frac{U_0}{r^2} - \frac{G}{r^3} + \frac{B}{r^5} \right] \mathcal{C}_3(s). \quad (55)$$

The boundary conditions on $\hat{\psi}_1$ and ψ_1 can be inferred from (8) to (10). The expansion for the temperature fields must proceed as $\hat{t} = \hat{t}_0 + Re\hat{t}_1 + o(Re)$ and $t = t_0 + Ret_1 + o(Re)$. The governing equations for \hat{t}_1 and t_1 are

$$\lambda \nabla^2 \hat{t}_1 = Pr[U_0 + \hat{\mathbf{v}}_0 \cdot \nabla \hat{t}_0], \quad (56)$$

$$\nabla^2 t_1 = Pr[U_0 + \mathbf{v}_0 \cdot \nabla t_0]. \quad (57)$$

The boundary conditions on these fields can be obtained from (12) and (13). The solutions for $\hat{\psi}_1$, ψ_1 , \hat{t}_1 , and t_1 are given in the Appendix. The result for U_0 remains unaltered from that given in (41). The result for the correction U_1 at $O(Re)$ in the case when $U_0 > 0$, that is, when the drop moves upward, is

$$U_1 = \frac{Pr - 1 - G(1 + \alpha)(2 + \beta)}{2(2 + 3\alpha)(2 + \beta)} G. \quad (58)$$

In the opposite situation when the drop moves downward, $U_0 < 0$, the sign of U_1 in (58) must be reversed.

The next correction to the migration velocity appears at $O(Re^2 \log Re)$ for reasons that were clearly established by Proudman & Pearson (1957). The interested reader will find the logic outlined in Zhang (2000). In the Appendix, we give the results for the corresponding temperature and streamfunction correction fields at $O(Re^2 \log Re)$. Zhang (2000) provides the details of the lengthy solution for the inner temperature field at $O(Re^2)$, obtained using the computer software *Maple*. Unfortunately, the labour involved in obtaining the outer streamfunction correction at $O(Re^2)$, following the procedure outlined by Chester & Breach (1969), proved to be too formidable. This is needed to calculate the correction to the scaled migration velocity at $O(Re^2)$. Even though we provide the following correction U_{2L} at $O(Re^2 \log Re)$ in the scaled migration velocity because it is available, we add the note of caution that the result is incomplete without having the correction U_2 at $O(Re^2)$ in hand:

$$U_{2L} = \frac{5Pr^2 + 3Pr - 6}{30(2 + 3\alpha)(2 + \beta)} G^2 - \frac{1 + \alpha}{5(2 + 3\alpha)} G^3. \quad (59)$$

This correction remains the same regardless of the sign of U_0 .

3. Concluding remarks

The principal results of this analysis can be found in (40), (53), (58) and (59). Equation (40), obtained by ignoring inertia, shows that the correction to the undisturbed linear temperature field far from the drop grows in magnitude logarithmically with distance from the drop, and therefore is ill-behaved. We trace this behaviour to the interaction of the $1/r$ decay of the Stokeslet velocity field with the uniform temperature gradient field far from the drop. Inclusion of a slight amount of inertia has a critical influence on the outer temperature field, as can be observed from (53). This temperature field is well-behaved, and the correction to the undisturbed linear

temperature field far from the drop decays inversely with distance from the drop. This is a consequence of the more rapid spatial rate of decay of the Oseen velocity field that is brought about by the inclusion of inertia. Equations (58) and (59) provide the results for the migration velocity at higher orders that result from the present analysis.

We note that the solutions for the fields and the migration velocity behave properly when one takes the limit $Pr \rightarrow 1$. Also, it is worthy of mention that the analysis becomes significantly more complex when the Prandtl number is no longer of $O(1)$. In the present analysis, in a natural manner, the velocity field that appears in the inner energy equations is the inner velocity field, and that in the outer energy equations is the outer velocity field. This happens because we assume $Pr \sim O(1)$. This is not a physically correct approach when the Prandtl number is very small or very large. For example, consider the case when the Prandtl number is very small, implying $Re \gg Ma$. In this case, the inertia terms in the Navier–Stokes equation become comparable to the viscous terms at $r \sim O(1/Re)$, but conduction continues to be the dominant mechanism for heat transport until one reaches $r \sim O(1/Ma)$. Therefore, the Oseen velocity field must be used in the inner temperature field equations in the region $(1/Re) < r < (1/Ma)$, instead of the Stokes solution and inner corrections to it that appear naturally in these equations. In the opposite case when the Prandtl number is very large, $Re \ll Ma$. Here, even though the inertial correction to Stokes flow needs to be accommodated only at $r \sim O(1/Re)$, in the energy transport problem, the Oseen solution is used beginning at $r \sim O(1/Ma)$. This is perhaps more tolerable, because the Oseen solution provides a uniform representation of the velocity field at leading order. In any case, the situation involving very large or very small Prandtl numbers involves an additional length scale, and requires more detailed analysis, which we do not undertake in this work. Finally, it is noteworthy that the leading-order inner temperature fields are different from the solution of Young *et al.* (1959) by an additive constant that is of $O(1)$, a result that has no effect on the migration velocity which only depends on the temperature gradient at the surface of the drop.

Appendix

The solutions for the first corrections to the streamfunction fields at $O(Re)$, $\hat{\psi}_1$ and ψ_1 , can be written as

$$\hat{\psi}_1 = \hat{F}_1(r^4 - r^2)\mathcal{C}_2(s) + \hat{F}_2(r^5 - r^3)\mathcal{C}_3(s), \quad (\text{A } 1)$$

$$\psi_1 = \left[F_1 r^2 + \frac{F_2}{r} \right] \mathcal{C}_2(s) + \left[\frac{F_3}{r^3} + F_4 \right] \mathcal{C}_3(s) + 6G \left[\frac{U_0}{24} r^2 - \frac{F_5}{24r} - \frac{G}{24} r \right] \mathcal{C}_3(s), \quad (\text{A } 2)$$

where $\mathcal{C}_n(s)$ is the Gegenbauer polynomial of order n and degree $-\frac{1}{2}$, and

$$\hat{F}_1 = \frac{3}{4} H_1 Pr G, \quad (\text{A } 3)$$

$$\hat{F}_2 = H_2 H_3 H_4 Pr + \left[\frac{7}{20(2 + \beta)} H_2 - H_2 H_3 H_6 Pr \right] G + \frac{\alpha(5 + 4\alpha)}{20} H_2 G^2, \quad (\text{A } 4)$$

$$F_1 = \frac{1}{2} H_1 Pr G, \quad F_2 = -\frac{1}{2} H_1 Pr G, \quad (\text{A } 5), (\text{A } 6)$$

$$F_3 = -H_2 H_3 H_4 Pr + \left[\frac{5\alpha - 2}{20(2 + \beta)} H_2 + H_2 H_3 H_6 Pr \right] G + \frac{\alpha(5\alpha + 6)}{20} H_2 G^2, \quad (\text{A } 7)$$

$$F_4 = -H_2H_3H_4Pr - \left[\frac{25\alpha + 18}{20(2 + \beta)} H_2 + H_2H_3H_6Pr \right] G + \frac{\alpha(5\alpha + 4)}{20} H_2G^2, \quad (\text{A } 8)$$

$$F_5 = [G\alpha(2 + \beta) - 2] H_1. \quad (\text{A } 9)$$

The solutions for the first corrections to the temperature fields at $O(Re)$, \hat{t}_1 and t_1 , can be written as

$$\hat{t}_1 = \hat{K}_1 + \hat{K}_2rP_1(s) + \hat{K}_3r^2P_2(s) + \frac{Pr}{\lambda} \left(\frac{\hat{K}_4}{6}r^2 + \frac{\hat{K}_5}{20}r^4 + \frac{\hat{K}_6}{14}r^4P_2(s) \right), \quad (\text{A } 10)$$

$$t_1 = K_1 + \frac{K_2}{r} + \left(K_3r + \frac{K_4}{r^2} \right) P_1(s) + \frac{K_5}{r^3} P_2(s) + \frac{K_6}{12r^4} + \frac{K_7}{2r^2} + \frac{K_8}{2}r + \left(\frac{K_9}{6r^4} - \frac{K_{10}}{4r^2} - \frac{K_{11}}{6r} - \frac{K_{12}}{4}r \right) P_2(s). \quad (\text{A } 11)$$

The constant K_1 cannot be determined at this level. The remaining constants are

$$\hat{K}_1 = K_1 - H_1Pr(H_8 + H_9G), \quad (\text{A } 12)$$

$$\hat{K}_2 = \frac{3}{4(2 + \beta)} PrG, \quad (\text{A } 13)$$

$$\hat{K}_3 = -\frac{H_1}{21\lambda} Pr \left(H_3 [9(3 + 4\beta) + 7\lambda(7 - \beta)] + \frac{35\lambda H_6 + 9 + 6\beta}{3 + 2\beta} G \right), \quad (\text{A } 14)$$

$$\hat{K}_4 = H_7[13 + 2\beta + (2 + \beta)\{2\alpha(2 + \beta) + 2\beta + 7\}G], \quad (\text{A } 15)$$

$$\hat{K}_5 = -5H_7[3 + (2 + \beta)G], \quad \hat{K}_6 = 2H_7[3 + (2 + \beta)G], \quad (\text{A } 16), (\text{A } 17)$$

$$K_2 = -Pr[H_7H_{10} + H_1H_{11}G], \quad K_3 = \frac{1}{4}PrG, \quad K_4 = -3H_{12}PrG, \quad (\text{A } 18), (\text{A } 19), (\text{A } 20)$$

$$K_5 = Pr(H_{13} + H_{14}G), \quad (\text{A } 21)$$

$$K_6 = -H_7Pr [2(1 - \beta) - \alpha(2 - \beta - \beta^2)G], \quad (\text{A } 22)$$

$$K_7 = 4H_{12}PrG, \quad K_8 = \frac{2}{3}PrG, \quad K_9 = K_6, \quad (\text{A } 23), (\text{A } 24), (\text{A } 25)$$

$$K_{10} = 20H_{12}PrG, \quad (\text{A } 26)$$

$$K_{11} = H_7Pr(8 - 2\beta + H_{15}G), \quad (\text{A } 27)$$

$$K_{12} = \frac{1}{3}PrG. \quad (\text{A } 28)$$

The solutions for the corrections to the streamfunction fields at $O(Re^2 \log Re)$, $\hat{\psi}_{2L}$ and ψ_{2L} , can be written as

$$\hat{\psi}_{2L} = \hat{L}_1(r^2 - r^4)C_2(s), \quad \psi_{2L} = (L_1r^2 + L_2/r)C_2(s), \quad (\text{A } 29), (\text{A } 30)$$

where

$$\hat{L}_1 = -\frac{H_1}{20} Pr(3 + 5Pr)G^2, \quad (\text{A } 31)$$

$$L_1 = -\frac{2}{3}\hat{L}_1, \quad L_2 = -L_1. \quad (\text{A } 32), (\text{A } 33)$$

The solutions for the first corrections to the temperature fields at $O(Re^2 \log Re)$, \hat{t}_{2L} and t_{2L} , can be written as

$$\hat{t}_{2L} = W_1 + \frac{W_2}{2 + \beta} r P_1(s), \quad (\text{A } 34)$$

$$t_{2L} = W_1 + \frac{W_2}{3} \left(r + \frac{1 - \beta}{2 + \beta} \frac{1}{r^2} \right) P_1(s). \quad (\text{A } 35)$$

The constant W_1 cannot be determined at this level. The other constant is

$$W_2 = \frac{1}{20} Pr(3 + 5Pr)G^2. \quad (\text{A } 36)$$

In the above results

$$H_1 = \frac{1}{(2 + 3\alpha)(2 + \beta)}, \quad H_2 = \frac{1}{(1 + \alpha)(2 + 3\alpha)}, \quad (\text{A } 37), (\text{A } 38)$$

$$H_3 = \frac{1}{(2 + \beta)(3 + 2\beta)}, \quad H_4 = \frac{7\lambda\beta - 49\lambda - 18\beta}{35\lambda(2 + \beta)}, \quad (\text{A } 39), (\text{A } 40)$$

$$H_5 = \frac{7}{20}(3 + 2\beta), \quad (\text{A } 41)$$

$$H_6 = \frac{35\lambda\alpha\beta + 133\lambda\alpha + 154\lambda + 24\beta + 14\lambda\beta}{140\lambda}, \quad (\text{A } 42)$$

$$H_7 = \frac{1}{(2 + 3\alpha)(2 + \beta)^2}, \quad H_8 = \frac{17 + 20\beta + 8\beta^2 + 6\beta\lambda - 6\lambda}{12\lambda(2 + \beta)}, \quad (\text{A } 43), (\text{A } 44)$$

$$H_9 = \frac{3 - 3\lambda(12 + 17\alpha + 4\beta + 7\alpha\beta) + 4(1 + \alpha)(2 + 5\beta + 2\beta^2)}{12\lambda}, \quad (\text{A } 45)$$

$$H_{10} = \frac{2(2\beta + \beta^2 + \lambda\beta - \lambda)}{3\lambda}, \quad (\text{A } 46)$$

$$H_{11} = \frac{4\alpha\beta + 2\alpha\beta^2 - 8\alpha\lambda - \alpha\beta\lambda + 4\beta + 2\beta^2 - 6\lambda}{3\lambda}, \quad (\text{A } 47)$$

$$H_{12} = \frac{\beta - 1}{12(2 + \beta)}, \quad (\text{A } 48)$$

$$H_{13} = \frac{56\lambda + 35\lambda\beta - 18\beta - 28\lambda\beta^2}{21\lambda(2 + 3\alpha)(12 + 20\beta + 11\beta^2 + 2\beta^3)}, \quad (\text{A } 49)$$

$$H_{14} = -\frac{24\beta + 112\lambda + 280\alpha\lambda - 98\beta\lambda - 77\alpha\beta\lambda - 56\beta^2\lambda - 140\alpha\beta^2\lambda}{84\lambda(2 + 3\alpha)(2 + \beta)(3 + 2\beta)}, \quad (\text{A } 50)$$

$$H_{15} = 8 + 4\alpha - 4\beta - 4\beta^2 - 8\alpha\beta - 5\alpha\beta^2. \quad (\text{A } 51)$$

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