

The structure of the Ricci tensor on locally homogeneous Lorentzian gradient Ricci solitons

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We describe the structure of the Ricci tensor on a locally homogeneous Lorentzian gradient Ricci soliton. In the non-steady case, we show that the soliton is rigid in dimensions 3 and 4. In the steady case we give a complete classification in dimension 3.

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1. Introduction

Let (M, g) be a Lorentzian manifold of dimension $n + 2$ for $n \geq 1$ and let ρ be the Ricci tensor. Let Ric be the Ricci operator; $\rho(X, Y) = g(\text{Ric } X, Y)$. If $f \in C^\infty(M)$, let Hess_f be the Hessian; f is often called the *potential function*. Then

$$\text{Hess}_f(X, Y) = (\nabla_X \text{d}f)(Y) = XY(f) - (\nabla_X Y)(f).$$

Let ∇f be the vector field dual to the exterior derivative $\text{d}f$ of f ; this will also be denoted by $\text{grad}\{f\}$ for notational clarity when convenient. The *Hessian operator*

$$\mathcal{H}_f(X) := \nabla_X(\nabla f)$$

satisfies

$$\text{Hess}_f(X, Y) = g(\mathcal{H}_f X, Y).$$

Note that $\|\rho\|^2 = \|\text{Ric}\|^2$ and $\|\mathcal{H}_f\|^2 = \|\text{Hess}_f\|^2$.

The triple (M, g, f) is said to be a *Lorentzian gradient Ricci soliton* if f satisfies the *gradient Ricci soliton equation*:

$$\text{Hess}_f + \rho = \lambda g \quad \text{for some } \lambda \in \mathbb{R}. \quad (1.1)$$

Setting $f = 0$ yields the *Einstein equation* $\rho = \lambda g$; thus, (1.1) is a natural generalization of the Einstein equation and a gradient Ricci soliton can be thought of as a generalized Einstein manifold. Gradient Ricci solitons also correspond to self-similar solutions of the *Ricci flow* $\partial_t g(t) = -2\rho_{g(t)}$. For these reasons, gradient Ricci solitons have been extensively investigated in the literature – see, for example, the discussion in [6, 10, 13, 23] and the references therein. If $\lambda > 0$ (respectively, $\lambda = 0$ or $\lambda < 0$), then (M, g, f) is said to be *shrinking* (respectively, *steady* or *expanding*). We shall assume for the most part that (M, g) is locally homogeneous. This implies that the scalar curvature is constant.

One has canonical examples that play a central role in the theory. Let (N, g_N) be an Einstein manifold with *Einstein constant* λ , i.e. $\rho_N = \lambda g_N$. Let $M = N \times \mathbb{R}^k$ have the product metric g_M and let $f(x) := \frac{1}{2}\lambda\|\pi(x)\|^2$, where π is projection on the second factor. Then (M, g_M, f) is a gradient Ricci soliton and is said to be *rigid*. Since we are interested in questions of local geometry, by an abuse of notation we shall also say that (M, g_M, f) is *rigid* if (M, g_M, f) is isomorphic to an open subset of a product $N \times \mathbb{R}^k$ that is rigid. We shall use the following results of Petersen and Wylie [24]. Assertion (2) was first proved in the Riemannian setting but extends easily to arbitrary signature.

THEOREM 1.1 (Petersen and Wylie [24]).

- (1) *Any locally homogeneous Riemannian gradient Ricci soliton is rigid.*
- (2) *Let $(M, g) = (M_1 \times M_2, g_1 \oplus g_2)$ be the direct product of two pseudo-Riemannian manifolds. If f satisfies the gradient Ricci soliton equation on (M, g) , then $f(x_1 + x_2) = f_1(x_1) + f_2(x_2)$, where f_1 and f_2 satisfy the gradient Ricci soliton equation on (M_1, g_1) and on (M_2, g_2) separately.*

Assertion (1) was originally proven for homogeneous manifolds, but the assumption of homogeneity can be weakened to local homogeneity by modifying the argument in [24, proposition 1], as in the proof of lemma 1.2(2)(c). Since any locally homogeneous Riemannian gradient Ricci soliton is rigid, the classification is complete in this context. However the possible geometries are much richer in the Lorentzian setting owing to the existence of degenerate parallel line fields. For example, in example 4.1 we present results of [1] showing that Cahen–Wallach symmetric spaces admit steady non-rigid gradient Ricci solitons.

1.1. Outline of the paper and summary of results

In § 1.2 we state lemma 1.2. This lemma, which will be proved in § 2, summarizes the relevant results we shall need concerning gradient Ricci solitons with constant scalar curvature; many of these results rely upon earlier papers. The analysis there will be local in nature and will rely on the investigation of the gradient Ricci soliton equation (1.1) as this links the geometry of the manifold, through its Ricci curvature, with the extrinsic geometry of the level sets of the potential function by

means of their second fundamental form. The signature of the manifold plays no role in lemma 1.2 and is completely general. We shall see that if the scalar curvature is constant, then any solution of (1.1) is an isoparametric function, i.e.

$$\|\nabla f\|^2 = b(f) \quad \text{and} \quad \Delta f = a(f) \quad \text{for } a, b \text{ smooth on Range}(f).$$

For the remainder of the paper we shall assume (unless otherwise noted) that the underlying manifold (M, g) is a locally homogeneous Lorentzian manifold and that (M, g, f) is a gradient Ricci soliton. In § 1.3 we present our results in theorems 1.3–1.5 concerning non-steady solitons ($\lambda \neq 0$); these results will be proved in § 3. In low dimensions, such solitons are rigid; in arbitrary dimensions, the eigenvalue structure of the Ricci operator agrees with the corresponding eigenvalue structure of a rigid soliton, i.e. there are only two eigenvalues $\{0, \lambda\}$. In § 1.4 we present our results concerning steady solitons ($\lambda = 0$) in theorems 1.8–1.9; these will be proved in § 4. Theorem 1.8 gives a complete classification if $\|\nabla f\|^2 < 0$. In theorem 1.9 we examine the situation when $\|\nabla f\|^2 = 0$ and show that the Ricci tensor is either two- or three-step nilpotent; the metrics in question are pure radiation metrics with parallel rays [21]. If we further restrict the geometry, stronger results are available. In § 1.5 we give a complete classification of symmetric Lorentzian gradient Ricci solitons in theorem 1.12. This result is proved in § 5. In theorem 1.16 of § 1.6 we give a complete classification of three-dimensional Lorentzian locally homogeneous gradient Ricci solitons; there are three non-trivial families of examples. Theorem 1.16 will be proved in § 6.

The fact that (M, g) is Lorentzian plays a crucial role in many arguments. For example, when we study the non-steady case, there exists a distinguished null parallel vector field and there do not exist orthogonal null vector fields – this is a Lorentzian phenomenon not present in the Riemannian or the higher signature setting. The fact that (M, g) is locally homogeneous is not simply used to ensure that the scalar curvature is constant, it plays a role in many proofs where we take frame fields consisting at least in part of Killing vector fields. As our discussion is local in nature, it is not necessary to impose global conditions such as global homogeneity or completeness.

1.2. Consequences of the gradient Ricci soliton equation

Let τ be the scalar curvature. Let ∇f be the vector field that is dual to the 1-form df . It is characterized by the identity

$$g(\nabla f, X) = X(f) \quad \text{for any vector field } X. \tag{1.2}$$

Let \mathcal{L} be the Lie derivative; a vector field X on (M, g) is Killing if $\mathcal{L}_X g = 0$; X is Killing if and only if

$$g(\nabla_X Z, X) = 0 \quad \text{for any vector field } X. \tag{1.3}$$

We say that (M, g, f) is isotropic if $\|\nabla f\|^2 = 0$. The proof of the following quite general result concerning gradient Ricci solitons with constant scalar curvature in arbitrary signature is given in § 2.

LEMMA 1.2. *Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature.*

- (1) *We have the following relations:*
 - (a) $\text{Ric}(\nabla f) = 0$;
 - (b) $\|\nabla f\|^2 - 2\lambda f = \text{const.}$;
 - (c) $R(X, Y, Z, \nabla f) = (\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z)$;
 - (d) $(\nabla_{\nabla f} \text{Ric}) + \text{Ric} \circ \mathcal{H}_f = R(\nabla f, \cdot) \nabla f$.
- (2) *Let X be a Killing vector field. Then*
 - (a) $\mathcal{L}_X(\text{Hess}_f) = \text{Hess}_{X(f)}$;
 - (b) $\text{grad}\{X(f)\}$ *is a parallel vector field*;
 - (c) *if $\lambda \neq 0$, then $\text{grad}\{X(f)\} = 0$ if and only if $X(f) = 0$.*
- (3) *We have $\lambda((n+2)\lambda - \tau) = \|\text{Hess}_f\|^2$.*
- (4) *If (M, g, f) is isotropic and non-steady, then (M, g) is Einstein.*
- (5) *If (M, g, f) is steady, then $\|\text{Hess}_f\|^2 = 0$ and $\|\nabla f\|^2 = \mu$ is constant.*

In what follows we shall apply different techniques to study the steady and the non-steady cases since setting $\lambda \neq 0$ or $\lambda = 0$ in lemma 1.2 gives significantly different information about the potential function f . By lemma 1.2, any isotropic non-steady gradient Ricci soliton with constant scalar curvature is Einstein. However, there exist isotropic steady gradient Ricci solitons that are not Einstein [1].

1.3. Non-steady locally homogeneous Lorentzian gradient Ricci solitons

We say that a Lorentzian manifold (M, g) is *irreducible* if the holonomy representation has no non-trivial invariant subspace. We say that (M, g) is *indecomposable* if the metric on any non-trivial subspace fixed by the holonomy representation is degenerate, and thus the holonomy representation does not decompose as a non-trivial direct sum of subrepresentations. The distinction between irreducible and indecomposable is only relevant in the indefinite setting. We shall establish the following results in §3.

THEOREM 1.3. *Let (M, g, f) be a locally homogeneous Lorentzian non-steady gradient Ricci soliton. Then one of the following holds.*

- (1) *(M, g) is irreducible and Einstein.*
- (2) *(M, g, f) is rigid, that is, there is a local splitting*

$$(M, g, f) = (N \times \mathbb{R}_\nu^k, g_N + g_e, f_N + f_e),$$

where (N, g_N) is Einstein with Einstein constant λ and $(\mathbb{R}_\nu^k, g_e, f_e)$ is pseudo-Euclidean space, $\nu = 0, 1$, with $f_e(x) := \frac{1}{2}\lambda\|x\|^2$.

(3) (M, g, f) locally splits as

$$(M, g, f) = (N_0 \times N_1 \times \mathbb{R}^k, g_0 + g_1 + g_e, f_0 + f_1 + f_e),$$

where (N_0, g_0, f_0) is an indecomposable locally homogeneous Lorentzian gradient Ricci soliton, (N_1, g_1) is a Riemannian Einstein manifold with Einstein constant λ and (\mathbb{R}^k, g_e, f_e) is Euclidean space with $f_e(x) := \frac{1}{2}\lambda\|x\|^2$.

We now focus on the situation in assertion (3) above and study the indecomposable factor. Recall that a Lorentzian manifold is said to be *Walker* if it admits a parallel null line field, and *strict Walker* if this distribution is spanned by a parallel null vector field; we refer the reader to [4] for further details. We shall say that (M, g) has *harmonic Weyl tensor* if the *Schouten tensor* S is *Codazzi*. This means (see [3]) that

$$\nabla_X S_{YZ} = \nabla_Y S_{XZ}, \quad \text{where } S = \rho - \frac{\tau}{2(n+1)}g.$$

THEOREM 1.4. *Let (M, g, f) be a locally homogeneous indecomposable Lorentzian non-steady gradient Ricci soliton that is not Einstein.*

- (1) *Locally, there exists a Killing vector field X so $U := \text{grad}\{X(f)\}$ is a non-trivial parallel null vector field; thus, (M, g) is strict Walker.*
- (2) *U is unique up to scale, $\mathcal{V} := \{U, \nabla f\} \subset \ker\{\text{Ric}\}$ is a U -parallel Lorentzian distribution, and $\text{grad}\{U(f)\} = \lambda U$.*
- (3) *$\nabla_U \text{Ric} = \nabla_U \mathcal{H}_f = 0$, $\text{Spec}\{\text{Ric}\} = \text{Spec}\{\mathcal{H}_f\} = \{0, \lambda\}$, Ric and \mathcal{H}_f are diagonalizable, $\ker\{\text{Ric}\} = \text{Image}\{\mathcal{H}_f\}$, and $\ker\{\mathcal{H}_f\} = \text{Image}\{\text{Ric}\}$.*
- (4) *The Weyl tensor of (M, g) is harmonic if and only if (M, g, f) is rigid.*
- (5) *If $\dim(\ker\{\text{Ric}\}) = 2$, then (M, g, f) is rigid.*

This leads to the following classification result in low dimensions.

THEOREM 1.5. *Let (M, g, f) be a locally homogeneous Lorentzian non-steady gradient Ricci soliton of dimension $m \leq 4$. Then (M, g, f) is rigid.*

REMARK 1.6. What is indeed proven in theorem 1.5 is that if the factor N_0 of the decomposition given in theorem 1.3 is of dimension $n_0 \leq 4$, then the gradient Ricci soliton is rigid.

1.4. Steady locally homogeneous Lorentzian gradient Ricci solitons

The geometry of the level sets of the potential function plays an essential role in our analysis; the norm $\|\nabla f\|^2$ is important as this controls the nature of the metric on the level sets. The two-dimensional case is trivial; see [6, 14].

THEOREM 1.7. *A steady locally homogeneous Ricci soliton of dimension 2 either in the Riemannian or in the Lorentzian setting is flat.*

The following two results will be established in § 4.

THEOREM 1.8. *Let (M, g, f) be a locally homogeneous steady gradient Lorentzian Ricci soliton. If $\|\nabla f\|^2 < 0$, then (M, g) splits locally as an isometric product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a flat Riemannian manifold and f is the orthogonal projection on \mathbb{R} .*

The cases in which $\|\nabla f\|^2 \geq 0$ are less rigid in the steady setting. Several examples in the spacelike case $\|\nabla f\|^2 > 0$ are known [1, 6], but little more of a general nature is known about this case. In the isotropic case one has some restrictions on the Ricci operator; in particular, it must be nilpotent. Recall that a tensor T is said to be *recurrent* if there is a smooth 1-form ω such that $\nabla_X T = \omega(X)T$.

THEOREM 1.9. *Let (M, g, f) be an isotropic locally homogeneous Lorentzian steady gradient Ricci soliton. One of the following two possibilities pertains.*

- (1) $\mathcal{H}_f = -\text{Ric}$ has rank 2 and is three-step nilpotent.
- (2) $\mathcal{H}_f = -\text{Ric}$ has rank 1 and is two-step nilpotent. In this case (M, g) is locally a strict Walker manifold and, more specifically, the following hold.
 - (a) $\ker\{\mathcal{H}_f\} = \nabla f^\perp$ and $\text{Image}\{\mathcal{H}_f\} = \nabla f$.
 - (b) ∇f is a recurrent vector field and ∇f^\perp is an integrable totally geodesic distribution with leaves the level sets of f .
 - (c) Let $P \in M$. At least one of the following possibilities holds near P .
 - (i) There exists a Killing vector field F so $\text{grad}\{F(f)\}$ is a null parallel vector field.
 - (ii) There exists a smooth function ψ defined near P so $\psi\nabla f$ is a null parallel vector field.

We shall illustrate possibility (2) in example 4.1 presently.

1.5. Symmetric Lorentzian gradient Ricci solitons

Stronger results are available if (M, g) is *locally symmetric*; this implies that $\nabla R = 0$.

DEFINITION 1.10. We say that (N, g_N) is a *Cahen–Wallach symmetric space* if there are coordinates (t, y, x_1, \dots, x_n) such that

$$g = 2 dt dy + \left(\sum_{i=1}^n \kappa_i x_i^2 \right) dy^2 + \sum_{i=1}^n dx_i^2 \quad \text{for } 0 \neq \kappa_i \in \mathbb{R}. \quad (1.4)$$

We shall always assume that all $\kappa_i \neq 0$ to ensure that (N, g_N) is indecomposable.

For the proofs of assertions (1) and (2) in the following result, we respectively refer the reader to [7, 8] and [1].

THEOREM 1.11.

- (1) Let (M, g) be a Lorentzian locally symmetric space.
 - (a) If (M, g) is irreducible, then (M, g) has constant sectional curvature.

- (b) If (M, g) is indecomposable but reducible, then (M, g) is a Cahen–Wallach symmetric space.
- (2) If (M, g, f) is a Cahen–Wallach gradient Ricci soliton, then (M, g, f) is steady, $f = a_0 + a_1y + \frac{1}{4} \sum_i \kappa_i y^2$, and $\nabla f = (a_0 + \frac{1}{2} \sum_i \kappa_i y) \partial_t$ is null.

Theorem 1.11 will play a crucial role in the proof that we shall give of the following result in § 5.

THEOREM 1.12. *Let (M, g, f) be a locally symmetric Lorentzian gradient Ricci soliton. Then (M, g) splits locally as a product $M = N \times \mathbb{R}^k$, where*

- (1) if (M, g, f) is not steady, then (N, g_N) is Einstein and the soliton is rigid;
- (2) if (M, g, f) is steady, then (N, g_N, f_N) is locally isometric to a Cahen–Wallach symmetric space.

1.6. Three-dimensional locally homogeneous gradient Ricci solitons

We will establish the following two results in three-dimensional geometry in § 6. Let (M, g) be a Lorentzian manifold of dimension 3. We suppose first that (M, g) is strict Walker, i.e. admits a null parallel vector field. We may then (see, for example, [4]) find local adapted coordinates (t, x, y) such that

$$g = 2 dt dy + dx^2 + \phi(x, y) dy^2. \tag{1.5}$$

The following is of independent interest; we drop for the moment the assumption that the metric is locally homogeneous and focus on Walker geometry.

THEOREM 1.13. *Let (M, g) be a non-flat three-dimensional Lorentzian strict Walker manifold. Then (M, g, f) is a gradient Ricci soliton if and only if there exists a cover of M by coordinate systems where the metric has the form given in (1.5) and where one of the following occurs.*

- (1) We have

$$\phi(x, y) = \frac{1}{\alpha^2} a(y) e^{\alpha x} + xb(y) + c(y) \quad \text{and} \quad f(x, y) = x\alpha + \gamma(y),$$

where $\alpha \in \mathbb{R}$ and $\gamma''(y) = -\frac{1}{2} \alpha b(y)$. In this setting, $\nabla f = \alpha \partial_x + \gamma'(y) \partial_t$ is spacelike.

- (2) We have

$$\phi(x, y) = x^2 a(y) + xb(y) + c(y) \quad \text{and} \quad f(x, y) = \gamma(y),$$

where $\gamma''(y) = \frac{1}{4} a(y)$. In this setting $\nabla f = \gamma' \partial_t$ is null.

Moreover, in both cases the Ricci soliton is steady.

DEFINITION 1.14. Adopt the notation of (1.5).

- (1) Let $\phi(x, y) = b^{-2} e^{bx}$ for $0 \neq b \in \mathbb{R}$ define \mathcal{N}_b .
- (2) Let $\phi(x, y) = \frac{1}{2} x^2 \alpha(y)$, where $\alpha(y) = c \alpha^{3/2}(y)$ and $\alpha(y) > 0$, define \mathcal{P}_c .
- (3) Let $\phi(x, y) = \pm x^2$ define the Cahen–Wallach symmetric space \mathcal{CW}_\pm .

The following result was established in [18].

THEOREM 1.15. *Let (M, g) be a locally homogeneous Lorentzian strict Walker manifold of dimension 3. Then (M, g) is locally isometric to one of the manifolds given in definition 1.14.*

We can now state our classification result.

THEOREM 1.16. *Let (M, g, f) be a Lorentzian locally homogeneous gradient Ricci soliton of dimension 3. If (M, g, f) is non-trivial, then either it is rigid or (M, g) is locally isometric to either CW_{\pm} , \mathcal{P}_c or \mathcal{N}_b , as defined above, and the soliton is steady. Moreover, ∇f is null if $(M, g) = \mathcal{P}_c$ or if $(M, g) = CW_{\pm}$, and ∇f is spacelike if $(M, g) = \mathcal{N}_b$.*

**2. Consequences of the gradient Ricci soliton equation:
the proof of lemma 1.2**

The proof of lemma 1.2(1). If (M, g, f) is a gradient Ricci soliton, then $\nabla\tau = 2 \operatorname{Ric}(\nabla f)$ [15,25]. Assertion (1)(a) now follows as $\nabla\tau = 0$. We also have [6,13,15,25] that $\tau + \|\nabla f\|^2 - 2\lambda f = \text{const.}$; assertion (1)(b) now follows. We refer the reader to [6, 16] for the proof of assertion (1)(c), which holds without assuming that $\tau = \text{const.}$ The identity

$$(\nabla_{\nabla f} \operatorname{Ric}) + \operatorname{Ric} \circ \mathcal{H}_f = R(\nabla f, \cdot)\nabla f + \frac{1}{2}\nabla\nabla\tau$$

was proved in the Riemannian setting in [25]. One can use analytic continuation to extend this identity to the indefinite setting (or simply observe that the proof goes through without change in the higher signature context). Assertion (1)(d) now follows once again using the fact that τ is constant. □

The proof of lemma 1.2(2). Let X be a Killing vector field. Fix a point P of M so that $X(P) \neq 0$; assertion (2) for P where $X(P) = 0$ will then follow by continuity. Choose a system of local coordinates (x_1, \dots, x_{n+2}) so that $X = \partial_{x_1}$. Set $g_{ij} := g(\partial_{x_i}, \partial_{x_j})$ and observe that

$$\begin{aligned} \partial_{x_1} g_{ij} &= g(\nabla_{\partial_{x_1}} \partial_{x_i}, \partial_{x_j}) + g(\partial_{x_i}, \nabla_{\partial_{x_1}} \partial_{x_j}) = g(\nabla_{\partial_{x_i}} \partial_{x_1}, \partial_{x_j}) + g(\partial_{x_i}, \nabla_{\partial_{x_j}} \partial_{x_1}) \\ &= (\mathcal{L}_{\partial_{x_1}} g)(\partial_{x_i}, \partial_{x_j}). \end{aligned}$$

Thus, $\partial_{x_1} g_{ij} = 0$, so $\partial_{x_1} \Gamma_{ij}^k = 0$ as well. We establish assertion (2)(a) by computing as follows:

$$\begin{aligned} (\mathcal{L}_{\partial_{x_1}} \operatorname{Hess}_f)(\partial_{x_i}, \partial_{x_j}) &= \mathcal{L}_{\partial_{x_1}} \operatorname{Hess}_f(\partial_{x_i}, \partial_{x_j}) \\ &= \mathcal{L}_{\partial_{x_1}} (\partial_{x_i x_j}^2(f) - \Gamma_{ij}^k \partial_{x_k}(f)) \\ &= \partial_{x_1 x_i x_j}^3(f) - \partial_{x_1} (\Gamma_{ij}^k) \partial_{x_k}(f) - \Gamma_{ij}^k \partial_{x_1 x_k}^2(f) \\ &= \partial_{x_i x_j}^2 \partial_{x_1}(f) - \Gamma_{ij}^k \partial_{x_k} \partial_{x_1}(f) \\ &= \operatorname{Hess}_{\partial_{x_1}}(f)(\partial_{x_i}, \partial_{x_j}). \end{aligned}$$

Since $\mathcal{L}_X g = 0$ and since ρ is natural, $\mathcal{L}_X \rho = 0$. Equation (1.1) implies that $\mathcal{L}_X \operatorname{Hess}_f = 0$, and therefore, by assertion (2)(a), $\operatorname{Hess}_{X(f)} = 0$. Consequently,

$\text{grad}\{X(f)\}$ is parallel. This establishes assertion (2)(b). Assume now that $\lambda \neq 0$. It is clear that $\text{grad}\{X(f)\} = 0$ if $X(f) = 0$. Conversely, if $\text{grad}\{X(f)\} = 0$, then $X(f) = \kappa$ for some constant κ . Since the scalar curvature is constant, assertion (1) implies that $\text{Ric}(\nabla f) = 0$. Since X is a Killing vector field,

$$\begin{aligned} 0 = \nabla f(\kappa) &= \nabla f(X(f)) = \nabla f g(\nabla f, X) = g(\nabla_{\nabla f} \nabla f, X) + g(\nabla f, \nabla_{\nabla f} X) \\ &= \text{Hess}_f(\nabla f, X) + \frac{1}{2}(\mathcal{L}_X g)(\nabla f, \nabla f) \\ &= -\rho(\nabla f, X) + \lambda g(\nabla f, X) \\ &= \lambda \kappa. \end{aligned}$$

Thus, $\kappa = 0$. Consequently, $\text{grad}\{X(f)\} = 0$ if and only if $X(f) = 0$. This establishes assertion (2)(c). □

The proof of lemma 1.2(3). We have the Bochner identity:

$$\frac{1}{2}\Delta g(\nabla f, \nabla f) = \|\text{Hess}_f\|^2 + \rho(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f). \tag{2.1}$$

By assertion (1), $\text{Ric}(\nabla f) = 0$ and $\|\nabla f\|^2 - 2\lambda f = \text{const}$. Thus, the left-hand side of (2.1) becomes $\frac{1}{2}\Delta g(\nabla f, \nabla f) = \lambda \Delta f - \frac{1}{2}\Delta \tau$. Taking the trace in (1.1) shows that $\Delta f = (n + 2)\lambda - \tau$, and hence $\frac{1}{2}\Delta g(\nabla f, \nabla f) = \lambda((n + 2)\lambda - \tau)$. On the other hand, since $\text{Ric}(\nabla f) = 0$ and $\nabla \Delta f = -\nabla \tau = 0$, the right-hand side in the Bochner formula reduces to $\|\text{Hess}_f\|^2$. □

The proof of lemma 1.2(4). If $\|\nabla f\|^2 = 0$, we may apply assertion (1) to see that $2\lambda f = \text{const}$. Since $\lambda \neq 0$, f is constant and (M, g) is Einstein. □

The proof of lemma 1.2(5). If $\lambda = 0$, then $\|\text{Hess}_f\|^2 = 0$. By (1.1), $\mathcal{H}_f = -\text{Ric}$ and thus $\text{Ric}(\nabla f) = 0$ implies that $\mathcal{H}_f(\nabla f) = 0$. Consequently, ∇f is a geodesic vector field. Next, using the identity $\tau + \|\nabla f\|^2 - 2\lambda f = \text{const}$., one has that $\|\nabla f\|^2$ is constant and therefore f is a solution of the *Eikonal equation* $\|\nabla f\|^2 = \mu$. □

**3. Non-steady locally homogeneous gradient Ricci solitons:
the proof of theorems 1.3–1.5**

By lemma 1.2, isotropic non-steady locally homogeneous gradient Ricci solitons are Einstein. Consequently, we shall concentrate henceforth on the study of non-isotropic non-steady locally homogeneous gradient Ricci solitons. In § 3.1 we will prove theorem 1.3, in § 3.2 we will establish theorem 1.4, and in § 3.3 we will establish theorem 1.5. We shall use lemma 1.2 repeatedly and without further reference in what follows. Throughout § 3 we shall let (M, g, f) be a locally homogeneous non-steady gradient Ricci soliton.

3.1. The proof of theorem 1.3

Assume that (M, g) is irreducible or, equivalently, that there are no non-trivial parallel distributions on M . Consequently, any parallel vector field is trivial. Let X be a Killing vector field. Then $\text{grad}\{X(f)\}$ is a parallel vector field and thus $\text{grad}\{X(f)\} = 0$, so $X(f)$ is constant and hence $X(f) = 0$. Since the underlying

Lorentzian structure (M, g) is locally homogeneous, there are $(n + 2)$ linearly independent Killing vector fields X_1, \dots, X_{n+2} locally. Consequently, f is constant and the metric is Einstein. This establishes assertion (1) of theorem 1.3.

We now apply the local splitting result of assertion (2) in theorem 1.1. Let X be a Killing vector field on (M, g) . If $\text{grad}\{X(f)\}$ is spacelike or timelike, then we may split, at least locally, a one-dimensional factor from (M, g) and decompose locally

$$(M, g, f) = (N \times \mathbb{R}, g_N \oplus g_e, f_N + f_e).$$

If $\text{grad}\{X(f)\}$ is timelike, then (N, g_N) is Riemannian and, by assertion (1) of theorem 1.1, rigid, which would finish the discussion. Thus, we may assume that (N, g_N) is Lorentzian, so $\text{grad}\{X(f)\}$ is spacelike and the factor (\mathbb{R}, g_e) is positive definite. We proceed inductively to decompose $(M, g, f) = (N \times \mathbb{R}^k, g_N \oplus g_e, f_N + f_e)$ (at least locally) so that (N, g_N, f_N) is a locally homogeneous Lorentzian Ricci soliton with $\text{grad}\{X(f)\}$ null or zero for all Killing vector fields X . Now two possibilities may occur. If N is indecomposable, assertion (3) follows with trivial N_1 . If N is decomposable, then either N is Einstein and assertion (2) holds (this is the case if $\text{grad}\{X(f)\} = 0$ for all Killing vector fields in N) or N decomposes as $N = N_0 \times N_1$, where N_0 is Lorentzian and indecomposable (the latter happens if there exists a Killing vector field X so that $\text{grad}\{X(f)\}$ is null). (N_1, g_1, f_1) is a Riemannian locally homogeneous gradient Ricci soliton that, as a consequence of theorem 1.1, is Einstein. This establishes theorem 1.3. \square

3.2. The proof of theorem 1.4

We establish assertions (1)–(5) of theorem 1.4 *seriatim*. We suppose that (M, g) is not decomposable and is not Einstein.

The proof of theorem 1.4(1). We must show that there exists an X such that $U = \text{grad}\{X(f)\}$ is a parallel null vector field. Let Z be any Killing vector field. Since (M, g) is not decomposable and since $\text{grad}\{Z(f)\}$ is parallel, $\text{grad}\{Z(f)\}$ must be isotropic. If $\text{grad}\{Z(f)\}$ vanishes for all such Z , then f is constant and hence (M, g) is Einstein, which is contrary to our assumption. Thus, $U := \text{grad}\{Z(f)\}$ has the desired properties for some Killing vector field Z . \square

The proof of theorem 1.4(2). We must show that U is unique up to scale, that $U \in \ker\{\text{Ric}\}$, and that $\text{grad}\{U(f)\} = \lambda U$. Suppose that there are two Killing vector fields Z_1 and Z_2 on (M, g) such that $\text{grad}\{Z_1(f)\}$ and $\text{grad}\{Z_2(f)\}$ are linearly independent. Since the signature is Lorentzian, $\text{Span}\{\text{grad}\{Z_1(f)\}, \text{grad}\{Z_2(f)\}\}$ cannot be a null distribution. Consequently, there exists a linear combination $Z = a_1 Z_1 + a_2 Z_2$ such that $\text{grad}\{Z(f)\}$ is either timelike or spacelike. This implies that (M, g) is decomposable, which is false. Thus, the vector field $U = \text{grad}\{Z(f)\}$ is unique up to scale.

Since U is parallel, it is Killing and hence $\text{grad}\{U(f)\} = \alpha U$ for some $\alpha \in \mathbb{R}$. We must now show that $\text{Ric}(U) = 0$. Let $\{Z_1, Z_2, \dots, Z_{n+2}\}$ be a local basis of Killing vector fields. Choose the notation so $Z = Z_1$. We then have $\text{grad}\{Z_i(f)\} = \mu_i U$ for $i \geq 2$. Since $\text{grad}\{Z_i(f)\}$ is parallel, necessarily μ_i is constant. By replacing Z_i by $Z_i - \mu_i Z_1$, we may assume therefore that $\text{grad}\{Z_i(f)\} = 0$ for $i \geq 2$. Since $\lambda \neq 0$,

lemma 1.2 implies that $Z_i(f) = 0$ for $i \geq 2$. We use (1.1) and (1.2) to see that

$$\begin{aligned} g(U, \nabla f) &= g(\text{grad}\{Z_1(f)\}, \nabla f) = g(\text{grad}\{g(Z_1, \nabla f)\}, \nabla f) \\ &= \nabla f g(Z_1, \nabla f) = g(\nabla_{\nabla f} Z_1, \nabla f) + g(Z_1, \nabla_{\nabla f} \nabla f) \\ &= \text{Hess}_f(Z_1, \nabla f) = \lambda g(Z_1, \nabla f) \\ &= \lambda Z_1(f) \neq 0, \end{aligned} \tag{3.1}$$

where, by (1.3), $g(\nabla_{\nabla f} Z_1, \nabla f) = 0$ since Z_1 is Killing. As $g(U, \nabla f) \neq 0$ and as U is a null vector, $\mathcal{V} := \text{Span}\{U, \nabla f\}$ has Lorentzian signature. We have that $\text{grad}\{U(f)\} \neq 0$ due to lemma 1.2, so $\alpha \neq 0$.

If X is an arbitrary vector field, we study $\mathcal{H}_f(U)$ by computing as follows:

$$\text{Hess}_f(X, U) = g(U, \nabla_X \nabla f) = Xg(U, \nabla f) = g(X, \text{grad}\{U(f)\}) = \alpha g(X, U).$$

This shows that $\mathcal{H}_f(U) = \alpha U$. Since $\mathcal{H}_f(\nabla f) = \lambda \nabla f$, we also have

$$\alpha g(\nabla f, U) = \text{Hess}_f(\nabla f, U) = \lambda g(\nabla f, U),$$

so $\alpha = \lambda$. By (1.1), $\text{Ric}(U) = 0$. Since $\nabla_U U = 0$ and $\nabla_U \nabla f = \lambda U$, ∇_U preserves $\mathcal{V} \subset \ker\{\text{Ric}\}$. This proves assertion (2). \square

The proof of theorem 1.4(3). We have shown that $\mathcal{V} := \text{Span}\{U, V\} \subset \ker\{\text{Ric}\}$ is a U -parallel Lorentzian distribution. Consequently, \mathcal{V}^\perp is a Ric invariant distribution with a positive definite signature. Since Ric is self-adjoint, there exists an orthonormal basis $\{E_1, \dots, E_n\}$ of \mathcal{V}^\perp so $\text{Ric}(E_i) = \alpha_i E_i$; the α_i are constant since (M, g) is locally homogeneous. This proves in particular that Ric and $\mathcal{H}_f = \lambda \text{Id} - \text{Ric}$ are diagonalizable. We now show that ∇_U preserves the eigenspaces in \mathcal{V}^\perp . For $i \neq j$, since U is parallel, $R(U, E_i, E_j, \nabla f) = 0$. By lemma 1.2(1),

$$\begin{aligned} 0 &= R(U, E_i, E_j, \nabla f) = (\nabla_U \rho)(E_i, E_j) - (\nabla_{E_i} \rho)(U, E_j) \\ &= U\rho(E_i, E_j) - \rho(\nabla_U E_i, E_j) - \rho(E_i, \nabla_U E_j) \\ &\quad - E_i \rho(U, E_j) + \rho(\nabla_{E_i} U, E_j) + \rho(U, \nabla_{E_i} E_j) \\ &= -\alpha_j g(\nabla_U E_i, E_j) - \alpha_i g(E_i, \nabla_U E_j) \\ &= (\alpha_i - \alpha_j) g(\nabla_U E_i, E_j). \end{aligned}$$

We conclude that if E_i and E_j belong to different eigenspaces, $\nabla_U E_i$ is orthogonal to E_j . Hence, ∇_U commutes with Ric and, as a consequence of the Ricci soliton equation (1.1), it also commutes with \mathcal{H}_f . Consequently, as desired, $\nabla_U \text{Ric} = 0$ and $\nabla_U \mathcal{H}_f = 0$.

We must show that 0 and λ are the only eigenvalues of Ric. Normalize V to be a multiple of ∇f so that $g(V, V) = \varepsilon = \pm 1$. Let S be any level set of f . The integral curves of U are transversal to S because $g(U, \nabla f) \neq 0$. Use parallel transport along the integral curves of U to extend the local frame $\{E_1, \dots, E_n\}$ from S to a neighbourhood of S to define a local frame field $\{F_1, \dots, F_n\}$ for \mathcal{V}^\perp such that $\nabla_U F_i = 0$. Since $\nabla_U \text{Ric} = 0$, the vector fields F_i are still eigenvectors of the Ricci operator Ric. We shall use this local frame field to see that Ric has only two

eigenvalues $\{0, \lambda\}$. First note that

$$\begin{aligned} (\nabla_{\nabla f} \rho)(F_i, F_i) &= \nabla f \rho(F_i, F_i) - 2\rho(\nabla_{\nabla f} F_i, F_i) \\ &= \alpha_i \nabla f g(F_i, F_i) - 2\alpha_i g(\nabla_{\nabla f} F_i, F_i) \\ &= \alpha_i (\nabla_{\nabla f} g)(F_i, F_i) \\ &= 0. \end{aligned}$$

We use lemma 1.2 to compute as follows:

$$\begin{aligned} \rho(F_i, F_i) &= \varepsilon R(F_i, V, F_i, V) + \sum_{j \neq i} R(F_i, F_j, F_i, V) g(F_j, V) + \sum_{j \neq i} R(F_i, F_j, F_i, F_j) \\ &= \frac{\varepsilon}{\|\nabla f\|^2} ((\nabla_{F_i} \rho)(\nabla f, F_i) - (\nabla_{\nabla f} \rho)(F_i, F_i)) \\ &\quad + \sum_{j \neq i} R(F_i, F_j, F_i, V) g(F_j, V) + \sum_{j \neq i} R(F_i, F_j, F_i, F_j) \\ &= \frac{\varepsilon}{\|\nabla f\|^2} (F_i \rho(\nabla f, F_i) - \rho(\nabla_{F_i} \nabla f, F_i) - \rho(\nabla f, \nabla_{F_i} F_i)) \\ &\quad + \sum_{j \neq i} R(F_i, F_j, F_i, V) g(F_j, V) + \sum_{j \neq i} R(F_i, F_j, F_i, F_j) \\ &= -\frac{\varepsilon}{\|\nabla f\|^2} \rho(\mathcal{H}_f F_i, F_i) + \sum_{j \neq i} R(F_i, F_j, F_i, V) g(F_j, V) \\ &\quad + \sum_{j \neq i} R(F_i, F_j, F_i, F_j). \end{aligned}$$

Since we have shown that $\nabla_U \rho = 0$, we have that $U\rho(F_i, F_i) = 2\rho(\nabla_U F_i, F_i)$, which vanishes. We now differentiate the three summands in the previous expression with respect to U :

$$\begin{aligned} &U\left(-\frac{1}{\|\nabla f\|^2} \rho(\mathcal{H}_f F_i, F_i)\right) \\ &= \frac{Ug(\nabla f, \nabla f)}{\|\nabla f\|^4} \rho(\mathcal{H}_f F_i, F_i) - \frac{1}{\|\nabla f\|} U\rho(\mathcal{H}_f F_i, F_i) \\ &= \frac{2\lambda g(U, \nabla f)}{\|\nabla f\|^4} \rho(\mathcal{H}_f F_i, F_i) - \frac{1}{\|\nabla f\|} (\rho(\nabla_U \mathcal{H}_f F_i, F_i) + \rho(\nabla_{F_i} \nabla f, \nabla_U F_i)) \\ &= \frac{2\lambda g(U, \nabla f)}{\|\nabla f\|^4} \rho(\mathcal{H}_f F_i, F_i) - \frac{1}{\|\nabla f\|} (\rho(\mathcal{H}_f(\nabla_U F_i), F_i) + \rho(\nabla_{F_i} \nabla f, \nabla_U F_i)) \\ &= \frac{2\lambda g(U, \nabla f)}{\|\nabla f\|^4} \alpha_i (\lambda - \alpha_i), \end{aligned}$$

$$\begin{aligned} &U(R(F_i, F_j, F_i, \nabla f)g(F_j, \nabla f)) \\ &= \{(\nabla_U R)(F_i, F_j, F_i, \nabla f) + R(\nabla_U F_i, F_j, F_i, \nabla f) \\ &\quad + R(F_i, \nabla_U F_j, F_i, \nabla f) + R(F_i, F_j, \nabla_U F_i, \nabla f) \\ &\quad\quad\quad + R(F_i, F_j, F_i, \nabla_U \nabla f)\}g(F_j, \nabla f) \\ &\quad + R(F_i, F_j, F_i, \nabla f)(g(\nabla_U F_j, \nabla f) + g(F_j, \nabla_U \nabla f)) \end{aligned}$$

$$\begin{aligned}
 &= \{-(\nabla_{F_i}R)(F_j, U, F_i, \nabla f) - (\nabla_{F_j}R)(U, F_i, F_i, \nabla f) \\
 &\quad + R(\nabla_U F_i, F_j, F_i, \nabla f) + R(F_i, \nabla_U F_j, F_i, \nabla f) \\
 &\quad\quad + R(F_i, F_j, \nabla_U F_i, \nabla f) + R(F_i, F_j, F_i, \lambda U)\}g(F_j, \nabla f) \\
 &\quad + R(F_i, F_j, F_i, \nabla f)(g(\nabla_U F_j, \nabla f) + \lambda g(F_j, U)) \\
 &= \{R(\nabla_U F_i, F_j, F_i, \nabla f) + R(F_i, \nabla_U F_j, F_i, \nabla f) \\
 &\quad\quad + R(F_i, F_j, \nabla_U F_i, \nabla f)\}g(F_j, \nabla f) \\
 &\quad + R(F_i, F_j, F_i, \nabla f)g(\nabla_U F_j, \nabla f) \\
 &= 0.
 \end{aligned}$$

Consequently, along the slice S we have

$$\begin{aligned}
 U(R(F_i, F_j, F_i, V)g(F_j, V)) &= U\|\nabla f\|^{-2}R(F_i, F_j, F_i, \nabla f)g(F_j, \nabla f) \\
 &\quad + \|\nabla f\|^{-2}U(R(F_i, F_j, F_i, \nabla f)g(F_j, \nabla f)) \\
 &= 0, \\
 UR(F_i, F_j, F_i, F_j) &= (\nabla_U R)(F_i, F_j, F_i, F_j) + 2R(\nabla_U F_i, F_j, F_i, F_j) \\
 &\quad + 2R(F_i, \nabla_U F_j, F_i, F_j) \\
 &= -(\nabla_{F_i}R)(F_j, U, F_i, F_j) - (\nabla_{F_j}R)(U, F_i, F_i, F_j) \\
 &\quad + 2R(\nabla_U F_i, F_j, F_i, F_j) + 2R(F_i, \nabla_U F_j, F_i, F_j) \\
 &= 2R(\nabla_U F_i, F_j, F_i, F_j) + 2R(F_i, \nabla_U F_j, F_i, F_j) \\
 &= 0.
 \end{aligned}$$

Hence, the following equation holds:

$$0 = 2\lambda g(U, \nabla f)\|\nabla f\|^{-4}\alpha_i(\lambda - \alpha_i).$$

Since λ and $g(U, \nabla f)$ are different from 0, either $\alpha_i = 0$ or $\alpha_i = \lambda$ for $i = 1, \dots, n$. Since the level set S of f that was chosen was arbitrary, this is true on all of M . By (1.1) we have $\mathcal{H}_f + \text{Ric} = \lambda \text{Id}$. The remaining conclusions of assertion (3) are now immediate from the discussion above. \square

The proof of theorem 1.4(4). Recall that (M, g) has a harmonic Weyl tensor if its Schouten tensor $S = \rho - (\tau/2(n + 1))g$ is Codazzi, i.e. $\nabla_X S_{YZ} = \nabla_Y S_{XZ}$ (see [3]). If the Weyl tensor is harmonic, then $(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) = 0$ since the scalar curvature is constant. Choose $E_1, E_2 \in \text{Image}\{\mathcal{H}_f\}$ and $F \in \text{Image}\{\text{Ric}\}$. We use assertion (3) to compute

$$0 = (\nabla_{E_1} \rho)(F, E_2) - (\nabla_F \rho)(E_1, E_2) = \rho(F, \nabla_{E_1} E_2) = \lambda g(F, \nabla_{E_1} E_2).$$

Choose $E \in \text{Image}\{\mathcal{H}_f\}$ and $F_1, F_2 \in \text{Image}\{\text{Ric}\}$. We show that the two eigenspaces are parallel and that the soliton is rigid by computing as follows:

$$\begin{aligned}
 0 &= (\nabla_{F_1} \rho)(E, F_2) - (\nabla_E \rho)(F_1, F_2) \\
 &= \rho(\nabla_{F_1} E, F_2) - E\rho(F_1, F_2) + \rho(\nabla_E F_1, F_2) + \rho(F_1, \nabla_E F_2) \\
 &= \lambda g(\nabla_{F_1} E, F_2) - \lambda E g(F_1, F_2) + \lambda g(\nabla_E F_1, F_2) + \lambda g(F_1, \nabla_E F_2) \\
 &= \lambda g(\nabla_{F_1} E, F_2).
 \end{aligned}$$

\square

The proof of theorem 1.4(5). We apply theorem 1.3. If $\dim(\ker\{\text{Ric}\}) = 2$, then $\mathcal{V} = \ker\{\text{Ric}\}$. Since U is parallel, we have that $\mathcal{H}_f(X) = \nabla_X \nabla f = \lambda X$ if $X \in \mathcal{V}$ and that $\mathcal{H}_f(X) = \nabla_X \nabla f = 0$ if $X \in \ker\{\mathcal{H}_f\} = \text{Image}\{\text{Ric}\}$. Consequently, the distribution \mathcal{V} is parallel. Since the metric is not degenerate on \mathcal{V} , this implies that the manifold locally decomposes as a product $B \times F$ so that B is Ricci flat and hence flat. On the other hand, F is Einstein satisfying $\rho^F = \lambda g^F$. Therefore, the soliton is rigid. This completes the proof of theorem 1.4. \square

3.3. The proof of theorem 1.5

If $\dim(M) = 3$, the result follows from the discussion above since $\dim(\ker\{\text{Ric}\}) = 2$. Assume that $\dim(M) = 4$ henceforth. Using the previous discussion, we need only examine the case in which $\dim(\ker\{\text{Ric}\}) = 3$. We are going to use theorem 1.4 to show that $\text{Image}\{\text{Ric}\}$ is a non-null parallel distribution. We consider the adapted basis $\{U, \nabla f, E, F\}$, where $\{U, \nabla f, E\}$ is a basis of $\ker\{\text{Ric}\}$ and $F \cdot \mathbb{R} = \text{Image}\{\text{Ric}\}$. We show that the Weyl tensor is harmonic and (M, g, f) is rigid by examining the components of the curvature tensor that have ∇f as an argument:

$$\begin{aligned} R(E, \nabla f, E, \nabla f) &= (\nabla_E \rho)(\nabla f, E) - (\nabla_{\nabla f} \rho)(E, E) = 0, \\ R(F, \nabla f, F, \nabla f) &= (\nabla_F \rho)(\nabla f, F) - (\nabla_{\nabla f} \rho)(F, F) = 0, \\ R(F, \nabla f, E, \nabla f) &= \rho(F, E) \|\nabla f\|^2 = 0, \\ R(F, E, F, \nabla f) &= \rho(\nabla f, E) = 0, \\ R(E, F, E, \nabla f) &= \rho(\nabla f, F) = 0. \end{aligned}$$

\square

4. Steady locally homogeneous Lorentzian gradient Ricci solitons: the proof of theorems 1.8 and 1.9

Again, we shall use lemma 1.2 throughout the section without further mention. Let (M, g, f) be a steady locally homogeneous Lorentzian gradient Ricci soliton. Then $\|\text{Hess}_f\|^2 = 0$ and $\|\nabla f\|^2 = \mu$ is constant. In what follows we will consider the possibilities $\mu < 0$ and $\mu = 0$ separately.

4.1. The proof of theorem 1.8

Assume that $\mu < 0$. As $\mathcal{H}_f(\nabla f) = 0$, we may restrict \mathcal{H}_f to ∇f^\perp . As ∇f^\perp inherits a positive definite metric and since $\|\text{Hess}_f\|^2 = 0$, $\mathcal{H}_f = 0$. This shows that ∇f is a parallel vector field, and thus (M, g) is locally a product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a locally homogeneous Riemannian manifold (see, for example, [17]). Additionally, (N, g_N) is a steady gradient Ricci soliton, and therefore Ricci flat. Following [26], locally homogeneous Ricci flat Riemannian manifolds are locally isometric to Euclidean space. This completes the proof of theorem 1.8. \square

4.2. The proof of theorem 1.9(1)

Assume that $\|\nabla f\|^2 = 0$, so ∇f is a null vector. Choose an orthonormal basis $\{E_1, \dots, E_{n+2}\}$ for the tangent space at a point so that E_1 is timelike, so that $\{E_2, \dots, E_{n+2}\}$ are spacelike, and so that $\nabla f = c(E_1 + E_2)$ for some $c \neq 0$. We further normalize the basis so that $\mathcal{H}_f E_1 \in \text{Span}\{E_1, E_2, E_3\}$. Let $\mathcal{H}_f E_i = \mathcal{H}_i^j E_j$.

Since $E_1 + E_2 \in \ker\{\mathcal{H}_f\}$, $\mathcal{H}_1^i + \mathcal{H}_2^i = 0$ for all i . Furthermore, $\mathcal{H}_1^i = \mathcal{H}_2^i = 0$ for $i \geq 4$ since $\mathcal{H}_f E_1 \in \text{Span}\{E_1, E_2, E_3\}$. Finally, since \mathcal{H}_f is self-adjoint, $\mathcal{H}_1^i = -\mathcal{H}_i^1$ for $2 \leq i$ and $\mathcal{H}_i^j = \mathcal{H}_j^i$ for $2 \leq i, j$. We summarize these relations as follows:

$$\left. \begin{aligned} \mathcal{H}_1^i &= -\mathcal{H}_i^1 \quad \text{for } i \geq 2, & \mathcal{H}_i^j &= \mathcal{H}_j^i \quad \text{for } 2 \leq i, j, \\ \mathcal{H}_1^i &= \mathcal{H}_2^i = 0 \quad \text{for } i \geq 4, & \mathcal{H}_1^i + \mathcal{H}_2^i &= 0 \quad \text{for all } i. \end{aligned} \right\} \tag{4.1}$$

Since $\mathcal{H}_f = \mathcal{H}_i^j E^i \otimes E_j$ and $\|\text{Hess}_f\|^2 = \lambda((n + 2)\lambda - \tau) = 0$, we have

$$0 = \|\text{Hess}_f\|^2 = \|\mathcal{H}_f\|^2 = (\mathcal{H}_1^1)^2 - 2 \sum_{i \geq 2} (\mathcal{H}_i^1)^2 + \sum_{2 \leq j, k} (\mathcal{H}_j^k)^2. \tag{4.2}$$

The relations of (4.1) then permit us to rewrite (4.2) in the form

$$0 = \sum_{3 \leq j, k} (\mathcal{H}_j^k)^2.$$

This implies that $\mathcal{H}_j^k = 0$ for $3 \leq j, k$ and thus, by (4.1), $\mathcal{H}_f E_i = 0$ for $i \geq 4$. Thus, the relevant portion of the matrix \mathcal{H} becomes

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_1^1 & \mathcal{H}_2^1 & \mathcal{H}_3^1 \\ \mathcal{H}_1^2 & \mathcal{H}_2^2 & \mathcal{H}_3^2 \\ \mathcal{H}_1^3 & \mathcal{H}_2^3 & \mathcal{H}_3^3 \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1^1 & -\mathcal{H}_1^1 & \mathcal{H}_3^1 \\ \mathcal{H}_1^1 & -\mathcal{H}_1^1 & \mathcal{H}_3^1 \\ -\mathcal{H}_3^1 & \mathcal{H}_3^1 & 0 \end{pmatrix}.$$

We compute that

$$\mathcal{H}^2 = (\mathcal{H}_1^3)^2 \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{H}^3 = 0.$$

This shows that \mathcal{H} is either two- or three-step nilpotent, which proves assertion (1). □

4.3. The proof of theorem 1.9(2)

Let \mathcal{H}_f be two-step nilpotent. The analysis above shows that $\nabla f \in \text{Image}\{\mathcal{H}_f\}$. Since \mathcal{H}_f has rank 1, $\text{Image}\{\mathcal{H}_f\} = \nabla f \cdot \mathbb{R}$. We use the Fredholm alternative and the fact that \mathcal{H}_f is self-adjoint to establish assertion (2)(a) using the following equivalencies:

$$\begin{aligned} \mathcal{H}_f Z = 0 &\iff g(\mathcal{H}_f Z, Y) = 0 \quad \forall Y \\ &\iff g(Z, \mathcal{H}_f Y) = 0 \quad \forall Y \\ &\iff Z \perp \text{Range}\{\mathcal{H}_f\} \\ &\iff Z \perp \nabla f. \end{aligned}$$

Choose a vector field U such that $g(U, \nabla f) = 1$. Since $\text{Range}\{\mathcal{H}_f\} = \nabla f$ and since $g(U, \nabla f) = 1$, the fact that ∇f is recurrent follows from

$$\nabla_X(\nabla f) = \mathcal{H}_f(X) = \theta(X) \cdot \nabla f, \quad \text{where } \theta(X) = g(U, \mathcal{H}_f(X)). \tag{4.3}$$

Let X and Y be smooth vector fields in ∇f^\perp . We show that $[X, Y]$ belongs to ∇f^\perp and thus ∇f^\perp is an integrable distribution by computing that

$$\begin{aligned} g([X, Y], \nabla f) &= g(\nabla_X Y - \nabla_Y X, \nabla f) \\ &= Xg(Y, \nabla f) - g(Y, \nabla_X \nabla f) - Yg(X, \nabla f) + g(X, \nabla_Y \nabla f) \\ &= X\{0\} - \text{Hess}_f(Y, X) - Y\{0\} + \text{Hess}_f(X, Y) \\ &= 0. \end{aligned}$$

Let $\gamma(t)$ be a geodesic with $\dot{\gamma}(0) \perp \nabla f$. We compute

$$\partial_t g(\dot{\gamma}, \nabla f) = g(\ddot{\gamma}, \nabla f) + g(\dot{\gamma}, \nabla_{\partial_t} \nabla f) = \theta(\partial_t)g(\dot{\gamma}, \nabla f).$$

Since $g(\dot{\gamma}, \nabla f)(0) = 0$, the fundamental theorem of ordinary differential equations implies that $g(\dot{\gamma}, \nabla f)$ vanishes identically, and thus $\dot{\gamma} \in \nabla f^\perp$. Since $g(\dot{\gamma}, \nabla f) = \partial_t f$, the geodesic lies entirely in the level set of f . Assertion (2)(b) follows.

We proceed by induction on the dimension to establish assertion (2)(c). Fix a point $P \in M$. Let $\mathcal{V} := \text{Span}\{U, \nabla f\}$. The metric on \mathcal{V} is non-degenerate and contains a null vector; consequently, \mathcal{V} has Lorentzian signature. We can choose complementary Killing vector fields $\{F_1, \dots, F_n\}$ so that $\{U, \nabla f, F_1, \dots, F_n\}$ is a local frame field near P and so that

$$g(U, F_i)|_P = g(\nabla f, F_i)|_P = 0. \quad (4.4)$$

Consequently, $\text{Span}\{F_1, \dots, F_n\}$ is spacelike near P . Let $\xi_i := \text{grad}\{F_i(f)\}$; these are parallel vector fields by lemma 1.2. Let $\mathcal{W} := \text{Span}\{\xi_1, \dots, \xi_n\}$. Since the ξ_i are parallel, $r(x) := \text{Rank}\{\mathcal{W}(x)\}$ is locally constant. Suppose that $r > 0$. By reordering the collection $\{F_1, \dots, F_n\}$ if necessary, we may assume that $\{\xi_1, \dots, \xi_r\}$ is a local frame field for \mathcal{W} . Let $\varepsilon_{ij} := g(\xi_i, \xi_j)$ describe the induced metric on \mathcal{W} . Again we use the fact that the ξ_i are parallel; this implies that the ε_{ij} are constant. We can diagonalize ε or equivalently renormalize the choice of the Killing vector fields F_i to assume that ε is in fact diagonal. If $\det(\varepsilon) = 0$, then ξ_i is a parallel null vector field for some i and assertion (2)(c)(i) holds. Thus, we may assume that the inner product restricted to \mathcal{W} is non-degenerate. We may use theorem 1.1 to decompose, at least locally, $M = N^{2+n-r} \times \mathbb{R}_r^r$. If the metric on N is Riemannian, we may apply theorem 1.1 to see that the soliton is trivial. Thus, N is Lorentzian. If $\dim(N) = 2$, then theorem 1.7 shows that N is flat and $\mathcal{H}_f = 0$, which is false. This shows that $\dim(N) \geq 3$ and we may use our induction hypothesis on N . Thus, we may assume without loss of generality that $r = 0$, so $\mathcal{W} = \{0\}$, and assume henceforth that

$$\text{grad}\{F_i(f)\} = 0 \quad \text{for all } i. \quad (4.5)$$

By (4.5), $\kappa_i := F_i(f)$ is constant for all i . By (4.4),

$$\kappa_i = F_i(f)|_P = g(F_i, \nabla f)|_P = 0.$$

Consequently, $g(F_i, \nabla f)$ vanishes identically and we have

$$F_i \in \ker\{\mathcal{H}_f\} = \ker\{\text{Ric}\} = \nabla f^\perp. \quad (4.6)$$

We may thus further normalize the choice of U so that

$$g(U, F_i) = 0 \quad \text{for } 1 \leq i \leq n.$$

We may use (4.3) and (4.6) to see that

$$\left. \begin{aligned} \nabla_{\nabla f} \nabla f &= \mathcal{H}_f(\nabla f) = 0, & \nabla_{F_i} \nabla f &= \mathcal{H}_f(F_i) = 0 \quad \text{for all } i, \\ \nabla_U \nabla f &= \mathcal{H}_f(U) = \Xi \nabla f, & \text{where } \Xi &:= g(\mathcal{H}_f(U), U) = -\rho(U, U). \end{aligned} \right\} \quad (4.7)$$

We use (4.7) to see that

$$\nabla_Y \nabla f = 0 \quad \text{if } Y \perp \nabla f. \quad (4.8)$$

Thus, the only covariant derivative at issue is $\nabla_U \nabla f$. We shall let $\Psi := \psi \cdot \nabla f$. This is a null vector field. By (4.8), Ψ will be parallel if and only if ψ satisfies the equations

$$Y(\psi) = 0 \quad \text{if } Y \perp \nabla f \quad \text{and} \quad U(\psi) + \psi \Xi = 0. \quad (4.9)$$

Since F_i is a Killing vector field, $\nabla_{F_i} \rho = 0$. Since $F_i \in \ker\{\text{Ric}\}$, $\rho(F_i, \cdot)$ vanishes identically. Consequently, lemma 1.2 yields

$$\begin{aligned} R(F_i, U, F_j, \nabla f) &= (\nabla_{F_i} \rho)(U, F_j) - (\nabla_U \rho)(F_i, F_j) \\ &= -U\rho(F_i, F_j) + \rho(\nabla_U F_i, F_j) + \rho(\nabla_U F_j, F_i) \\ &= 0. \end{aligned} \quad (4.10)$$

Let $g_{ij} = g(F_i, F_j)$. Since $U \in \ker\{\text{Ric}\}$ and $\{U, \nabla f\}$ span a hyperbolic pair, (4.10) implies that

$$\begin{aligned} 0 = \rho(U, \nabla f)|_P &= R(U, \nabla f, \nabla f, U)|_P + \sum_{i,j=1}^n g^{ij} R(U, F_i, \nabla f, F_j)|_P \\ &= R(U, \nabla f, \nabla f, U)|_P. \end{aligned}$$

Since P was arbitrary and the only condition on U was that $g(U, \nabla f) = 1$, this holds for arbitrary P and we have

$$0 = R(U, \nabla f, \nabla f, U) \quad \text{if } g(U, \nabla f) = 1. \quad (4.11)$$

Also, in general, if X is a Killing vector field, then for arbitrary vector fields, we have (see, for example, [20, 22]) that

$$R(X, Y)Z = -\nabla_Y \nabla_Z X + \nabla_{\nabla_Y Z} X.$$

Let Ξ be as defined in (4.7). We use (4.6) to see that

$$g(\nabla_U F_i, \nabla f) = Ug(F_i, \nabla f) - g(F_i, \nabla_U \nabla f) = -g(F_i, \Xi \nabla f) = 0.$$

Since the F_i are Killing vector fields, since $g(F_i, \nabla f) = 0$, and since ∇f is recurrent,

$$\begin{aligned} R(F_i, U, U, \nabla f) &= -g(\nabla_U \nabla_U F_i, \nabla f) + g(\nabla_{\nabla_U U} F_i, \nabla f) \\ &= -Ug(\nabla_U F_i, \nabla f) + g(\nabla_U F_i, \nabla_U \nabla f) + (\nabla_U U)g(F_i, \nabla f) \\ &\quad - g(F_i, \nabla_{\nabla_U U} \{\nabla f\}) \\ &= -U\{Ug(F_i, \nabla f) - g(F_i, \nabla_U \nabla f)\} + g(\nabla_U F_i, \Xi \nabla f) \\ &= Ug(F_i, \Xi \nabla f) + \Xi g(\nabla_U F_i, \nabla f) \\ &= 0. \end{aligned}$$

By lemma 1.2, if $\{X, Y, Z\}$ are vector fields on a gradient Ricci soliton, then

$$R(X, Y, Z, \nabla f) = (\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z).$$

Consequently, we have that

$$\begin{aligned} 0 &= R(U, \nabla f, U, \nabla f) = (\nabla_U \rho)(\nabla f, U) - (\nabla_{\nabla f} \rho)(U, U), \\ 0 &= R(F_i, U, U, \nabla f) = (\nabla_{F_i} \rho)(U, U) - (\nabla_U \rho)(F_i, U). \end{aligned}$$

By (4.7), $\Xi = -\rho(U, U)$. Thus, we may compute as follows:

$$\begin{aligned} -\nabla f(\Xi) &= \nabla f \rho(U, U) = (\nabla_{\nabla f} \rho)(U, U) + 2\rho(\nabla_{\nabla f} U, U) \\ &= (\nabla_U \rho)(\nabla f, U) - 2g(\nabla_{\nabla f} U, \Xi \nabla f) \\ &= U\rho(\nabla f, U) - \rho(\nabla_U \nabla f, U) - \rho(\nabla f, \nabla_U U) \\ &\quad - 2\Xi(\nabla f g(U, \nabla f) - g(U, \nabla_{\nabla f} \nabla f)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} -F_i(\Xi) &= F_i \rho(U, U) = (\nabla_{F_i} \rho)(U, U) + 2\rho(\nabla_{F_i} U, U) \\ &= (\nabla_U \rho)(F_i, U) - 2g(\nabla_{F_i} U, \Xi \nabla f) \\ &= U\rho(F_i, U) - \rho(\nabla_U F_i, U) - \rho(F_i, \nabla_U U) \\ &\quad - 2\Xi(F_i g(U, \nabla f) - g(U, \nabla_{F_i} \nabla f)) \\ &= g(\nabla_U F_i, \Xi \nabla f) \\ &= \Xi U g(F_i, \nabla f) - \Xi g(F_i, \Xi \nabla f) \\ &= 0. \end{aligned}$$

This shows that $X(\Xi) = 0$ if $X \in \nabla f^\perp$. Since the distribution ∇f^\perp is integrable, the Frobenius theorem means that we can introduce local coordinates (u, x^2, \dots, x^{n+2}) so that $U = \partial_u$ and $\nabla f^\perp = \text{Span}\{\partial_{x_2}, \dots, \partial_{x_{n+2}}\}$. Thus, (4.9) becomes an ordinary differential equation that can be solved. This completes the proof of theorem 1.9. \square

EXAMPLE 4.1. We follow the discussion in [1]. A Cahen–Wallach space has the following metric, given locally by (1.4):

$$g = 2 dt dy + \left(\sum_{i=1}^n \kappa_i x_i^2 \right) dy^2 + \sum_{i=1}^n dx_i^2 \quad \text{for } 0 \neq \kappa_i \in \mathbb{R}.$$

The Levi-Civita connection is determined by the non-zero Christoffel symbols

$$\nabla_{\partial_y} \partial_y = - \sum_i \kappa_i x_i \partial_{x_i} \quad \text{and} \quad \nabla_{\partial_y} \partial_{x_i} = \nabla_{\partial_{x_i}} \partial_y = \kappa_i x_i \partial_v.$$

Thus, the only non-zero entries in the curvature tensor are given by

$$R(\partial_y, \partial_{x_i}, \partial_y, \partial_{x_i}) = -\kappa_i,$$

and thus (possibly) non-zero entries in the Ricci tensor are

$$\rho(\partial_y, \partial_y) = -\kappa, \quad \text{where } \kappa := \kappa_1 + \dots + \kappa_n.$$

Assuming that $\kappa \neq 0$, we then have $\text{Ric}(\partial_y) = -\kappa\partial_t$ and $\text{Ric}(\partial_t) = 0$. Thus, the Ricci tensor is two-step nilpotent. The f defines a gradient Ricci soliton if and only if $f(t, y, x_1, \dots, x_n) = f(y)$, where $f(y) = a_0 + a_1y + \frac{1}{4}\kappa y^2$; $\lambda = 0$ in this instance. Note that $df = (a_1 + \frac{1}{2}\kappa y) dy$, and hence $\nabla f = (a_1 + \frac{1}{2}\kappa y)\partial_t$ is a null parallel vector field.

5. Symmetric gradient Ricci solitons: the proof of theorem 1.12

Let (M, g) be a locally symmetric Lorentzian manifold. If (M, g, f) is a non-steady gradient Ricci soliton, then, by theorem 1.3, M splits, at least locally, as a product $M = N_0 \times N_1 \times \mathbb{R}^k$, where (N_0, g_0) is indecomposable but reducible and (N_1, g_1) is Einstein. If N_0 does not appear in the decomposition, then the soliton is rigid. Otherwise, (N_0, g_0) is an indecomposable but not irreducible Lorentzian symmetric space, and hence a Cahen–Wallach symmetric space [7] (see also [2]). Theorem 1.11 rules out this latter possibility since if (N, g_N, f_N) is a Cahen–Wallach gradient Ricci soliton, then it is steady.

Next suppose that (M, g, f) is a locally symmetric Lorentzian steady gradient Ricci soliton. We can use the de Rham–Wu decomposition of the manifold to split (M, g) locally as a product $M = N \times M_1 \times \dots \times M_l \times \mathbb{R}^k_\nu$, where (N, g_N) is a Cahen–Wallach symmetric space, where the M_i are irreducible symmetric spaces, and where \mathbb{R}^k_ν is either Euclidean or Minkowskian space. Since irreducible symmetric spaces are Einstein, the induced soliton is either trivial or the scalar curvature vanishes, which implies that M_i is Ricci flat. If M_i is Riemannian, then it is flat since Ricci flat locally symmetric spaces are flat in the Riemannian setting [3, 19]. Moreover, if M_i is Lorentzian, then it is flat since irreducible Lorentzian locally symmetric spaces are of constant sectional curvature [8]. Hence, if the gradient Ricci soliton is steady, then the decomposition above reduces to $M = N \times \mathbb{R}^k$, where (N, g_N) is a Cahen–Wallach symmetric space. Theorem 1.12 now follows. \square

6. Three-dimensional locally homogeneous gradient Ricci solitons

6.1. The proof of theorem 1.13

Let (M, g) be a three-dimensional Lorentzian strict Walker metric. There exist local coordinates such that the metric is given by (1.5):

$$g = 2 dt dy + dx^2 + \phi(x, y) dy^2.$$

Let $f(t, x, y)$ be a smooth real-valued function. To simplify the notation, set $f_t = \partial f / \partial t$, $f_{tx} = \partial^2 f / \partial t \partial x$, and so forth. One computes easily that the soliton equation $\text{Hess}_f + \rho = \lambda g$ is equivalent to the following relations:

$$\left. \begin{aligned} 0 &= f_{tt} = f_{tx}, & 0 &= f_{xx} - \lambda = f_{ty} - \lambda, \\ 0 &= 2f_{xy} - \phi_x f_t, & 0 &= 2\lambda\phi + \phi_{xx} - 2f_{yy} - \phi_x f_x + \phi_y f_t. \end{aligned} \right\} \tag{6.1}$$

We use the first identities in (6.1) to see that

$$f(t, x, y) = t(\lambda y + \kappa) + \frac{1}{2}\lambda x^2 + \alpha(y)x + \gamma(y) \quad \text{for } \kappa \in \mathbb{R}.$$

Hence, the equations of (6.1) simplify to become

$$0 = 2\alpha'(y) - (\lambda y + \kappa)\phi_x, \quad (6.2)$$

$$0 = 2\lambda\phi - 2\gamma''(y) - 2x\alpha''(y) + (\lambda y + \kappa)\phi_y - (\lambda x + \alpha(y))\phi_x + \phi_{xx}. \quad (6.3)$$

We differentiate (6.2) with respect to x to conclude that

$$0 = (\lambda y + \kappa)\phi_{xx}. \quad (6.4)$$

Since the Ricci operator is given by

$$\text{Ric} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\phi_{xx} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the metric is flat if and only if $\phi_{xx} = 0$. Since we assume that the Walker metric is not flat, we may use (6.4) to see that $\lambda = \kappa = 0$ and conclude that the gradient Ricci soliton is steady. Consequently, (6.2) and (6.4) imply that $f(t, x, y) = \alpha x + \gamma(y)$, so (6.3) becomes

$$2\gamma''(y) + \alpha\phi_x - \phi_{xx} = 0. \quad (6.5)$$

We take the derivative with respect to x to see that $\alpha\phi_{xx} = \phi_{xxx}$. We examine the two cases *seriatim*.

CASE I (suppose that $\alpha \neq 0$). We then have

$$\phi(x, y) = \frac{1}{\alpha^2}a(y)e^{\alpha x} + xb(y) + c(y)$$

for some arbitrary functions $a(y) \neq 0$, $b(y)$ and $c(y)$. Moreover, the potential function of the soliton is given by $f(t, x, y) = \alpha x + \gamma(y)$, where $\gamma''(y) = -\frac{1}{2}ab(y)$. In this case $\nabla f = \gamma'(y)\partial_t + \alpha\partial_x$ is spacelike. This gives rise to the first possibility in theorem 1.13.

CASE II (suppose that $\alpha = 0$). We then have

$$\phi(x, y) = x^2a(y) + xb(y) + c(y)$$

for some arbitrary functions $a(y) \neq 0$, $b(y)$ and $c(y)$. Moreover, the potential function of the soliton is given by $f(t, x, y) = \gamma(y)$, where $\gamma''(y) = \frac{1}{4}a(y)$. In this case $\nabla f = \gamma'(y)\partial_t$ is a null and recurrent vector field. This gives rise to the second possibility in theorem 1.13. \square

6.2. The proof of theorem 1.16

Let (M, g, f) be a locally homogeneous Lorentzian gradient Ricci soliton of dimension 3.

CASE I (suppose that (M, g, f) is non-steady). By theorem 1.5 the soliton is rigid.

CASE II (suppose that (M, g, f) is steady). Consequently, by lemma 1.2, the potential function is a solution of the Eikonal equation $\|\nabla f\|^2 = \mu$. We distinguish three subcases.

CASE II(a) ((M, g) is steady and $\mu < 0$). We apply theorem 1.8 to see that (M, g) splits locally as a product and hence the soliton is rigid.

CASE II(b) ((M, g) is steady and $\mu = 0$). We use theorem 1.9 to see that the Ricci operator is either two- or three-step nilpotent. It follows from work of [11] that there do not exist locally homogeneous three-dimensional manifolds with three-step nilpotent Ricci operator. Consequently, the Ricci operator is two-step nilpotent and (M, g) admits a locally defined parallel null vector field by theorem 1.9. Consequently, (M, g) is locally a strict Walker manifold. Consequently, the underlying geometry of (M, g) is given by theorem 1.15; the function f is now determined by theorem 1.13.

CASE II(c) ((M, g) is steady and $\mu > 0$). Since the scalar curvature is constant, the Ricci operator satisfies $\text{Ric}(\nabla f) = 0$, which shows that either f is constant, or otherwise the Ricci operator has a zero eigenvalue. We now consider the different possibilities for the kernel of Ric .

Assume that $\dim(\ker\{\text{Ric}\}) = 1$. It follows from [9] that (M, g) is either a symmetric space or a Lie group. If (M, g) is symmetric, then it is one of the following: a manifold of constant sectional curvature, a product $\mathbb{R} \times N$, where (N, g_N) is of constant curvature, or a three-dimensional Cahen–Wallach symmetric space. Hence, in all the cases, any gradient Ricci soliton is trivial, rigid or the underlying manifold admits a null parallel vector field (and we have already examined that case). Now we concentrate on Lie groups. Since the eigenspaces of the Ricci operator are left-invariant, since ∇f has constant norm $\mu > 0$, and since $\dim(\ker\{\text{Ric}\}) = 1$, we have that ∇f is a left-invariant vector field. Left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups were considered in [5], showing that they exist if and only if the Ricci operator has exactly one single eigenvalue, which must be zero since $\text{Ric}(\nabla f) = 0$. This shows that the Ricci operator is three-step nilpotent, but that is not possible due to the analysis carried out in [11].

Finally, assume that $\dim(\ker\{\text{Ric}\}) = 2$. In this case the Ricci operator is either diagonalizable or two-step nilpotent. The latter implies that the manifold admits locally a null parallel vector field [12], and again this case has been treated. If the Ricci operator is diagonalizable, then $\|\text{Ric}\|^2 = \pm\tau^2 = \|\text{Hess}_f\|^2$ and lemma 1.2(3) shows that $\tau = 0$, from where it follows that (M, g) is flat and the soliton is trivial. This completes the proof of theorem 1.16. \square

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