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METABELIAN LIE POWERS OF GROUP REPRESENTATIONS

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Abstract

Any representation of a group G on a vector space V extends uniquely to a representation of G on the free metabelian Lie algebra on V. In this paper we study such representations and make some group-theoretic applications.

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1. Introduction

Let K be a commutative ring with identity and let X be a set. We write $S^K(X)$ for the free commutative associative K-algebra with identity on X (equal to the polynomial ring $K[X]$), $L^K(X)$ for the free Lie algebra over K on X, and $M^K(X)$ for the free metabelian Lie algebra over K on X. (Thus $M^K(X)$ is isomorphic to $L^K(X)$ factored out by its second derived algebra. See [2, 4, 5, 15] for basic material concerning Lie algebras.) Let V be a free K-module such that X is a basis of V. (All our modules will be left unital modules.) Then V can be identified with the K-submodule of $S^K(X)$ spanned by X; and the same can be done for $L^K(X)$ and $M^K(X)$. We define $\mathbf{S}V = S^K(X)$, $\mathbf{L}V = L^K(X)$ and $\mathbf{M}V = M^K(X)$. (With V identified in the way described, these algebras are independent of the choice of basis of V.) $\mathbf{S}V$ has a K-module decomposition $\mathbf{S}V = \bigoplus_{n\geq 0} \mathbf{S}_n V$ where $\mathbf{S}_n V$ is the submodule spanned by all

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monomials $x_{i_1} \dots x_{i_n}$ with $x_{i_1}, \dots, x_{i_n} \in X$. Similarly $\mathbf{L}V = \bigoplus_{n\geq 1} \mathbf{L}_n V$ and $\mathbf{M}V = \bigoplus_{n\geq 1} \mathbf{M}_n V$ where $\mathbf{L}_n V$ and $\mathbf{M}_n V$ are spanned by the left-normed Lie monomials $[x_{i_1}, \dots, x_{i_n}]$ with $x_{i_1}, \dots, x_{i_n} \in X$. (We use square-bracket notation for Lie products.)

For any K-module V (not necessarily free) we write $\mathrm{GL}_K(V)$ or $\mathrm{GL}(V)$ for the group of all K-module automorphisms of V. If V is free then for each $g \in \mathrm{GL}(V)$ the action of g on V extends (uniquely) to give algebra automorphisms of $\mathbf{S}V$, $\mathbf{L}V$ and $\mathbf{M}V$: for example, $g[x_{i_1}, \dots, x_{i_n}] = [gx_{i_1}, \dots, gx_{i_n}]$. Thus $\mathbf{S}V$, $\mathbf{L}V$ and $\mathbf{M}V$ become modules for the group algebra $K\mathrm{GL}(V)$ in which each element of $\mathrm{GL}(V)$ acts as an algebra automorphism. Clearly each $\mathbf{S}_n V$, $\mathbf{L}_n V$ and $\mathbf{M}_n V$ is a $K\mathrm{GL}(V)$-submodule. More generally, if H is any group and V is a KH-module which is free as a K-module then the representation $H \to \mathrm{GL}(V)$ gives a KH-module structure to $\mathbf{S}V$, $\mathbf{L}V$ and $\mathbf{M}V$.

THEOREM A. *Let V be a finite dimensional vector space of dimension at least 2 over a field K and let G be a finite subgroup of* $\mathrm{GL}(V)$. *Then, for all $n \geq 1$, there exists $t \geq n$ such that $\mathbf{M}_n V \oplus \cdots \oplus \mathbf{M}_t V$ has a regular KG-submodule.*

A similar result for $\mathbf{S}V$ is well known. (We shall give a simple proof in Section 3.) An analogue of Theorem A for $\mathbf{L}V$ was proved by Bryant and Kovács [8] where the result was applied to the study of automorphism groups of finite p-groups (p a prime). If P is a finite p-group and $\Phi(P)$ denotes the Frattini subgroup of P then every automorphism of P induces an automorphism of $P/\Phi(P)$, giving a homomorphism $\pi : \mathrm{Aut}(P) \to \mathrm{Aut}(P/\Phi(P))$ from the automorphism group of P to that of $P/\Phi(P)$. We may regard $P/\Phi(P)$ as a vector space over $\mathbb{F}_p$, the field of p elements: thus $\pi(\mathrm{Aut}(P))$ is a linear group over $\mathbb{F}_p$. In [8] it was shown that every linear group of finite dimension at least 2 over $\mathbb{F}_p$ arises from some P in this way. Here we use Theorem A to show that P may be taken to be metabelian.

THEOREM B. *Let p be a prime number. For every linear group G of finite dimension at least 2 over $\mathbb{F}_p$ there exists a finite metabelian p-group P such that G is isomorphic as linear group to the image of* $\pi : \mathrm{Aut}(P) \to \mathrm{Aut}(P/\Phi(P))$.

Our other results are concerned with the case where K is a field of characteristic 0 and V is a finite dimensional vector space over K. Thus the $K\mathrm{GL}(V)$-modules $\mathbf{S}_n V$, $\mathbf{L}_n V$ and $\mathbf{M}_n V$ are all finite dimensional. For any finite dimensional $K\mathrm{GL}(V)$-module W we write χ_W for the character of $\mathrm{GL}(V)$

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on W. A formula for $\chi_{\mathbf{L}_n V}$ was given by Brandt [6]. Here we shall give formulae for the 'generating functions' of $\chi_{\mathbf{S}_n V}$ and $\chi_{\mathbf{M}_n V}$ (see Proposition 4.1).

We shall apply these formulae to the 'relation module' of a finite group. Let F be a free group of finite rank and let R be a normal subgroup of F such that $G = F/R$ is finite. The relation module for G is the derived factor group R/R' regarded as a $\mathbb{Z}G$-module ($\mathbb{Z}$ the ring of integers) by means of conjugation: $(fR)(uR') = (fuf^{-1})R'$ for all $f \in F$, $u \in R$. This module is faithful for G when F is non-cyclic (see [3]). So, with G regarded as a subgroup of $\mathrm{GL}_{\mathbb{Z}}(R/R')$, the algebras $\mathbf{S}(R/R')$, $\mathbf{L}(R/R')$ and $\mathbf{M}(R/R')$ have the structure of $\mathbb{Z}G$-modules. By tensoring with $\mathbb{Q}$ (the field of rational numbers) we obtain a finite dimensional $\mathbb{Q}G$-module $V = \mathbb{Q} \otimes_{\mathbb{Z}} (R/R')$ and $\mathbb{Q}G$-modules $\mathbf{S}V$, $\mathbf{L}V$ and $\mathbf{M}V$. Several results about $\mathbf{L}V$ as $\mathbb{Q}G$-module, derived from Brandt's character formula, were obtained by Gupta, Laffey and Thomson [13]. Here we obtain some analogous results concerning $\mathbf{S}V$ and $\mathbf{M}V$.

In the statement of the following theorem, $|G|$ denotes the order of the finite group G and $|g|$ denotes the order of the element g of G. For non-negative integers a and b, (a, b) denotes the greatest common divisor of a and b and, when $a \geq b$, $\binom{a}{b}$ denotes the binomial coefficient $a!/(b!(a-b)!)$.

THEOREM C. *Let F be a free group of finite rank $e \geq 2$ and let R be a normal subgroup of F such that $G = F/R$ is finite and $|G| \neq 1$. Let $V = \mathbb{Q} \otimes_{\mathbb{Z}} (R/R')$, and regard V as a $\mathbb{Q}G$-module by means of conjugation. Let $m = 1 + (e-1)|G|$.*

(i) *Let $g \in G$ and write $q = |g|$. For all $n \geq 0$,*

$$\chi_{\mathbf{S}_n V}(g) = \binom{((m-1)/q) + [n/q]}{[n/q]},$$

where $[n/q]$ is the greatest integer not exceeding n/q.

(ii) *Let $g \in G$ and write $q = |g|$. For all $n \geq 2$,*

$$\chi_{\mathbf{M}_n V}(g) = \begin{cases} 0 & \text{if } q \nmid n, \\ -\binom{(m+n-q-1)/q}{n/q} & \text{if } q \mid n \text{ and } q \neq 1, \\ (n-1)\binom{m+n-2}{n} & \text{if } q = 1. \end{cases}$$

(iii) *For $n \geq 2$, the multiplicity of the one-dimensional trivial $\mathbb{Q}G$-module*

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in $\mathbf{M}_n V$ *is*

$$d_n = \frac{1}{|G|}\left\{(n-1)\binom{m+n-2}{n} - \sum_{\substack{q|(n,|G|)\\ q\neq 1}} \nu(q)\binom{(m+n-q-1)/q}{n/q}\right\},$$

where $\nu(q)$ *denotes the number of elements of* G *with order* q. *Furthermore* d_n *is the multiplicity in* $\mathbf{M}_n V$ *of the regular* $\mathbb{Q}G$*-module (that is, the greatest rank of a free* $\mathbb{Q}G$*-submodule of* $\mathbf{M}_n V$*).*

(iv) *If* $n \geq 2$ *and* $(n, |G|) = 1$ *then* $\mathbf{M}_n V$ *is a free* $\mathbb{Q}G$*-module of rank*

$$\frac{1}{|G|}(n-1)\binom{m+n-2}{n}.$$

Theorem C has some group-theoretic consequences. Most of our notation for groups is standard but, to avoid confusion with Lie multiplication, we use round brackets for group commutators: $(g, h) = g^{-1}h^{-1}gh$. For $n \geq 1$, $\gamma_n H$ denotes the nth term of the lower central series of the group H: $\gamma_1 H = H$ and $\gamma_{n+1} H = (\gamma_n H, H)$. Also, for $n \geq 1$, we write $\mu_n H = (\gamma_n H)H''$ (where H'' is the second derived group of H). Thus $H/\mu_n H$ is the largest factor group of H which is both nilpotent of class $n-1$ and metabelian.

Let F, R and $G = F/R$ be as before. The factor groups $\gamma_n R/\gamma_{n+1} R$ and $\mu_n R/\mu_{n+1} R$ may be regarded as $\mathbb{Z}G$-modules by means of conjugation. Thus $(fR)(u\gamma_{n+1}R) = (fuf^{-1})\gamma_{n+1}R$ for all $f \in F$, $u \in \gamma_n R$, and similarly for $\mu_n R/\mu_{n+1} R$. The modules $\gamma_n R/\gamma_{n+1} R$ are the 'higher relation modules'. There are $\mathbb{Z}G$-module isomorphisms $\gamma_n R/\gamma_{n+1} R \cong \mathbf{L}_n(R/R')$ and $\mu_n R/\mu_{n+1} R \cong \mathbf{M}_n(R/R')$ (see Section 4). In [13] results about $\mathbf{L}_n(\mathbb{Q} \otimes (R/R'))$ were used to obtain information about $\gamma_n R/\gamma_{n+1} R$. Here we use Theorem C to obtain information about $\mu_n R/\mu_{n+1} R$. For any group H we write $Z(H)$ for the centre of H and, for any H-module W, we write $W^H = \{w \in W : hw = w$ for all $h \in H\}$.

THEOREM D. *Let* F, R *and* G *be as in Theorem* C. *Let* $d_1 = e$ *and, for* $n \geq 2$, *let* d_n *be defined as in Theorem* C (iii).

(i) *For* $n \geq 1$, $Z(F/\mu_{n+1}R) = (\mu_n R/\mu_{n+1}R)^G$ *and this is a free abelian group of rank* d_n.

(ii) *For* $n \geq 1$, $\mu_n R/(\mu_n R, F)\mu_{n+1} R$ *has torsion-free rank* d_n.

In [14] Hannebauer and Stöhr studied the factors $\mu_n R/(\mu_n R, F)\mu_{n+1} R$ in the general case where F/R is not necessarily finite. They showed in [14,

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Theorem 7.1] that, for $n \geq 2$, $\mu_n R/(\mu_n R, F)\mu_{n+1} R$ decomposes as the direct sum of a free abelian group D_n and a torsion group T_n which has exponent dividing n if n is odd and $2n$ if n is even. Theorem D (ii) above shows that, in the case where F/R is finite, D_n has rank d_n.

The organisation of the paper is as follows. Section 2 contains background information on Lie algebras and related topics. Theorems A and B will be proved in Section 3, and Theorems C and D in Section 4.

2. Central series of subgroups and Lie algebras

In this section we shall collect together information about the Lie algebras associated with certain descending series of subgroups of a group. Much of the material is well known, but we have re-cast it in a form suitable for our purposes. We have not attempted the difficult task of attributing individual results to their original sources. Suffice it to say that a major contribution was made by Lazard [16] and our account is greatly influenced by the account in Chapter VIII of Huppert and Blackburn's book [15].

Throughout this section p will denote a fixed prime number. For any group H and any positive integer m, H^m denotes the subgroup of H generated by all the mth powers h^m, $h \in H$. For any Lie algebra L, L' denotes the derived algebra $[L, L]$ and $L'' = [L', L']$.

Let H be any group. We regard the lower central factors $\gamma_n H/\gamma_{n+1} H$ as $\mathbb{Z}$-modules and form the (restricted) direct sum

$$\gamma H = \bigoplus_{n \geq 1} \gamma_n H/\gamma_{n+1} H.$$

As is well known, γH can be given the structure of a Lie ring (Lie algebra over $\mathbb{Z}$) by defining

$$[u\gamma_{m+1} H, v\gamma_{n+1} H] = (u, v)\gamma_{m+n+1} H$$

for all $u \in \gamma_m H$, $v \in \gamma_n H$ and all $m, n \geq 1$. (See, for example, [15, VIII.9.3].)

Similarly, the direct sum

$$\tilde{\gamma} H = \bigoplus_{n \geq 1} \gamma_n H/(\gamma_n H)^p \gamma_{n+1} H$$

can be given the structure of a Lie algebra over $\mathbb{F}_p$ by defining

$$[u(\gamma_m H)^p \gamma_{m+1} H,\ v(\gamma_n H)^p \gamma_{n+1} H] = (u, v)(\gamma_{m+n} H)^p \gamma_{m+n+1} H$$

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for all $u \in \gamma_m H$, $v \in \gamma_n H$ and all $m, n \geq 1$. In fact $\tilde{\gamma} H \cong \mathbb{F}_p \otimes_{\mathbb{Z}} \gamma H$. Furthermore we write

$$\tilde{\gamma}' H = \bigoplus_{n \geq 2} \gamma_n H / (\gamma_n H)^p \gamma_{n+1} H.$$

Thus $\tilde{\gamma}' H$ is a subalgebra of $\tilde{\gamma} H$.

For all $n \geq 1$ let

$$\lambda_n H = (\gamma_1 H)^{p^{n-1}} (\gamma_2 H)^{p^{n-2}} \dots (\gamma_n H).$$

Then (see [15, VIII.1.5]) $H = \lambda_1 H \geq \lambda_2 H \geq \dots$, and, for all $m, n \geq 1$,

$$(\lambda_m H, \lambda_n H) \leq \lambda_{m+n} H \qquad \text{and} \qquad \lambda_{n+1} H = (\lambda_n H)^p (\lambda_n H, H).$$

Thus $\lambda_n H / \lambda_{n+1} H$ is the largest factor group of $\lambda_n H$ which is an elementary abelian p-group centralized by H. The direct sum

$$\lambda H = \bigoplus_{n \geq 1} \lambda_n H / \lambda_{n+1} H$$

can be given the structure of a Lie algebra over $\mathbb{F}_p$ by defining

$$[u \lambda_{m+1} H,\ v \lambda_{n+1} H] = (u, v) \lambda_{m+n+1} H$$

for all $u \in \lambda_m H$, $v \in \lambda_n H$ and all $m, n \geq 1$. Furthermore we write

$$\lambda' H = \bigoplus_{n \geq 2} \left(H' \cap \lambda_n H\right) \lambda_{n+1} H / \lambda_{n+1} H.$$

Thus $\lambda' H$ is a subalgebra of λH.

Let $\mathbb{F}_p[\omega]$ be the ring of polynomials in an indeterminate ω over $\mathbb{F}_p$.

PROPOSITION 2.1. *Let H be any group.*

(i) *For $p \neq 2$, λH has the structure of an $\mathbb{F}_p[\omega]$-Lie algebra in which $\omega(u \lambda_{n+1} H) = u^p \lambda_{n+2} H$ for all $u \in \lambda_n H$, $n \geq 1$.*

(ii) *For all primes p, $\lambda' H$ has the structure of an $\mathbb{F}_p[\omega]$-Lie algebra in which $\omega(u \lambda_{n+1} H) = u^p \lambda_{n+2} H$ for all $u \in H' \cap \lambda_n H$, $n \geq 2$.*

PROOF. It is easy to verify that, for all $n \geq 1$, there is a well defined map $\omega_n : \lambda_n H / \lambda_{n+1} H \to \lambda_{n+1} H / \lambda_{n+2} H$ given by $\omega_n(u \lambda_{n+1} H) = u^p \lambda_{n+2} H$ for all $u \in \lambda_n H$. If $p \neq 2$ or $n \neq 1$ then $(uv)^p \lambda_{n+2} H = u^p v^p \lambda_{n+2} H$ for all u,

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$v \in \lambda_n H$: this follows from the congruence $(uv)^p \equiv u^p v^p \bmod (\gamma_2 A)^p(\gamma_p A)$, where $A = \langle u, v\rangle$ (see [15, VIII.1.1a]). Thus, if $p \neq 2$ or $n \neq 1$, ω_n is an $\mathbb{F}_p$-module homomorphism. Let $\bar{\omega} : \lambda H \to \lambda H$ be defined by $\bar{\omega}\left(\sum_n a_n\right) = \sum_n \omega_n(a_n)$, where $a_n \in \lambda_n H/\lambda_{n+1}H, n \geq 1$. Then, if $p \neq 2$, $\bar{\omega}$ is an $\mathbb{F}_p$-module homomorphism and, for all p, the restriction of $\bar{\omega}$ to $\lambda' H$ gives an $\mathbb{F}_p$-module homomorphism $\bar{\omega}' : \lambda' H \to \lambda' H$.

If $p \neq 2$ or $m \neq 1$, $(u^p, v)\lambda_{m+n+2}H = (u, v)^p \lambda_{m+n+2}H$ for all $u \in \lambda_m H$, $v \in \lambda_n H$: this follows from the congruence $(u^p, v) \equiv (u, v)^p \bmod (\gamma_2 B)^p(\gamma_p B)$, where $B = \langle u, (u, v)\rangle$ (see [15, VIII.1.1b]). Thus, in this case,

$$[\bar{\omega}\,(u\lambda_{m+1}H)\,,\, v\lambda_{n+1}H] = \bar{\omega}\,([u\lambda_{m+1}H, v\lambda_{n+1}H])\,.$$

Results (i) and (ii) follow by defining $\omega u = \bar{\omega}(u)$ for all $u \in \lambda H$.

PROPOSITION 2.2. *Let H be a group such that $H/\gamma_{n+1}H$ is torsion-free for all $n \geq 1$. Then there is a bijective map*

$$\alpha : \mathbb{F}_p[\omega] \otimes_{\mathbb{F}_p} \tilde{\gamma} H \to \lambda H$$

satisfying $\alpha(\omega^i \otimes u(\gamma_n H)^p \gamma_{n+1} H) = u^{p^i} \lambda_{n+i+1} H$ for all $u \in \gamma_n H, i \geq 0, n \geq 1$, such that (i) *for $p \neq 2$, α is an $\mathbb{F}_p[\omega]$-Lie algebra isomorphism, and* (ii) *for all p, α restricts to an $\mathbb{F}_p[\omega]$-Lie algebra isomorphism $\alpha' : \mathbb{F}_p[\omega] \otimes \tilde{\gamma}' H \to \lambda' H$.*

PROOF. Note firstly that $\mathbb{F}_p[\omega] \otimes \tilde{\gamma} H$ is a direct sum of $\mathbb{F}_p$-submodules $\omega^i \otimes (\gamma_n H/(\gamma_n H)^p \gamma_{n+1} H)$, $i \geq 0, n \geq 1$. For $s \geq 1$ let

$$A_s = \bigoplus_{n=1}^{s} \omega^{s-n} \otimes \left(\gamma_n H/\,(\gamma_n H)^p\,\gamma_{n+1} H\right).$$

Thus $\mathbb{F}_p[\omega] \otimes \tilde{\gamma} H = \bigoplus_{s \geq 1} A_s$. By [15, VIII.1.9b], for each $s \geq 1$, there is a bijective map $\alpha_s : A_s \to \lambda_s H/\lambda_{s+1}H$ defined by

$$\alpha_s\big((\omega^{s-1} \otimes \bar{u}_1) + (\omega^{s-2} \otimes \bar{u}_2) + \cdots + (\omega^0 \otimes \bar{u}_s)\big) = \left(u_1^{p^{s-1}} u_2^{p^{s-2}} \ldots u_s\right) \lambda_{s+1} H,$$

where $u_n \in \gamma_n H$ and $\bar{u}_n = u_n(\gamma_n H)^p \gamma_{n+1} H$ $(1 \leq n \leq s)$. Accordingly, let $\alpha : \mathbb{F}_p[\omega] \otimes \tilde{\gamma} H \to \lambda H$ be defined by $\alpha\left(\sum_s a_s\right) = \sum_s \alpha_s(a_s)$, where $a_s \in A_s$, $s \geq 1$. Thus α is bijective and

$$\alpha\big(\omega^i \otimes u\,(\gamma_n H)^p\,\gamma_{n+1} H\big) = u^{p^i} \lambda_{n+i+1} H$$

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for all $u \in \gamma_n H$, $i \geq 0$, $n \geq 1$. By [15, VIII.1.9c], for $p \neq 2$, α is an $\mathbb{F}_p$-module isomorphism, and, for all p, α restricts to an $\mathbb{F}_p$-module isomorphism $\alpha' : \mathbb{F}_p[\omega] \otimes \tilde{\gamma}' H \to \lambda' H$.

For all $u \in \gamma_n H$, $n \geq 1$, and all $i \geq 0$,

$$\begin{aligned}\alpha\left(\omega^{i+1} \otimes u\,(\gamma_n H)^p\,\gamma_{n+1} H\right) &= u^{p^{i+1}} \lambda_{n+i+2} H = \omega\left(u^{p^i} \lambda_{n+i+1} H\right)\\ &= \omega\alpha\left(\omega^i \otimes u\,(\gamma_n H)^p\,\gamma_{n+1} H\right).\end{aligned}$$

It follows that, for $p \neq 2$, α is an $\mathbb{F}_p[\omega]$-module isomorphism, and, for all p, α' is an $\mathbb{F}_p[\omega]$-module isomorphism. Now $\alpha(1 \otimes u(\gamma_n H)^p \gamma_{n+1} H) = u\lambda_{n+1} H$ for all $u \in \gamma_n H$, $n \geq 1$. Thus it is easily verified that α restricted to $1 \otimes \tilde{\gamma} H$ is a homomorphism of $\mathbb{F}_p$-Lie algebras. Results (i) and (ii) follow because $1 \otimes \tilde{\gamma} H$ spans $\mathbb{F}_p[\omega] \otimes \tilde{\gamma} H$ as $\mathbb{F}_p[\omega]$-module and $1 \otimes \tilde{\gamma}' H$ spans $\mathbb{F}_p[\omega] \otimes \tilde{\gamma}' H$.

For any group H we write $\mathrm{End}(H)$ for the set of endomorphisms of H. If $\beta \in \mathrm{End}(H)$ then, since the subgroups $\gamma_n H$ and $\lambda_n H$ are fully invariant, β induces endomorphisms of $\gamma_n H/\gamma_{n+1} H$, $\gamma_n H/(\gamma_n H)^p \gamma_{n+1} H$ and $\lambda_n H/\lambda_{n+1} H$, which we denote by $\bar{\gamma}_n \beta$, $\tilde{\gamma}_n \beta$ and $\bar{\lambda}_n \beta$, respectively. Recall that H is 'relatively free' if $H \cong F/V(F)$ for some free group F and some fully invariant subgroup $V(F)$ of F.

LEMMA 2.3. (i) *Let H be a group and let $\beta, \beta' \in \mathrm{End}(H)$. If $\bar{\gamma}_1 \beta = \bar{\gamma}_1 \beta'$ then $\bar{\gamma}_n \beta = \bar{\gamma}_n \beta'$ for all $n \geq 1$. If $\bar{\lambda}_1 \beta = \bar{\lambda}_1 \beta'$ then $\tilde{\gamma}_n \beta = \tilde{\gamma}_n \beta'$ and $\bar{\lambda}_n \beta = \bar{\lambda}_n \beta'$ for all $n \geq 1$.*

(ii) *If N is a normal subgroup of a relatively free group H and $\theta \in \mathrm{End}(H/N)$ then there exists $\beta \in \mathrm{End}(H)$ such that $\beta(N) \subseteq N$ and $\beta(h)N = \theta(hN)$ for all $h \in H$.*

(iii) *Let H be a relatively free group and let $\beta \in \mathrm{End}(H)$. If $\bar{\gamma}_1 \beta$ is an automorphism then so is $\bar{\gamma}_n \beta$ for all $n \geq 1$. If $\bar{\lambda}_1 \beta$ is an automorphism then so are $\tilde{\gamma}_n \beta$ and $\bar{\lambda}_n \beta$ for all $n \geq 1$.*

PROOF. The proofs of VIII.1.7a, VIII.13.3a and VIII.13.3b of [15] apply with only minor modifications.

PROPOSITION 2.4. *Let H be a relatively free group and write $U = H/H' = H/\gamma_2 H$. The action of $\mathrm{GL}_{\mathbb{Z}}(U)$ on U extends to γH so that γH is a $\mathbb{Z}\mathrm{GL}(U)$-module on which every element of $\mathrm{GL}(U)$ acts as a Lie ring automorphism.*

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PROOF. Let $g \in \mathrm{GL}(U)$. By Lemma 2.3 (ii) there exists $\beta_g \in \mathrm{End}(H)$ such that β_g induces g on U. Lemma 2.3 (i), (iii) show that, for all $n \geq 1$, $\bar{\gamma}_n\beta_g$ is an automorphism of $\gamma_n H/\gamma_{n+1}H$ depending only on g and not on the choice of β_g. The action of g on $\gamma_n H/\gamma_{n+1}H$ is defined by $g(u\gamma_{n+1}H) = \beta_g(u)\gamma_{n+1}H$ for all $u \in \gamma_n H$, and the rest of the proof is straightforward.

A similar proof gives the analogous results for $\tilde{\gamma}H$ and λH.

PROPOSITION 2.5. *Let H be a relatively free group and write $V = H/H^p\gamma_2 H = H/\lambda_2 H$. The action of $\mathrm{GL}_{\mathbb{F}_p}(V)$ on V extends to $\tilde{\gamma}H$ and λH so that $\tilde{\gamma}H$ and λH are $\mathbb{F}_p\mathrm{GL}(V)$-modules on which every element of $\mathrm{GL}(V)$ acts as a Lie algebra automorphism.*

In the case of λH we can do rather better.

PROPOSITION 2.6. *Let H be a relatively free group and write $V = H/\lambda_2 H$. Regard λH as a $\mathrm{GL}(V)$-module as in Proposition 2.5. Then* (i) *for $p \neq 2$, λH is an $\mathbb{F}_p[\omega]\mathrm{GL}(V)$-module in which every element of $\mathrm{GL}(V)$ acts as an $\mathbb{F}_p[\omega]$-Lie algebra automorphism, and* (ii) *for all p, a similar statement holds for $\lambda' H$.*

PROOF. It is easily verified that $\lambda' H$ is an $\mathbb{F}_p\mathrm{GL}(V)$-submodule of λH. Let $g \in \mathrm{GL}(V)$ and let β_g be an endomorphism of H which induces g on V. Let $\bar{\omega}$ be as in the proof of Proposition 2.1. Then for all $u \in \lambda_n H$, $n \geq 1$,

$$\begin{aligned}\bar{\omega}\left(g\left(u\lambda_{n+1}H\right)\right) &= \bar{\omega}\left(\beta_g(u)\lambda_{n+1}H\right) = \beta_g(u)^p\lambda_{n+2}H \\ &= \beta_g\left(u^p\right)\lambda_{n+2}H = g\bar{\omega}\left(u\lambda_{n+1}H\right).\end{aligned}$$

Results (i) and (ii) now follow easily.

PROPOSITION 2.7. *Let H be a relatively free group such that $H/\gamma_{n+1}H$ is torsion-free for all $n \geq 1$, and write $V = H/H^p\gamma_2 H = H/\lambda_2 H$. Then the $\mathbb{F}_p[\omega]$-Lie algebra isomorphisms*

$$\alpha : \mathbb{F}_p[\omega] \otimes \tilde{\gamma}H \to \lambda H \quad (p \neq 2), \qquad \alpha' : \mathbb{F}_p[\omega] \otimes \tilde{\gamma}' H \to \lambda' H,$$

of Proposition 2.2 are $\mathbb{F}_p[\omega]\mathrm{GL}(V)$-module isomorphisms.

PROOF. Let $g \in \mathrm{GL}(V)$ and let β_g be an endomorphism of H which induces g on V. Let $u \in \gamma_n H$, $n \geq 1$, and write $\bar{u} = u(\gamma_n H)^p\gamma_{n+1}H$. Then

$$\begin{aligned}\alpha\left(g\left(1 \otimes \bar{u}\right)\right) &= \alpha\left(1 \otimes \beta_g(u)\left(\gamma_n H\right)^p\gamma_{n+1}H\right) = \beta_g(u)\lambda_{n+1}H \\ &= g\left(u\lambda_{n+1}H\right) = g\left(\alpha\left(1 \otimes \bar{u}\right)\right).\end{aligned}$$

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The results follow because $1 \otimes \tilde{\gamma} H$ spans $\mathbb{F}_p[\omega] \otimes \tilde{\gamma} H$ as $\mathbb{F}_p[\omega]$-module and $1 \otimes \tilde{\gamma}' H$ spans $\mathbb{F}_p[\omega] \otimes \tilde{\gamma}' H$.

Let E be the free metabelian group on a set X and let $M^{\mathbb{Z}}(X)$ be the free metabelian Lie ring on X (as in Section 1). By Theorem 3.2 of Shmel'kin [19] there is an isomorphism of Lie rings $\xi_0 : M^{\mathbb{Z}}(X) \to \gamma E$ such that $\xi_0(x) = x\gamma_2 E$ for all $x \in X$. Let $U = E/E' = E/\gamma_2 E$. Then we can use ξ_0^{-1} to identify U with the $\mathbb{Z}$-submodule of $M^{\mathbb{Z}}(X)$ spanned by X. Thus $M^{\mathbb{Z}}(X) = \mathbf{M}U$. It is easily verified that $\xi_0(\mathbf{M}_n U) = \gamma_n E/\gamma_{n+1} E$ for all $n \geq 1$. As explained in Section 1, $\mathbf{M}U$ has the structure of a $\mathbb{Z}\mathrm{GL}(U)$-module. Also (see Proposition 2.4), γE has the structure of a $\mathbb{Z}\mathrm{GL}(U)$-module. For all $u \in U$ and all $g \in \mathrm{GL}(U)$ we have $\xi_0(gu) = gu = g\xi_0(u)$ because of the identification of U. Since U generates $\mathbf{M}U$ as a Lie ring, and $\mathrm{GL}(U)$ acts by Lie ring automorphisms on $\mathbf{M}U$ and γE, it follows that $\xi_0(gu) = g\xi_0(u)$ for all $u \in \mathbf{M}U$. Thus ξ_0 is an isomorphism of $\mathbb{Z}\mathrm{GL}(U)$-modules and we have proved the following result.

PROPOSITION 2.8. *Let E be a free metabelian group and write $U = E/E' = E/\gamma_2 E$. There is a bijective map $\xi_0 : \mathbf{M}U \to \gamma E$ which is an isomorphism of Lie rings and of $\mathbb{Z}\mathrm{GL}(U)$-modules such that $\xi(\mathbf{M}_n U) = \gamma_n E/\gamma_{n+1} E$ for all $n \geq 1$.*

Now $M^{\mathbb{F}_p}(X) \cong \mathbb{F}_p \otimes_{\mathbb{Z}} M^{\mathbb{Z}}(X)$ and $\tilde{\gamma} E \cong \mathbb{F}_p \otimes_{\mathbb{Z}} \gamma E$. Thus ξ_0 gives an $\mathbb{F}_p$-Lie algebra isomorphism $\xi : M^{\mathbb{F}_p}(X) \to \tilde{\gamma} E$. We can repeat the proof of Proposition 2.8 to obtain an analogous result for $\tilde{\gamma} E$.

PROPOSITION 2.9. *Let E be a free metabelian group and write $V = E/E^p\gamma_2 E = E/\lambda_2 E$. There is a bijective map $\xi : \mathbf{M}V \to \tilde{\gamma} E$ which is an isomorphism of $\mathbb{F}_p$-Lie algebras and of $\mathbb{F}_p\mathrm{GL}(V)$-modules such that $\xi(\mathbf{M}_n V) = \gamma_n E/(\gamma_n E)^p \gamma_{n+1} E$ for all $n \geq 1$.*

We can now prove the main result of this section.

PROPOSITION 2.10. *Let E be a free metabelian group and write $V = E/\lambda_2 E$.*
(i) *For $p \neq 2$, and $s \geq 1$, there is an $\mathbb{F}_p\mathrm{GL}(V)$-module isomorphism*

$$\lambda_s E/\lambda_{s+1} E \cong \bigoplus_{n=1}^{s} \mathbf{M}_n V.$$

(ii) *For all p, and $s \geq 2$, there is an $\mathbb{F}_p\mathrm{GL}(V)$-module isomorphism*

$$\left(E' \cap \lambda_s E\right) \lambda_{s+1} E/\lambda_{s+1} E \cong \bigoplus_{n=2}^{s} \mathbf{M}_n V.$$

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PROOF. Note first that Proposition 2.7 is applicable with $H = E$: the fact that $E/\gamma_{n+1}E$ is torsion-free for all $n \geq 1$ is contained in the work of Chen [9] although probably proved earlier by Magnus (see [18, 36.32]).

(i) By Proposition 2.7 the map $\alpha^{-1} : \lambda E \to \mathbb{F}_p[\omega]\otimes\tilde{\gamma} E$ is an $\mathbb{F}_p[\omega]\mathrm{GL}(V)$-module isomorphism, hence an $\mathbb{F}_p\mathrm{GL}(V)$-module isomorphism. But

$$\begin{aligned}\alpha^{-1}(\lambda_s E/\lambda_{s+1}E) &= \bigoplus_{n=1}^{s} \omega^{s-n} \otimes \left(\gamma_n E/(\gamma_n E)^p\,\gamma_{n+1}E\right) \\ &\cong \bigoplus_{n=1}^{s} \gamma_n E/(\gamma_n E)^p\,\gamma_{n+1}E\end{aligned}$$

as $\mathbb{F}_p\mathrm{GL}(V)$-module. The result follows by Proposition 2.9.

(ii) Similarly, using α', we obtain $\mathbb{F}_p\mathrm{GL}(V)$-module isomorphisms

$$\begin{aligned}\left(E' \cap \lambda_s E\right)\lambda_{s+1}E/\lambda_{s+1}E &\cong \bigoplus_{n=2}^{s} \omega^{s-n} \otimes \left(\gamma_n E/(\gamma_n E)^p\,\gamma_{n+1}E\right) \\ &\cong \bigoplus_{n=2}^{s} \gamma_n E/(\gamma_n E)^p\,\gamma_{n+1}E.\end{aligned}$$

The result follows by Proposition 2.9.

REMARK. Similar proofs give analogues of Propositions 2.8, 2.9 and 2.10 for a free group F, describing γF, $\tilde{\gamma} F$ and λF in terms of the free Lie algebras $\mathbf{L}(F/\gamma_2 F)$ and $\mathbf{L}(F/\lambda_2 F)$. Similarly (using results from [19]) we can obtain analogues for any free polynilpotent group H, describing γH, $\tilde{\gamma} H$ and λH in terms of associated free polynilpotent Lie algebras $\mathbf{P}(H/\gamma_2 H)$ and $\mathbf{P}(H/\lambda_2 H)$.

We conclude this section by recording a fact which will be used repeatedly : in the applications K' will either be an extension ring of K or a factor ring of K.

PROPOSITION 2.11. *Let $K \to K'$ be a homomorphism of non-zero commutative rings with identity (so that K' can be regarded as a (K', K)-bimodule). Let H be any group and let V be a KH-module which is free as a K-module. Let $\mathbf{D}$ denote $\mathbf{S}$, $\mathbf{L}$ or $\mathbf{M}$. Then there is an isomorphism $K' \otimes_K \mathbf{D}V \to \mathbf{D}(K' \otimes_K V)$ which is an isomorphism of K'-algebras and of $K'H$-modules such that $1 \otimes v \mapsto 1 \otimes v$ for all $v \in V$.*

PROOF. The result essentially follows from the universal properties of free algebras and tensor products. We omit the details (cf. [5, §2, 5.3]).

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3. Regular modules and automorphisms

In this section we shall prove Theorems A and B. We begin with an analogue of Theorem A for $\mathbf{S}V$. Although this result is essentially well known (see, for example, the proof of Theorem 7.1 of [1]), the simple proof given here seems worthy of note.

PROPOSITION 3.1. *Let V be a non-zero finite dimensional vector space over a field K and let G be a finite subgroup of* $\mathrm{GL}(V)$*, where $|G| = c$. Then, for all $n \geq 0$, $\mathbf{S}_n V \oplus \cdots \oplus \mathbf{S}_{n+c-1} V$ has a regular KG-submodule.*

PROOF. Let $G = \{g_1, \ldots, g_c\}$. By Proposition 2.11 and [15, VII.7.23], we may assume that K is infinite (by passing to an infinite extension field of K if necessary). For each $g \in G\backslash\{1\}$, $\{v \in V : gv = v\}$ is a proper subspace of V. Hence, since K is infinite, there exists a non-zero element v of V such that $gv \neq v$ for all $g \in G\backslash\{1\}$; thus the elements $g_1 v, \ldots, g_c v$ are distinct. Hence, in the integral domain $\mathbf{S}V$, the van der Monde determinant

$$\begin{vmatrix} (g_1 v)^n & \cdots & (g_c v)^n \\ (g_1 v)^{n+1} & \cdots & (g_c v)^{n+1} \\ \vdots & \ddots & \vdots \\ (g_1 v)^{n+c-1} & \cdots & (g_c v)^{n+c-1} \end{vmatrix}$$

is non-zero. Hence the columns of the van der Monde matrix are linearly independent regarded as vectors over the field of fractions of $\mathbf{S}V$. Therefore they are linearly independent over K. By comparing the homogeneous components, it follows that the elements $(g_i v)^n + (g_i v)^{n+1} + \cdots + (g_i v)^{n+c-1}$ of $\mathbf{S}V$ $(1 \leq i \leq c)$ are linearly independent over K. Let $u = v^n + v^{n+1} + \cdots + v^{n+c-1}$. Then, for each i, $g_i u = (g_i v)^n + \cdots + (g_i v)^{n+c-1}$. Hence the elements $g_1 u, \ldots, g_c u$ are linearly independent over K. It follows that u generates a regular KG-submodule of $\mathbf{S}_n V + \cdots + \mathbf{S}_{n+c-1} V$.

REMARK. From Proposition 3.1 we can easily deduce the corresponding result for $\mathbf{A}V$, the free associative algebra or tensor algebra on V. This is because $\mathbf{S}V$ is isomorphic to a factor algebra $\mathbf{A}V/I$, and if $u + I$ is an element of $\mathbf{A}V/I$ which generates a regular KG-module then u generates a regular KG-submodule of $\mathbf{A}V$. Thus we obtain a simple proof of a generalisation of the well known result of Burnside concerned with the occurrence of irreducible modules in tensor powers. (For further background see [7].)

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Let V be a vector space over a field K with finite dimension $d \geq 2$ and basis $X = \{x_1, \ldots, x_d\}$. We temporarily use the notation $S_{(1)} = \bigoplus_{j \geq 1} \mathbf{S}_j V$ and $M_{(2)} = \bigoplus_{j \geq 2} \mathbf{M}_j V$. Every element of $S_{(1)}$ can be written uniquely in the form

(1) $$w = \sum \rho_{(n_1, \ldots, n_d)} x_1^{n_1} \ldots x_d^{n_d}$$

where the sum is over all ordered d-tuples $(n_1, \ldots, n_d)$ of non-negative integers not all equal to 0 and where the $\rho_{(n_1, \ldots, n_d)}$ are elements of K all but finitely many of which are 0. If $u \in M_{(2)}$ and w is as given by (1) we write $[u; w]$ for the element of $M_{(2)}$ given by

$$[u; w] = \sum \rho_{(n_1, \ldots, n_d)} [u, \underbrace{x_1, \ldots, x_1}_{\leftarrow n_1 \rightarrow}, \ldots, \underbrace{x_d, \ldots, x_d}_{\leftarrow n_d \rightarrow}]$$

where, in the term corresponding to $(n_1, \ldots, n_d)$, there are n_i occurrences of x_i for each i. Since $\mathbf{M}V$ is metabelian,

(2) $$[u, x_i, x_j] = [u, x_j, x_i]$$

for all $u \in M_{(2)}$ and all $i, j \in \{1, \ldots, d\}$. Hence, for all $u \in M_{(2)}$ and w_1, $w_2 \in S_{(1)}$,

(3) $$[u; w_1 w_2] = \big[[u; w_1]; w_2\big].$$

Also

(4) $$[u; w_1 + w_2] = [u; w_1] + [u; w_2].$$

For all $g \in \mathrm{GL}(V)$, $u \in M_{(2)}$, $v \in V = \mathbf{S}_1 V$, we have

$$g[u; v] = g[u, v] = [gu, gv] = [gu; gv].$$

Hence, by (3), (4) and induction,

(5) $$g[u; w] = [gu; gw]$$

for all $g \in \mathrm{GL}(V)$, $u \in M_{(2)}$, $w \in S_{(1)}$. The tensor product $\mathbf{M}_2 V \otimes_K S_{(1)}$ has the structure of a $K\mathrm{GL}(V)$-module under the 'diagonal' action of $\mathrm{GL}(V)$: $g(u \otimes w) = gu \otimes gw$ for all $g \in \mathrm{GL}(V)$, $u \in \mathbf{M}_2 V$, $w \in S_{(1)}$. From (5) we obtain the following result.

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LEMMA 3.2. *There is a* $K\mathrm{GL}(V)$*-module homomorphism*

$$\eta : \mathbf{M}_2 V \otimes_K \left(\bigoplus_{j\geq 1} \mathbf{S}_j V \right) \to \mathbf{M}V$$

such that $\eta(u \otimes w) = [u; w]$ *for all* $u \in \mathbf{M}_2 V$, $w \in \bigoplus_{j\geq 1} \mathbf{S}_j V$.

We shall first prove Theorem A in the case where K has non-zero characteristic.

PROPOSITION 3.3. *Let* V *be a finite dimensional vector space of dimension at least 2 over a field* K *of characteristic* $p > 0$ *and let* G *be a finite subgroup of* $\mathrm{GL}(V)$. *For all* $n \geq 1$ *there exists* $t \geq n$ *such that* $\mathbf{M}_n V \oplus \cdots \oplus \mathbf{M}_t V$ *has a regular* KG*-submodule.*

PROOF. As before, let $\{x_1, \dots, x_d\}$ be a basis of V and let $G = \{g_1, \dots, g_c\}$ where $|G| = c$. As in the proof of Proposition 3.1 we may assume that K is infinite and take $v \in V$ so that the elements $g_1 v, \dots, g_c v$ are distinct. Let $n \geq 1$ and take a positive integer m so that $p^m \geq n$. Since $\mathbf{S}V$ has characteristic p,

$$(g_i v)^{p^m} - (g_j v)^{p^m} = (g_i v - g_j v)^{p^m} \neq 0$$

for all i, j $(i \neq j)$. Thus the argument used in the proof of Proposition 3.1 shows that the element

$$v^{p^m} + v^{2p^m} + \cdots + v^{cp^m}$$

generates a regular KG-submodule W of $\mathbf{S}V$.

Let Z be the subalgebra of $\mathbf{S}V$ generated by $x_1^p, \dots, x_d^p$. Clearly Z is a KG-submodule of $\mathbf{S}V$ and $W \subseteq Z \subseteq \bigoplus_{j\geq 1} \mathbf{S}_j V$. Let η be the KG-module homomorphism given by Lemma 3.2. Then

$$\eta\,(\mathbf{M}_2 V \otimes W) \subseteq \bigoplus_{n\leq j\leq t} \mathbf{M}_j V,$$

for some t, and $\mathbf{M}_2 V \otimes W$ is a free KG-module by [15, VII.7.19a]. Thus to prove the proposition it is sufficient to prove that η restricted to $\mathbf{M}_2 V \otimes W$ is injective. We do this by showing that η restricted to $\mathbf{M}_2 V \otimes Z$ is injective.

Now

$$Z = \bigoplus_{\substack{k\geq 1\\ p\mid k}} Z \cap \mathbf{S}_k V$$

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and, for each k,

$$\eta\big(\mathbf{M}_2V \otimes (Z \cap \mathbf{S}_kV)\big) \subseteq \mathbf{M}_{k+2}V.$$

Thus it is sufficient to prove that η restricted to $\mathbf{M}_2V \otimes (Z \cap \mathbf{S}_kV)$ is injective for each k.

Let A be the set of all elements of $\mathbf{S}V$ of the form $x_1^{\beta_1} \dots x_d^{\beta_d}$ where $\beta_1 + \dots + \beta_d = k$ and all of $\beta_1, \dots, \beta_d$ are divisible by p. Then A is a K-basis of $Z \cap \mathbf{S}_kV$. Furthermore, the sets

$$B_2 = \big\{[x_{i_1}, x_{i_2}] : i_1 > i_2\big\}$$

and

$$B_{k+2} = \big\{[x_{i_1}, \dots, x_{i_{k+2}}] : i_1 > i_2 \leq i_3 \leq \dots \leq i_{k+2}\big\}$$

are K-bases of $\mathbf{M}_2V$ and $\mathbf{M}_{k+2}V$, respectively (see [2, 15.3.2]). For each d-tuple $\alpha = (\alpha_1, \dots, \alpha_d)$ of non-negative integers satisfying $\alpha_1 + \dots + \alpha_d = k+2$ let N_α be the K-subspace of $\mathbf{M}_{k+2}V$ spanned by all those basis elements $[x_{i_1}, \dots, x_{i_{k+2}}]$ of B_{k+2} for which there are exactly α_j of the subscripts $i_1, \dots, i_{k+2}$ equal to j $(j = 1, \dots, d)$. Thus $\mathbf{M}_{k+2}V$ is the direct sum of the subspaces N_α.

The elements $b \otimes a$ $(b \in B_2, a \in A)$ form a basis for $\mathbf{M}_2V \otimes (Z \cap \mathbf{S}_kV)$. It is sufficient to show that the elements $\eta(b \otimes a)$ are linearly independent. Thus it is sufficient to show that, for each $b \otimes a$, $\eta(b \otimes a)$ is a non-zero element of N_α for some $\alpha = \alpha(b \otimes a)$ depending on $b \otimes a$, and that $\alpha(b \otimes a) \neq \alpha(b' \otimes a')$ for distinct basis elements $b \otimes a$ and $b' \otimes a'$.

Let $b = [x_{i_1}, x_{i_2}]$ and $a = x_1^{\beta_1} \dots x_d^{\beta_d}$. Write $\alpha = (\alpha_1, \dots, \alpha_d)$ where $\alpha_{i_1} = \beta_{i_1} + 1$, $\alpha_{i_2} = \beta_{i_2} + 1$ and $\alpha_j = \beta_j$ $(j \neq i_1, i_2)$. Let s be the least integer for which $\beta_s \neq 0$. Thus

$$\eta(b \otimes a) = \big[[x_{i_1}, x_{i_2}]; x_s^{\beta_s} \dots x_d^{\beta_d}\big] = \big[x_{i_1}, x_{i_2}, x_s, \dots, x_s, \dots, x_d, \dots, x_d\big].$$

If $i_2 \leq s$ then $\eta(b \otimes a)$ is an element of B_{k+2} and so $\eta(b \otimes a) \neq 0$. Clearly $\eta(b \otimes a) \in N_\alpha$ in this case. If $i_2 > s$ then, by the Jacobi identity and (2), we can write $\eta(b \otimes a)$ in the form

$$-\big[x_{i_2}, x_s, \dots, x_s, \dots, x_{i_1}, \dots\big] + \big[x_{i_1}, x_s, \dots, x_s, \dots, x_{i_2}, \dots\big],$$

the difference of two distinct elements of B_{k+2}. Thus $\eta(b \otimes a) \neq 0$ and $\eta(b \otimes a) \in N_\alpha$ in this case also.

It remains to prove that if $b \otimes a$ and $b' \otimes a'$ are basis elements such that $\alpha(b \otimes a) = \alpha(b' \otimes a')$ then $b \otimes a = b' \otimes a'$. In other words we have to show that $\alpha(b \otimes a)$ determines $b \otimes a$ uniquely. Suppose that $b = [x_{i_1}, x_{i_2}]$, $a = x_1^{\beta_1} \dots x_d^{\beta_d}$

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and $\alpha(b \otimes a) = (\alpha_1, \ldots, \alpha_d)$, as before. Then $\alpha_{i_1} = \beta_{i_1} + 1$, $\alpha_{i_2} = \beta_{i_2} + 1$ and $\alpha_j = \beta_j$ $(j \neq i_1, i_2)$. Since each β_i is divisible by p, $(\alpha_1, \ldots, \alpha_d)$ determines i_1 and i_2. Thus $(\alpha_1, \ldots, \alpha_d)$ determines $(\beta_1, \ldots, \beta_d)$. Hence a and b are uniquely determined by $\alpha(b \otimes a)$.

We can now deal with the characteristic 0 case of Theorem A. This is obtained from the prime characteristic case by the standard process of modular reduction.

PROPOSITION 3.4. *Let V be a finite dimensional vector space of dimension at least 2 over a field K of characteristic 0 and let G be a finite subgroup of* $\mathrm{GL}(V)$. *For all $n \geq 1$ there exists $t \geq n$ such that $\mathbf{M}_n V \oplus \cdots \oplus \mathbf{M}_t V$ has a regular KG-submodule.*

PROOF. By Proposition 2.11 and [15, VII.7.23] we may assume that K is an algebraically closed extension field of the rational field $\mathbb{Q}$. By [10, (1.11)] there exists a subfield $\mathscr{F}$ of K such that $\mathscr{F}$ is finite dimensional over $\mathbb{Q}$ and $\mathscr{F}$ is a splitting field for G. Hence there is a basis of V such that the matrices representing G with respect to this basis have their entries in $\mathscr{F}$. Since $KG \cong K \otimes_{\mathscr{F}} \mathscr{F}G$ it is enough to prove the result with K replaced by $\mathscr{F}$.

Let p be a prime such that $p \nmid |G|$ and let $\mathfrak{p}_0$ be a prime ideal in the ring of algebraic integers of $\mathscr{F}$ such that $p \in \mathfrak{p}_0$. Let $\mathscr{D}$ be the subring of $\mathscr{F}$ consisting of all elements a/b where a and b are integers of $\mathscr{F}$ such that $b \notin \mathfrak{p}_0$. As shown in [10, pp. 24-25], $\mathscr{D}$ has a unique maximal ideal and the factor ring by this ideal is a finite field $\overline{\mathscr{D}}$ of characteristic p. Let us say that $\mathscr{D}G$-modules I and J are $\mathscr{F}G$-equivalent if the $\mathscr{F}G$-modules $\mathscr{F} \otimes_{\mathscr{D}} I$ and $\mathscr{F} \otimes_{\mathscr{D}} J$ are isomorphic. For any $\mathscr{D}G$-module I, let $\bar{I}$ denote the corresponding $\overline{\mathscr{D}}G$-module: $\bar{I} = \overline{\mathscr{D}} \otimes_{\mathscr{D}} I$. By [10, (4.4)] the mapping $I \mapsto \bar{I}$ yields a bijective mapping from the set of $\mathscr{F}G$-equivalence classes of finitely generated $\mathscr{D}G$-modules which are free as $\mathscr{D}$-modules to the set of isomorphism classes of finitely generated $\overline{\mathscr{D}}G$-modules. It follows (by considering the same correspondence for a proper factor group of G) that if I is faithful as a G-module then so is $\bar{I}$. Clearly $\overline{\mathscr{D}G} \cong \overline{\mathscr{D}}G$. Thus if $\bar{I}$ has a regular submodule we can write $\bar{I} \cong \overline{\mathscr{D}G} \oplus \bar{J} \cong \overline{\mathscr{D}G \oplus J}$ for some $\mathscr{D}G$-module J which is free as $\mathscr{D}$-module, and so I is $\mathscr{F}G$-equivalent to $\mathscr{D}G \oplus J$.

By [10, (4.1)] there is an $\mathscr{F}$-basis Y of V such that the matrices representing G with respect to Y have their entries in $\mathscr{D}$. Let V_0 be the (free) $\mathscr{D}$-module spanned by Y. Thus $\mathbf{M}V \cong \mathscr{F} \otimes_{\mathscr{D}} \mathbf{M}V_0$. By the remarks above $\bar{V}_0$ is faithful as a G-module. Thus we may regard G as a subgroup of $\mathrm{GL}(\bar{V}_0)$. By Proposition 2.11,

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$\overline{\mathscr{D}} \otimes_{\mathscr{D}} \mathbf{M}V_0 \cong \mathbf{M}\bar{V}_0$ and, by Proposition 3.3, there exists $t \geq n$ such that $\mathbf{M}_n\bar{V}_0 \oplus \cdots \oplus \mathbf{M}_t\bar{V}_0$ has a regular $\overline{\mathscr{D}}G$-submodule. Let $I = \mathbf{M}_n V_0 \oplus \cdots \oplus \mathbf{M}_t V_0$ and regard I as a $\mathscr{D}G$-module. Then $\bar{I} \cong \mathbf{M}_n\bar{V}_0 \oplus \cdots \oplus \mathbf{M}_t\bar{V}_0$, so $\bar{I}$ has a regular $\overline{\mathscr{D}}G$-submodule. By the remarks above it follows that $\mathscr{F} \otimes_{\mathscr{D}} I$ has a regular $\mathscr{F}G$-submodule. In other words, $\mathbf{M}_n V \oplus \cdots \oplus \mathbf{M}_t V$ has a regular $\mathscr{F}G$-submodule, as required.

PROOF OF THEOREM B. Let p be a prime number and let G be a linear group of finite dimension $d \geq 2$ over $\mathbb{F}_p$. Let F be a free group of rank d and let $E = F/F''$, a free metabelian group of rank d. Then G can be identified as a linear group with a subgroup of $\hat{G}$ =GL(V), where $V = E/\lambda_2 E$. Note that $\hat{G}$ is a finite group. By Theorem A there exists $t \geq 2$ such that $\mathbf{M}_2 V \oplus \cdots \oplus \mathbf{M}_t V$ has a regular $\mathbb{F}_p\hat{G}$-submodule. By Proposition 2.10, $\lambda_t E/\lambda_{t+1}E$ has a submodule isomorphic to $\mathbf{M}_2 V \oplus \cdots \oplus \mathbf{M}_t V$ and so has a regular $\mathbb{F}_p\hat{G}$-submodule. Let $P^* = E/\lambda_{t+1}E$: thus P^* is a finite metabelian p-group. Take w to be an element of $\lambda_t E/\lambda_{t+1}E$ which generates a regular $\mathbb{F}_p\hat{G}$-submodule and let W_G be the $\mathbb{F}_p G$-submodule generated by w. Then W_G is a central subgroup of P^* since $\lambda_t E/\lambda_{t+1}E$ is central in P^*. Take $P = P^*/W_G$. Then the method of proof of [8, Theorem 1] shows that P has the required properties.

4. Characters and higher relation modules

Let K be a field of characteristic 0 and let V be a vector space over K of finite dimension d. We shall consider the characters $\chi_{\mathbf{S}_n V}$ and $\chi_{\mathbf{M}_n V}$ of GL(V). Let y be an indeterminate over K and let $K[[y]]$ be the ring of formal power series in y with coefficients from K. For $g \in$ GL(V) we define $H_g = \sum_{n\geq 0} \chi_{\mathbf{S}_n V}(g)y^n$ and $M_g = \sum_{n\geq 1} \chi_{\mathbf{M}_n V}(g)y^n$. (In other words, H_g and M_g are the 'generating functions' of $\chi_{\mathbf{S}_n V(g)}$ and $\chi_{\mathbf{M}_n V(g)}$.) We write $\chi(g)$ instead of $\chi_V(g)$. Theorems C and D will be derived using the following result, at least part of which is well known (see, for example, the closely related formulae in [17, Ch. I, Section 2]).

PROPOSITION 4.1. (i) *For all* $g \in$ GL(V),

$$H_g = \prod_{k\geq 1} \left(1 - y^k\right)^{e_g(k)} = \prod_{n\geq 1} \exp\left(\frac{1}{n}\chi\left(g^n\right) y^n\right),$$

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where

$$e_g(k) = -\frac{1}{k}\sum_{i \mid k} \mu(i)\chi\left(g^{k/i}\right).$$

(ii) *For all* $g \in \mathrm{GL}(V)$,

$$M_g = \big(\chi(g)y - 1\big)H_g + \chi(g)y + 1.$$

In the statement of Proposition 4.1, μ denotes the Möbius function and we have used the notation $(1+u)^a = \sum_{i\geq 0}\binom{a}{i}u^i$ for $u \in yK[[y]]$, $a \in K$, where $\binom{a}{i} = a(a-1)\dots(a-i+1)/i!$, and $\exp(u) = \sum_{i\geq 0} u^i/i!$ for $u \in yK[[y]]$.

PROOF. Choose a basis $X = \{x_1, \dots, x_d\}$ of V. Thus we regard each element of $\mathrm{GL}(V)$ as a $d \times d$ matrix $g = (g_{ij})$. Choose bases for each $\mathbf{S}_n V$ and $\mathbf{M}_n V$. Then it is easily seen that the entries of the matrices representing g on $\mathbf{S}_n V$ and $\mathbf{M}_n V$ are homogeneous polynomials over K of total degree n in the g_{ij}. In other words, $\mathbf{S}_n V$ and $\mathbf{M}_n V$ afford polynomial representations of $\mathrm{GL}(V)$ of degree n (see [12, §2.2]). Thus, by [12, (3.4e)], there are polynomials σ_n and τ_n in d indeterminates with integer coefficients such that, for all $g \in \mathrm{GL}(V)$, $\chi_{\mathbf{S}_n V}(g) = \sigma_n(\varepsilon_1, \dots, \varepsilon_d)$ and $\chi_{\mathbf{M}_n V}(g) = \tau_n(\varepsilon_1, \dots, \varepsilon_d)$, where $\varepsilon_1, \dots, \varepsilon_d$ are the eigenvalues of g (in some extension field of K).

Equations (i) and (ii) can be interpreted as giving $\chi_{\mathbf{S}_n V}(g)$ and $\chi_{\mathbf{M}_n V}(g)$ as polynomials over K in $\varepsilon_1, \dots, \varepsilon_d$ (because $\chi(g^i) = \varepsilon_1^i + \dots + \varepsilon_d^i$). These polynomials will coincide with σ_n and τ_n if they do so for all $\varepsilon_1, \dots, \varepsilon_d \in K\backslash\{0\}$, since K is infinite. Thus it is sufficient to verify (i) and (ii) in the case where g is a diagonal matrix,

$$g = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_d \end{pmatrix}, \varepsilon_1, \dots, \varepsilon_d \in K\backslash\{0\}.$$

Note that $\mathbf{S}_n V$ has the basis $\{x_{i_1}x_{i_2}\dots x_{i_n} : i_1 \leq i_2 \leq \dots \leq i_n\}$. Thus $\chi_{\mathbf{S}_n V}(g) = h_n(\varepsilon_1, \dots, \varepsilon_d)$ where h_n is the nth complete symmetric function (see [17, p. 14]). Thus

$$H_g = \sum_{n\geq 0} h_n(\varepsilon_1, \dots, \varepsilon_d)y^n = \prod_{i=1}^{d}(1-\varepsilon_i y)^{-1}.$$

By [17, Ch. I, (2.10)] we have

$$H_g'/H_g = \sum_{n\geq 1} p_n(\varepsilon_1, \dots, \varepsilon_d)\, y^{n-1} = \sum_{n\geq 1}\chi(g^n)\, y^{n-1} \tag{6}$$

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where p_n is the nth power sum and H_g' is the formal derivative of H_g with respect to y. For $k \geq 1$ define

$$z_g(k) = \sum_{i \mid k} \mu(i) \chi\left(g^{k/i}\right).$$

Möbius inversion gives, for $n \geq 1$,

$$\chi\left(g^n\right) = \sum_{k \mid n} z_g(k).$$

Thus

$$\begin{aligned} H_g'/H_g &= \sum_{n \geq 1} \sum_{k \mid n} \sum_{i \mid k} \mu(i) \chi\left(g^{k/i}\right) y^{n-1} \\ &= \sum_{\substack{i,k \geq 1 \\ i \mid k}} \mu(i) \chi\left(g^{k/i}\right) y^{k-1} \left(1 - y^k\right)^{-1}. \end{aligned} \tag{7}$$

(i) follows by formal integration of (6) and (7).

By [2, 15.3.2], for $n \geq 2$, $\mathbf{M}_n V$ has the basis

$$\left\{\left[x_{i_1}, x_{i_2}, \ldots, x_{i_n}\right] : i_1 > i_2 \leq i_3 \leq \ldots \leq i_n\right\}.$$

Thus

$$\begin{aligned} \chi_{\mathbf{M}_n V}(g) &= \sum_{i_1 > i_2 \leq i_3 \leq \ldots \leq i_n} \varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_n} \\ &= \sum_{i_1 = 1}^{d} \sum_{i_2 \leq i_3 \leq \ldots \leq i_n} \varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_n} - \sum_{i_1 \leq i_2 \leq i_3 \leq \ldots \leq i_n} \varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_n} \\ &= h_1\left(\varepsilon_1, \ldots, \varepsilon_d\right) h_{n-1}\left(\varepsilon_1, \ldots, \varepsilon_d\right) - h_n\left(\varepsilon_1, \ldots, \varepsilon_d\right). \end{aligned}$$

Clearly, $\chi_{\mathbf{M}_1 V}(g) = h_1(\varepsilon_1, \ldots, \varepsilon_d)$. Thus we obtain

$$M_g = \chi(g) y + \sum_{n \geq 2} \left(\chi(g) \chi_{\mathbf{S}_{n-1} V}(g) y^n - \chi_{\mathbf{S}_n V}(g) y^n\right),$$

and (ii) follows.

A simpler version of the last proof gives the following well known formula for the dimension of $\mathbf{M}_n V$ (see [9, Corollary 1] for example).

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PROPOSITION 4.2. *For* $n \geq 2$,

$$\dim \mathbf{M}_n V = (n-1)\binom{d+n-2}{n}.$$

PROOF. By the previous proof, the number of basis elements of $\mathbf{M}_n V$ is equal to $(\dim \mathbf{S}_1 V)(\dim \mathbf{S}_{n-1} V) - \dim \mathbf{S}_n V$. The result follows using the fact that $\dim \mathbf{S}_n V = \binom{d+n-1}{n}$.

We now turn to the proof of Theorems C and D. Let F be a free group of finite rank e, where $e \geq 2$. Let R be a normal subgroup of F such that $G = F/R$ is finite and $|G| \neq 1$. Let V be the $\mathbb{Q}G$-module $\mathbb{Q} \otimes (R/R')$ and write $m = 1 + (e-1)|G|$. Note that, by Schreier's formula, m is the rank of R. By [3, Theorem 1] (or by [11]), R/R' is faithful as G-module. Thus we can identify G with a subgroup of $\mathrm{GL}(V)$. As before we write $\chi(g)$ instead of $\chi_V(g)$.

PROOF OF THEOREM C. (i) It follows from the work of Gaschütz [11] that

$$\chi(g) = \begin{cases} m & \text{if } g = 1, \\ 1 & \text{if } g \in G\backslash\{1\}. \end{cases} \tag{8}$$

In other words, V is the direct sum of a one-dimensional trivial module and a free $\mathbb{Q}G$-module of rank $e-1$. Let $g \in G$ and let $q = |g|$. By (6),

$$\begin{aligned} H_g'/H_g &= \sum_{n\geq 1} \chi(g^n)\, y^{n-1} = \sum_{\substack{n\geq 1\\ q\mid n}} m y^{n-1} + \sum_{\substack{n\geq 1\\ q\nmid n}} y^{n-1} \\ &= \sum_{k\geq 1} (m-1) y^{kq-1} + \sum_{k\geq 1} y^{k-1} \\ &= (m-1) y^{q-1} (1-y^q)^{-1} + (1-y)^{-1}. \end{aligned}$$

Since H_g has 1 as its constant term, integration gives

$$H_g = (1-y^q)^{-(m-1)/q} (1-y)^{-1}. \tag{9}$$

Thus

$$\begin{aligned} H_g &= \left\{ \sum_{i\geq 0} \binom{-(m-1)/q}{i} (-y^q)^i \right\} \left\{ \sum_{j\geq 0} y^j \right\} \\ &= \left\{ \sum_{i\geq 0} \binom{((m-1)/q)+i-1}{i} y^{qi} \right\} \left\{ \sum_{j\geq 0} y^j \right\}. \end{aligned}$$

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Hence

$$\chi_{\mathbf{S}_n V}(g) = \sum_{i=0}^{[n/q]} \binom{((m-1)/q)+i-1}{i}.$$

But

$$\binom{j}{0} + \binom{j+1}{1} + \cdots + \binom{j+s}{s} = \binom{j+s+1}{s}$$

for all $j, s \geq 0$, as is easily proved by induction on s. Thus

$$\chi_{\mathbf{S}_n V}(g) = \binom{((m-1)/q)+[n/q]}{[n/q]}.$$

(ii) Suppose $n \geq 2$. If $g = 1$ then $\chi_{\mathbf{M}_n V}(g) = (n-1)\binom{m+n-2}{n}$ by Proposition 4.2. Thus we may assume that $q = |g| \neq 1$. By Proposition 4.1 (ii) with (8) and (9) we have

$$M_g = -(1-y^q)^{-(m-1)/q} + y + 1 = -\sum_{j \geq 0} \binom{((m-1)/q)+j-1}{j} y^{jq} + y + 1.$$

This gives the remaining statements of (ii).

(iii) Suppose $n \geq 2$. We may work over the complex field $\mathbb{C}$ because, by [15, VII.1.21], the multiplicities in $\mathbf{M}_n V$ of the one-dimensional trivial $\mathbb{Q}G$-module and the regular $\mathbb{Q}G$-module are respectively equal to the multiplicities in $\mathbb{C} \otimes \mathbf{M}_n V$ of the one-dimensional trivial $\mathbb{C}G$-module and the regular $\mathbb{C}G$-module. Let ϕ be a complex irreducible character of G. The multiplicity of the corresponding irreducible module in $\mathbb{C} \otimes \mathbf{M}_n V$ is

$$d_n(\phi) = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\chi_{\mathbf{M}_n V}(g)}$$

where the bar denotes complex conjugation. Thus, by (ii),

$$|G| d_n(\phi) = \phi(1)(n-1)\binom{m+n-2}{n} - \sum_{\substack{q | (n,|G|) \\ q \neq 1}} \Bigg(\sum_{\substack{g \in G \\ |g| = q}} \phi(g) \Bigg) \binom{(m+n-q-1)/q}{n/q}. \tag{10}$$

In particular, if d_n denotes the multiplicity of the one-dimensional trivial module,

$$|G| d_n = (n-1)\binom{m+n-2}{n} - \sum_{\substack{q | (n,|G|) \\ q \neq 1}} \nu(q) \binom{(m+n-q-1)/q}{n/q}, \tag{11}$$

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where $\nu(q)$ denotes the number of elements of G of order q. For an arbitrary ϕ, we have $|\phi(g)| \le \phi(1)$ for all $g \in G$ (see [10, (6.7)]) and so, from (10),

$$|G|d_n(\phi) \ge \phi(1)(n-1)\binom{m+n-2}{n} - \sum_{\substack{q|(n,|G|)\\ q\neq 1}} \nu(q)\phi(1)\binom{(m+n-q-1)/q}{n/q}. \tag{12}$$

Thus, from (11) and (12), $d_n(\phi) \ge \phi(1)d_n$ for all ϕ. But the multiplicity of the irreducible module corresponding to ϕ in the free $\mathbb{C}G$-module of rank d_n is exactly $\phi(1)d_n$. Thus $\mathbb{C} \otimes \mathbf{M}_n V$ has a free $\mathbb{C}G$-submodule of rank d_n. It has no larger free submodule because it contains the one-dimensional trivial module with multiplicity d_n.

(iv) Suppose that $n \ge 2$ and $(n, |G|) = 1$. Again we may work over $\mathbb{C}$. By (10) and (11), $d_n(\phi) = \phi(1)d_n$ for every irreducible character ϕ. Thus $\mathbb{C} \otimes \mathbf{M}_n V$ is a free $\mathbb{C}G$-module of rank d_n. Also, by (11),

$$d_n = \frac{1}{|G|}(n-1)\binom{m+n-2}{n}$$

in this case.

Proof of Theorem D. Let $d_1 = e$ and for $n \ge 2$ let d_n be defined as in Theorem C (iii). Since F is finitely generated and F/R is finite, R is finitely generated and the groups $\gamma_n R/\gamma_{n+1} R$ and $\mu_n R/\mu_{n+1} R$ are finitely generated abelian groups. Note also that $\mu_2 R = R'$.

(i) We first prove that $Z(F/\mu_{n+1}R) = (\mu_n R/\mu_{n+1}R)^G$. Clearly we have $(\mu_n R/\mu_{n+1}R)^G \subseteq Z(F/\mu_{n+1}R)$. For the reverse inclusion let $f\mu_{n+1}R \in Z(F/\mu_{n+1}R)$. Then, clearly, $fR' \in Z(F/R')$. Since R/R' is a faithful G-module it follows that $f \in R$, and so $f\mu_{n+1}R \in Z(R/\mu_{n+1}R)$. Since R is a free group of rank greater than 1, $R/\mu_{n+1}R$ is a free group of rank greater than 1 in the variety of all groups which are both nilpotent of class n and metabelian. The centre of such a group is equal to the nth term of its lower central series: see, for example, the remark after 35.22 of [18]. Thus

$$Z(R/\mu_{n+1}R) = \gamma_n(R/\mu_{n+1}R) = \mu_n R/\mu_{n+1}R.$$

It follows that $f\mu_{n+1}R \in \mu_n R/\mu_{n+1}R$ and so $f\mu_{n+1}R \in (\mu_n R/\mu_{n+1}R)^G$. Therefore $Z(F/\mu_{n+1}R) = (\mu_n R/\mu_{n+1}R)^G$.

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Let $E = R/R''$. Then $\mu_n R/\mu_{n+1}R \cong \gamma_n E/\gamma_{n+1}E$, and $\gamma_n E/\gamma_{n+1}E$ has the structure of a $\mathbb{Z}G$-module defined via conjugation. As noted in the proof of Proposition 2.10, $E/\gamma_{n+1}E$ is torsion-free. Thus $\gamma_n E/\gamma_{n+1}E$ and $(\gamma_n E/\gamma_{n+1}E)^G$ are free abelian groups (of finite rank). It remains to calculate the rank of $(\gamma_n E/\gamma_{n+1}E)^G$.

Let $U = E/\gamma_2 E \cong R/\gamma_2 R$. Since U is a faithful G-module we can regard G as a subgroup of $\mathrm{GL}(U)$. By Proposition 2.4, γE has the structure of a $\mathbb{Z}\mathrm{GL}(U)$-module defined via endomorphisms of E, giving a $\mathbb{Z}G$-module structure to each $\gamma_n E/\gamma_{n+1}E$. It is easily verified that this is identical to the $\mathbb{Z}G$-module structure of $\gamma_n E/\gamma_{n+1}E$ defined via conjugation (because inner automorphisms of F yield endomorphisms, indeed automorphisms, of E). By Proposition 2.8, $\gamma_n E/\gamma_{n+1}E \cong \mathbf{M}_n U$. Thus it suffices to find the rank of $(\mathbf{M}_n U)^G$. By a standard argument (see [13, Section 2]), this rank is equal to $\dim(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{M}_n U)^G$. Let $V = \mathbb{Q} \otimes_{\mathbb{Z}} U$: then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{M}_n U \cong \mathbf{M}_n V$, so it suffices to find $\dim(\mathbf{M}_n V)^G$. This is equal to the multiplicity of the one-dimensional trivial module in $\mathbf{M}_n V$. Hence, if $n \geq 2$, $\dim(\mathbf{M}_n V)^G = d_n$ by Theorem C (iii), and, if $n = 1$, $\dim(\mathbf{M}_n V)^G = e$ by (8).

(ii) Let $W = \mu_n R/\mu_{n+1}R$. Then

$$\mu_n R/(\mu_n R, F)\,\mu_{n+1}R \cong W/[W, G],$$

where $[W, G]$ denotes the subgroup of W generated by $\{(g-1)w : g \in G,\ w \in W\}$. By a standard argument (see [13, Lemma 2.1]), $W/[W, G]$ has the same torsion-free rank as W^G. Thus the result follows from (i).

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