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COMPARING BIRNBAUM IMPORTANCE MEASURE OF SYSTEM COMPONENTS

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This article provides us with some simple criteria to compare Birnbaum reliability importance measure of components in a general binary coherent system. Such criteria are particularly useful in the absence of information concerning component reliabilities. We also find several simple (necessary and sufficient) conditions concerning system structure, under which such comparison is possible. Examples are given to illustrate our results.

1. INTRODUCTION

Consider a binary coherent system (C,ϕ) of n independent components, where $C = \{1,2,\ldots,n\}$ is the index set of the n components and $\phi:\{0,1\}^n \mapsto \{0,1\}$ denotes the nondecreasing structure function of the system. The reliability of the ith component is denoted by $p_i = \Pr\{X_i = 1\}(i = 1,\ldots,n)$; the reliability function of the system is denoted by $h(\mathbf{p}) = \Pr\{\phi(\mathbf{X}) = 1\}$, where $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{X} = (X_1,\ldots,X_n)$. Throughout this article, we assume that $0 < p_i < 1$ for each $1 \le i \le n$ to avoid trivial cases. The p minimal path sets and the c minimal cut sets of the system are respectively denoted by $P_1,\ldots,P_p$ and $C_1,\ldots,C_c$ (see [1] for unspecified notations).

Among various measures of component importance, the most fundamental and widely used one is the Birnbaum importance measure. The Birnbaum reliability importance measure of component i, denoted by $I_B(i;\mathbf{p})$, is defined by (see [1] and [2])

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$$I_B(i;\mathbf{p}) = \frac{\partial h(\mathbf{p})}{\partial p_i} = h(1_i,\mathbf{p}) - h(0_i,\mathbf{p}), \tag{1}$$

where $(\cdot_i,\mathbf{p}) = (p_1,\ldots,p_{i-1},\cdot,p_{i+1},\ldots,p_n)$. The Birnbaum structural importance measure of component i, denoted by $I_B(i)$, is the Birnbaum reliability importance measure $I_B(i;\mathbf{p})$ evaluated at $p_i = \frac{1}{2}, i = 1,2,\ldots,n$; that is,

$$I_B(i) = I_B\left(i;\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right) = \frac{1}{2^{n-1}}\,|\{(\cdot_i,\mathbf{x}) : \phi(1_i,\mathbf{x}) > \phi(0_i,\mathbf{x})\}|, \tag{2}$$

where $|\cdot|$ denotes the cardinality of a set. Because *structural importance of component i* represents the importance of node i in the system, the terms *importance of node* and *structural importance of component* are used interchangeably in this article without ambiguity.

Instead of quantitative measures, some structural importance (partial) ordering among system components have been introduced and studied by researchers. For example, the following structural criticality (partial) ordering and the cut-importance ordering of components are due to Boland, Proschan, and Tong [4] and Butler [5], respectively.

Definition 1 (Boland et al. [4]): *Node i is* more critical *than node j for ϕ, denoted by $i \overset{c}{>} j$, if $\phi(1_i,0_j,\mathbf{x}) \geq \phi(0_i,1_j,\mathbf{x})$ for all $\mathbf{x}$ and strict inequality holds for some $\mathbf{x}$. Nodes i and j are* permutation equivalent, *denoted by $i \overset{c}{=} j$, if $\phi(1_i,0_j,\mathbf{x}) = \phi(0_i,1_j,\mathbf{x})$ for all $\mathbf{x}$.*

Definition 2 (Butler [5]): *For each node s, let $d_{ij}^{(s)}$ denote the number of collections of i distinct min cut sets such that the union of each collection contains exactly j nodes and the union includes node s. Let $b_j^{(s)} = \sum_{i=1}^{c}(-1)^{i-1}d_{ij}^{(s)}$ and let $\mathbf{b}^{(s)} = (b_1^{(s)},\ldots,b_n^{(s)})$. Node s is said to be* more cut-important *than node t, denoted by $s >_c t$, if and only if $\mathbf{b}^{(s)} > \mathbf{b}^{(t)}$, where $>$ denotes lexicographic ordering.*

Because of its wide applications and merits, the Birnbaum importance measure has been extensively studied by researchers, and many importance measures of components introduced by them are either motivated by the Birnbaum measure or closely related to it; for example, the two structural ordering of components introduced earlier (see Meng [8]):

$$i \overset{c}{>} j \Leftrightarrow I_B(i;\mathbf{p}) > I_B(j;\mathbf{p}) \quad \text{for all } \mathbf{p} \text{ satisfying } p_i = p_j, \tag{3}$$

$$i >_c j \Leftrightarrow I_B(i;\mathbf{p}) > I_B(j;\mathbf{p}) \quad \text{for all } \mathbf{p} \text{ satisfying } p_i = p_j, p_k = p, \; \forall k \neq i,j, \text{ and } p \to 1. \tag{4}$$

It then follows that $i \overset{c}{>} j \Rightarrow I_B(i) > I_B(j)$ and $i \overset{c}{>} j \Rightarrow i >_c j$. The fact that $i \overset{c}{>} j \Rightarrow i >_c j$ was first proved (directly) in [7] by discussing the minimal cut (path) sets of the structure. From the relationships they have with the Birnbaum reliability importance measure (Eqs. (3) and (4)), this implication is easily concluded.

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Based on the criticality ordering, Boland et al. [4] introduced a principle for pairwise rearrangement of components. Since we assume that $0 < p_i < 1$ for each i, this principle is restated in the following theorem in a more simplified manner. Also, the vector $(\alpha_i, \beta_j, \mathbf{p})$ denotes that the component with reliability $\alpha(\beta)$ is assigned to node $i(j)$.

THEOREM 1 (Boland et al. [4]): *$i \overset{c}{>} j \Leftrightarrow h(\alpha_i, \beta_j, \mathbf{p}) > h(\beta_i, \alpha_j, \mathbf{p})$ for all $\beta < \alpha$ and all $\mathbf{p}$.*

Consider a situation, often encountered in practice, in which exact values of component reliabilities are unknown (e.g., during design stages), but the system structure is known to us. Suppose that there are two components to be allotted to two nodes i and j, one for each node. Theorem 1 of Boland et al. states that if node i is more critical than node j, then the more reliable component should be allotted to node i to achieve higher system reliability, irrespective of the reliabilities of other components. Since the Birnbaum reliability importance measure is an important index in analyzing a reliability system, it is then of natural interest to consider the problem, in the case that $i \overset{c}{>} j$, whether (and when) the implication $p_i \geq (\leq)\, p_j \Rightarrow I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ holds, irrespective of the reliabilities of other components. The following result due to Meng [10] represents a partial answer to the problem raised.

THEOREM 2 (Meng [10]): *Suppose that $i \overset{c}{>} j$ and $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \geq 0\ (\leq 0)$ for all $\mathbf{p}$. Then, $I_B(i;\mathbf{p}) > I_B(j;\mathbf{p})$ for all $\mathbf{p}$ satisfying $p_i \leq p_j\ (\geq p_j)$.*

The structure of a coherent system is generally represented in terms of its minimal cut (path) sets, and reliability analysts usually know how to use them to analyze a system. Thus, it is desirable to find some equivalent conditions, in terms of minimal cut (path) sets, to the left-hand side condition stated in Theorem 2. In this article, Theorem 2 is enhanced in two ways: (1) we obtain such equivalent structural conditions and (2) we show that the conditions are also necessary, under which such comparisons can be made. The case that nodes i and j are permutation equivalent is also studied and analogous results are obtained.

We now briefly summarize the present article. In Section 2, we first compare the Birnbaum reliability importance of two components located in two permutation equivalent nodes of a general coherent system. We show that under some assumptions on system structure, their relative Birnbaum reliability importance can be easily determined. The assumptions are also necessary, under which such comparisons are possible in the absence of information concerning component reliabilities. We then treat the case that the two nodes are asymmetric but are ordered by their structural criticality. Similar to the symmetry case, the criticality ordering is divided into three cases, and only two cases allow us to make such comparisons. Examples are given to illustrate our results.

2. RESULTS

Suppose that two components in a system are structurally permutation equivalent (i.e., $i \overset{c}{=} j$). It is easy to see that they possess equal Birnbaum structural importance

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measures; however, their Birnbaum reliability importance measures $I_B(i;\mathbf{p})$ and $I_B(j;\mathbf{p})$ may not be the same when $p_i \neq p_j$. A k-out-of-n system is the only system in which all components are symmetric. Let $h_k(\mathbf{p})$ denote the reliability function of a k-out-of-n system. Boland and Proschan [3] obtained that if $p_i \leq (k-1)/(n-1)$ for each i, then the reliability function $h_k(\mathbf{p})$ is Schur-concave and, hence, $I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p}) \Leftrightarrow p_i \leq p_j$; however, if $p_i \geq (k-1)/(n-1)$ for each i, then $h_k(\mathbf{p})$ is Schur-convex and, hence, $I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p}) \Leftrightarrow p_i \geq p_j$ $(1 \leq i, j \leq n)$ (see [3] for the definitions of Schur-concavity and Schur-convexity).

For the case that $i \stackrel{c}{=} j$, the following result is presented in [10], which provides a criterion to compare $I_B(i;\mathbf{p})$ and $I_B(j;\mathbf{p})$ for a general system, not necessarily a k-out-of-n system. Note that in the following theorem, the strict inequality "$>$" in Theorem 2 is replaced by "$\geq$."

THEOREM 3 (Meng [10]): *Suppose that* $i \stackrel{c}{=} j$ *and* $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \geq 0$ (≤ 0) *for all* $\mathbf{p}$. *Then,* $I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ *for all* $\mathbf{p}$ *satisfying* $p_i \leq (\geq)\ p_j$.

A real-valued function f defined on R^n is called L-superadditive (-subadditive) if f satisfies the following condition:

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq (\leq)\, f(\mathbf{x}) + f(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in R^n,$$

where $\mathbf{x} \vee (\wedge)\ \mathbf{y}$ is the vector of componentwise maxima (minima). It is known that if f has second partial derivatives, then f is L-superadditive (-subadditive) if and only if $\partial^2 f(\mathbf{x})/\partial x_i \partial x_j \geq 0$ (≤ 0) for all $\mathbf{x} \in R^n, 1 \leq i, j \leq n$ (see Marshall and Olkin [6]). Boland and Proschan [3] established Schur-concavity (-convexity) of a *symmetric reliability function* on a region of component reliabilities and obtained results for comparing the Birnbaum reliability importance of components in k-out-of-n systems. In Theorem 3, by restricting the L-superadditivity (-subadditivity) property on two components, such comparison results are obtained for more general coherent systems, with less restrictions on component reliabilities. In Theorem 4, we observe that the condition stated in Theorem 3 is also necessary, and this property will be utilized later in this article to derive further results.

THEOREM 4: *Suppose that* $i \stackrel{c}{=} j$. *Then,* $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \geq 0$ (≤ 0) *for all* $\mathbf{p} \Leftrightarrow I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ *for all* $\mathbf{p}$ *satisfying* $p_i \leq (\geq)\ p_j$.

PROOF: $(\Rightarrow)$ This part has been shown in [10].

$(\Leftarrow)$ By expressing $I_B(i;\mathbf{p})$ (similarly for $I_B(j;\mathbf{p})$) as

$$\begin{aligned} I_B(i;\mathbf{p}) &= h(1_i,\mathbf{p}) - h(0_i,\mathbf{p}) \\ &= p_j[h(1_i,1_j,\mathbf{p}) - h(1_i,0_j,\mathbf{p}) - h(0_i,1_j,\mathbf{p}) + h(0_i,0_j,\mathbf{p})] \\ &\quad + h(1_i,0_j,\mathbf{p}) - h(0_i,0_j,\mathbf{p}) \\ &= p_j \frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_j} + h(1_i,0_j,\mathbf{p}) - h(0_i,0_j,\mathbf{p}), \end{aligned}$$

we obtain that

$$I_B(i;\mathbf{p}) - I_B(j;\mathbf{p}) = (p_j - p_i)\,\frac{\partial^2 h(\mathbf{p})}{\partial p_i\,\partial p_j} + h(1_i,0_j,\mathbf{p}) - h(0_i,1_j,\mathbf{p}). \tag{5}$$

Thus, when $i \stackrel{c}{=} j, h(1_i,0_j,\mathbf{p}) = h(0_i,1_j,\mathbf{p})$ and hence

$$I_B(i;\mathbf{p}) - I_B(j;\mathbf{p}) = (p_j - p_i)\,\frac{\partial^2 h(\mathbf{p})}{\partial p_i\,\partial p_j}. \tag{6}$$

The right-hand side condition in Theorem 4 then implies that $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \geq 0$ (≤ 0) for all $\mathbf{p}$. ■

We are now ready to present one of the main results of this article: We replace the *restricted L-superadditivity* (*-subadditivity*) property stated in Theorem 4 by equivalent conditions in terms of minimal cut (path) sets, which is much more convenient for engineers to use in practice. First, some notations are needed. Denote by $C(i)(P(i))$ the collection of minimal cut (path) sets in which each minimal cut (path) set contains i. Also denote by $C(ij)(P(ij))$ the collection of minimal cut (path) sets in which each minimal cut (path) set contains both i and j, and by $C(i\bar{j})(P(i\bar{j}))$ the collection of minimal cut (path) sets in which each minimal cut (path) set contains i but not j. It is known that if $i \stackrel{c}{=} j$, then $C_k \in C(i\bar{j}) \Rightarrow \{j\} \cup C_k - \{i\} \in C(j\bar{i})$ (see [7]). Thus, $C(i\bar{j}) = \{\emptyset\} \Rightarrow C(j\bar{i}) = \{\emptyset\}$. The condition that $i \stackrel{c}{=} j$ can be divided into three cases: (i) $C(i) = C(j)$, (ii) $P(i) = P(j)$, and (iii) neither (i) nor (ii) holds. Note that in this case ($i \stackrel{c}{=} j$), (i) is equivalent to $C(i\bar{j}) = \{\emptyset\}$ and (ii) is equivalent to $C(ij) = \{\emptyset\}$.

THEOREM 5: *Suppose that $i \stackrel{c}{=} j$. Then, the following hold:*

(i) $C(i) = C(j) \Leftrightarrow I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ for all $\mathbf{p}$ satisfying $p_i \geq p_j$.
(ii) $P(i) = P(j) \Leftrightarrow I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ for all $\mathbf{p}$ satisfying $p_i \leq p_j$.

PROOF: Case (i). ($\Rightarrow$) Recall the *minimax* representation of the binary structure ϕ and that $h(\mathbf{p}) = \Pr\{\min_{1\leq k\leq c}\max_{i\in C_k} X_i = 1\}$ (see [1]). Since the assumption is equivalent to $i \in C_k \Leftrightarrow j \in C_k$, it is easy to see that the structure allows for a modular decomposition with a modular set, $z = \max\{x_i, x_j\}$ and that $h(\mathbf{p}) = p_z h(1_z,\mathbf{p}) + (1 - p_z) h(0_z,\mathbf{p})$, where the reliability of the module $p_z = \Pr\{Z = 1\} = p_i + p_j - p_i p_j$. Hence,

$$\frac{\partial h(\mathbf{p})}{\partial p_i} = (1 - p_j) h(1_z,\mathbf{p}) - (1 - p_j) h(0_z,\mathbf{p}),$$

$$\frac{\partial^2 h(\mathbf{p})}{\partial p_i\,\partial p_j} = h(0_z,\mathbf{p}) - h(1_z,\mathbf{p}) \leq 0 \quad \text{for all } \mathbf{p}.$$

The conclusion then follows from Theorem 4.

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($\Leftarrow$) Suppose that the right-hand side of (i) holds. Then, from Theorem 4, $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \le 0$ holds for all $\mathbf{p}$. Note that

$$\frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_j} = E\{\phi(1_i,1_j,\mathbf{X}) - \phi(1_i,0_j,\mathbf{X}) - \phi(0_i,1_j,\mathbf{X}) + \phi(0_i,0_j,\mathbf{X})\}. \quad \textbf{(7)}$$

Thus, the assumption implies that

$$\phi(1_i,1_j,\mathbf{x}) - \phi(1_i,0_j,\mathbf{x}) - \phi(0_i,1_j,\mathbf{x}) + \phi(0_i,1_j,\mathbf{x}) \le 0, \quad \forall \mathbf{x}. \quad \textbf{(8)}$$

Now suppose that $C(i) \ne C(j)$ (i.e., $C(i\bar{j}) \ne \{\emptyset\}$). We will derive a contradiction. Let $C_k \in C(i\bar{j})$; then, $\{j\} \cup C_k - \{i\}$ is also a minimal cut set. Hence, $\phi(0_i,1_j,0^{C_k-i},\mathbf{1}) = \phi(1_i,0_j,0^{C_k-i},\mathbf{1}) = 0$ and $\phi(1_i,1_j,0^{C_k-i},\mathbf{1}) = 1$, where 0^A means that $x_i = 0$ for all $i \in A$. It is seen that Eq. (8) fails to hold when $\mathbf{x} = (0^{C_k-i},\mathbf{1})$.

Case (ii). ($\Rightarrow$) The assumption is equivalent to $i \in P_r \Leftrightarrow j \in P_r$. Recalling the maxmin representation $h(\mathbf{p}) = \Pr\{\max_{1\le r\le p} \min_{i\in P_r} x_i = 1\}$, we can let $z = \min\{x_i, x_j\}$ be a modular set with reliability, $p_z = p_i p_j$. It is then easily obtained that

$$\frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_j} = h(1_z,\mathbf{p}) - h(0_z,\mathbf{p}) \ge 0 \quad \text{for all } \mathbf{p}.$$

The conclusion follows from Theorem 4.

($\Leftarrow$) In this case, by Theorem 4, $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \ge 0$ for all $\mathbf{p}$. Suppose that $C(ij) \ne \{\emptyset\}$ and let $C_k \in C(ij)$. Then, $\phi(0_i,1_j,0^{C_k-i-j},\mathbf{1}) = \phi(1_i,0_j,0^{C_k-i-j},\mathbf{1}) = 1$ and $\phi(0_i,0_j,0^{C_k-i-j},\mathbf{1}) = 0$. Hence, $\phi(1_i,1_j,\mathbf{x}) - \phi(1_i,0_j,\mathbf{x}) - \phi(0_i,1_j,\mathbf{x}) + \phi(0_i,0_j,\mathbf{x}) < 0$, when $\mathbf{x} = (0^{C_k-i-j},\mathbf{1})$. Thus, from Eq. (7), we see that the condition $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \ge 0$ for all $\mathbf{p}$ fails to hold. ■

Example 1: Let $\phi(\mathbf{x}) = x_1 \wedge (x_2 \vee x_3)$, $x_1, x_2, x_3 \in \{0,1\}$. In this example, there are two minimal cut sets: $\{1\}$ and $\{2,3\}$. Clearly, $2 \overset{c}{=} 3$ and Theorem 5(i) applies to the two nodes. Hence, $I_B(2;\mathbf{p}) \ge I_B(3;\mathbf{p})$ if $p_2 \ge p_3$.

Example 2: Let $\phi(\mathbf{x}) = x_1 \vee (x_2 \wedge x_3)$, $x_1, x_2, x_3 \in \{0,1\}$. In this example, the minimal cut sets are $\{1,2\}$ and $\{1,3\}$; Theorem 5(ii) applies to nodes 2 and 3. Hence, $I_B(2;\mathbf{p}) \ge I_B(3;\mathbf{p})$ if $p_2 \le p_3$.

Next, we consider the case that nodes i and j are not symmetric but are ordered by their criticality. First, the following theorem is analogous to Theorem 4, but inequality is replaced by strict inequality.

THEOREM 6: *Suppose that* $i \overset{c}{>} j$. *Then,* $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \ge 0$ (≤ 0) *for all* $\mathbf{p} \Leftrightarrow I_B(i;\mathbf{p}) > I_B(j;\mathbf{p})$ *for all* $\mathbf{p}$ *satisfying* $p_i \le p_j$ $(\ge p_j)$.

PROOF: ($\Rightarrow$) This part has been shown in [10].

($\Leftarrow$) To show that $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \ge 0$ for all $\mathbf{p}$, we will prove that $\phi(1_i,1_j,\mathbf{x}) - \phi(1_i,0_j,\mathbf{x}) - \phi(0_i,1_j,\mathbf{x}) + \phi(0_i,0_j,\mathbf{x}) \ge 0$ holds for all $\mathbf{x}$. Suppose that our claim is not true. Then, there exists an $\mathbf{x}^*$ such that $\phi(1_i,0_j,\mathbf{x}^*) = \phi(0_i,1_j,\mathbf{x}^*) = 1$

and $\phi(0_i, 0_j, \mathbf{x}^*) = 0$. Choose a probability vector $\mathbf{p}^*$ such that $\Pr\{(\cdot_i, \cdot_j, \mathbf{X}) = (\cdot_i, \cdot_j, \mathbf{x}^*)\} \to 1$. Then, when $\mathbf{p} = \mathbf{p}^*$ in Eq. (5), $h(1_i, 0_j, \mathbf{p}) - h(0_i, 1_j, \mathbf{p}) \to 0$ and $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \to -1$. Hence, $I_B(i;\mathbf{p}) < I_B(j;\mathbf{p})$ holds for some $p_i < p_j$, which contradicts our assumption. (The case $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \leq 0$ can be treated in a similar manner; the details are omitted.) ■

Now, similar to the symmetry case, we divide the ordering $i \overset{c}{>} j$ into three cases: (i) $P(j) \subset P(i)$, (ii) $C(j) \subset C(i)$, and (iii) neither (i) nor (ii) holds, where "$\subset$" denotes strict containment relation.

THEOREM 7: *Suppose that* $i \overset{c}{>} j$. *Then, the following hold:*

(i) $P(j) \subset P(i) \Leftrightarrow I_B(i;\mathbf{p}) > I_B(j;\mathbf{p})$ *for all* $\mathbf{p}$ *satisfying* $p_i \leq p_j$.
(ii) $C(j) \subset C(i) \Leftrightarrow I_B(i;\mathbf{p}) > I_B(j;\mathbf{p})$ *for all* $\mathbf{p}$ *satisfying* $p_i \geq p_j$.

PROOF: Case (i). ($\Leftarrow$) Suppose that there is a minimal path set P_r such that $j \in P_r$ and $i \notin P_r$. Then, since $i \overset{c}{>} j$, $\{i\} \cup P_r - \{j\}$ is a path set (not necessarily minimal). Hence, $\phi(0_i, 1_j, 1^{P_r - j}, \mathbf{0}) = 1$, $\phi(1_i, 0_j, 1^{P_r - j}, \mathbf{0}) = 1$, and $\phi(0_i, 0_j, 1^{P_r - j}, \mathbf{0}) = 0$. We then choose a probability vector such that $\Pr\{(\cdot_i, \cdot_j, \mathbf{X}) = (\cdot_i, \cdot_j, 1^{P_r - j}, \mathbf{0})\}$ is sufficiently close to one. Then, from Eqs. (5) and (7), $I_B(i;\mathbf{p}) < I_B(j;\mathbf{p})$ for some $p_i < p_j$, which contradicts our assumption. Thus, $P(j) \subseteq P(i)$ holds. Clearly, since $i \overset{c}{>} j$, there is a minimal path set P_r such that $i \in P_r$ and $j \notin P_r$; hence, $P(j) \subset P(i)$.

($\Rightarrow$) Suppose that $P(j) \subset P(i)$. It suffices to show that, by Theorem 6, $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j \geq 0$ for all $\mathbf{p}$. Let A_{ij} be the event that at least one minimal path set, containing both i and j, is functioning; let A_i be the event that at least one minimal path set, containing i but not j, is functioning; and let A be the event that at least one minimal path set, containing neither i nor j, is functioning. Further, let B_{ij} be the event that there exists a minimal path set, containing both i and j and in which the components other than i and j are functioning. Similarly, let B_i be the event that there exists a minimal path set, containing i but not j in which the components other than i are functioning. Then,

$$
\begin{aligned}
h(\mathbf{p}) &= \Pr\{A_{ij} \cup A_i \cup A\} \\
&= \Pr\{A_{ij}\} + \Pr\{A_i\} + \Pr\{A\} - \Pr\{A_{ij} A_i\} - \Pr\{A_{ij} A\} - \Pr\{A_i A\} \\
&\quad + \Pr\{A_{ij} A_i A\} \\
&= p_i p_j \Pr\{B_{ij}\} + p_i \Pr\{B_i\} + \Pr\{A\} - p_i p_j \Pr\{B_{ij} B_i\} - p_i p_j \Pr\{B_{ij} A\} \\
&\quad - p_i \Pr\{B_i A\} + p_i p_j \Pr\{B_{ij} B_i A\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_j} &= \Pr\{B_{ij}\} - \Pr\{B_{ij} B_i\} - \Pr\{B_{ij} A\} + \Pr\{B_{ij} B_i A\} \\
&= \Pr\{B_{ij}\} - \Pr\{B_{ij} B_i \cup B_{ij} A\} \\
&= \Pr\{B_{ij}\} - \Pr\{B_{ij} \cap (B_i \cup A)\} \\
&\geq 0, \quad \forall \mathbf{p}.
\end{aligned}
$$

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Case (ii). ($\Leftarrow$) Suppose that there is a minimal cut set C_k such that $j \in C_k$ and $i \notin C_k$. Then, since $i \overset{c}{>} j$, $\{i\} \cup C_k - \{j\}$ is a cut set (not necessarily minimal). Hence, $\phi(0_i, 1_j, 0^{C_k - j}, \mathbf{1}) = 0$, $\phi(1_i, 0_j, 0^{C_k - j}, \mathbf{1}) = 0$, and $\phi(1_i, 1_j, 0^{C_k - j}, \mathbf{1}) = 1$. Choose a probability vector such that $\Pr\{(\cdot_i, \cdot_j, \mathbf{X}) = (\cdot_i, \cdot_j, 0^{C_k - j}, \mathbf{1})\}$ is sufficiently large. Then, by employing similar arguments to that in (i), a contradiction is derived.

($\Rightarrow$) Suppose that $C(j) \subset C(i)$. We then let A_{ij} be the event at which at least one minimal cut set, containing both i and j, is working, where a minimal cut set working means that all of its components have failed. Define A_i, A, B_{ij}, and B_i similar to that in case (i), except that minimal path sets are replaced by minimal cut sets. Then,

$$\begin{aligned}
1 - h(\mathbf{p}) &= \Pr\{A_{ij} \cup A_i \cup A\} \\
&= \Pr\{A_{ij}\} + \Pr\{A_i\} + \Pr\{A\} - \Pr\{A_{ij} A_i\} - \Pr\{A_{ij} A\} - \Pr\{A_i A\} \\
&\quad + \Pr\{A_{ij} A_i A\} \\
&= (1 - p_i)(1 - p_j)\Pr\{B_{ij}\} + (1 - p_i)\Pr\{B_i\} + \Pr\{A\} \\
&\quad - (1 - p_i)(1 - p_j)\Pr\{B_{ij} B_i\} - (1 - p_i)(1 - p_j)\Pr\{B_{ij} A\} \\
&\quad - (1 - p_i)\Pr\{B_i A\} + (1 - p_i)(1 - p_j)\Pr\{B_{ij} B_i A\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^2 h(\mathbf{p})}{\partial p_i \, \partial p_j} &= -\Pr\{B_{ij}\} + \Pr\{B_{ij} B_i\} + \Pr\{B_{ij} A\} - \Pr\{B_{ij} B_i A\} \\
&= -\Pr\{B_{ij}\} + \Pr\{B_{ij} \cap (B_i \cup A)\} \\
&\leq 0, \quad \forall \mathbf{p}.
\end{aligned}$$

The conclusion then follows from Theorem 6. ■

Example 1 (continued): Consider nodes 1 and 2 in this example. Clearly, $P(2) \subset P(1)$ holds. Hence, $I_B(1;\mathbf{p}) > I_B(2;\mathbf{p})$ for all $p_1 \leq p_2$.

Example 2 (continued): Consider nodes 1 and 2 in this example. Clearly, $C(2) \subset C(1)$ holds. Hence, $I_B(1;\mathbf{p}) > I_B(2;\mathbf{p})$ for all $p_1 \geq p_2$.

Example 3: Consider the bridge structure shown in Figure 1. The minimal path sets of the system are $P_1 = \{1,4\}$, $P_2 = \{2,5\}$, $P_3 = \{1,3,5\}$, and $P_4 = \{2,3,4\}$; the minimal cut sets are $C_1 = \{1,2\}$, $C_2 = \{4,5\}$, $C_3 = \{1,3,5\}$, and $C_4 = \{2,3,4\}$. Consider nodes

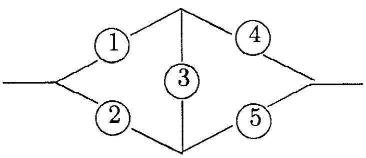

Figure 1. Bridge structure for Example 3.

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1 and 3. Clearly, $1 \overset{c}{>} 3$ holds, but neither $P(3) \subset P(1)$ nor $C(3) \subset C(1)$. Thus, the third case of the ordering "$\overset{c}{>}$" holds between nodes 1 and 3. In this example,

$$h(\mathbf{p}) = p_1 p_4 + p_2 p_5 + p_1 p_3 p_5 + p_2 p_3 p_4 - p_1 p_2 p_4 p_5 - p_1 p_3 p_4 p_5 - p_1 p_2 p_3 p_4 - p_1 p_2 p_3 p_5 - p_2 p_3 p_4 p_5 + 2 p_1 p_2 p_3 p_4 p_5,$$

$$\frac{\partial^2 h(\mathbf{p})}{\partial p_1 \partial p_3} = p_5(1 - p_2)(1 - p_4) - p_2 p_4 (1 - p_5).$$

Thus, $\partial^2 h(\mathbf{p})/\partial p_1 \partial p_3 > 0$ if $p_5 \to 1$ and $p_2, p_4 \to 0$, whereas $\partial^2 h(\mathbf{p})/\partial p_i \partial p_j < 0$ if $p_5 \to 0$ and $p_2, p_4 \to 1$.

3. CONCLUSION

Suppose that there exists a vector $(\cdot_i, \cdot_j, \mathbf{x})$ such that $\phi(1_i, 0_j, \mathbf{x}) < \phi(0_i, 1_j, \mathbf{x})$. Let $p(\mathbf{x}) = \Pr\{(\cdot_i, \cdot_j, \mathbf{X}) = (\cdot_i, \cdot_j, \mathbf{x})\}$. Then, from Eq. (5), we see that $\mathrm{I}_B(i;\mathbf{p}) < I_B(j;\mathbf{p})$ if $p(\mathbf{x})$ is sufficiently large and $|p_i - p_j|$ sufficiently small. Hence, $I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ for all $\mathbf{p}$ satisfying $p_i \leq (\geq)\, p_j$ holds only when nodes i and j are ordered by their criticality (i.e., $i \overset{c}{=} j$ or $i \overset{c}{>} j$). Theorems 5 and 7 further specify four structural conditions under which such comparisons can be made.

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