SOME STUDIES ON GROUP CHARACTERS

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Introduction

The theory of group characters was originated by G. Frobenius and has been studied by many authors including, above all, I. Schur and W. Burnside. As to the modular theory we owe it to recent works by R. Brauer and his collaborators, including T. Nakayama and C. Nesbitt, which clarified also the connections between the structure theory and the representation theory.

By using these results, especially those of R. Brauer we want to discuss in the present paper the correspondence between the blocks of characters of G and those of N, where N is a normal maximal subgroup of G (§2). Further we will prove several propositions on soluble groups which connect the structure theory with the representation theory to a certain degree and are proved by the arguments of §2 (§3).

§1. Preliminaries

Let us first cite some known results, mostly due to R. Brauer and C. Nesbitt. Let G be a group of finite order g and let K be the field of the primitive g-th root of unity over the rational number field. Then all the irreducible representations of G are realizable in K. Let $K_1, K_2, \ldots, K_n$ be the classes of conjugate elements in G and let $\chi_1, \chi_2, \ldots, \chi_n$ be the irreducible characters of G. Let p be a rational prime number and let $\mathfrak{p}$ be a prime divisor of p in K. Then we say that χ_i and χ_j belong to the same block (for $\mathfrak{p}$) if and only if we have $g_\nu \chi_i(K_\nu)/z_i \equiv g_\nu \chi_j(K_\nu)/z_j \pmod{\mathfrak{p}}$ for $\nu = 1, 2, \ldots, n$, where z_i and z_j are the degrees of χ_i and χ_j respectively, and g_ν is the index of the centralizer of the element in K_ν in G, and $\chi_i(K_\nu)$ is the value which χ_i takes at K_ν. Let $g = p^a g'$, where $(p, g') = 1$. If in a block all the degrees of characters belonging to it are divisible by p^α and at least one of them is not divisible by $p^{\alpha+1}$, then the block is said to be of defect $a - \alpha$. On the other hand if ρ_ν is the maximal exponent of p dividing g/g_ν then we call K_ν a class of defect ρ_ν. Further K_ν is called p-regular if, and only if, it contains an element which has an order prime to p.

The following four papers will often be quoted and refered to as BN_1, BN_2, B_1 and B_2, respectively.

R. Brauer and C. Nesbitt, On the modular representations of groups of finite

order, Univ. Toronto, Studies, Math. Series 4 (1937).

R. Brauer and C. Nesbitt, On the modular characters of groups, Ann. Math. 42, pp. 556-590 (1941).

R. Brauer, On the arithmetic in a group ring, Proc. Nat. Acad. Sci. 30, pp. 109-114 (1944).

R. Brauer, Investigations on group-characters, Ann. Math. 42, pp. 936-950 (1941).

Now let S_p be a p-Sylow subgroup of G. And we reduce χ_i to a sum of irreducible characters of S_p:

$$\chi_i = \sum r_{\kappa i}\, \varphi_\kappa$$

where $r_{\kappa i}$ is a non-negative rational integer. According to a theorem of Frobenius we have contragradiently

$$\varphi_\kappa{}^* = \sum r_{\kappa i}\, \chi_i$$

where $\varphi_\kappa{}^*$ is the character of G which is induced from the character φ_κ of S_p. Therefore it vanishes identically for all classes of G which do not intersect with S_p. Now all the p-regular classes except e do not meet with S_p. Therefore $\varphi_\kappa{}^*$ vanishes for such classes. We consider the following expression

$$\sum' r_{\kappa i}\chi_i - (e_\kappa/p^a) \sum z_i \chi_i$$

and see, by the well known orthogonal property and the definition of induced characters, that this expression vanishes identically for all the p-regular classes. Therefore, by a fundamental results in BN_1 §8, we have

$$\sum{}' (r_{\kappa i} - z_i e_\kappa/p^a)\, \chi_i = 0$$

for all the p-regular classes, where the summation extends over all the characters which belong to a certain arbitrarily fixed block. Especially for $K_\lambda = e$ we have that:

(*) $(p^a/e_\kappa) \sum{}' r_{\kappa i} z_i = \sum{}' z_i{}^2$.

This equation tells us many interesting facts, I will quote few of them in the following; the argument having been applied by R. Brauer. We first notice that though we have here started with a Sylow subgroup, the same results remain valid with an arbitrary p-subgroup. Namely, in (*) the right hand side is independent from κ and this implies that each block contains a character which contains an arbitrary given character of S_p. Consequently if the starting p-subgroup is normal we can conclude that each block contains a character belonging to such a normal p-subgroup. [Cf. B_1 §3]. So, if G has a normal subgroup of order p^a we can conclude that the defects of all the blocks of G

are at least d.

Secondly if χ_i is a character of defect 0 then χ_i is equal to a certain integral multiple of the character of the regular representation of S_p.

§ 2. Results

Let G be a group of finite order and let N be a normal subgroup of G with a prime index in G. Our concern is to establish a correspondence between the blocks of G and those of N. Firstly we consider a correspondence of the blocks of G with those of N (Reduction Part).

Now let G have a block B of defect d. By B_1 § 4, G has a defect group P of order p^r such that $\mathfrak{N}_G(P)$ has a block B^* of defect d corresponding to B. Setting $\widetilde{P} = P \cap N$ we consider $\mathfrak{N}_G(\widetilde{P})$ and $\mathfrak{N}_N(\widetilde{P})$. Concerning the relations between $\mathfrak{N}_G(\widetilde{P})$ and $\mathfrak{N}_N(\widetilde{P})$ the following four cases are clearly distinguished:

(1) $\qquad P = \widetilde{P}, \qquad \mathfrak{N}_G(P) = \mathfrak{N}_N(P)$.

(2) $\qquad P = \widetilde{P}, \qquad \mathfrak{N}_G(P) : \mathfrak{N}_N(P) = q$, with a prime q distinct from p.

(3) $\qquad P = \widetilde{P}, \qquad \mathfrak{N}_G(P) : \mathfrak{N}_N(P) = p$.

(4) $\qquad P : \widetilde{P} = p, \quad \mathfrak{N}_G(\widetilde{P}) : \mathfrak{N}_N(\widetilde{P}) = p$.

(The relation $\mathfrak{N}_G(\widetilde{P}) : \mathfrak{N}_N(\widetilde{P}) = p$ in the case (4) can be seen readily; for $P \subseteqq \mathfrak{N}_G(\widetilde{P})$ and $P \nsubseteqq \mathfrak{N}_N(\widetilde{P})$). We treat each case individually.

Case (1). This case is trivial. Since $\mathfrak{N}_G(P) = \mathfrak{N}_N(P)$ has a block B^* of defect d, by B_1 § 4, N has a block $\widetilde{B}$ of defect d corresponding to B^*. We associate B with $\widetilde{B}$. Both are blocks of defect d.

Case (2). Let χ be a character in the block B^* of $\mathfrak{N}_G(P)$ belonging to P; such a character actually exists because of our arguments in §1. We reduce χ to a sum of irreducible characters of $\mathfrak{N}_N(P)$:

$$\chi = (f/e_\kappa s_\kappa) \sum \varphi_\kappa$$

where f and e_κ are the degrees of χ and φ_κ respectively and s_κ is the number of conjugate characters of φ_κ in $\mathfrak{N}_G(P)$. (It is obvious that φ_κ belongs to P). Since χ is a character of defect d, by B_1 § 3, there exists at least a p-regular class of defect d in $\mathfrak{Z}_G(P) \cdot P$, say K_λ, such that $\chi(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$. And if such a class belongs to $\mathfrak{N}_N(P)$, then, as we can easily see, $f/e_\kappa s_\kappa \not\equiv 0 \pmod{\mathfrak{p}}$ and $\varphi_\kappa(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$[1] for at least one of the conjugate characters of φ_κ. Further it is obvious that the defects of K_λ in $\mathfrak{N}_G(P)$ and in $\mathfrak{N}_N(P)$ are coincident. Therefore, by B_1 § 3, φ_κ is a character of defect d. Let $\widetilde{B}$ be the block of characters of N corresponding to the block of characters of $\mathfrak{N}_N(P)$ to which φ_κ belongs. We associate B with $\widetilde{B}$. Both B and $\widetilde{B}$ are blocks of defect d.

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[1] Strictly speaking K_λ must be replaced by a suitable class of conjugate elements in $\mathfrak{N}_N(P)$ which is contained in K_λ.

Now we assume that none of such classes belongs to $\mathfrak{N}_N(P)$. Then we have necessarily $s_\kappa = 1$; for, otherwise, we have $\chi = \varphi_\kappa{}^*$ where $\varphi_\kappa{}^*$ is the character of $\mathfrak{N}_G(P)$ induced by φ_κ, and thus χ vanishes identically for all classes except those of $\mathfrak{N}_N(P)$, which contradicts with the fact that $\chi(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$. Then we have $\chi = (f/e)\varphi$. And since χ takes a value distinct from 0 at K_λ, the q characters $\chi_1 = \chi$, χ_2, ..., χ_q obtained from χ by multiplying the q linear characters of $\mathfrak{N}_G(P)$ belonging to $\mathfrak{N}_N(P)$ are all distinct and all equal to χ on $\mathfrak{N}_N(P)$. Therefore, by a theorem of Frobenius, φ^* contains these q characters and we have $\chi = \varphi$. Now we reduce χ of $\mathfrak{N}_G(P)$ to a sum of irreducible characters of $\mathfrak{Z}_G(P) \cdot P$:

$$\chi = (f/d_\kappa r_\kappa) \sum \psi_\kappa$$

where d_κ is the degree of ψ_κ and r_κ is the number of conjugate characters of ψ_κ in $\mathfrak{N}_G(P)$. Here K_λ belongs to $\mathfrak{Z}_G(P) \cdot P$ and this implies that $f/d_\kappa r_\kappa \not\equiv 0 \pmod{\mathfrak{p}}$ and that for at least one of the conjugate characters of ψ_κ, say ψ_κ itself, $\psi_\kappa(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$.[2] Now we consider $\mathfrak{N}_G(P)/P$ and $\mathfrak{Z}_G(P) \cdot P/P$ in place of $\mathfrak{N}_G(P)$ and $\mathfrak{Z}_G(P) \cdot P$. Then we see that χ and the all conjugate characters of ψ_κ are characters of defect 0 in the former and in the latter respectively. Next we reduce ψ_κ to a sum of irreducible characters of $\mathfrak{Z}_N(P) \cdot P/P$. Again we see readily that ψ_κ remains irreducible, for, otherwise, we have $\psi_\kappa(K_\lambda) = 0$ as above, and this is a contradiction. That is, when we reduce χ of $\mathfrak{N}_N(P)$ to the sum of irreducible characters of $\mathfrak{Z}_N(P) \cdot P$ we also have the same form:

$$\chi = (f/d_\kappa r_\kappa) \sum \psi_\kappa.$$

Then we consider $\chi^* = (f/d_\kappa r_\kappa) \sum \psi^*_\kappa$ which is the character of $\mathfrak{Z}_G(P) \cdot P$ induced by the character χ of $\mathfrak{Z}_N(P) \cdot P$. Since this is a sum of modular characters of $\mathfrak{Z}_G(P) \cdot P$ which are linearly independent mod $\mathfrak{p}$ by BN_1 §5, this takes a value $\not\equiv 0 \pmod{\mathfrak{p}}$ for at least one of the p-regular classes in $\mathfrak{Z}_N(P) \cdot P$. In fact this vanishes identically for all classes which does not belong to $\mathfrak{Z}_N(P) \cdot P$, for this is a character induced from a character of $\mathfrak{Z}_N(P) \cdot P$. And the defect of this class in $\mathfrak{N}_N(P)$ obviously equals to d. This contradicts our assumptions.

Case (3). Let χ be a character in the block B^* of $\mathfrak{N}_G(P)$ belonging to P. We reduce χ to the sum of irreducible characters of $\mathfrak{N}_N(P)$:

$$\chi = (f/e_\kappa s_\kappa) \sum \varphi_\kappa$$

where e_κ is the degree of φ_κ and s_κ is the number of conjugate characters of φ_κ in $\mathfrak{N}_G(P)$. Since χ is a character of defect d of $\mathfrak{N}_G(P)$, there exists by B_1

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[2] See [1].

§§ 3–4, at least one p-regular classes, say K_λ, of defect d in $\mathfrak{Z}_G(P) \cdot P$ such that $\chi(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$. Since all the p-regular classes are contained in $\mathfrak{N}_N(P)$ in this case this implies that $f/e_\kappa s_\kappa \not\equiv 0 \pmod{\mathfrak{p}}$ and that for at least one of the conjugate characters of φ_κ, say φ_κ itself, $\varphi_\kappa(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$.[3] Besides we easily see that in this case $s_\kappa = p$, for $f \equiv 0 \pmod{p^{a-d}}$ and $\mathfrak{N}_G(P) : P \not\equiv 0 \pmod{p^{a-d+1}}$ where p^a is the highest power of p which divides the order of $\mathfrak{N}_G(P)$. Thus we have $\chi = \sum \varphi_\kappa$. Again K_λ has the defect d in $\mathfrak{N}_N(P)$, too. Therefore, by $B_1 \S 3$, φ_κ belongs to a block $B\dagger$ of defect d of $\mathfrak{N}_N(P)$ and N has a block $\tilde{B}$ of defect d corresponding to $B\dagger$. We associate B with $\tilde{B}$. Both B and $\tilde{B}$ are blocks of defect d.

Case (4). As $\tilde{P}$ is normally contained in $\mathfrak{N}_G(P)$ we have $\mathfrak{N}_G(\tilde{P}) \supseteq \mathfrak{N}_G(P)$. Since $\mathfrak{N}_G(P) = \mathfrak{N}_{\mathfrak{N}_G(\tilde{P})}(P)$, as we see readily, $\mathfrak{N}_G(\tilde{P})$ has a block $B\dagger$ of defect d corresponding to B^* by $B_1 \S 4$. Let χ be a character of $B\dagger$ of $\mathfrak{N}_G(\tilde{P})$ belonging to P. We reduce χ to the sum of irreducible characters of $\mathfrak{N}_N(\tilde{P})$:

$$\chi = (f/e_\kappa s_\kappa) \sum \varphi_\kappa$$

where e_κ is the degree of φ_κ and s_κ is the number of conjugate characters of φ_κ in $\mathfrak{N}_G(\tilde{P})$. Now we assume that we have $\chi = \varphi$, that is, χ remains irreducible on $\mathfrak{N}_N(\tilde{P})$. Since χ is a character of defect d of $\mathfrak{N}_G(\tilde{P})$ there exists at least one p-regular class, say K_λ, of defect d in $\mathfrak{Z}_G(\tilde{P}) \cdot \tilde{P}$ such that $\chi(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$. Now this class belongs to $\mathfrak{N}_N(\tilde{P})$ and the defect of this class in $\mathfrak{N}_N(\tilde{P})$ is equal to $d-1$. Therefore, by $B_1 \S 3$, φ belongs to a block $B\dagger\dagger$ of defect $d-1$ of $\mathfrak{N}_N(\tilde{P})$. Next we assume that we have $s_\kappa = 1$, that is, $\chi = (f/e)\varphi$. Then since $\chi(K_\lambda) = (f/e)\varphi(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$[4] we can readily see that φ is a character of defect $d-1$. Let $B\dagger\dagger$ be the block of $\mathfrak{N}_N(\tilde{P})$ to which φ belongs. In these two cases N has a block $\tilde{B}$ of defect $d-1$ corresponding to $B\dagger\dagger$ again by $B_1 \S 3$. We associate B with $\tilde{B}$. Here $\tilde{B}$ is a block of defect $d-1$. Finally we assume further, that $s_\kappa = p$, $\chi = \varphi^*$, then χ vanishes identically on classes except those in $\mathfrak{N}_N(\tilde{P})$. Consequently if we take an element A of P not belonging to $\tilde{P}$ we have $\chi(AK_\lambda) = 0$ but this is congruent to $\chi(K_\lambda) \pmod{\mathfrak{p}}$, and the latter $\not\equiv 0 \pmod{\mathfrak{p}}$. This is a contradiction.

Next we consider a correspondence of the blocks of N with those of G. (Induction Part).

Let N have a block B of defect d. By $B_1 \S 4$, N has a defect group P of order p^a such that $\mathfrak{N}_N(P)$ has a block $\tilde{B}$ of defect d corresponding to B. We consider $\mathfrak{N}_G(P)$ and denote by P^* its maximal normal subgroup having a power

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[3] See [1].
[4] See [1].

of p as its order. Then $\mathfrak{N}_G(P^*) = \mathfrak{N}_G(P)$. For $\mathfrak{N}_G(P^*) \supseteqq \mathfrak{N}_G(P)$ is implied from the definition of P^*. On the other hand, $P = P^* \cap N$, and consequently, P is normally contained in $\mathfrak{N}_G(P^*)$ and this implies that $\mathfrak{N}_G(P) \supseteqq \mathfrak{N}_G(P^*)$. Concerning the relations between $\mathfrak{N}_G(P)$ and $\mathfrak{N}_N(P)$ the following four cases are clearly distinguished:

(1) $P = P^*$, $\mathfrak{N}_G(P) = \mathfrak{N}_N(P)$.

(2) $P = P^*$, $\mathfrak{N}_G(P) : \mathfrak{N}_N(P) = q$, with a prime q distinct from p.

(3) $P = P^*$, $\mathfrak{N}_G(P) : \mathfrak{N}_N(P) = p$.

(4) $P^* : P = p$, $\mathfrak{N}_G(P) : \mathfrak{N}_N(P) = p$.

We treat each case individually. The arguments are parallel to the above "Reduction Part." But we must repeat them for the sake of completeness.

Case (1). This case is rather trivial. since $\mathfrak{N}_N(P) = \mathfrak{N}_G(P)$ has a block $\tilde{B}$ of defect d, by B_1 §4, G has a block B^* of defect d corresponding to $\tilde{B}$. We associate B with B^*. B and B^* are blocks of defect d.

Case (2). Let φ_κ be a character of $\tilde{B}$ of $\mathfrak{N}_N(P)$ belonging to P and let χ be a character of $\mathfrak{N}_G(P)$ containing φ_κ on $\mathfrak{N}_N(P)$. (It is obvious that χ belongs to P):

$$\chi = (f/e_\kappa s_\kappa) \sum \varphi_\kappa$$

where e_κ is the degree of φ_κ and s_κ is the number of conjugate characters of φ_κ in $\mathfrak{N}_G(P)$. Firstly if we have $\chi = \varphi$ then χ takes a value $\not\equiv 0$ (mod $\mathfrak{p}$) for at least one, say K_λ, of the p-regular classes of defect d in $\mathfrak{Z}_N(P) \cdot P$ and the defect of K_λ in $\mathfrak{N}_G(P)$ equals to d, too. Therefore, in this case, by B_1 §3, $\mathfrak{N}_G(P)$ has a block $B\dagger$ of defect d, and by B_1 §4, G has a block B^* of defect d corresponding to $B\dagger$. We associate B with B^*. Both B and B^* are blocks of defect d. Secondly, if we have $s_\kappa = 1$, then $\chi = (f/e)\varphi$. If we consider this as a character of $\mathfrak{N}_G(P)/P$, then it becomes a character of defect 0, as we easily see. This implies that $f/e \not\equiv 0$ (mod $\mathfrak{p}$). Lastly if we have $s_\kappa = q$ then we have that $\chi = \sum \varphi_\kappa$, $\varphi_\kappa^* = \chi$ and $f = qe$, where φ_κ^* is the character of $\mathfrak{N}_G(P)$ induced by a character φ_κ of $\mathfrak{N}_N(P)$. In the last two cases, there exists, by B_1 §§3–4, at least one p-regular class, say K_λ, in $\mathfrak{Z}_N(P) \cdot P$ such that $\varphi_\kappa(K_\lambda) \not\equiv 0$ (mod $\mathfrak{p}$),[5] since φ_κ is a character of defect d of $\mathfrak{N}_N(P)$. Therefore, when we reduce φ_κ to the sum of irreducible characters of $\mathfrak{Z}_N(P) \cdot P$:

$$\varphi_\kappa = (e_\kappa/d_\lambda r_\lambda) \sum \psi_{\kappa\lambda}$$

where d_λ is the degree of $\psi_{\kappa\lambda}$ and r_λ is the number of conjugate characters of $\psi_{\kappa\lambda}$ in $\mathfrak{N}_N(P)$, we can conclude that $e_\kappa/d_\lambda r_\lambda \not\equiv 0$ (mod $\mathfrak{p}$). Now if any two conju-

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5) See.[1]

gate characters of φ_κ are equivalent to each other on $\mathfrak{Z}_N(P) \cdot P$ then we can readily see that all the conjugate characters of φ_κ are equivalent one another on $\mathfrak{Z}_N(P) \cdot P$. Therefore, we can conclude that χ is a sum of irreducible characters ψ of $\mathfrak{Z}_N(P) \cdot P$ with p-regular coefficients. On the other hand, considering $\mathfrak{N}_N(P)/P$ and $\mathfrak{Z}_N(P) \cdot P/P$ we can easily see, by $BN_2 \S 12$, that all the ψ's are modular characters, because they are characters of defect 0 of $\mathfrak{Z}_N(P) \cdot P/P$, and since again they are linearly independent mod $\mathfrak{p}$ by $BN_1 \S 5$, χ takes a value $\not\equiv 0 \pmod{\mathfrak{p}}$ at K_λ in $\mathfrak{Z}_N(P) \cdot P$ and it is obvious that the defect of K_λ in $\mathfrak{N}_G(P)$ are equal to d, too. This implies, by $B_1 \S 3$, that χ is a character of defect d of $\mathfrak{N}_G(P)$. Consequently $\mathfrak{N}_G(P)$ and, therefore by $B_1 \S 4$, G has a block B^* of defect d. We associate B with B^*. They are blocks of defect d.

Case (3). Let φ_κ be a character of $\tilde{B}$ of $\mathfrak{N}_N(P)$ belonging to P and let χ be a character of $\mathfrak{N}_G(P)$ containing φ_κ:

$$\chi = (f/e_\kappa s_\kappa) \sum \varphi_\kappa$$

where e_κ is the degree of φ_κ and s_κ is the number of conjugate characters of φ_κ in $\mathfrak{N}_G(P)$. Here the following three subcases are distinguished.

(i) $s_\kappa = p$, $\varphi_\kappa^* = \chi$ and all the conjugate characters of φ_κ are distinct even on $\mathfrak{Z}_N(P) \cdot P$.

(ii) $s_\kappa = p$, $\varphi_\kappa^* = \chi$ and the conjugate characters of φ_κ are all equal on $\mathfrak{Z}_N(P) \cdot P$.

(iii) $s_\kappa = 1$, $\chi = \chi_1$, χ_2, ..., χ_p are all equal to φ_κ on $\mathfrak{N}_N(P)$, where φ_κ^* is the character of $\mathfrak{N}_G(P)$ induced by φ_κ and χ_1, χ_2, ..., χ_p are characters which obtained from χ by multiplying with p linear characters of $\mathfrak{N}_G(P)$ belonging to $\mathfrak{N}_N(P)$. In fact if χ vanishes identically outside of $\mathfrak{N}_N(P)$ then χ is a character of defect 0 of $\mathfrak{N}_G(P)/P$ and we can readily see that $s_\kappa = p$, $\chi = \varphi_\kappa^*$. If this is not the case then the p characters χ_1, χ_2, ..., χ_p are all distinct and φ_κ^* contains all these characters by a theorem of Frobenius. Therefore $s_\kappa = 1$, and $\chi = \chi_1$, χ_2, ..., χ_p are all equivalent to φ_κ on $\mathfrak{N}_N(P)$. Each subcase is treated separatedly.

Subcase (i). In this case we reduce φ_κ to the sum of irreducible characters of $\mathfrak{Z}_N(P) \cdot P$:

$$\varphi_\kappa = (e_\kappa/d_\lambda r_\lambda) \sum \psi_{\kappa\lambda}$$

where d_λ is the degree of $\psi_{\kappa\lambda}$ and r_λ the number of conjugate characters of $\psi_{\kappa\lambda}$ in $\mathfrak{N}_N(P)$. Under the assumptions in this case those irreducible characters of $\mathfrak{Z}_N(P) \cdot P$ which are obtained by reducing φ_κ on $\mathfrak{Z}_N(P) \cdot P$ are all distinct. Now

6) See.[1]

since $\psi_{\kappa\lambda}$ are characters of defect 0 of $\mathfrak{Z}_N(P) \cdot P/P$ as we see readily, they are modular characters by BN_2 §12, and therefore, by BN_1 §5 they are linearly independent mod $\mathfrak{p}$. Now as a character of defect d of $\mathfrak{N}_N(P)$ φ_κ takes, by B_1 §§3-4, a value, $\not\equiv 0 \pmod{\mathfrak{p}}$ for at least one, say K_λ,[6] of the p-regular classes of defect d in $\mathfrak{Z}_N(P) \cdot P$ and this implies that $e_\kappa/d_\lambda r_\lambda \not\equiv 0 \pmod{p}$ (and it is obvious that $e_\kappa/d_\lambda r_\lambda$ are all equal independent from κ). In this case χ can be written, after all, as a sum of (modular) irreducible characters $\psi_{\kappa\lambda}$ with p-regular coefficients on $\mathfrak{Z}_N(\mathfrak{p}) \cdot P$. Therefore χ takes a value $\not\equiv 0 \pmod{\mathfrak{p}}$ at K_λ and χ becomes a character of defect d of $\mathfrak{N}_G(P)$. Finally G has a block B^* of defect d corresponding to that of χ and we associate B with B^*. Both are blocks of defect d.

In the next place we observe a certain fact from which the remaining subcases (ii) and (iii) readily follow. Now we denote by θ a p-automorphism of $\mathfrak{N}_N(P)$ by an element of $\mathfrak{N}_G(P)$ not belonging to $\mathfrak{N}_N(P)$ and assume that $\varphi_\kappa = \varphi_\kappa{}^\theta$. Then there exists at least one p-regular class, say K_λ, of defect d of $\mathfrak{N}_N(P)$ such that $\varphi_\kappa(K_\lambda) \not\equiv 0 \pmod{\mathfrak{p}}$ and $K_\lambda = K_\lambda{}^\theta$. To see this we reduce φ_κ to the sum of irreducible characters of $\mathfrak{Z}_N(P) \cdot P$:

$$\varphi_\kappa = (e_\kappa/d_\lambda r_\lambda) \sum' \psi_{\kappa\lambda}$$

where d_λ is the degree of $\psi_{\kappa\lambda}$ and r_λ is the number of conjugate characters of $\psi_{\kappa\lambda}$ in $\mathfrak{N}_N(P)$. Then we may assume that $\psi_{\kappa\lambda} = \psi_{\kappa\lambda}{}^\theta$. For we have, at any rate, $\psi_{\kappa\lambda}{}^\theta = \psi_{\kappa\lambda}{}^A$ where A is some element of $\mathfrak{N}_N(P)$, and so we may replace a suitable power of θA^{-1} for θ. If we consider $\psi_{\kappa\lambda}$ on $\mathfrak{Z}_N(P) \cdot P/P$ then it becomes a character of defect 0. And if we assume that all such classes in question are altered by θ, we have by BN_1 §5, denoting by g_ν the index of K_ν in $\mathfrak{Z}_N(P) \cdot P/P$,

$$0 \equiv \sum g_\nu \varphi_\kappa(K_\nu) \psi_{\kappa\lambda}(K_\nu{}^*) \equiv \sum_{\varphi_\kappa \not\equiv 0} g_\nu \varphi_\kappa(K_\nu) \psi_{\kappa\lambda}(K_\nu{}^*) \pmod{p^a\mathfrak{p}}$$

where p^a is the highest power of p dividing the order of $\mathfrak{Z}_N(P) \cdot P/P$. But from our assumption follows that the latter $\equiv 0 \pmod{p^{a+1}}$. This is, however, a contradiction.

Subcase (ii). From above, we have by B_1 §3, in the same manner as before, that χ is a character of defect $d+1$ of $\mathfrak{N}_G(P)$. Now if we denote by P^* the defect group of χ in $\mathfrak{N}_G(P)$ then we can readily see that $P^* \not\subseteq \mathfrak{N}_N(P)$ and $P = P^* \cap \mathfrak{N}_N(P)$. Therefore $\mathfrak{N}_G(P^*) \subseteq \mathfrak{N}_G(P)$ and $\mathfrak{N}_{\mathfrak{N}_G(P)}(P^*) = \mathfrak{N}_G(P^*)$. Consequently $\mathfrak{N}_G(P^*)$ has a block of defect $d+1$ and, by B_1 §4, G has a block B^* of defect $d+1$ corresponding to it. We associate B with B^*. The latter is a block of defect $d+1$.

Subcase (iii). This case may be treated in exactly the same manner as

above.

Case (4). Here obviously $\mathfrak{N}_G(P^*)/P^* \cong \mathfrak{N}_N(P)/P$ by the second isomorphism theorem. Therefore a character of $\tilde{B}$ of $\mathfrak{N}_N(P)$ belonging to P may be considered as a character of $\mathfrak{N}_G(P^*)$ belonging to P^*. Now let h_p be an arbitrary p-regular element of $\mathfrak{Z}_N(P) \cdot P$ and set $P^* = \{P, A\}$. We consider an element $h_p^{-1}A^{-1}h_pA$. Then this belongs to both P^* and $\mathfrak{Z}_N(P) \cdot P$, whence to P, and since h_p is commutative with every element of P, it coincides with e, that is, h_p is commutative with A. Therefore, $\mathfrak{N}_G(P^*)$ has a block of defect $d+1$, by B_1 §3. Then G has also a block B^* of defect $d+1$ by B_1 §4. We associate B to B^*. The latter is a block of defect $d+1$.

Thus we can assert the following theorem:

THEOREM. *Let G be a group of finite order and let N be its normal subgroup with a prime index in G. Then we can associate with in a definite manner each block of defect d of G that of N which is a block of defect d or $d-1$. And conversely we can associate with each block of defect d of N that of G which is a block of defect d or $d+1$.*

§3. Applications

As a consequence of the arguments in the preceding section we may expect that we can analyze the connection between the representation——especially the block of characters——and the structure——especially the Sylow structure——of a *soluble* group. For the present we have, however, only some rather fragmental results and we hope we can come to a complete theory in future.

On the other hand, in BN_2 §29, there is the following description: If in a given group a Sylow subgroup is a normal subgroup then all the blocks of characters of this group are blocks of defect a, where a is the highest exponent of a prime which divides the order of the group. And the converse is unknown. Now we can answer the problem to a certain degree for soluble groups. Even for soluble groups we cannot conclude in general that the proposition "all the blocks are blocks of defect a" implies the proposition "a Sylow subgroup is a normal subgroup." With $p = 2$ $\mathfrak{S}_4$ gives such an instance. Further we can construct such a group in a simple, typical manner for an arbitrarily given prime p, as we describe in the following.

Example: Let p be a given prime number. Take a prime number q such that $q \equiv 1 \pmod{p}$. Construct the non-cyclic group H of order pq. Then H has a faithful representation of degree p in the field of the q-th roots of unity over the rational field. Let $\mathfrak{p}$ be any prime divisor of p in this field. If f is the exponent of p with regard to q then $\mathfrak{p}$ has the degree f. Then by BN_2 §12

H may be considered as the group of linear transformations over the p-dimensional vector space V_p over the finite field $GF(p^f)$. Let G be the holomorph of V_p by H. Then G is the group of order $p^{fp+1}\,q$ and has only one (p-) block as we can verify easily. But in G every Sylow subgroup is non-normal.

In this group there is a normal subgroup whose order is a power of p (distinct from e) and this fact, in general, can be proved for soluble groups. But for insoluble groups this fact is not valid, for example, the well known *simple* permutation group of degree 24 of Mathieu type has only one block for $p=2$.

From now on we assume that all the groups are *soluble*. Then the following propositions are valid.

PROPOSITION 1. If G has no normal subgroup whose order is a power of prime p then there exists a block of defect $< a$ in G, where a is the highest exponent of p which divides the order of G. (We assume that the order of G is p-singular.)

PROPOSITION 2. If in G a Sylow subgroup is abelian and nonnormal then there exists a block of defect $< a$ in G, where a is the same as above.

PROPOSITION 3. If in G all Sylow subgroups are disjoint then there exists a block of defect 0 in G.

PROPOSITION 4. If the order of G is $g = p^a g'$ where $(p, g') = 1$ and $g \leqq p^{2a}$ then there is no block of defect 0 in G.

PROPOSITION 5. If the degrees of irreducible representations of G are all p-regular then a Sylow subgroup is normal in G.

Proof of proposition 1: Assume that the assertion is true for groups of smaller orders. Let N be a normal subgroup of G with a prime index in G. Then N has also no normal subgroup whose order is a power of prime p. Therefore there is by induction hypothesis a block of defect $< a$ or $< a-1$ in N according as the index of N in G is distinct from p or not. Then by the arguments in §2 there is in G itself a block of defect $< a$. However as for the case $a = 1$ we treat it in the next proof of proposition 2.

Proof of proposition 2: Assume that the assertion is true for groups of smaller orders. Let N be a normal subgroup of G with a prime index in G. If the index of N in G is distinct from p then there is a block of defect $< a$ in N by induction hypothesis. Further by the arguments in §2 there is in G itself a block of defect $< a$. Consequently we may assume that the index of N in G is equal to p. In this case, too, if there is a block of defect $< a-1$ in N then by the arguments in §2 there is in G itself a block of defect $< a$.

Then we may assume that the p-Sylow subgroup $S_p(N)$ of N is a normal subgroup again by induction hypothesis. In this case if $S_p(N)$ is contained in the center of G then we consider $G/S_p(N)$. If we have here $S_p(N) > e$ then by induction hypothesis $G/S_p(N)$ has a block of defect $< a$ and then G itself has a block of defect $< a$ by B_1 §3. If $S_p(N)$ is not contained in the center of G then we consider $\mathfrak{Z}_G(S_p(N)) \neq G$. By induction hypothesis there is in $\mathfrak{Z}_G(S_p(N))$ a block of defect $a - 1$. Since S_p is contained in $\mathfrak{Z}_G(S_p(N))$, further, in this case, by the arguments in §2, G itself has a block of defect $a - 1$. Now there remains the case $S_p(N) = e$, that is, the case where the order of N is p-regular. In this last case we consider a minimal normal subgroup M of G with p-singular order and a maximal normal subgroup M_1 of G which is contained properly in the p-complement of M. Then the group M/M_1 is a group of Frobeniusean type and therefore it has a block of defect 0. Consequently G/M_1, whence G itself has a block of defect 0, because of $a = 1$. Thus the proof of this proposition is complete.

Proof of proposition 3:[7] Assume again that the assertion is true for groups of smaller orders. Let N be a normal subgroup of G with a prime index in G. If the index of N in G is distinct from p then N has a block of defect 0 by induction hypothesis and further G itself has a block of defect 0. Therefore we may assume that the index of N in G is equal to p. If there is no block of defect 0 in G there is a block of defect 1 in G by B_2 §3. Let P be the defect group of this block. Then $\mathfrak{N}_G(P)$ has a block of defect 1. But under the assumption of this proposition in $\mathfrak{N}_G(P)$ a Sylow subgroup is normal and this implies that the order of N is p-regular that is, $a = 1$ which can be seen from the elementary property of p-groups. We can prove the remainder in the same way as in the proof of proposition 2.

Proof of proposition 4: Assume again that the assertion is true for groups of smaller orders. If G has a block of defect 0 then N also has a block of defect 0 by the arguments in §2 where N is a normal subgroup of G with a prime index in G. Here if the index of N in G is distinct from p then we have $g/q > p^a$ by induction hypothesis. Then we may assume that the index of N in G is equal to p. Then a character of defect 0 of G is reduced on N to the sum of p conjugate characters of defect 0 of N. From a well known property of group characters we have $g/p > p \cdot p^{(a-1)2}$, that is, $g > p^{2a}$.

Proof of proposition 5: Assume again that the assertion is true for groups

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[7] Mr M. Suzuki kindly communicated to me that this is an immediate consequence of Proposition 1 since a defect group is apparently the meet of some Sylow subgroups. (Added on Sep. 3, 1950).

of smaller orders. Let N be a normal subgroup of G with a prime index in G. Then N also satisfies the condition which is fullfilld by G. Therefore in N a Sylow subgroup is normal by induction hypothesis. Denoting it by $S_p(N)$ we consider $G/S_p(N)$. Now, if $S_p(N)$ is a Sylow subgroup of G no problem remains. We assume that $S_p(N)$ is not a Sylow subgroup of G. Then $G/S_p(N)$ also satisfies the same condition which is fullfilled by G. Therefore in $G/S_p(N)$ a Sylow subgroup is normal by induction hypothesis. But this argument fails for the case $S_p(N) = e$, that is, the case $a = 1$. The remainder can be obtained by the arguments used.

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