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On Arnol'd's and Kazhdan's equidistribution problems

ALEXANDER GORODNIK† and AMOS NEVO‡

† *School of Mathematics, University of Bristol, Bristol, UK*
(e-mail: a.gorodnik@bristol.ac.uk)
‡ *Department of Mathematics, Technion, Haifa, Israel*
(e-mail: anevo@tx.technion.ac.il)

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Abstract. We consider isometric actions of lattices in semisimple algebraic groups on (possibly non-compact) homogeneous spaces with (possibly infinite) invariant Radon measure. We assume that the action has a dense orbit, and demonstrate two novel and non-classical dynamical phenomena that arise in this context. The first is the existence of a mean ergodic theorem even when the invariant measure is infinite; this implies the existence of an associated limiting distribution, which can be different from the underlying invariant measure. The second is uniform quantitative equidistribution of all orbits in the space, which follows from a quantitative mean ergodic theorem for such actions. In turn, these results imply quantitative ratio ergodic theorems for isometric actions of lattices. This sheds some unexpected light on certain equidistribution problems posed by Arnol'd [*Arnol'd's Problems*. Springer, Berlin, 2004] and also on the ratio equidistribution conjecture for dense subgroups of isometries formulated by Kazhdan [Uniform distribution on a plane. *Tr. Mosk. Mat. Obs.* **14** (1965), 299–305]. We briefly mention the general problem regarding ergodic theorems for actions of lattices on homogeneous spaces and its solution given by Gorodnik and Nevo [Duality principle and ergodic theorems, in preparation], and present a number of examples to demonstrate our results. Finally, we also prove results on quantitative equidistribution for absolutely continuous averages in transitive actions.

1. *Equidistribution beyond amenable groups*

Let G be a locally compact second countable (lcsc) group acting continuously on an lcsc space X. Assume that X carries a σ-finite G-invariant Radon measure μ of full support. Consider the following three natural conditions that often arise in practice.

(1) The action of G on X is transitive.
(2) The action of G on X has a unique invariant Radon measure.
(3) The action of G on X is isometric.

Letting Γ be a countable dense subgroup of G, consider the problem of formulating and establishing equidistribution results for the Γ-orbits in X. This problem has been studied in the past mainly in the case where the group Γ is amenable and the measure μ is finite. It is a compelling challenge to generalize the theory to the case where the group is non-amenable and the measure may be infinite. In particular, one would like to establish that a limiting distribution for the Γ-orbits exists, and study what its properties might be. This challenge can be generalized further to the case where X is any standard Borel space with σ-finite measure, where now the results sought are mean ergodic theorems in L^p and pointwise ergodic theorems that hold almost everywhere.

The possible choices of G, X and Γ include a large set of important examples arising naturally in various branches of dynamics. We will discuss some of these examples below.

In the present paper we concentrate on establishing equidistribution results with an effective rate of convergence for certain non-amenable groups and, in particular, lattice subgroups of semisimple algebraic groups. A major ingredient in our considerations will be mean ergodic theorems for actions of these groups.

We note that a mean ergodic theorem for spaces with infinite measure is a novel and distinctly non-classical phenomenon. Indeed, it is well-known (see [**Aa**, Ch. 2]) that for an action of a single transformation on a space with infinite measure, no formulation of such a result is possible. Likewise, an effective rate of orbit equidstribution is a phenomenon that does not arise in the ergodic theory of amenable groups, since the ergodic averages may converge arbitrarily slowly.

1.1. *Some background.* The problem of extending ergodic theory to general countable groups was raised half a century ago by Arnol'd and Krylov [**AK**]. They established equidistribution of dense free subgroups of $SO_3(\mathbb{R})$ acting on S^2, with respect to word length, and formulated the problem of establishing ergodic theorems for balls with respect to word length for actions of general countable finitely generated groups. Motivated by the problems raised in [**AK**], Kazhdan [**K**] established that the orbits of certain dense 2-generator subsemigroups of the isometry group of the plane satisfy a ratio ergodic theorem, namely that for every $x \in X$ and any two bounded open sets A_1 and A_2 (with nice boundary),

$$\lim_{t\to\infty} \frac{|\{\gamma \in B_t : \gamma^{-1}x \in A_1\}|}{|\{\gamma \in B_t : \gamma^{-1}x \in A_2\}|} = \frac{m(A_1)}{m(A_2)}.$$

Here B_t denotes the ball of radius t with respect to the word length on the free semigroup, so that the counting is in effect governed by the weights given by convolution powers, and m is Lebesgue measure on the plane. Kazhdan raised in [**K**] the question of extending this result to other dense subgroups of a Lie group G acting on a homogeneous space $X = G/H$, particularly when the action is isometric. Motivated by [**AK, K**], Guivarc'h proved in [**Gu1**] the mean ergodic theorem for actions of free groups on a probability space, generalizing von-Neumann's classical result, and established in [**Gu2**] a generalization of Kazhdan's ratio ergodic theorem for certain dense subsemigroups of isometries of Euclidean spaces, the weights being given again by the convolution powers of a fixed probability measure on Γ. Guivarc'h also raised the problem of establishing equidistribution results in the generality of actions with a unique invariant measure.

1.2. *Ergodic theorems on homogeneous spaces: Arnol'd's problems.* Subsequently (see [**A**, problems 1996-15 on page 115 and 2002-16 on page 148]), Arnol'd revisited this topic and posed the following challenges. Consider the standard Lorentzian form $Q(x, y, z) = x^2 + y^2 - z^2$ on $\mathbb{R}^{2,1}$ together with the identity component of the group of isometries of the form, denoted by $G = \mathrm{SO}^0(2, 1)$. Under the standard linear action of G, the space $\mathbb{R}^{2,1}$ decomposes into three invariant subsets of different types, as follows:

(1) the light cone $\mathcal{C}$, namely the set where the form vanishes;
(2) the two-sheeted hyperboloid $\mathcal{H}$ given by $\{x^2 + y^2 - z^2 = -1\}$, each of whose components inherits a G-invariant Riemannian structure of constant negative curvature isometric to the hyperbolic plane; thus $\mathcal{H}$ is a homogeneous space G/K with compact stability group conjugate to $K = \mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$;
(3) the one-sheeted hyperboloid known as the de-Sitter space $\mathcal{S}$ and given by $\{x^2 + y^2 - z^2 = 1\}$, which inherits a G-invariant two-dimensional Lorentzian structure and is a homogeneous space G/H with stability group conjugate to $H = \mathrm{SO}^0(1, 1) \cong \mathbb{R}$.

The group G has a natural action on the projectivization of the positive light cone, namely the usual action by fractional linear transformations of the circle. The circle forms the boundary of the hyperbolic plane and is denoted by $\mathcal{B}$.

Now consider any lattice subgroup $\Gamma \subset G$. Then all orbits of the lattice in hyperbolic space are discrete, and all orbits of the lattice on the boundary are dense. On the de-Sitter space, almost all Γ-orbits are dense, but not all. For example, the Γ-orbit of a point is discrete if the intersection of its stability group with the lattice is a lattice in the stability group. Thus, two problems that arise naturally and which were formulated by Arnol'd are the following.

(1) Establish equidistribution of the lattice orbits on the boundary $\mathcal{B}$.
(2) Establish ergodic theorems for dense lattice orbits in the de-Sitter space $\mathcal{S}$.

The first problem was solved in [**G1**] (see also [**GM, GO**] for generalizations); the second problem was solved by Maucourant (unpublished) and in greater generality with explicit rate in [**GN2**]. The distribution of orbits for the action of Γ on the positive light cone $\mathcal{C}$ was computed in [**G2, L, No**] and [**GW**, §12].

1.3. *Ergodic theorems beyond amenable groups: some surprises.* We now turn to explicating the results alluded to above and to describing their general context.

The study of the distribution of G-orbits in a general σ-finite measure space X proceeds by fixing a family of bounded Borel measures β_t on G for $t \in \mathbb{Z}^+$ or $t \in \mathbb{R}^+$. The measures β_t are not necessarily probability measures. For example, one important special case is where $B_t \subset G$ is a family of bounded sets of positive Haar measure; we then fix a choice of growth rate function $V(t)$ and define β_t to be Haar measure on B_t divided by $V(t)$. The growth function $V(t)$ may be of lower order of magnitude than $m_G(B_t)$, for instance.

We consider the operators defined on a compactly supported test function $f : X \to \mathbb{R}$ by

$$\pi_X(\beta_t) f(x) = \int_G f(g^{-1}x)\, d\beta_t(g).$$

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Thus, in the special case noted above,

$$\pi_X(\beta_t)f(x) = \frac{1}{V(t)}\int_{g\in B_t} f(g^{-1}x)\,dg.$$

The properties of this family of operators provide the key to analyzing the distribution of the orbit $G\cdot x$. Of course, in the classical case of amenable groups acting by measure-preserving transformations on a probability space, we have for a family of sets $B_t \subset G$ that:

(i) the right choice of growth function $V(t)$ is the total measure $m_G(B_t)$, so that the operators above become averaging (i.e. Markov) operators;

(ii) the limit of the time averages as $t \to \infty$ is the space average of the function, when the probability measure is ergodic; in particular, the limiting distribution is G-invariant.

It turns out that the ergodic theory of non-amenable groups is full of surprises, and reveals several phenomena that have no analogues in classical amenable ergodic theory.

(1) The operators $\pi_X(\beta_t)$ may fail to converge, even when the β_t are normalized ball averages with respect to a word metric and the action is a dense isometric action on a compact homogeneous space preserving Haar measure.

(2) The operators $\pi_X(\beta_t)$ may converge to a limit operator, but the limit may be different from the ergodic mean, even when the β_t are normalized ball averages with respect to a word metric and the action is a dense isometric action on a compact homogeneous space preserving Haar measure.

(3) When the invariant measure is infinite, the operator $\pi_X(\beta_t)$ associated with a family B_t may converge for a choice of growth function $V(t)$ which is of lower order of growth than $m_G(B_t)$, with convergence for almost all points or even for all points x outside a countable set:

$$\lim_{t\to\infty}\frac{1}{V(t)}\int_{g\in B_t} f(g^{-1}x)\,dg = \int_X f\,d\nu_x, \qquad \nu_x \neq 0.$$

(4) The limit measure ν_x appearing in (3) may be non-invariant and depend non-trivially on the initial point x. Furthermore, the limit measure may be completely different if the family of sets B_t which are taken as the support of the measures β_t is changed.

(5) The expression in (3) may converge for *each and every* $x \in X$, and yet the measure ν_x may still be non-invariant and may depend on the initial point x as well as the family B_t. This can happen even when the invariant measure is unique and even when the action is isometric.

(6) The operators $\pi_X(\beta_t)$ in (3) may converge with a *uniform rate of convergence*, valid for almost all points, or even for all points; that is,

$$\left|\frac{1}{V(t)}\int_{g\in B_t} f(g^{-1}x)\,dg - \int_X f\,d\nu_x\right| \le C(x, f)V(t)^{-\delta}.$$

This can happen in compact spaces and also in non-compact spaces.

(7) As a result, convergence of the ratios:

$$\frac{|\{\gamma \in B_t : \gamma^{-1}x \in A_1\}|}{|\{\gamma \in B_t : \gamma^{-1}x \in A_2\}|}$$

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may take place at a uniform rate for almost all points. As before, even for an isometric action (with infinite invariant measure) the uniform rate may apply to all points, but the ν_x appearing in the limiting expression $\nu_x(A_1)/\nu_x(A_2)$ may depend on x and B_t.

We remark that (1) and (2) are implicit already in [**AK, Gu1**], and that (1) has been noted explicitly in [**B**] (see also Theorem 1.1 below).

The phenomena described in (3) and (4) were first demonstrated by a pioneering result of Ledrappier [**L**] on the distribution of orbits of lattice subgroups of $\mathrm{SL}_2(\mathbb{R})$ in the real plane; see also [**LP1, LP2**]. Equivalently, the result applies to the action of a lattice in $\mathrm{SO}^0(2, 1)$ on the light cone $\mathcal{C}$ above (see [**GW**, Theorem 12.2] for more information).

The phenomenon described in (5) for isometric actions was first demonstrated in [**GW**, Corollary 1.4(ii)] (see also Theorem 1.6 below).

Regarding (6) and (7), note that mean and pointwise ergodic theorems with uniform rates for semisimple Kazhdan groups acting on probability spaces have been established in [**GN1, MNS, Ne2**]. The question of whether an ergodic theorem implies equidistribution with rates for all points forms one of the main subjects of the present paper. The solution to this problem gives the phenomena described in (6) and (7) as immediate corollaries.

1.4. *The mean ergodic theorem and equidistribution in compact spaces.* Assume now that the space X is equipped with a G-invariant probability measure μ. We say that $\pi_X(\beta_t)$ satisfies the *mean ergodic theorem* in L^2 (with limit operator $\mathcal{P}$) if for every $f \in L^2(X, \mu)$,

$$\left\| \int_G f(g^{-1}x)\, d\beta_t(g) - \mathcal{P}f(x) \right\|_2 = E(f, t) \longrightarrow 0 \quad \text{as } t \to \infty, \tag{1.1}$$

where $\mathcal{P} : L^2(X, \mu) \to L^2(X, \mu)$ is a linear operator, which may be different from the orthogonal projection on the space of G-invariant functions; see, for instance, Theorem 1.1 below.

The mean ergodic theorem is known to hold for several large classes of lcsc groups, including general amenable groups (see [**Ne3**] for a survey) and also semisimple S-algebraic groups and their lattice subgroups (see [**GN1**] for a comprehensive discussion). The next obvious question regarding the distribution of orbits concerns pointwise convergence of the averages, i.e. whether for every $f \in L^2(X, \mu)$ and almost all $x \in X$ it is true that

$$\left| \int_G f(g^{-1}x)\, d\beta_t(g) - \mathcal{P}f(x) \right| \longrightarrow 0 \quad \text{as } t \to \infty. \tag{1.2}$$

When the space X is a compact metric space, one can consider sharpening the pointwise ergodic theorem in two material ways. The first is to establish *pointwise everywhere* convergence when f is continuous, namely that (1.2) holds for every point $x \in X$ without exception, in which case we say that the orbits of G in X are equidistributed. It was noted in [**GN1**] (based on an earlier argument due to Guivarc'h [**Gu1**]) that for isometric actions on compact spaces with an invariant ergodic probability measure of full support, pointwise everywhere convergence of the averages for continuous functions *follows* from the mean ergodic theorem. This result has as a consequence the fact that such actions are in fact uniquely ergodic. Thus unique ergodicity can be established via spectral arguments.

The second way of sharpening the pointwise ergodic theorem is to establish that the convergence in (1.2) proceeds at a fixed rate, uniformly for every starting point, if the function f is Hölder continuous; that is,

$$\left|\int_G f(g^{-1}x)\,d\beta_t(g) - \mathcal{P}f(x)\right| \le C(f, x)E(t) \tag{1.3}$$

where $E(t) \to 0$ as $t \to \infty$. In this case, we say that the orbits have a uniform rate of equidistribution, and we wish to estimate this rate.

In the present paper, we will establish equidistribution results with an effective uniform rate in two significant cases, namely the isometric and transitive cases. To establish a quantitative version of these results for actions of general groups G, we will require the spectral assumption of the existence of a spectral gap. Recall that a unitary representation has a spectral gap if it has no almost invariant sequence of unit vectors. We emphasize that this assumption is necessary for the conclusion to hold, and that its validity is a very common phenomenon. For example, all actions of groups with property T have a spectral gap (in the orthogonal complement of the invariants).

We also show that the convergence rate is uniform on the sets

$$C^a(X)_1 = \left\{ f \in C(X) : \sup_{x\in X} |f(x)| + \sup_{x\ne y} \frac{|f(x) - f(y)|}{d(x, y)^a} \le 1 \right\}$$

of Hölder continuous functions with Hölder norm bounded by one.

Let us now describe some concrete instances of these results, where the action preserves a probability measure.

1.5. *Uniform rate of equidistribution.*

1.5.1. *Free groups.* Let $\mathbb{F}_r$ be a free group with r generators, where $r \ge 2$. We denote by $\ell(\gamma)$ the length of an element $\gamma \in \Gamma$ with respect to the free generating set and by B_{2n} the ball of radius $2n$. We denote by $\varepsilon_0 : \mathbb{F}_r \to \{\pm 1\}$ the sign character of the free group, taking the value 1 on words of even length and -1 on words of odd length.

Given a unitary representation π of $\mathbb{F}_r$ on a Hilbert space $\mathcal{H}$, let $\mathcal{H}^1$ denote the space of invariants and let $\mathcal{H}^{\varepsilon_0}$ denote the space realizing the character ε_0. Any vector $f_0 \in \mathcal{H}^{\varepsilon_0}$ satisfies $\pi(\gamma) f_0 = (-1)^{\ell(\gamma)} f_0$.

Given a decreasing family of finite-index subgroups Γ_i of $\mathbb{F}_r$, we denote by $\hat{\mathbb{F}}_r$ the profinite completion equipped with an invariant metric defined by

$$d(\gamma_1, \gamma_2) = \max\{|\mathbb{F}_r : \Gamma_i|^{-1} : \gamma_1^{-1}\gamma_2 \notin \hat{\Gamma}_i\}$$

for $\gamma_1, \gamma_2 \in \hat{\mathbb{F}}_r$.

THEOREM 1.1.

(1) *Consider an isometric action of $\mathbb{F}_r$ on a compact manifold X. Let μ be an ergodic smooth probability measure on X with full support such that the representation of $\mathbb{F}_r$ on $L_0^2(X, \mu)$ has a spectral gap. Then for every Hölder-continuous $f \in C^a(X)_1$ and every $x \in X$,*

$$\frac{1}{\#B_{2n}} \sum_{\gamma\in B_{2n}} f(\gamma^{-1}x) = \mathcal{P}f(x) + O(e^{-\theta_a n}) \tag{1.4}$$

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where:

- $\theta_a > 0$ *depends only on the spectral gap and* $\dim(X)$*;*
- *the operator* $\mathcal{P}$ *is given by*

$$\mathcal{P}f = \int_X f\,d\mu + \frac{r-1}{r}\left(\int_X f\bar{f}_0\,d\mu\right)f_0,$$

with $f_0 \equiv 0$ *when* $\mathcal{H}^{\varepsilon_0} = 0$*, which is always the case when* X *is connected; otherwise,* $\mathcal{H}^{\varepsilon_0}$ *is one-dimensional and* f_0 *denotes a unit vector that spans* $\mathcal{H}^{\varepsilon_0}$*.*

(2) *Let* $\Gamma_i \subset \mathbb{F}_r$ *be a decreasing family of finite-index subgroups such that:*

- $|\Gamma_i : \Gamma_{i+1}|$ *is uniformly bounded;*
- *the family of representations* $L^2_0(\mathbb{F}_r/\Gamma_i)$ *of* $\mathbb{F}_r$ *satisfies property* (τ) *(i.e. has uniform spectral gap).*

Then for every $f \in C^a(\hat{\mathbb{F}}_r)_1$ *and* $x \in \hat{\mathbb{F}}_r$*, (1.4) holds as well, with respect to the Haar probability measure* μ *on* $\hat{\mathbb{F}}_r$*, the profinite completion associated with the family* Γ_i*.*

We recall that, taking $X = S^2$ to be the unit sphere in $\mathbb{R}^3$ and $G = \mathrm{SO}_3(\mathbb{R})$, it was shown in [**LPS1, LPS2**] (see also [**C, O**] for generalizations to higher-dimensional spheres) that certain dense subgroups $\Gamma \subset G$ admit a spectral gap in their representation on $L^2_0(S^2)$. The class of such subgroups was significantly enlarged recently in [**BG**]. It may even be the case that *every* dense finitely generated subgroup of G admits a spectral gap, without exception, but this remains a challenging open problem. In any case, whenever a spectral gap exists, *every* orbit of the dense free group becomes equidistributed on the sphere at a uniform exponential rate, depending on the size of the spectral gap (as well as on the parameters r and a, of course).

We remark that in [**Do**], a property weaker than a spectral gap is established for dense subgroups, and this result is sufficient to derive quantitative equidistribution, albeit at a subexponential rate.

1.5.2. *Lattices in simple algebraic groups.* Let $\mathsf{G} \subset \mathrm{GL}_d$ be a simply connected absolutely simple algebraic group defined over a local field K of characteristic zero which is isotropic over K (for example, $\mathsf{G} = \mathrm{SL}_d$). Let $G = \mathsf{G}(K)$, and let Γ be a lattice in G. We fix a norm on $\mathrm{Mat}_d(K)$ which is the Euclidean norm if K is Archimedean and the max-norm otherwise. Let B_t denote the ball $\{g \in G : \log\|g\| < t\}$.

THEOREM 1.2.

(1) *Consider an isometric action of* Γ *on a compact manifold* X*. Let* μ *be a smooth probability measure on* X *with full support such that the action of* Γ *in* $L^2_0(X,\mu)$ *has a spectral gap. Then for every* $f \in C^a(X)_1$ *and every* $x \in X$*,*

$$\frac{1}{\#(\Gamma \cap B_t)} \sum_{\gamma \in B_t} f(\gamma^{-1}x) = \int_X f\,d\mu + O(e^{-\theta_a t}) \tag{1.5}$$

with explicit $\theta_a > 0$*.*

(2) *Let* $\Gamma_i \subset \Gamma$ *be a decreasing family of finite-index subgroups such that:*

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- $|\Gamma_i : \Gamma_{i+1}|$ *is uniformly bounded;*
- *the family of representations* $L^2_0(\Gamma/\Gamma_i)$ *of* Γ *satisfies property* (τ).

Then for every $f \in C^a(\hat{\Gamma})_1$ *and every* $x \in \hat{\Gamma}$, *(1.5) holds as well, with respect to the Haar probability measure* μ *on the profinite completion* $\hat{\Gamma}$ *associated with the family* Γ_i.

Remark 1.3. One can also formulate a version of Theorem 1.2 for general semisimple S-arithmetic groups, but then one needs to impose the condition that the unitary representation of G^+ induced from the representation of Γ on $L^2_0(X, \mu)$ has a strong spectral gap (see [**GN1**] for the terminology).

1.5.3. *Transitive actions.* Another significant case where pointwise everywhere convergence holds with a uniform rate is that of transitive actions, for which this property holds for every bounded Borel function. We will consider below general homogenous spaces with σ-finite invariant measure. Here we note only the following consequence in the case where the measure is finite.

Let $\mathsf{G} \subset \mathrm{GL}_d$, the norm and the balls β_t be as described in §1.5.2.

THEOREM 1.4. *Consider a transitive continuous action of G on a homogeneous space X that supports invariant Borel probability measure* μ. *Then, for every bounded Borel function* f *on* X *of compact support and for every* $x \in X$,

$$\pi_X(\beta_t) f(x) = \int_X f \, d\mu + O\left(\left(\sup_{s \in (t-1, t+1)} E(f, s)\right)^\theta\right)$$

with an explicit $\theta \in (0, 1)$ *which is independent of* f *and* x.

The proofs of Theorems 1.1, 1.2 and 1.4 will be given in §5.

Remark 1.5. Let us note two other general approaches to deriving equidistribution results in isometric actions from estimates on L^2-norms. The first approach, which can be found in [**CO, GN1, Gu1**] and is originally due to Guivarc'h, applies only in the case of compact spaces and does not produce a rate of convergence. The second approach (see [**CU**, §8]) uses the theory of elliptic operators, so it can only be applied in the setting of Lie groups and sufficiently smooth functions.

1.6. *Ergodic theorems: spaces with infinite measure.* Let us now turn to spaces with infinite invariant measure, and consider the problem of establishing mean and pointwise ergodic theorems and quantitative equidistribution of orbits for general group actions on such spaces. In general, this basic challenge is largely unexplored, and here we take up the important case of dense subgroups $\Gamma \subset G$ acting isometrically by translations, where we can establish *pointwise everywhere* convergence at a uniform rate. To that end, we introduce a natural generalization of the mean ergodic theorem in this setting (see Definition 2.1 below). In particular, we obtain the following equidistribution result that provides a quantitative version of [**GW**, Corollary 1.4].

Let $\mathsf{G} \subset \mathrm{GL}_d$ be a semisimple simply connected algebraic group which is defined over a number field K and is K-simple. Let T and S be finite sets of Archimedean valuations

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of K with $T \subset S$. For $v \in S$, we denote by K_v the corresponding completions. Let O_S denote the ring of S-integers and let $\Gamma = \mathsf{G}(O_S)$.

Let

$$H(g) = \prod_{v \in S} \|g_v\|_v \quad \text{for } g = (g_v) \in \prod_{v \in S} \mathsf{G}(K_v),$$

where $\| \cdot \|_v$ are norms on $\mathrm{Mat}_d(K_v)$ as in §1.5.2.

THEOREM 1.6. *Assume that Γ is dense in $G = \prod_{v \in T} \mathsf{G}(K_v)$ with respect to the diagonal embedding. Then there exist $\alpha \in \mathbb{Q}^+$ and $\beta \in \mathbb{N}$ such that for every Hölder function on G with exponent a and compact support and for every $x \in G$,*

$$\frac{1}{t^{\beta-1} e^{\alpha t}} \sum_{\gamma \in \Gamma : \log H(\gamma) < t} f(\gamma x) = \int_G f(g) \frac{dm_G(g)}{H(gx)^\alpha} + O_{f,x}(e^{-\theta_a t})$$

uniformly for x in compact sets, where m_G is a suitably normalized Haar measure on G and $\theta_a > 0$.

We note that the L^2-convergence for the operators appearing in Theorem 1.6 is a special case of the results of [**GN2**]. Hence, since the action of Γ on X is isometric, Theorem 1.6 follows from Theorem 2.5(2) below.

We refer to [**GW**, page 107] for the identification of α and also for a proof of the following fact. Taking the family B_t associated with the distance function given (for example) by a power of the height function will change the power of the density function appearing in the limiting distribution. Thus, as we have already mentioned, this result demonstrates that in the infinite-measure setting the limit measure does not have to be invariant and may depend non-trivially on the initial point x and the family B_t, even if the action is an isometric action with a spectral gap.

Finally, it is interesting to compare Theorem 1.6 with the results on equidistribution of dense subgroups of nilpotent Lie groups established in [**Br2**]. In that case, the averages constitute a Følner sequence and the limit distribution is Haar measure.

1.7. *Dense groups of isometries: Kazhdan's conjecture.* Let (X, d) be an lcsc metric space, and let $G = \mathrm{Isom}(X)$ be its group of isometries. Assume that the action of G on X is transitive, and let m_X be the unique isometry-invariant Radon measure on X. Fix two bounded open sets A_1 and A_2 with boundary of zero measure. Consider a countable dense subgroup $\Gamma \subset G$ and a family of sets $B_t \subset \Gamma$, for example balls with respect to a left-invariant metric. For each $x \in X$, the orbit $\Gamma \cdot x$ is dense in X and we can form the ratios

$$\frac{|\{\gamma \in B_t : \gamma^{-1} x \in A_1\}|}{|\{\gamma \in B_t : \gamma^{-1} x \in A_2\}|}.$$

Consider the question of whether the ratios satisfy a ratio ergodic theorem, i.e. whether the limit as $t \to \infty$ exists and, furthermore, whether it is given by

$$\lim_{t \to \infty} \frac{\sum_{\gamma \in B_t} \chi_{A_1}(\gamma^{-1} x)}{\sum_{\gamma \in B_t} \chi_{A_2}(\gamma^{-1} x)} = \frac{m_X(A_1)}{m_X(A_2)}. \tag{1.6}$$

This problem was raised by Kazhdan in [**K**], where the case of certain dense 2-generator subsemigroups of $\mathrm{Isom}(\mathbb{R}^2)$ acting on the plane was studied. Upon assuming that one of the generators is an irrational rotation, a version of (1.6) was established, but with B_t taken to be balls in the free group or semigroup, not balls with respect to the word metric. This amounts to considering weighted averages on Γ, with the weights being given by convolution powers. This result was generalized by Guivarc'h [**Gu2**], who considered weighted averages given by convolution powers on certain dense subsemigroups of $\mathrm{Isom}(\mathbb{R}^n)$ acting on $\mathbb{R}^n$ (note also that a gap in the argument in [**K**] was closed in [**Gu2**]). For further results in this direction see [**V1, V2**] and also [**Br1**], where the original convergence theorem in the plane is sharpened to a local limit theorem.

Theorem 1.6 has, of course, a direct bearing on this problem. In principle, to show that ratios converge, one does not need to establish the much stronger result that both the numerator and the denominator converge at a common rate and find an explicit expression for the rate. However, that is precisely the conclusion of Theorem 1.6; so, as an immediate corollary, we obtain the following result.

COROLLARY 1.7. *Keeping the notation and assumptions of Theorem 1.6, we have:*

(1) *if f_1 and f_2 are continuous of compact support and $f_2 \geq 0$ (and not identically zero), then for every $x_1, x_2 \in X$,*

$$\lim_{t\to\infty} \frac{\sum_{\gamma\in\Gamma:\log H(\gamma)<t} f_1(\gamma x_1)}{\sum_{\gamma\in\Gamma:\log H(\gamma)<t} f_2(\gamma x_2)} = \frac{\int_X f_1(g) H(gx_1)^{-\alpha}\, dm_X(g)}{\int_X f_2(g) H(gx_2)^{-\alpha}\, dm_X(g)};$$

(2) *if, in addition, f_1 and f_2 are Hölder continuous with exponent a, then for every $x_1, x_2 \in X$,*

$$\frac{\sum_{\gamma\in\Gamma:\log H(\gamma)<t} f_1(\gamma x_1)}{\sum_{\gamma\in\Gamma:\log H(\gamma)<t} f_2(\gamma x_2)} = \frac{\int_X f_1(g) H(gx_1)^{-\alpha}\, dm_X(g)}{\int_X f_2(g) H(gx_2)^{-\alpha}\, dm_X(g)} + O_{f_1,f_2,x_1,x_2}(e^{-\theta_a t})$$

uniformly over x_1 and x_2 in compact sets.

Thus the ratios converge for every point, with uniform rate; however, the limit is *not* the ratio of the integrals with respect to the isometry-invariant measure, but with respect to a *different* measure.

We also remark that if f_1 and f_2 are bounded measurable functions with bounded support, with $f_2 \geq 0$ not identically zero, then the ratios converge to the stated limit at almost every point, with uniform rate. This is a consequence of the results of [**GN2**].

2. *Formulation of quantitative equidistribution results*

Let G be an lcsc group acting measurably on a measurable space X equipped with a σ-finite quasi-invariant measure μ. We fix an increasing filtration of X by measurable sets X_r, $r \in \mathbb{N}$, of finite measure. We denote by $\|\cdot\|_{p,r}$ the L^p-norm with respect to the measure $\mu|_{X_r}$.

We consider families β_t of bounded Borel measures on G; in particular, given a family of sets B_t on G for $t \geq t_0$ and a positive growth function $V(t)$, we consider the operators

$$\pi_X(\beta_t) f(x) = \frac{1}{V(t)} \int_{g\in B_t} f(g^{-1}x)\, dg$$

for measurable $f : X \to \mathbb{R}$.

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Definition 2.1. The operators $\pi_X(\beta_t)$ satisfy the *mean ergodic theorem* in L^p for the action of G on X if for every $r \in \mathbb{N}$ and $f \in L^p(X, \mu|_{X_r})$, the sequence $\pi_X(\beta_t)f$ converges in $L^p(X, \mu|_{X_r})$.

It is clear from the definition that for $1 \leq p < \infty$ there exist linear operators

$$\mathcal{P}_r : L^p(X, \mu|_{X_r}) \to L^p(X, \mu|_{X_r})$$

such that

$$E_{p,r}(f, t) := \|\pi_X(\beta_t)f - \mathcal{P}_r f\|_{p,r} \to 0 \quad \text{as } t \to \infty, \tag{2.1}$$

and since

$$\mathcal{P}_{r+1}|_{L^p(X,\mu|_{X_r})} = \mathcal{P}_r,$$

it is consistent to denote these operators by $\mathcal{P}$. We shall then say that the $\pi_X(\beta_t)$ satisfy the mean ergodic theorem in L^p with limit operator $\mathcal{P}$.

Remark 2.2. We note that our notion of mean ergodic theorem depends on the choice of the filtration $X_r \subset X$ and the normalization $V(t)$. The crucial point is to choose the normalization so that the operator $\mathcal{P}$ is non-trivial, in which case the mean ergodic theorem yields significant information about the limiting distribution of the orbits.

As noted above, the fact that the foregoing formulation of the mean ergodic theorem for spaces with *infinite* measure is meaningful is an indication of a novel and distinctly non-classical phenomenon. Indeed, it is well-known (see [**Aa**, Ch. 2]) that for an action of a single transformation on a space with infinite measure, no normalization $V(t)$ for which (2.1) holds can be found. Nonetheless, gradually the realization has grown that mean ergodic theorems and equidistribution results do hold for some classes of action on infinite measure spaces (see [**G1, G2, GW, L, LP1, LP2**]). In fact, in the forthcoming paper [**GN2**] we establish the mean ergodic theorem for lattices in S-algebraic semisimple groups acting on general algebraic homogeneous spaces. This result is part of a systematic approach to ergodic theory on homogeneous spaces via the duality principle.

2.1. *Isometric actions.* Let us assume now that X is a locally compact second countable metric space equipped with a Radon measure μ, so that the measures of balls are finite. We fix a filtration of X by balls X_r of radius r centered at some fixed $x_0 \in X$. We denote by $D_\varepsilon(x)$ the closed ball in X of radius ε centered at x.

Definition 2.3.

(1) We say that the measure μ is *uniformly of full support* if for every $r \in \mathbb{N}$ and $\varepsilon \in (0, 1]$,

$$\inf_{x \in X_r} \mu(D_\varepsilon(x)) > 0.$$

(2) We say that the measure μ has *local dimension at most* ρ if for every $r \in \mathbb{N}$, $\varepsilon \in (0, 1]$ and $x \in X_r$,

$$\mu(D_\varepsilon(x)) \geq m_r \varepsilon^\rho.$$

Remark 2.4. If the sets X_r are compact, then every measure μ of full support is uniformly of full support. Moreover, if X is a compact manifold and μ is a smooth measure of full support, then μ has local dimension at most $\dim(X)$.

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The following theorem is our main technical result concerning equidistribution for isometric actions.

THEOREM 2.5. *Consider an isometric action of an lcsc group G on the lcsc metric space X equipped with a quasi-invariant Radon measure* μ. *Assume that the mean ergodic theorem in* L^p *holds for the operators* $\pi_X(\beta_t)$ *in the action of G on X, for some* $1 \leq p < \infty$, *and let* $f \in L^p$. *Then:*

(1) *if the measure* μ *is uniformly of full support, then for every uniformly continuous function* f *such that* $\mathrm{supp}(f) \subset X_r$ *and* $(\mathcal{P}f)|_{X_r}$ *is uniformly continuous, we have*

$$\max_{x \in X_r} |\pi_X(\beta_t)f(x) - \mathcal{P}f(x)| = o_{p,r,f}(1)$$

as $t \to \infty$;

(2) *if the measure* μ *has local dimension at most* ρ, *then for every* $f \in C^a(X)_1$ *such that* $\mathrm{supp}(f) \subset X_r$ *and* $(\mathcal{P}f)|_{X_r} \in C^a(X)_1$, *we have*

$$\max_{x \in X_r} |\pi_X(\beta_t)f(x) - \mathcal{P}f(x)| \ll_{p,r} E_{p,r}(f,t)^{a/(a+\rho/p)}$$

for all sufficiently large t.

2.2. *Transitive actions.* As noted above, the behavior of the operators $\pi_X(\beta_t)$ may, in general, depend quite sensitively on the initial point. Nonetheless, when the action is transitive it is still possible to obtain a uniform result, provided that a certain regularity property of the measures β_t holds.

Let d be a right-invariant metric on G compatible with the topology of G such that the closed balls with respect to d are compact (such a metric always exists; see, e.g., [**HP**]). We denote the closed ball of radius ε centered at $g \in G$ by $\mathcal{O}_\varepsilon(g)$.

Definition 2.6. The family of measures β_t is said to be *coarsely monotone* if there exist monotone functions $\kappa : (0,1] \to (0,\infty)$ and $\delta : (0,1] \to (1,\infty)$ such that

$$\delta_\varepsilon \to 1 \quad \text{and} \quad \kappa_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+$$

and, for every $\varepsilon \in (0,1]$, $g \in \mathcal{O}_\varepsilon(e)$ and $t \geq t_0$,

$$g \cdot \beta_t \leq \delta_\varepsilon \beta_{t+\kappa_\varepsilon}.$$

If, in addition, we have $\delta_\varepsilon = 1 + O(\varepsilon^{a_0})$ for some $a_0 > 0$, then the family of measures is said to be *Hölder coarsely monotone* with exponent a_0.

Let X be a lcsc space on which the group G acts transitively and continuously. Since X is locally compact, the topology on X coincides with the topology defined on X by viewing it as a factor space of G. We equip X with a G-quasi-invariant Radon measure μ. The space X is equipped with the natural metric (see [**HR**, §8]), which is defined by

$$d(x_1, x_2) = \inf\{d(g_1, g_2) : g_1, g_2 \in G,\ g_1x_0 = x_1,\ g_2x_0 = x_2\} \tag{2.2}$$

where x_0 is a fixed element of X. We use the filtration on X such that X_r are balls of radius r in X centered at some fixed $x_0 \in X$.

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THEOREM 2.7. *Assume that the mean ergodic theorem in L^p holds for the family of operators $\pi_X(\beta_t)$ in the transitive G-action on X, for some $1 \le p < \infty$.*

(1) *If β_t is a coarsely monotone family of measures, then for every non-negative bounded Borel function $f : X \to \mathbb{R}$ such that* $\operatorname{supp}(f) \subset X_r$ *and $\mathcal{P}f$ is uniformly continuous on X_r, we have*

$$\max_{x \in X_r} |\pi_X(\beta_t) f(x) - \mathcal{P}f(x)| = o_{p,r,f}(1)$$

as $t \to \infty$.

(2) *If, in addition, β_t is Hölder coarsely monotone with exponent a_0 and μ has local dimension at most ρ, then for every non-negative bounded Borel function $f : X \to \mathbb{R}$ such that* $\operatorname{supp}(f) \subset X_r$ *and $(\mathcal{P}f)|_{X_r} \in C^a(X)_1$, we have*

$$\max_{x \in X_r} |\pi_X(\beta_t) f(x) - \mathcal{P}f(x)| \ll_{p,r} \left(\sup_{s \in (t-\kappa_1, t+\kappa_1)} E_{p,r}(f, s) \right)^{\min(a_0,a)/(\min(a_0,a)+\rho/p)}$$

for all sufficiently large t.

Remark 2.8. It is often the case that the operator $\mathcal{P}$ maps compactly supported bounded functions to uniformly continuous ones, or even to Hölder functions. In that case, we can of course decompose every bounded real function into its positive and negative parts and apply Theorem 2.7, so that the same conclusions are valid for all bounded functions.

3. *Proof of equidistribution for isometric actions*

In this section we prove Theorem 2.5. We start the proof with the following lemma.

LEMMA 3.1. *Assume that the mean ergodic theorem in L^p holds for the family of operators $\pi_X(\beta_t)$, for some $1 \le p < \infty$. Assume that the action of G on the space X is isometric and equipped with a quasi-invariant Radon measure μ which is uniformly of full support. Then, for all sufficiently large t, $r \in \mathbb{N}$ and $y \in X_r$,*

$$\beta_t(\{g \in G : g^{-1} y \in X_r\}) = O_r(1).$$

Proof. It follows from the mean ergodic theorem that

$$\int_{X_r} |\beta_t(\{g \in G : g^{-1}x \in X_r\}) - \mathcal{P}\chi_{X_r}(x)|^p \, d\mu(x) = o_{p,r}(1)$$

as $t \to \infty$; hence

$$\|\pi_X(\beta_t)\chi_{X_r}\|_p^p = \int_{X_r} (\beta_t\{g \in G : g^{-1}x \in X_r\})^p \, d\mu(x) = O_{p,r}(1).$$

Let $\delta > 0$. Clearly, for the set

$$\Omega_r(\delta, t) := \{x \in X_r : \beta_t(\{g \in G : g^{-1}x \in X_r\}) > \delta\}$$

we have

$$\mu(\Omega_r(\delta, t)) \le \frac{\|\pi_X(\beta_t)\chi_{X_r}\|_p^p}{\delta^p}.$$

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Therefore, if we choose δ so that $\delta^p = \|\pi_X(\beta_t)\chi_{X_r}\|_p^p / m_{r-1}$ where

$$m_{r-1} := \inf_{y \in X_{r-1}} \mu(D_1(y)) > 0,$$

then

$$\mu(\Omega_r(\delta, t)) < \mu(D_1(y)) \quad \text{for all } y \in X_{r-1}.$$

Hence, for every $y \in X_{r-1}$ there exists $x \in D_1(y) \subset X_r$ such that $x \notin \Omega_r(\delta, t)$, i.e.

$$\beta_t(\{g \in G : g^{-1}x \in X_r\}) \leq \delta = O_r(1).$$

Since the action is isometric, we have $d(x_0, g^{-1}x) \leq d(x_0, g^{-1}y) + 1$. Hence, if $g^{-1}y \in X_{r-1}$, then $g^{-1}x \in X_r$, and so

$$\beta_t(\{g \in G : g^{-1}y \in X_{r-1}\}) \leq \beta_t(\{g \in G : g^{-1}x \in X_r\}).$$

This implies the claim. □

Proof of Theorem 2.5. In the proof we shall use parameters $\varepsilon \in (0, 1)$ and $\delta > 0$ that will be specified later. Again, let

$$\Omega_r(\delta, t) = \{x \in X_r : |\pi_X(\beta_t) f(x) - \mathcal{P} f(x)| > \delta\}. \tag{3.1}$$

Then

$$\mu(\Omega_r(\delta, t)) \leq E_{p,r}^p(f, t)/\delta^p.$$

Hence, if we assume that

$$m_{r-1}(\varepsilon) := \inf_{y \in X_{r-1}} \mu(D_\varepsilon(y)) > E_{p,r}^p(f, t)/\delta^p, \tag{3.2}$$

then for every $y \in X_{r-1}$ there exists $x \in D_\varepsilon(y) \subset X_r$ such that $x \notin \Omega_r(\delta, t)$, i.e.

$$|\pi_X(\beta_t) f(x) - \mathcal{P} f(x)| \leq \delta. \tag{3.3}$$

Let

$$\omega_r(f, \varepsilon) = \sup\{|f(z) - f(w)| : z, w \in X_r, d(z, w) < \varepsilon\}. \tag{3.4}$$

Since f is uniformly continuous, $\omega_r(f, \varepsilon) \to 0$ as $\varepsilon \to 0^+$. Using that the action of G on X is isometric and $\operatorname{supp}(f) \subset X_r$, we deduce that

$$\begin{aligned} &|\pi_X(\beta_t) f(x) - \pi_X(\beta_t) f(y)| \\ &\quad \leq \omega_r(f, \varepsilon)\beta_t(\{g \in G : g^{-1}x \in X_r \text{ or } g^{-1}y \in X_r\}) \ll_r \omega_r(f, \varepsilon), \end{aligned}$$

where the last inequality follows from Lemma 3.1. Hence, it follows from (3.3) that for every $y \in X_{r-1}$,

$$|\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \ll_r \delta + \omega_r(f, \varepsilon) + \omega_r(\mathcal{P} f, \varepsilon).$$

This estimate holds provided that δ satisfies (3.2). Therefore, it follows that for every $r \in \mathbb{N}$,

$$\max_{y \in X_{r-1}} |\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \ll_r E'_{p,r}(f, t)$$

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where

$$E'_{p,r}(f, t) = \inf_{\varepsilon\in(0,1)} \{E_{p,r}(f, t)/m_{r-1}(\varepsilon)^{1/p} + \omega_r(f, \varepsilon) + \omega_r(\mathcal{P}f, \varepsilon)\}.$$

Using the fact that $\omega_r(f, \varepsilon)$, $\omega_r(\mathcal{P}f, \varepsilon) \to 0$ as $\varepsilon \to 0^+$ and that $E_{p,r}(f, t) \to 0$ as $t \to \infty$, we conclude that $E'_{p,r}(f, t) = o_{p,r,f}(1)$ as $t \to \infty$ as well. This proves the first part of the theorem.

To prove the second part of the theorem, observe that under the additional assumptions we have

$$E'_{p,r}(f, t) \ll_r \inf_{\varepsilon\in(0,1)} \{\varepsilon^{-\rho/p} E_{p,r}(f, t) + \varepsilon^a\}.$$

We therefore take $\varepsilon = E_{p,r}(f, t)^{1/(a+\rho/p)}$ and note that since $E_{p,r}(f, t) \to 0$ as $t \to \infty$, we have $\varepsilon \in (0, 1)$ for all sufficiently large t. It then follows that

$$E'_r(f, t) \ll_r E_r(f, t)^{a/(a+\rho/p)},$$

as required. □

4. *Proof of equidistribution for transitive actions*

Our proof of Theorem 2.7 follows the same strategy as for the proof of Theorem 2.5. We start with the following lemma, which establishes directly that in the transitive case the quasi-invariant measure is uniformly of full support. We use the metric d on the homogeneous space X defined in (2.2).

LEMMA 4.1. *For every $r \in \mathbb{N}$ and $\varepsilon > 0$,*

$$m_r(\varepsilon) := \inf_{x\in X_r} \mu(D_\varepsilon(x)) > 0.$$

Proof. It follows from the definition of the metric on X that

$$D_s(x) = \mathcal{O}_s(e) \cdot x \quad \text{for every } s > 0 \text{ and } x \in X.$$

Since the balls $\mathcal{O}_r(e)$ are compact, there exists $\varepsilon' = \varepsilon'(\varepsilon, r) > 0$ such that $g\mathcal{O}_{\varepsilon'}(e)g^{-1} \subset \mathcal{O}_\varepsilon(e)$ for every $g \in \mathcal{O}_r(e)$. Then, since the measure μ is quasi-invariant, for every $g \in \mathcal{O}_r(e)$ we have

$$\mu(D_\varepsilon(gx_0)) \ge \mu(\mathcal{O}_\varepsilon(e)gx_0) \ge \mu(g\mathcal{O}_{\varepsilon'}(e)x_0) \gg_r \mu(\mathcal{O}_{\varepsilon'}(e)x_0).$$

This proves the claim. □

Proof of Theorem 2.7. Let $\varepsilon \in (0, 1)$ and $\delta > 0$. As in the proof of Theorem 2.5, we introduce the set

$$\Omega_r(\delta, t) = \{x \in X_r : |\pi_X(\beta_t) f(x) - \mathcal{P}f(x)| > \delta\}$$

and observe that

$$\mu(\Omega_r(\delta, t)) \le E^p_{p,r}(f, t)/\delta^p.$$

Let $\bar{E}_{p,r}(f, t) := \sup_{s\in(t-\kappa_1,t+\kappa_1)} E_{p,r}(f, s)$, where κ_ε is as defined in Definition 2.6. Let

$$\delta^p > \bar{E}^p_{p,r}(f, t)/m_{r-1}(\varepsilon). \tag{4.1}$$

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Then for every $y \in X_{r-1}$ and $s \in (t-\kappa_1, t+\kappa_1)$ there exists $x_s \in D_\varepsilon(y) \subset X_r$ such that $x_s \notin \Omega_r(\delta, s)$, i.e.

$$|\pi_X(\beta_s) f(x_s) - \mathcal{P} f(x_s)| \le \delta. \tag{4.2}$$

We set $x_1 = x_{t-\kappa_\varepsilon}$ and $x_2 = x_{t+\kappa_\varepsilon}$. Then, since $y = hx_1$ for some $h \in \mathcal{O}_\varepsilon(e)$, it follows from the coarsely monotone property of β_t that

$$\pi_X(\beta_t) f(y) = \pi_X(h \cdot \beta_t) f(x_1) \ge \delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1)$$

and

$$\begin{aligned} |\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \le\ & (\pi_X(\beta_t) f(y) - \delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1)) \\ & + |\delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1) - \mathcal{P} f(y)|. \end{aligned}$$

Similarly,

$$\pi_X(\beta_t) f(y) \le \delta_\varepsilon \pi_X(\beta_{t+\kappa_\varepsilon}) f(x_2)$$

and hence

$$\begin{aligned} |\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \le\ & (\delta_\varepsilon \pi_X(\beta_{t+\kappa_\varepsilon}) f(x_2) - \delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1)) \\ & + |\delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1) - \mathcal{P} f(y)|. \end{aligned}$$

Now we estimate each of the above terms separately.

It follows from (4.2), uniform continuity of $\mathcal{P} f$ on X_r and the boundedness of f that

$$\begin{aligned} & |\delta_\varepsilon \pi_X(\beta_{t+\kappa_\varepsilon}) f(x_2) - \delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1)| \\ & \quad \le \delta_\varepsilon |\pi_X(\beta_{t+\kappa_\varepsilon}) f(x_2) - \mathcal{P} f(x_2)| + |\delta_\varepsilon \mathcal{P} f(x_2) - \delta_\varepsilon^{-1} \mathcal{P} f(x_1)| \\ & \qquad + \delta_\varepsilon^{-1} |\pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1) - \mathcal{P} f(x_1)| \\ & \quad \le (\delta_\varepsilon + \delta_\varepsilon^{-1})\delta + \delta_\varepsilon |\mathcal{P} f(x_2) - \mathcal{P} f(x_1)| + (\delta_\varepsilon - \delta_\varepsilon^{-1}) |\mathcal{P} f(x_1)| \\ & \quad \ll_r \delta + \omega_r(\mathcal{P} f, 2\varepsilon) + (\delta_\varepsilon - \delta_\varepsilon^{-1}), \end{aligned}$$

where the function ω_r is as defined in (3.4). Also,

$$\begin{aligned} & |\delta_\varepsilon^{-1} \pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1) - \mathcal{P} f(y)| \\ & \quad \le \delta_\varepsilon^{-1} |\pi_X(\beta_{t-\kappa_\varepsilon}) f(x_1) - \mathcal{P} f(x_1)| + |\delta_\varepsilon^{-1} \mathcal{P} f(x_1) - \mathcal{P} f(y)| \\ & \quad \le \delta_\varepsilon^{-1} \delta + \delta_\varepsilon^{-1} |\mathcal{P} f(x_1) - \mathcal{P} f(y)| + (1 - \delta_\varepsilon^{-1}) |\mathcal{P} f(y)| \\ & \quad \ll_r \delta + \omega_r(\mathcal{P} f, \varepsilon) + (1 - \delta_\varepsilon^{-1}). \end{aligned}$$

Therefore, we conclude that

$$|\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \ll_r \delta + \omega_r(\mathcal{P} f, 2\varepsilon) + \delta_\varepsilon - 2\delta_\varepsilon^{-1} + 1.$$

Since this estimate holds for all $\varepsilon \in (0, 1)$, $y \in X_{r-1}$ and δ satisfying (4.1), we have

$$\max_{y \in X_{r-1}} |\pi_X(\beta_t) f(y) - \mathcal{P} f(y)| \ll_r E''_{p,r}(f, t)$$

where

$$E''_{p,r}(f, t) = \inf_{\varepsilon \in (0,1)} \{\bar{E}_{p,r}(f, t)/m_{r-1}(\varepsilon)^{1/p} + \omega_r(\mathcal{P} f, 2\varepsilon) + \delta_\varepsilon - 2\delta_\varepsilon^{-1} + 1\}.$$

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Since $\delta_\varepsilon \to 1$ and $\omega_r(\mathcal{P}f, 2\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and $\bar{E}_{p,r}(f,t) \to 0$ as $t \to \infty$, it follows that $E''_{p,r}(f,t) \to 0$ as $t \to \infty$ too. This implies the first part of the theorem.

To prove the second part of the theorem, we observe that under the additional assumptions,

$$E''_{p,r}(f,t) \ll_r \inf_{\varepsilon \in (0,1)} (\varepsilon^{-\rho/p} \bar{E}_{p,r}(f,t) + \varepsilon^{\min(a_0,a)}).$$

Since $E_{p,r}(f,t) \to 0$ as $t \to \infty$, it follows that $\bar{E}_{p,r}(f,t) \in (0,1)$ for sufficiently large t. Taking $\varepsilon = \bar{E}_{p,r}(f,t)^{1/(\min(a_0,a)+\rho/p)}$, we deduce the second claim. □

5. *Completion of the proofs*

Proof of Theorem 1.1. We deduce Theorem 1.1 from Theorem 2.5(2). We recall that the mean ergodic theorem for the free group $\mathbb{F}_r$ was established in [**Gu1, Ne1**]. Moreover, under the spectral gap assumption, the method of the proof of [**Ne1**, Theorem 1] implies that

$$\left\| \frac{1}{\#B_{2n}} \sum_{\gamma \in B_{2n}} f(\gamma^{-1}x) - \mathcal{P}f(x) \right\|_2 = O(e^{-\theta n}\|f\|_2)$$

for some $\theta > 0$ determined by the spectral gap.

Let G be the closure of $\mathbb{F}_r$ in the isometry group of X. Then the measure μ is invariant and ergodic with respect to G. Since X is compact, G is compact, and it follows that μ is supported on a single orbit of G. Hence, G acts transitively on X. Let G_0 be the closure in G of the subgroup of $\mathbb{F}_r$ generated by the words of even length. Since G_0 has index at most two in G, the subgroup G_0 is open in G, and X consists of at most two open orbits of G_0. This implies that $L^2(X)^{\varepsilon_0}$ has dimension at most one and is trivial when X is connected. Moreover, it is clear that f_0 is locally constant and, in particular, $\mathcal{P}f \in C^a(X)_1$.

Finally, we note that in case (1) the measure μ has local dimension at most $\dim(X)$ (cf. Remark 2.4). In case (2), we have

$$\mu(D_\varepsilon(x)) = \mu(D_\varepsilon(e)) = \varepsilon$$

when $\varepsilon = |\Gamma : \Gamma_i|^{-1}$. Since $|\Gamma_i : \Gamma_{i+1}|$ is uniformly bounded, this implies that μ has local dimension at most one. Now Theorem 1.1 follows from Theorem 2.5(2). □

Proof of Theorem 1.2. We note that L^2-convergence for (1.5) with exponential rate follows from the results of [**GN1**]. Indeed, the balls B_t are Hölder admissible by [**GN1**, Ch. 7]. Since in both cases we have a lower estimate on the local dimension (see the proof of Theorem 1.1), Theorem 1.2 follows from Theorem 2.5(2). □

Proof of Theorem 1.4. It follows from [**GN1**, Ch. 7] that the family of measures β_t is Hölder coarsely monotone. The fact that the $\pi_X(\beta_t)$ satisfy the quantitative mean ergodic theorem in $L^2(X)$, when X is homogeneous and has finite invariant measure, is well-known (see [**GN1**, Theorem 4.3] for more details and for the general case). Here, of course, $\mathcal{P}(f) = \int_X f\,d\mu$. Hence, Theorem 1.4 is a consequence of Theorem 2.7(2), taking Remark 2.8 into account as well. □

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