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MIXED HODGE STRUCTURES WITH MODULUS

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Abstract We define a notion of mixed Hodge structure with modulus that generalizes the classical notion of mixed Hodge structure introduced by Deligne and the level one Hodge structures with additive parts introduced by Kato and Russell in their description of Albanese varieties with modulus. With modulus triples of any dimension, we attach mixed Hodge structures with modulus. We combine this construction with an equivalence between the category of level one mixed Hodge structures with modulus and the category of Laumon 1-motives to generalize Kato–Russell's Albanese varieties with modulus to 1-motives.

Keywords: enriched Hodge structure; formal Hodge structure; Laumon 1-motives

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Contents

1. Introduction

1.1. Background

Unlike K-theory, classical cohomology theories, such as Betti cohomology, étale cohomology or motivic cohomology (in particular Chow groups) are not able to distinguish a smooth variety from its nilpotent thickenings. This inability to detect nilpotence makes those cohomologies not the right tool to study non-homotopy invariant phenomena. One very important situation where these kinds of phenomena occur is at the boundary of a smooth variety. More precisely, if $\overline{X}$ is a smooth proper variety and D is an effective Cartier divisor on $\overline{X}$, then D can be seen as the non-reduced boundary at infinity of the smooth variety $X := \overline{X} \setminus D$. Since the works of Rosenlicht and Serre (see [25]), it is known that the cohomology groups that admit a geometrical interpretation in terms of Jacobian varieties or Albanese varieties do admit generalizations able to see the non-reducedness of the boundary (unipotent groups appear in those generalized Jacobians).

In recent years, most of the developments, following the work of Bloch–Esnault [5], have focused on the algebraic cycle part of the story. In this work, we focus on the Hodge theoretic counterpart of these developments.

1.2. Main results

In the present paper, we introduce a notion of mixed Hodge structure with modulus (see Definition 1) that generalizes the classical notion of mixed Hodge structure introduced by Deligne [7]. It is closely related to the notion of enriched Hodge structure introduced by Bloch–Srinivas [6] and the notion of formal Hodge structure introduced by Barbieri-Viale [1] and studied by Mazzari [18]. However, the relationship is not trivial; see § 7. Our main results are summarized as follows:

(1) The category **MHSM** of mixed Hodge structures with modulus is Abelian. It contains the usual category of mixed Hodge structures **MHS** as a full subcategory. Duality and Tate twists extend to mixed Hodge structures with modulus.

(2) The category **MHSM** of mixed Hodge structures with modulus contains a full subcategory $\mathbf{MHSM}_1$ which is equivalent to the category of Laumon 1-motives (the duality functor on mixed Hodge structures with modulus corresponding via this equivalence to Cartier duality).

(3) Given a smooth proper variety X and two effective simple normal crossing divisors Y, Z on X such that $|Y| \cap |Z| = \emptyset$, we associate functorially an object $\mathcal{H}^n(X, Y, Z)$ of **MHSM** for each $n \in \mathbb{Z}$. Its underlying mixed Hodge structure is given by the relative cohomology $H^n(X \setminus Z, Y, \mathbb{Z})$.

(4) For (X, Y, Z) as above, if further X is equidimensional of dimension d, then we have a duality theorem (fr denotes the free part; see § 2.7):

$$\mathcal{H}^n(X, Y, Z)^\vee \cong \mathcal{H}^{2d-n}(X, Z, Y)(d)_{\mathrm{fr}}.$$

Our construction is closer to Kato–Russell's category $\mathcal{H}_1$ from [15]. It is also motivated by the recent developments of the theory of algebraic cycles with modulus (such as

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additive Chow group [5], higher Chow groups with modulus [4], and Suslin homology with modulus [20]), to which our theory might be considered as the Hodge theoretic counterpart. We hope to study their relationship in a future work. We also leave as a future problem a construction of an object of **MHSM** that overlays Deligne's mixed Hodge structure on $H^n(X, \mathbb{Z})$ for non-proper X.

1.3. Application to Albanese 1-motives

For a pair (X, Y) consisting of a smooth proper variety X and an effective divisor Y on X, Kato and Russell constructed in [15] the Albanese variety with modulus $\mathrm{Alb}^{KR}(X, Y)$ as a higher dimensional analogue of the generalized Jacobian variety of Rosenlicht–Serre. Our theory yields an extension of their construction to 1-motives. This goes as follows.

Given a triple (X, Y, Z) as in (3) and (4) above, it is easy to see that the mixed Hodge structure with modulus $\mathcal{H}^{2d-1}(X, Y, Z)(d)_{\mathrm{fr}}$ belongs to the subcategory $\mathbf{MHSM}_1$. Therefore, it produces a Laumon 1-motive $\mathrm{Alb}(X, Y, Z)$ corresponding to $\mathcal{H}^{2d-1}(X, Y, Z)(d)_{\mathrm{fr}}$ under the equivalence (2) above. When $Z = \emptyset$, it turns out that $\mathrm{Alb}(X, Y, \emptyset) = [0 \to \mathrm{Alb}^{KR}(X, Y)]$. When $d = 1$, $\mathrm{Alb}(X, Y, Z)$ agrees with the Laumon 1-motive $\mathsf{LM}(X, Y, Z)$ constructed in [14, Definition 25].

1.4. Organization of the paper

The definition of the mixed Hodge structures with modulus is given in § 2. Its connection with Laumon 1-motives is studied in § 3. We construct $\mathcal{H}^n(X, Y, Z)$ in § 4 and prove the duality in § 5. In § 6, we construct Albanese 1-motives. We compare our theory with the enriched and formal Hodge structures in § 7.

2. Mixed Hodge structures with modulus

2.1.

Let $\mathbf{Vec}_{\mathbb{C}}$ be the category of finite dimensional $\mathbb{C}$-vector spaces. Let $\mathbf{Z}$ be the category associated with the ordered set $\mathbb{Z}$ and consider the category $\mathbf{Z}^{\mathrm{op}}\mathbf{Vec}_{\mathbb{C}}$ of functors $\mathbf{Z}^{\mathrm{op}} \to \mathbf{Vec}_{\mathbb{C}}$, that is, sequences in $\mathbf{Vec}_{\mathbb{C}}$ (which may be neither injective nor surjective and may not form a complex)

$$\cdots \to V^k \xrightarrow{\tau_V^k} V^{k-1} \xrightarrow{\tau_V^{k-1}} V^{k-2} \to \cdots . \tag{1}$$

We denote by $\mathbf{Vec}_{\mathbb{C}}^{\bullet}$ the strictly full subcategory of $\mathbf{Z}^{\mathrm{op}}\mathbf{Vec}_{\mathbb{C}}$ formed by the objects $V^{\bullet}$ such that $V^k = 0$ for all but finitely many elements $k \in \mathbb{Z}$.

We denote by **MHS** the category of mixed Hodge structures. For an object H of **MHS**, we denote by $H_{\mathbb{Z}}$ its underlying finitely generated $\mathbb{Z}$-module, by $W_{\bullet}H_{\mathbb{Q}}$ the weight filtration on $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$, and by $F^{\bullet}H_{\mathbb{C}}$ the Hodge filtration on $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$.

Given an object $\mathcal{H} := (H, H_{\mathrm{add}}^{\bullet}, H_{\mathrm{inf}}^{\bullet})$ in the product category $\mathbf{MHS} \times \mathbf{Vec}_{\mathbb{C}}^{\bullet} \times \mathbf{Vec}_{\mathbb{C}}^{\bullet}$, we set

$$\begin{aligned}
\mathcal{H}^k &:= H_{\mathbb{C}} \oplus H_{\mathrm{add}}^k \oplus H_{\mathrm{inf}}^k, \\
\tau^k &:= \mathrm{Id} \oplus \tau_{\mathrm{add}}^k \oplus \tau_{\mathrm{inf}}^k : \mathcal{H}^k \to \mathcal{H}^{k-1},
\end{aligned}$$

where $\tau_{\mathrm{add}}^k : H_{\mathrm{add}}^k \to H_{\mathrm{add}}^{k-1}$ and $\tau_{\mathrm{inf}}^k : H_{\mathrm{inf}}^k \to H_{\mathrm{inf}}^{k-1}$ are the structural maps.

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Definition 1. A mixed Hodge structure with modulus is a tuple

$$\mathcal{H} := (H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet}),$$

consisting of a mixed Hodge structure H, two objects $H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}$ in $\mathbf{Vec}^{\bullet}_{\mathbb{C}}$, and for every $k \in \mathbb{Z}$ a linear subspace $\mathcal{F}^k$ of $\mathcal{H}^k$ such that the following conditions are satisfied:

(1-a) $\tau^k(\mathcal{F}^k) \subseteq \mathcal{F}^{k-1}$;

(1-b) an element $x \in H_{\mathbb{C}}$ is in $F^k H_{\mathbb{C}}$ if and only if there exists $v \in H^k_{\mathrm{add}}$ such that $x + v \in \mathcal{F}^k$;

(1-c) $\mathcal{H}^k = \mathcal{F}^k + H_{\mathbb{C}} + H^k_{\mathrm{add}}$;

(1-d) $H^k_{\mathrm{add}} \cap \mathcal{F}^k = 0$.

By abuse of terminology, we call $\mathcal{F}^{\bullet}$ the Hodge filtration on $\mathcal{H}$. A morphism between two mixed Hodge structures with modulus is a morphism of $\mathbf{MHS} \times \mathbf{Vec}^{\bullet}_{\mathbb{C}} \times \mathbf{Vec}^{\bullet}_{\mathbb{C}}$ that respects Hodge filtrations. The category of mixed Hodge structures with modulus is denoted by **MHSM**. A mixed Hodge structure with modulus is said to be polarizable if its underlying mixed Hodge structure is graded polarizable, that is, the graded pieces for the weight filtration are polarizable Hodge structures.

Remark 2. The conditions **(1-c)** and **(1-d)** can be rewritten in a more symmetric way (in the sense of the opposite category). Indeed, they are equivalent to requiring that the linear map $\mathcal{F}^k \hookrightarrow \mathcal{H}^k \twoheadrightarrow H^k_{\mathrm{inf}}$ is surjective and the linear map $H^k_{\mathrm{add}} \hookrightarrow \mathcal{H}^k \twoheadrightarrow \mathcal{H}^k/\mathcal{F}^k$ is injective.

Our Definition 1 is motivated by preceding works [1, 6, 15, 18] as well as the geometric example described in § 4.

2.2.

Let $\mathcal{H} = (H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet})$ be an object of **MHSM**. For each integer k, put

$$\mathcal{H}^k_{\mathrm{inf}} := H_{\mathbb{C}} \oplus H^k_{\mathrm{inf}},$$
$$\mathcal{F}^k_{\mathrm{inf}} := \{x \in \mathcal{H}^k_{\mathrm{inf}} \mid x + v \in \mathcal{F}^k \text{ for some } v \in H^k_{\mathrm{add}}\} = \mathrm{Im}(\mathcal{F}^k \subset \mathcal{H}^k \twoheadrightarrow \mathcal{H}^k_{\mathrm{inf}}).$$

This definition and condition **(1-d)** implies that the projection map $\mathcal{H}^k \to \mathcal{H}^k_{\mathrm{inf}}$ restricts to an isomorphism $\mathcal{F}^k \cong \mathcal{F}^k_{\mathrm{inf}}$, and we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}^k & \xrightarrow{\simeq} & \mathcal{F}^k_{\mathrm{inf}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H^k_{\mathrm{add}} & \to & \mathcal{H}^k & \to & \mathcal{H}^k_{\mathrm{inf}} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & H^k_{\mathrm{add}} & \to & \mathcal{H}^k/\mathcal{F}^k & \to & \mathcal{H}^k_{\mathrm{inf}}/\mathcal{F}^k_{\mathrm{inf}} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (2)$$

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made of short exact sequences. It follows from this diagram and (**1-c**) that $\mathcal{H}^k_{\mathrm{inf}} = \mathcal{F}^k_{\mathrm{inf}} + H_{\mathbb{C}}$. Therefore, we find that

$$\mathcal{H}_{\mathrm{inf}} := (H, 0, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet_{\mathrm{inf}}) \tag{3}$$

is an object of **MHSM**. We obtain a functor

$$\pi_{\mathrm{inf}} : \mathbf{MHSM} \to \mathbf{MHSM}_{\mathrm{inf}}, \quad \pi_{\mathrm{inf}}(\mathcal{H}) = \mathcal{H}_{\mathrm{inf}},$$

where $\mathbf{MHSM}_{\mathrm{inf}}$ is the full subcategory of **MHSM** consisting of $(H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ such that $H^\bullet_{\mathrm{add}}$ is trivial. This is a left adjoint of the inclusion functor $i_{\mathrm{inf}} : \mathbf{MHSM}_{\mathrm{inf}} \to \mathbf{MHSM}$.

2.3.

Similarly, for an object $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ of **MHSM**, put

$$\mathcal{H}^k_{\mathrm{add}} := H_{\mathbb{C}} \oplus H^k_{\mathrm{add}}, \quad \mathcal{F}^k_{\mathrm{add}} := \mathcal{F}^k \cap \mathcal{H}^k_{\mathrm{add}} = \ker(\mathcal{H}^k_{\mathrm{add}} \subset \mathcal{H}^k \twoheadrightarrow \mathcal{H}^k/\mathcal{F}^k).$$

This definition and condition (**1-c**) implies that the inclusion map $\mathcal{H}^k_{\mathrm{add}} \to \mathcal{H}^k$ induces an isomorphism $\mathcal{H}^k_{\mathrm{add}}/\mathcal{F}^k_{\mathrm{add}} \cong \mathcal{H}^k/\mathcal{F}^k$ and we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{F}^k_{\mathrm{add}} & \to & \mathcal{F}^k & \to & H^k_{\mathrm{inf}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{H}^k_{\mathrm{add}} & \to & \mathcal{H}^k & \to & H^k_{\mathrm{inf}} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{H}^k_{\mathrm{add}}/\mathcal{F}^k_{\mathrm{add}} & \xrightarrow{\cong} & \mathcal{H}^k/\mathcal{F}^k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4}$$

made of short exact sequences. It follows from (**1-d**) that $H^k_{\mathrm{add}} \cap \mathcal{F}^k_{\mathrm{add}} = 0$, and

$$\mathcal{H}_{\mathrm{add}} := (H, H^\bullet_{\mathrm{add}}, 0, \mathcal{F}^\bullet_{\mathrm{add}}) \tag{5}$$

is an object of **MHSM**. We obtain a functor

$$\pi_{\mathrm{add}} : \mathbf{MHSM} \to \mathbf{MHSM}_{\mathrm{add}}, \quad \pi_{\mathrm{add}}(\mathcal{H}) = \mathcal{H}_{\mathrm{add}},$$

where $\mathbf{MHSM}_{\mathrm{add}}$ is the full subcategory of **MHSM** consisting of $(H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ such that $H^\bullet_{\mathrm{inf}}$ is trivial. This is a right adjoint of the inclusion functor $i_{\mathrm{add}} : \mathbf{MHSM}_{\mathrm{add}} \to \mathbf{MHSM}$.

2.4.

We identify **MHS** with the intersection of $\mathbf{MHSM}_{\mathrm{inf}}$ and $\mathbf{MHSM}_{\mathrm{add}}$ in **MHSM**. Then π_{inf} and π_{add} restrict to

$$\pi^0_{\mathrm{inf}} : \mathbf{MHSM}_{\mathrm{add}} \to \mathbf{MHS}, \quad \pi^0_{\mathrm{add}} : \mathbf{MHSM}_{\mathrm{inf}} \to \mathbf{MHS},$$

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and they are left and right adjoints of the inclusion functors

$$i^0_{\mathrm{inf}} : \mathbf{MHS} \to \mathbf{MHS}_{\mathrm{add}}, \quad i^0_{\mathrm{add}} : \mathbf{MHS} \to \mathbf{MHS}_{\mathrm{inf}},$$

respectively (see (8)). Let $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ be an object of **MHSM**. We have $\pi^0_{\mathrm{add}}\pi_{\mathrm{inf}}\mathcal{H} = \pi^0_{\mathrm{inf}}\pi_{\mathrm{add}}\mathcal{H} = (H, 0, 0, F^\bullet H_{\mathbb{C}})$. We may apply the results of § 2.3 and § 2.2 to $\mathcal{H}_{\mathrm{inf}}$ and $\mathcal{H}_{\mathrm{add}}$, respectively, yielding commutative diagrams

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^k H_{\mathbb{C}} & \longrightarrow & \mathcal{F}^k_{\mathrm{inf}} & \longrightarrow & H^k_{\mathrm{inf}} \to 0 \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & H_{\mathbb{C}} & \longrightarrow & \mathcal{H}^k_{\mathrm{inf}} & \longrightarrow & H^k_{\mathrm{inf}} \to 0 \\ & & \downarrow & & \downarrow & & \\ & & H_{\mathbb{C}}/F^k H_{\mathbb{C}} & \xrightarrow{\simeq} & \mathcal{H}^k_{\mathrm{inf}}/\mathcal{F}^k_{\mathrm{inf}} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \qquad \begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{F}^k_{\mathrm{add}} & \xrightarrow{\simeq} & F^k H_{\mathbb{C}} \\ & & & & \downarrow & & \downarrow \\ 0 & \to & H^k_{\mathrm{add}} & \longrightarrow & \mathcal{H}^k_{\mathrm{add}} & \longrightarrow & H_{\mathbb{C}} \longrightarrow 0 \\ & & \| & & \downarrow & & \downarrow \\ 0 & \to & H^k_{\mathrm{add}} & \to & \mathcal{H}^k_{\mathrm{add}}/\mathcal{F}^k_{\mathrm{add}} & \to & H_{\mathbb{C}}/F^k H_{\mathbb{C}} \to 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \tag{6}$$

made of short exact sequences. In particular, there exists a unique $\mathbb{C}$-linear map $\mathcal{H}^k/\mathcal{F}^k \to H_{\mathbb{C}}/F^k H_{\mathbb{C}}$ which makes the following diagram

$$\begin{array}{ccc} \mathcal{H}^k/\mathcal{F}^k & \longrightarrow & \mathcal{H}^k_{\mathrm{inf}}/\mathcal{F}^k_{\mathrm{inf}} \\ \simeq \uparrow & \searrow & \uparrow \simeq \\ \mathcal{H}^k_{\mathrm{add}}/\mathcal{F}^k_{\mathrm{add}} & \longrightarrow & H^k_{\mathbb{C}}/F^k H_{\mathbb{C}} \end{array} \tag{7}$$

commute (the vertical maps are induced by inclusions and the horizontal ones by projections).

2.5.

As the following proposition shows, the category of mixed Hodge structures with modulus is Abelian.

Proposition 3.

(1) *Any morphism* $f : \mathcal{H} \to \mathcal{H}'$ *in* **MHSM** *is strict with respect to the Hodge filtration, that is,* $f(\mathcal{F}^k) = \mathcal{F}'^k \cap f(\mathcal{H}^k)$ *for any* k *where* $\mathcal{H} := (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ *and* $\mathcal{H}' := (H', H'^\bullet_{\mathrm{add}}, H'^\bullet_{\mathrm{inf}}, \mathcal{F}'^\bullet)$.

(2) *The category* **MHSM** *is an Abelian category.*

Proof. The category **MHSM** is an additive category that has kernels and cokernels. Let $\mathrm{Im}\, f$ be the kernel of the canonical morphism $\mathcal{H}' \to \mathrm{Coker}\, f$ and $\mathrm{Coim}\, f$ the cokernel of the canonical morphism $\ker f \to \mathcal{H}$. Note that $\mathrm{Coim}\, f$ is the tuple

$$(H_{\mathbb{Z}}/\operatorname{Ker} f_{\mathbb{Z}}, H^\bullet_{\mathrm{add}}/\operatorname{Ker} f^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}/\operatorname{Ker} f^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet/(\mathcal{F}^\bullet \cap \operatorname{Ker} f^\bullet),$$

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while $\operatorname{Im} f$ is the tuple

$$(\operatorname{Im} f_{\mathbb{Z}}, \operatorname{Im} f^{\bullet}_{\mathrm{add}}, \operatorname{Im} f^{\bullet}_{\mathrm{inf}}, \mathcal{F}'^{\bullet} \cap \operatorname{Im} f^{\bullet}).$$

Recall that for every object $\mathcal{H}$ in **MHSM**, we have a (functorial) exact sequence and an isomorphism from (4) and (6):

$$0 \to \mathcal{F}^k_{\mathrm{add}} \to \mathcal{F}^k \to H^k_{\mathrm{inf}} \to 0, \quad \mathcal{F}^k_{\mathrm{add}} \xrightarrow{\cong} F^k H_{\mathbb{C}}.$$

By applying these to both $\operatorname{Coim} f$ and $\operatorname{Im} f$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^k H_{\mathbb{C}}/(F^k H_{\mathbb{C}} \cap \operatorname{Ker} f_{\mathbb{C}}) & \longrightarrow & \mathcal{F}^k/\mathcal{F}^k \cap \operatorname{Ker} f^k & \longrightarrow & H^k_{\mathrm{inf}}/\operatorname{Ker} f^k_{\mathrm{inf}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & F^k H'_{\mathbb{C}} \cap \operatorname{Im} f_{\mathbb{C}} & \longrightarrow & \mathcal{F}'^k \cap f(\mathcal{H}^k) & \longrightarrow & \operatorname{Im} f^k_{\mathrm{inf}} & \longrightarrow & 0. \end{array}$$

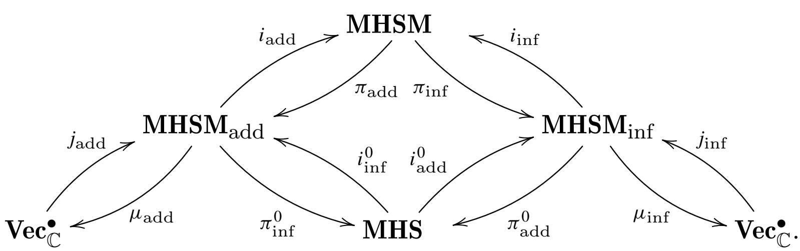
Thus, (1) is reduced to showing that the left vertical map is an isomorphism. This follows from the fact that **MHS** is an Abelian category (that is, every morphism of mixed Hodge structures is strict with respect to the Hodge filtration). (2) follows from (1). □

2.6.

Let us consider functors

$$\begin{aligned} &\mu_{\mathrm{add}} : \mathbf{MHSM}_{\mathrm{add}} \to \mathbf{Vec}^{\bullet}_{\mathbb{C}}, \quad \mu_{\mathrm{add}}(H, H^{\bullet}_{\mathrm{add}}, 0, \mathcal{F}^{\bullet}_{\mathrm{add}}) = H^{\bullet}_{\mathrm{add}}, \\ &\mu_{\mathrm{inf}} : \mathbf{MHSM}_{\mathrm{inf}} \to \mathbf{Vec}^{\bullet}_{\mathbb{C}}, \quad \mu_{\mathrm{inf}}(H, 0, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet}_{\mathrm{inf}}) = H^{\bullet}_{\mathrm{inf}}, \\ &j_{\mathrm{add}} : \mathbf{Vec}^{\bullet}_{\mathbb{C}} \to \mathbf{MHSM}_{\mathrm{add}} \quad j_{\mathrm{add}}(V^{\bullet}) = (0, V^{\bullet}, 0, 0), \\ &j_{\mathrm{inf}} : \mathbf{Vec}^{\bullet}_{\mathbb{C}} \to \mathbf{MHSM}_{\mathrm{inf}} \quad j_{\mathrm{inf}}(V^{\bullet}) = (0, 0, V^{\bullet}, V^{\bullet}). \end{aligned}$$

Then μ_{inf} is a left adjoint of j_{inf} and μ_{add} is a right adjoint of j_{add}. We summarize the functors we have introduced so far:



(8)

Remark 4.

(1) Let $\mathcal{H}$ be an object in **MHSM**. We write $\mathcal{H}_{\mathrm{inf}} = i_{\mathrm{inf}}\pi_{\mathrm{inf}}(\mathcal{H})$ and $\mathcal{H}_{\mathrm{add}} = i_{\mathrm{add}}\pi_{\mathrm{add}}(\mathcal{H})$; see (3) and (5). Let us also abbreviate $H_{\mathrm{inf}} = i_{\mathrm{inf}} j_{\mathrm{inf}} \mu_{\mathrm{inf}} \pi_{\mathrm{inf}}(\mathcal{H})$, $H_{\mathrm{add}} = i_{\mathrm{add}} j_{\mathrm{add}} \mu_{\mathrm{add}} \pi_{\mathrm{add}}(\mathcal{H})$, and $H = \pi^0_{\mathrm{inf}}\pi_{\mathrm{add}}(\mathcal{H}) = \pi^0_{\mathrm{add}}\pi_{\mathrm{inf}}(\mathcal{H})$. Various (co)unit maps

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make a commutative diagram in **MHSM**

$$\begin{array}{ccccccccc} & & & & \mathcal{H} & & & & \\ & & & \nearrow & & \searrow & & & \\ & & \mathcal{H}_{\mathrm{add}} & & & & \mathcal{H}_{\mathrm{inf}} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ H_{\mathrm{add}} & & & & H & & & & H_{\mathrm{inf}}, \end{array}$$

which produces the following (functorial) short exact sequences

$$0 \to \mathcal{H}_{\mathrm{add}} \to \mathcal{H} \to H_{\mathrm{inf}} \to 0, \quad 0 \to H_{\mathrm{add}} \to \mathcal{H} \to \mathcal{H}_{\mathrm{inf}} \to 0$$

and

$$0 \to H_{\mathrm{add}} \to \mathcal{H}_{\mathrm{add}} \to H \to 0, \quad 0 \to H \to \mathcal{H}_{\mathrm{inf}} \to H_{\mathrm{inf}} \to 0.$$

(2) Any morphism $f : \mathcal{H} \to \mathcal{H}'$ in **MHSM** is strict with respect to the filtration

$$H_{\mathrm{add}} \subset \mathcal{H}_{\mathrm{add}} \subset \mathcal{H}, \quad H'_{\mathrm{add}} \subset \mathcal{H}'_{\mathrm{add}} \subset \mathcal{H}',$$

that is, $f(\mathcal{H}) \cap \mathcal{H}'_{\mathrm{add}} = f(\mathcal{H}_{\mathrm{add}})$, $f(\mathcal{H}) \cap H'_{\mathrm{add}} = f(H_{\mathrm{add}})$.

2.7.

Recall that a mixed Hodge structure H is called free if $H_{\mathbb{Z}}$ is free as a $\mathbb{Z}$-module. In this case, it makes sense to define $W_k H_{\mathbb{Z}} := W_k H_{\mathbb{Q}} \cap H_{\mathbb{Z}}$. For general H, we define its free part by $H_{\mathrm{fr}} := (H_{\mathbb{Z}}/H_{\mathbb{Z},\mathrm{Tor}}, W_\bullet H_{\mathbb{Q}}, F^\bullet H_{\mathbb{C}})$. A mixed Hodge structure with modulus $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ is called free if H is. For general $\mathcal{H}$, we define its free part by $\mathcal{H}_{\mathrm{fr}} := (H_{\mathrm{fr}}, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$.

2.8.

Let H be a mixed Hodge structure. The dual mixed Hodge structure $H^\vee = (H^\vee_{\mathbb{Z}}, W_\bullet H^\vee_{\mathbb{Q}}, F^\bullet H^\vee_{\mathbb{C}})$ of H is defined by

$$H^\vee_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}), \quad W_k H^\vee_{\mathbb{Q}} = (H_{\mathbb{Q}}/W_{-1-k}H_{\mathbb{Q}})^\vee, \quad F^k H^\vee_{\mathbb{C}} = (H_{\mathbb{C}}/F^{1-k}H_{\mathbb{C}})^\vee,$$

where $\vee$ on the right-hand side denotes linear dual (see [7, 1.1.6], [12, 1.6.2]). Let $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ be an object in **MHSM**. We define the dual $\mathcal{H}^\vee$ of $\mathcal{H}$ as the tuple

$$\mathcal{H}^\vee := (H^\vee, H^{\vee,\bullet}_{\mathrm{add}}, H^{\vee,\bullet}_{\mathrm{inf}}, \mathcal{F}^{\vee,\bullet}).$$

Here $H^\vee$ is the dual of the mixed Hodge structure H and for every $k \in \mathbb{Z}$,

$$H^{\vee,k}_{\mathrm{add}} := (H^{1-k}_{\mathrm{inf}})^\vee, \quad H^{\vee,k}_{\mathrm{inf}} := (H^{1-k}_{\mathrm{add}})^\vee, \quad \mathcal{F}^{\vee,k} := (\mathcal{H}^{1-k}/\mathcal{F}^{1-k})^\vee.$$

It is straightforward to see that the tuple $\mathcal{H}^\vee$ belongs to **MHSM**. By definition, $\mathcal{H}^\vee$ is always free, and we have

$$\mathcal{H}^\vee = (\mathcal{H}_{\mathrm{fr}})^\vee, \quad (\mathcal{H}^\vee)^\vee \cong \mathcal{H}_{\mathrm{fr}}. \tag{9}$$

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2.9.

Let m be an integer. Recall that the Tate twist $H(m)$ of a mixed Hodge structure H is defined by

$$H(m)_{\mathbb{Z}} = (2\pi i)^m H_{\mathbb{Z}}, \ \ W_k H(m)_{\mathbb{Q}} = (2\pi i)^m W_{k+2m} H_{\mathbb{Q}}, \ \ F^k H(m)_{\mathbb{C}} = F^{k+m} H_{\mathbb{C}}.$$

Let $\mathcal{H} = (H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet})$ be an object in **MHSM**. We define the Tate twist $\mathcal{H}(m) = (H(m), H(m)^{\bullet}_{\mathrm{add}}, H(m)^{\bullet}_{\mathrm{inf}}, \mathcal{F}(m)^{\bullet}) \in \mathbf{MHSM}$ of $\mathcal{H}$ by

$$H(m)^k_{\mathrm{add}} = H^{k+m}_{\mathrm{add}}, \quad H(m)^k_{\mathrm{inf}} = H^{k+m}_{\mathrm{inf}}, \quad \mathcal{F}(m)^k = \mathcal{F}^{k+m}.$$

2.10.

Let $\mathbf{mod}(\mathbb{Z})$ be the category of finitely generated Abelian groups. There is a faithful exact functor

$$R : \mathbf{MHSM} \to \mathbf{mod}(\mathbb{Z}) \times \mathbf{Vec}_{\mathbb{C}},$$
$$R(H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet}) = \left(H_{\mathbb{Z}}, \ \bigoplus_{k \in \mathbb{Z}} \left(H^k_{\mathrm{add}} \oplus H^k_{\mathrm{inf}} \right) \right).$$

Remark 5.

(1) A sequence in **MHSM** is exact if and only if its image by R is exact in $\mathbf{mod}(\mathbb{Z}) \times \mathbf{Vec}_{\mathbb{C}}$.

(2) Since $H^k_{\mathrm{add}} = H^k_{\mathrm{inf}} = 0$ for almost all k, for any object $\mathcal{H}$ of **MHSM**, we have a canonical isomorphism $R(\mathcal{H}^{\vee}) \cong R(\mathcal{H})^{\vee}$, where, on the right-hand side, $\vee$ denotes the dual functor given by $(A, V)^{\vee} = (\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}), \ \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}))$.

3. Laumon 1-motives

In [15, §4.1], Kato and Russell have defined a category $\mathcal{H}_1$ which provides a Hodge theoretic description of the category $\mathscr{M}_1^{\mathrm{Lau}}$ of Laumon 1-motives over $\mathbb{C}$ that extends Deligne's description [8, §10] of the category $\mathscr{M}_1^{\mathrm{Del}}$ of Deligne 1-motives over $\mathbb{C}$ in terms of the full subcategory $\mathbf{MHS}_1$ of **MHS** (see § 3.1 for its definition and § 3.6 for Laumon 1-motives). In this section, we define a subcategory of $\mathbf{MHSM}_1$ of **MHSM** which is equivalent to $\mathcal{H}_1$, yielding an equivalence between $\mathbf{MHSM}_1$ and $\mathscr{M}_1^{\mathrm{Lau}}$ (Corollary 8). This will be used for our construction of Picard and Albanese 1-motives in § 6. There is another Hodge theoretic description of $\mathscr{M}_1^{\mathrm{Lau}}$, due to Barbieri-Viale [1], in terms of the category $\mathbf{FHS}_1^{\mathrm{fr}}$ of torsion-free formal Hodge structures of level $\leqslant 1$. As is explained in [15, §4.6], two categories $\mathcal{H}_1$ and $\mathbf{FHS}_1^{\mathrm{fr}}$ are equivalent.

3.1.

Let $\mathbf{MHS}_1$ be the full subcategory of **MHS** formed by the free mixed Hodge structures of Hodge type

$$\{(0, 0), (-1, 0), (0, -1), (-1, -1)\}$$

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such that Gr^W_{-1} is polarizable (see [8, Construction (10.1.3)]). Recall that such a mixed Hodge structure is simply a free Abelian group of finite rank $H_{\mathbb{Z}}$ with two filtrations (on $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ and $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$)

$$0 = W_{-3}H_{\mathbb{Q}} \subseteq W_{-2}H_{\mathbb{Q}} \subseteq W_{-1}H_{\mathbb{Q}} \subseteq W_0 H_{\mathbb{Q}} = H_{\mathbb{Q}}$$
$$0 = F^1 H_{\mathbb{C}} \subseteq F^0 H_{\mathbb{C}} \subseteq F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$$

such that $F^0 \mathrm{Gr}^W_0 H_{\mathbb{C}} = \mathrm{Gr}^W_0 H_{\mathbb{C}}$ (that is, $F^0 H_{\mathbb{C}} + W_{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$), $F^0 W_{-2} H_{\mathbb{C}} = 0$ and $\mathrm{Gr}^W_{-1} H_{\mathbb{Z}}$ is a polarizable pure Hodge structure of weight -1. (See § 2.7 for $W_\bullet H_{\mathbb{Z}}$.)

3.2.

Let $\mathcal{H}_1$ be the Abelian category defined by Kato and Russell in [15, §4.1]. Recall that an object in $\mathcal{H}_1$ is a pair $(H_{\mathbb{Z}}, H_V)$ consisting of a free Abelian group of finite rank $H_{\mathbb{Z}}$ and a $\mathbb{C}$-vector space H_V together with

(a) two two-step filtrations (called weight filtrations)

$$0 = W_{-3}H_{\mathbb{Q}} \subseteq W_{-2}H_{\mathbb{Q}} \subseteq W_{-1}H_{\mathbb{Q}} \subseteq W_0 H_{\mathbb{Q}} = H_{\mathbb{Q}}$$
$$0 = W_{-3}H_V \subseteq W_{-2}H_V \subseteq W_{-1}H_V \subseteq W_0 H_V = H_V$$

on $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ and H_V;

(b) a one-step filtration (called Hodge filtration)

$$0 = F^1 H_V \subseteq F^0 H_V \subseteq F^{-1} H_V = H_V;$$

(c) two $\mathbb{C}$-linear maps $a : H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \to H_V$ and $b : H_V \to H_{\mathbb{C}}$ which are compatible with the weight filtrations (that is, a maps $W_k H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Q}} W_k H_{\mathbb{Q}}$ to $W_k H_V$ and b maps $W_k H_V$ to $W_k H_{\mathbb{C}}$) and such that $b \circ a = \mathrm{Id}$;

(d) a splitting of the weight filtration on $\mathrm{Ker}(b : H_V \to H_{\mathbb{C}})$, that is, a decomposition of $\mathrm{Ker}(b)$ as the direct sum of the graded pieces of the filtration induced by the given weight filtration on H_V;

such that the following conditions are satisfied:

(i) the map a induces an isomorphism $\mathrm{Gr}^W_{-1} H_{\mathbb{C}} \to \mathrm{Gr}^W_{-1} H_V$ and the filtration on $\mathrm{Gr}^W_{-1} H_V$ induced by the Hodge filtration on H_V induces via this isomorphism a polarizable pure Hodge structure of weight -1 on $\mathrm{Gr}^W_{-1} H_{\mathbb{Z}}$ (here $H_{\mathbb{Z}}$ is endowed with the filtration induced by the weight filtration on $H_{\mathbb{Q}}$);

(ii) $F^0 \mathrm{Gr}^W_0 H_V = \mathrm{Gr}^W_0 H_V$ and $F^0 W_{-2} H_V = 0$.

Let us observe that one can attach to an object in $\mathcal{H}_1$ a canonical (graded polarizable) mixed Hodge structure. To see this, set

$$H^0_{\mathrm{add}} := \mathrm{Ker}(W_{-2} H_V \to W_{-2} H_{\mathbb{C}}),$$
$$H^0_{\mathrm{inf}} := \mathrm{Ker}(\mathrm{Gr}^W_0 H_V \to \mathrm{Gr}^W_0 H_{\mathbb{C}}). \tag{10}$$

The condition (i) implies $\mathrm{Ker}(\mathrm{Gr}^W_{-1} H_V \to \mathrm{Gr}^W_{-1} H_{\mathbb{C}}) = 0$ and thus the given splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ provides a direct sum decomposition $H_V = H_{\mathbb{C}} \oplus$

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$H^0_{\mathrm{add}} \oplus H^0_{\mathrm{inf}}$ in which the weight filtration on H_V becomes

$$\begin{aligned} W_0 H_V &:= H_V, \\ W_{-1} H_V &:= W_{-1} H_{\mathbb{C}} \oplus H^0_{\mathrm{add}}, \\ W_{-2} H_V &:= W_{-2} H_{\mathbb{C}} \oplus H^0_{\mathrm{add}}, \\ W_{-3} H_V &:= 0. \end{aligned}$$

One can then consider the one-step filtration

$$0 = F^1 H_{\mathbb{C}} \subseteq F^0 H_{\mathbb{C}} \subseteq F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}, \tag{11}$$

where $F^0 H_{\mathbb{C}}$ is defined as the linear subspace of $H_{\mathbb{C}}$ formed by the elements such that there exists $v \in H^0_{\mathrm{add}}$ for which $x + v$ is contained in $F^0 H_V$. The conditions (i) and (ii) have the following consequences.

Lemma 6. *We have* $F^0 H_{\mathbb{C}} + W_{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$ *and* $F^0 W_{-2} H_{\mathbb{C}} = 0$*. Moreover, the map* a *induces an isomorphism* $F^0 \mathrm{Gr}^W_{-1} H_{\mathbb{C}} = F^0 \mathrm{Gr}^W_{-1} H_V$*.*

In particular, this attaches to an object in $\mathcal{H}_1$ a canonical (graded polarizable) mixed Hodge structure of type $\{(0,0), (-1,0), (0,-1), (-1,-1)\}$ with Hodge filtration given by (11).

3.3.

Let us denote by $\mathbf{MHSM}_1$ the strictly full subcategory of $\mathbf{MHSM}$ formed by the mixed Hodge structures with modulus $(H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ such that the underlying mixed Hodge structure H belongs to $\mathbf{MHS}_1$ and such that $H^k_{\mathrm{inf}} = H^k_{\mathrm{add}} = 0$ if $k \neq 0$. The proof of the following proposition will be given in § 3.4 and § 3.5.

Proposition 7. *The categories* $\mathbf{MHSM}_1$ *and* $\mathcal{H}_1$ *are equivalent.*

3.4.

Let us explain the construction of a functor from $\mathcal{H}_1$ to $\mathbf{MHSM}_1$. Let $(H_{\mathbb{Z}}, H_V)$ together with the data described in § 3.2 be an object in the category $\mathcal{H}_1$. We associate with it the tuple $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$. Here H is the mixed Hodge structure constructed in § 3.2; $H^\bullet_{\mathrm{add}}$ and $H^\bullet_{\mathrm{inf}}$ are the sequences defined by $H^k_{\mathrm{add}} = H^k_{\mathrm{inf}} = 0$ if $k \neq 0$ and (10); the Hodge filtration $\mathcal{F}^\bullet$ is defined by $\mathcal{F}^0 := F^0 H_V$ via the canonical direct sum decomposition $H_V = H_{\mathbb{C}} \oplus H^0_{\mathrm{add}} \oplus H^0_{\mathrm{inf}} =: \mathcal{H}^0$ given by the splitting of the weight filtration (see § 3.2) and $\mathcal{F}^k = F^k H_{\mathbb{C}}$ if $k \neq 0$. All conditions are obviously satisfied unless $k = 0$. In that case, **(1-a)** is obvious and **(1-b)** is a consequence of the definition of the Hodge filtration in § 3.2. Condition **(1-c)** is implied by $F^0 \mathrm{Gr}^W_0 H_V = \mathrm{Gr}^W_0 H_V$ and **(1-d)** by $F^0 W_{-2} H_V = 0$.

3.5.

We now construct a functor from $\mathbf{MHSM}_1$ to the category $\mathcal{H}_1$. It is easy to see that this functor and the one constructed in § 3.4 are quasi-inverse one to another proving

Proposition 7. Given an object $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ in $\mathbf{MHSM}_1$, we set

$$H_V := \mathcal{H}^0 = H_\mathbb{C} \oplus H^0_{\mathrm{add}} \oplus H^0_{\mathrm{inf}}$$

and

$$\begin{aligned} F^{-1}H_V &:= H_V, & W_0 H_V &:= H_V,\\ F^0 H_V &:= \mathcal{F}^0, & W_{-1}H_V &:= W_{-1}H_\mathbb{C} \oplus H^0_{\mathrm{add}},\\ F^1 H_V &:= 0, & W_{-2}H_V &:= W_{-2}H_\mathbb{C} \oplus H^0_{\mathrm{add}},\\ & & W_{-3}H_V &:= 0. \end{aligned}$$

The map $a : H_\mathbb{C} \to H_V$ is given by the inclusion and the map $b : H_V \to H_\mathbb{C}$ by the projection. The weight filtration on $\mathrm{Ker}(b) = H^0_{\mathrm{add}} \oplus H^0_{\mathrm{inf}}$ is given by

$$0 = W_{-3}\,\mathrm{Ker}(b) \subseteq W_{-2}\,\mathrm{Ker}(b) = W_{-1}\,\mathrm{Ker}(b) = H^0_{\mathrm{add}} \subseteq W_0\,\mathrm{Ker}(b) = \mathrm{Ker}(b)$$

and the splitting is the obvious one. We have to check that the two conditions (i) and (ii) are satisfied.

We start with (ii). Note that (see (4)) $F^0W_{-2}H_V$ is a linear subspace of $\mathcal{F}^0_{\mathrm{add}}$ and its image under the projection onto $H_\mathbb{C}$ is contained in $F^0W_{-2}H_\mathbb{C}$ which is zero (because of the restriction on the Hodge type of H). Since the projection maps $\mathcal{F}^0_{\mathrm{add}}$ isomorphically onto $F^0H_\mathbb{C}$ by **(1-b)** and **(1-d)**, we have $F^0W_{-2}H_V = 0$. To show that $F^0\mathrm{Gr}^W_0 H_V = \mathrm{Gr}^W_0 H_V$, we have to show that $H_V = \mathcal{F}^0 + W_{-1}H_\mathbb{C} + H^0_{\mathrm{add}}$. We know that $F^0\mathrm{Gr}^W_0 H_\mathbb{C} = \mathrm{Gr}^W_0 H_\mathbb{C}$ (because of the restriction on the Hodge type of H); hence, $H_\mathbb{C} = F^0H_\mathbb{C} + W_{-1}H_\mathbb{C}$. By **(1-b)**, we also have $F^0H_\mathbb{C} \subseteq \mathcal{F}^0 + H^0_{\mathrm{add}}$. Hence, using **(1-c)**, we obtain

$$H_V = \mathcal{F}^0 + H_\mathbb{C} + H^0_{\mathrm{add}} \subseteq \mathcal{F}^0 + F^0H_\mathbb{C} + W_{-1}H_\mathbb{C} + H^0_{\mathrm{add}} \subseteq \mathcal{F}^0 + W_{-1}H_\mathbb{C} + H^0_{\mathrm{add}}$$

and therefore we have $H_V = \mathcal{F}^0 + W_{-1}H_\mathbb{C} + H^0_{\mathrm{add}}$ as desired.

To prove (i), note that (see (4)) the restriction of b to the linear subspace $W_{-1}H_V$ is induced by the projection $\mathcal{H}^0_{\mathrm{add}} \to H_\mathbb{C}$ which maps $\mathcal{F}^0_{\mathrm{add}}$ isomorphically onto $F^0H_\mathbb{C}$. Therefore, b maps $F^0W_{-1}H_V$ isomorphically onto $F^0W_{-1}H_\mathbb{C}$. Since $F^0W_{-2}H_V = 0$ and $F^0W_{-2}H_\mathbb{C} = 0$, we have a commutative square

$$\begin{array}{ccc} F^0W_{-1}H_V & \hookrightarrow & W_{-1}H_V/W_{-2}H_V = \mathrm{Gr}^W_{-1}H_V \\ \downarrow \simeq & & \downarrow \simeq \\ F^0W_{-1}H_\mathbb{C} & \hookrightarrow & W_{-1}H_\mathbb{C}/W_{-2}H_\mathbb{C} = \mathrm{Gr}^W_{-1}H_\mathbb{C} \end{array}$$

in which the vertical morphisms are the isomorphisms induced by b. This shows that the filtration on $\mathrm{Gr}^W_{-1}H_V$ deduced from the Hodge filtration on H_V is the Hodge filtration on $\mathrm{Gr}^W_{-1}H_\mathbb{Z}$ which is thus a polarizable pure Hodge structure of weight -1.

3.6.

We briefly recall the category $\mathscr{M}_1^{\mathrm{Lau}}$ of Laumon 1-motives over $\mathbb{C}$ in the sense of [16]. Let $\mathscr{S}$ be the category of fppf sheaves [26, Example 2.32] on the category of affine $\mathbb{C}$-schemes.

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We write $\mathrm{Lie}(F) := \ker(F(\mathbb{C}[\epsilon]/(\epsilon^2)) \to F(\mathbb{C}))$ for $F \in \mathscr{S}$. We consider the category $\mathscr{S}_0$ of connected commutative algebraic groups over $\mathbb{C}$ as a full subcategory of $\mathscr{S}$. Recall that any $G \in \mathscr{S}_0$ is an extension of an Abelian variety G_{ab} by a linear algebraic group G_{lin}, and we have a decomposition $G_{\mathrm{lin}} \cong G_{\mathrm{mul}} \times G_{\mathrm{add}}$ with $G_{\mathrm{mul}} \cong \mathbf{G}_m^r$ and $G_{\mathrm{add}} \cong \mathbf{G}_a^s$ for some $r, s \in \mathbb{Z}_{\geqslant 0}$. Denote by $\mathscr{S}_{-1}$ the category of formal groups F over $\mathbb{C}$ such that $F = F_{\text{ét}} \times F_{\mathrm{inf}}$ with $F_{\text{ét}} \cong \mathbb{Z}^r$ and $F_{\mathrm{inf}} \cong \hat{\mathbf{G}}_a^s$ for some $r, s \in \mathbb{Z}_{\geqslant 0}$. We also consider $\mathscr{S}_{-1}$ as a full subcategory of $\mathscr{S}$.

Recall that an object of $\mathscr{M}_1^{\mathrm{Lau}}$ is a two-term complex $[F \to G]$ in $\mathscr{S}$ where $F \in \mathscr{S}_{-1}$ and $G \in \mathscr{S}_0$.

It is proved in [15, Theorem 4.1] that $\mathcal{H}_1$ is equivalent to $\mathscr{M}_1^{\mathrm{Lau}}$. Therefore, Proposition 7 implies the following corollary.

Corollary 8. *The categories* $\mathbf{MHSM}_1$ *and* $\mathscr{M}_1^{\mathrm{Lau}}$ *are equivalent.*

Explicit description of the equivalence functors $\mathcal{H}_1 \to \mathscr{M}_1^{\mathrm{Lau}}$ and $\mathscr{M}_1^{\mathrm{Lau}} \to \mathcal{H}_1$ are given in [15, §4.3 and §4.4]. By composing them with those in § 3.4 and § 3.5, we can explicitly describe the equivalence functors in Corollary 8 as follows.

The functor $\mathbf{MHSM}_1 \to \mathscr{M}_1^{\mathrm{Lau}}$ sends an object $\mathcal{H} = (H, H_{\mathrm{add}}^\bullet, H_{\mathrm{inf}}^\bullet, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}_1$ to the Laumon 1-motive $[F_{\text{ét}} \times F_{\mathrm{inf}} \to G]$ described as follows. First, set

$$\begin{aligned} G &= W_{-1}H_{\mathbb{Z}} \backslash W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0 / \mathcal{F}^0 \cap (W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0), \\ F_{\text{ét}} &= \mathrm{Gr}_0^W H_{\mathbb{Z}}, \quad \mathrm{Lie}\, F_{\mathrm{inf}} = H_{\mathrm{inf}}^0. \end{aligned}$$

Next, we describe the map $F_{\text{ét}} \to G$. Consider a commutative diagram

$$\begin{array}{ccccc} \mathrm{Gr}_0^W H_{\mathbb{Z}} & \longrightarrow & \mathcal{H}/(W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0) & \xleftarrow{\cong} & \mathcal{F}^0/\mathcal{F}^0 \cap (W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0) \\ \uparrow & & \uparrow & & \uparrow \\ H_{\mathbb{Z}} & \hookrightarrow & \mathcal{H} = H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0 \oplus H_{\mathrm{inf}}^0 & \hookleftarrow & \mathcal{F}^0. \end{array}$$

(See § 3.2 (ii) for the bijectivity of the upper right arrow.) Given $x \in F_{\text{ét}} = \mathrm{Gr}_0^W H_{\mathbb{Z}}$, we choose its lift $y \in H_{\mathbb{Z}}$ and an element z of $\mathcal{F}^0$ having the same image as x in $\mathcal{H}/(W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0)$. Then $y - z \in \mathcal{H}$ belongs to $W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0$ and its class in G is independent of choices of y and z. Therefore, we get a well-defined map $F_{\text{ét}} \to G$. Finally, we describe the map $F_{\mathrm{inf}} \to G$, or what amounts to the same, $\mathrm{Lie}\, F_{\mathrm{inf}} \to \mathrm{Lie}\, G$. Consider a commutative diagram

$$\begin{array}{ccccc} H_{\mathrm{inf}}^0 & \longrightarrow & \mathcal{H}/(W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0) & \xleftarrow{\cong} & \mathcal{F}^0/\mathcal{F}^0 \cap (W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0) \\ \| & & \uparrow & & \uparrow \\ H_{\mathrm{inf}}^0 & \hookrightarrow & \mathcal{H} = H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0 \oplus H_{\mathrm{inf}}^0 & \hookleftarrow & \mathcal{F}^0. \end{array}$$

Given $x \in F_{\mathrm{inf}} = H_{\mathrm{inf}}^0$, we choose an element z of $\mathcal{F}^0$ having the same image as x in $\mathcal{H}/(W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0)$. Then $x - z \in \mathcal{H}$ belongs to $W_{-1}H_{\mathbb{C}} \oplus H_{\mathrm{add}}^0$ and its class in $\mathrm{Lie}\, G =$

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$W_{-1}H_{\mathbb{C}} \oplus H^0_{\rm add}/\mathcal{F}^0 \cap (W_{-1}H_{\mathbb{C}} \oplus H^0_{\rm add})$ is independent of choices of z. Therefore, we get a well-defined map $\operatorname{Lie} F_{\rm inf} \to \operatorname{Lie} G$.

In the other direction, the functor $\mathscr{M}_1^{\rm Lau} \to \mathbf{MHSM}_1$ sends a Laumon 1-motive $[u_{\rm \acute{e}t} \times u_{\rm inf} : F_{\rm \acute{e}t} \times F_{\rm inf} \to G]$ to the object $\mathcal{H} = (H, H^\bullet_{\rm add}, H^\bullet_{\rm inf}, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}_1$ described as follows. Let $H_{\mathbb{Z}}$ be the fiber product of $u_{\rm \acute{e}t} : F_{\rm \acute{e}t} \to G$ and $\exp : \operatorname{Lie} G \to G$, which comes equipped with $\alpha : H_{\mathbb{Z}} \to F_{\rm \acute{e}t}$ and $\beta : H_{\mathbb{Z}} \to \operatorname{Lie} G$. We set

$$\begin{aligned} W_0 H_{\mathbb{Z}} = H_{\mathbb{Z}} &\supseteq W_{-1}H_{\mathbb{Z}} = \ker(\alpha) = \ker(\exp) = H_1(G, \mathbb{Z}) \\ &\supseteq W_{-2}H_{\mathbb{Z}} = \ker(H_1(G, \mathbb{Z}) \to H_1(G_{\rm ab}, \mathbb{Z})) \\ &\supseteq W_{-3}H_{\mathbb{Z}} = 0, \end{aligned}$$

where $G_{\rm ab}$ is the maximal Abelian quotient of G. Put $H^0_{\rm inf} := \operatorname{Lie} F_{\rm inf}$ and $H^0_{\rm add} := \operatorname{Lie} G_{\rm add}$, where $G_{\rm add}$ is the additive part of G. Finally, we set

$$\mathcal{F}^0 := \ker(H_{\mathbb{C}} \oplus H^0_{\rm add} \oplus H^0_{\rm inf} \to \operatorname{Lie} G),$$

where $H_{\mathbb{C}} \to \operatorname{Lie} G$ is induced by β, $H^0_{\rm add} = \operatorname{Lie} G_{\rm add} \to \operatorname{Lie} G$ is the inclusion map, and $H^0_{\rm inf} = \operatorname{Lie} F_{\rm inf} \to \operatorname{Lie} G$ is induced by $u_{\rm inf}$. The Hodge filtration $F^\bullet H_{\mathbb{C}}$ is determined by the condition Definition 1 **(1-b)**.

3.7.

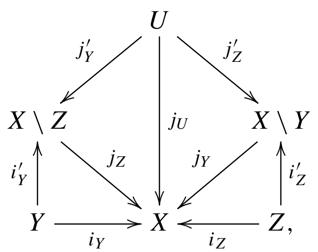
Let us consider the Cartier duality functor $(\mathscr{M}_1^{\rm Lau})^{\rm op} \to \mathscr{M}_1^{\rm Lau}$. The corresponding functor $\mathcal{H}_1^{\rm op} \to \mathcal{H}_1$ admits a simple description as $\underline{\operatorname{Hom}}(-, \mathbb{Z})(1)$ [15, 4.1]. By rewriting it through § 3.4 and § 3.5, we find that the functor

$$\mathbf{MHSM}_1^{\rm op} \to \mathbf{MHSM}_1, \quad \mathcal{H} \mapsto \mathcal{H}^\vee(1)$$

gives a duality that is compatible with the Cartier duality via the equivalence in Corollary 8. Here $\vee$ and (1) denotes the dual and Tate twist (see § 2.8 and § 2.9).

4. Cohomology of a variety with modulus

In this section, X is a connected smooth proper variety of dimension d over $\mathbb{C}$, and Y, Z are effective divisors on X such that $|Y| \cap |Z| = \emptyset$ and such that $(Y + Z)_{\rm red}$ is a simple normal crossing divisor. Put $U = X \setminus (Y \cup Z)$ and let us consider the following commutative diagram



(12)

where all the maps are embeddings. The aim of this section is to construct an object $\mathcal{H}^n(X, Y, Z)$ of **MHSM** for each integer n.

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4.1.

In this subsection, we assume that Y and Z are reduced. We consider the relative cohomology

$$H^n := H^n(X \setminus Z, Y, \mathbb{Z}) = H^n(X, \mathbb{Z}_{X|Y,Z}) \tag{13}$$

for each integer n, where $\mathbb{Z}_{X|Y,Z} := R(j_Z)_*(j'_Y)_!\mathbb{Z}_U$. It carries a mixed Hodge structure as in [8, 8.3.8] or [19, Proposition 5.46]. By applying $R(j_Z)_*$ to an exact sequence

$$0 \to (j'_Y)_!\mathbb{Z}_U \to \mathbb{Z}_{X\setminus Z} \to (i'_Y)_*\mathbb{Z}_Y \to 0,$$

we get a canonical distinguished triangle

$$\mathbb{Z}_{X|Y,Z} \to R(j_Z)_*\mathbb{Z}_{X\setminus Z} \to (i_Y)_*\mathbb{Z}_Y \xrightarrow{[+1]} \tag{14}$$

and therefore a long exact sequence

$$\cdots \to H^n \to H^n(X \setminus Z, \mathbb{Z}) \to H^n(Y, \mathbb{Z}) \to H^{n+1} \to \cdots. \tag{15}$$

The assumption $|Y| \cap |Z| = \emptyset$ immediately implies

$$\mathbb{Z}_{X|Y,Z} = R(j_Z)_*(j'_Y)_!\mathbb{Z}_U \cong (j_Y)_!R(j'_Z)_*\mathbb{Z}_U. \tag{16}$$

By applying $(j_Y)_!$ to a distinguished triangle

$$(i'_Z)_*(Ri'_Z)^!\mathbb{Z}_{X\setminus Y} \to \mathbb{Z}_{X\setminus Y} \to (Rj'_Z)_*\mathbb{Z}_U \xrightarrow{[+1]},$$

we get a canonical distinguished triangle

$$(i_Z)_*(Ri_Z)^!\mathbb{Z}_X \to (j_Y)_!\mathbb{Z}_{X\setminus Y} \to \mathbb{Z}_{X|Y,Z} \xrightarrow{[+1]}. \tag{17}$$

(Note that $(i_Z)_* = (i_Z)_!$ and $(Ri'_Z)^!\mathbb{Z}_{X\setminus Y} = (Ri_Z)^!\mathbb{Z}_X$ by excision.) Therefore, we get a long exact sequence

$$\cdots \to H^n_Z(X, \mathbb{Z}) \to H^n_c(X \setminus Y, \mathbb{Z}) \to H^n \to H^{n+1}_Z(X, \mathbb{Z}) \to \cdots. \tag{18}$$

Both (15) and (18) are long exact sequence of **MHS** (see [12]).

We set

$$\Omega^p_{X|Y,Z} := \Omega^p_X(\log(Y+Z)) \otimes \mathscr{O}_X(-Y),$$

where $\Omega^p_X(\log(Y+Z))$ is the sheaf on the analytic site of X of p-forms with logarithmic poles along $(Y+Z)$. It defines a subcomplex $\Omega^\bullet_{X|Y,Z}$ of $(j_Z)_*\Omega^\bullet_{X\setminus Z}$. For the definition of $\Omega^p_{X|Y,Z}$ in the case where Y or Z is non-reduced, we refer to § 4.2.

We recall the construction of the mixed Hodge structure on H^n and show that its Hodge filtration can be described in terms of the complex $\Omega^\bullet_{X|Y,Z}$ as follows.

Proposition 9. *Suppose that Y and Z are reduced, and let n be an integer.*

(1) *There is a canonical isomorphism*

$$H^n_{\mathbb{C}} \xrightarrow{\cong} H^n(X, \Omega^\bullet_{X|Y,Z}). \tag{19}$$

For every integer p, the induced map

$$H^n(X, \Omega^{\bullet \geqslant p}_{X|Y,Z}) \to H^n_{\mathbb{C}}$$

is injective, and its image agrees with the Hodge filtration $F^p H^n_{\mathbb{C}} \subset H^n_{\mathbb{C}}$.

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(2) *Let p, q be integers. If $\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_{\mathbb{C}}$ is non-trivial, then we have $p, q \in [0, n]$. If further $n > d$, then we have $p, q \in [n-d, d]$.*

Note that the isomorphism (19) is the one induced by the canonical quasi-isomorphism $\mathbb{C}_{X|Y,Z} \to \Omega^\bullet_{X|Y,Z}$ (see, for example, [9, Remarques 4.2.2 (c)]).

The mixed Hodge structure on H^n will be constructed from the cohomological mixed Hodge complex $K = (K_{\mathbb{Z}}, K_{\mathbb{Q}}, K_{\mathbb{C}})$ on X (in the sense of [8, 8.1.6]), and K is constructed as a cone of $K^Z \to K^Y$ where K^Z and K^Y are cohomological mixed complexes that produce the mixed Hodge structures on $H^n(X \setminus Z, \mathbb{Z})$ and $H^n(Y, \mathbb{Z})$, respectively. Therefore, we first need to recall the construction of K^Z and K^Y.

We recall from [8, 8.1.8] the description of K^Z and use, to do so, the notation for filtered derived categories from [8, 7.1.1]. Let $K^Z_{\mathbb{Z}} := Rj_{Z*}\mathbb{Z}_{X\setminus Z} \in D^+(X, \mathbb{Z})$. Define a filtered object $(K^Z_{\mathbb{Q}}, W) \in D^+F(X, \mathbb{Q})$ by

$$K^Z_{\mathbb{Q}} := Rj_{Z*}\mathbb{Q}_{X\setminus Z}, \quad W_q K^Z_{\mathbb{Q}} := \tau_{\leqslant q} K^Z_{\mathbb{Q}},$$

where $\tau_{\leqslant q}$ denotes the canonical truncation. Define a bifiltered object $(K^Z_{\mathbb{C}}, W, F) \in D^+F_2(X, \mathbb{C})$ by

$$K^Z_{\mathbb{C}} := \Omega^\bullet_X(\log Z), \quad F^p K^Z_{\mathbb{C}} := \Omega^{\bullet \geqslant p}_X(\log Z), \quad W_q K^Z_{\mathbb{C}} := W_q \Omega^\bullet_X(\log Z),$$

$$W_q \Omega^p_X(\log Z) = \begin{cases} 0 & (q < 0) \\ \Omega^{p-q}_X \wedge \Omega^q_X(\log Z) & (0 \leqslant q \leqslant p) \\ \Omega^p_X(\log Z) & (p \leqslant q). \end{cases}$$

We have an obvious isomorphism $K^Z_{\mathbb{Z}} \otimes \mathbb{Q} \cong K^Z_{\mathbb{Q}}$ as well as an isomorphism $(K^Z_{\mathbb{Q}}, W) \otimes \mathbb{C} \cong (K^Z_{\mathbb{C}}, W)$ deduced from the Poincaré lemma. The triple $K^Z = (K^Z_{\mathbb{Z}}, K^Z_{\mathbb{Q}}, K^Z_{\mathbb{C}})$ together with these isomorphisms is a cohomological mixed Hodge complex that produces the mixed Hodge structure on $H^n(X \setminus Z, \mathbb{Z})$ (see [8, 8.1.7 and 8.1.9 (ii)]).

Next we recall from [10, 3.2.4.2] the description of K^Y. Let I be the set of irreducible components of Y. For each $k \geqslant 0$, we write $Y^{[k]}$ for the disjoint union of $\cap_{T \in J} T$, where J ranges over subsets of I with cardinality $k+1$. Write $\pi^{[k]} : Y^{[k]} \to X$ for the canonical map. Fixing an ordering on I, we obtain a complex $\mathbb{Q}_{Y[\bullet]}$ and a double complex $\Omega^*_{Y[\bullet]}$ of sheaves on X:

$$\mathbb{Q}_{Y[\bullet]} := [\pi^{[0]}_* \mathbb{Q}_{Y[0]} \to \pi^{[1]}_* \mathbb{Q}_{Y[1]} \to \cdots \to \pi^{[q]}_* \mathbb{Q}_{Y[q]} \to \cdots],$$
$$\Omega^*_{Y[\bullet]} := [\pi^{[0]}_* \Omega^*_{Y[0]} \to \pi^{[1]}_* \Omega^*_{Y[1]} \to \cdots \to \pi^{[q]}_* \Omega^*_{Y[q]} \to \cdots].$$

Let $K^Y_{\mathbb{Z}} := i_{Y*}\mathbb{Z}_Y \in D^+(X, \mathbb{Z})$. Define a filtered object $(K^Y_{\mathbb{Q}}, W) \in D^+F(X, \mathbb{Q})$ by

$$K^Y_{\mathbb{Q}} := \mathbb{Q}_{Y[\bullet]} \quad W_q K^Y_{\mathbb{Q}} := \sigma_{\geqslant -q} K^Y_{\mathbb{Q}},$$

where $\sigma_{\geqslant -q}$ denotes the brutal truncation. Define a bifiltered object $(K^Y_{\mathbb{C}}, W, F) \in D^+F_2(X, \mathbb{C})$ by

$$K^Y_{\mathbb{C}} := \mathrm{Tot}(\Omega^*_{Y[\bullet]}), \quad F^p K^Y_{\mathbb{C}} := \mathrm{Tot}(\sigma_{* \geqslant p} \Omega^*_{Y[\bullet]}), \quad W_q K^Y_{\mathbb{C}} := \mathrm{Tot}(\sigma_{\bullet \geqslant -q} \Omega^*_{Y[\bullet]}).$$

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We have a Mayer–Vietoris isomorphism $K^Y_{\mathbb{Z}} \otimes \mathbb{Q} \cong K^Y_{\mathbb{Q}}$ as well as an isomorphism $(K^Y_{\mathbb{Q}}, W) \otimes \mathbb{C} \cong (K^Y_{\mathbb{C}}, W)$ deduced from the Poincaré lemma. The triple $K^Y = (K^Y_{\mathbb{Z}}, K^Y_{\mathbb{Q}}, K^Y_{\mathbb{C}})$ together with these isomorphisms is a cohomological mixed Hodge complex that produces the mixed Hodge structure on $H^n(Y, \mathbb{Z})$.

We construct a morphism $K^Z \to K^Y$ of cohomological mixed Hodge complexes. By applying Rj_{Z*} to the restriction map $\mathbb{Z}_{X\setminus Z} \to i'_{Y*}\mathbb{Z}_Y$, we get

$$K^Z_{\mathbb{Z}} = Rj_{Z*}\mathbb{Z}_{X\setminus Z} \to Rj_{Z*}i'_{Y*}\mathbb{Z}_Y = i_{Y*}\mathbb{Z}_Y = K^Y_{\mathbb{Z}}.$$

Similarly, we have

$$K^Z_{\mathbb{Q}} = Rj_{Z*}\mathbb{Q}_{X\setminus Z} \to i_{Y*}\mathbb{Q}_{Y^{[0]}} \to \pi^{[0]}_*\mathbb{Q}_{Y^{[\bullet]}} = K^Y_{\mathbb{Q}},$$

$$K^Z_{\mathbb{C}} = \Omega^*_X(\log Z) \xrightarrow{i_Y^*} \pi^{[0]}_*\Omega^*_{Y^{[0]}} \to \mathrm{Tot}(\Omega^*_{Y^{[\bullet]}}) = K^Y_{\mathbb{C}}.$$

They respect filtrations and define a morphism $\phi : K^Z \to K^Y$. We then apply the mixed cone construction [19, 3.22] [10, 3.3.24] to obtain $K := \mathrm{Cone}(\phi)[-1]$ which produces the mixed Hodge structure on $H^n = H^n(X \setminus Z, Y, \mathbb{Z})$. Proposition 9 (1), (2) is a consequence of [8, 8.1.9 (v)] and Lemma 10, while Proposition 9 (3) follows from [8, 8.2.4] and (15).

Lemma 10. *Set* $\Omega^p_{X|Y,Z} := \Omega^p_X(\log(Y+Z)) \otimes \mathcal{O}_X(-Y)$. *Define a bifiltered object* $(K'_{\mathbb{C}}, W', F') \in D^+F_2(X, \mathbb{C})$ *by*

$$K'_{\mathbb{C}} := \Omega^\bullet_{X|Y,Z}, \quad F^pK'_{\mathbb{C}} := \Omega^{\bullet \geqslant p}_{X|Y,Z}, \quad W_qK'_{\mathbb{C}} := W_q\Omega^\bullet_{X|Y,Z},$$

$$W_q\Omega^p_{X|Y,Z} = \begin{cases} 0 & (q < -p) \\ \Omega^{-q}_X \wedge \Omega^{p+q}_X(\log Y) \otimes \mathcal{O}_X(-Y) & (-p \leqslant q < 0) \\ \Omega^{p-q}_X \wedge \Omega^q_X(\log(Y+Z)) \otimes \mathcal{O}_X(-Y) & (0 \leqslant q \leqslant p) \\ \Omega^p_{X|Y,Z} & (p \leqslant q). \end{cases}$$

Then there is a canonical map $K'_{\mathbb{C}} \to K_{\mathbb{C}}$ *inducing an isomorphism* $(K'_{\mathbb{C}}, W', F') \cong (K_{\mathbb{C}}, W, F)$ *in* $D^+F_2(X, \mathbb{C})$.

Proof. The map exists since the composition of $K'_{\mathbb{C}} \hookrightarrow K^Z_{\mathbb{C}} \to K^Y_{\mathbb{C}}$ is the zero map. Let us verify that $K'_{\mathbb{C}} \to K_{\mathbb{C}}$ is a quasi-isomorphism. This is a local statement; thus, it suffices to show it over $X \setminus Y$ and $X \setminus Z$. The assertion becomes obvious over $X \setminus Y$, and over $X \setminus Z$, it follows from a standard fact

$$\mathrm{Cone}(\mathbb{C}_X \to i_{Y*}\mathbb{C}_Y)[-1] \cong j_{Y!}\mathbb{C}_{X\setminus Y} \cong \Omega^\bullet_X(\log Y) \otimes \mathcal{O}_X(-Y).$$

A direct calculation shows that the filtrations are transformed as described. □

Remark 11. Let $a : X \to \mathrm{Spec}(\mathbb{C})$ be the structural morphism. By [22, 23], the mixed Hodge structure on H^n can also be described using the six operations in the theory of mixed Hodge modules as the nth cohomology group of the object $a^{\mathcal{H}}_*(j_Z)^{\mathcal{H}}_*(j'_Y)^{\mathcal{H}}_!\mathbb{Q}^{\mathcal{H}}_U$ of the derived category $\mathrm{D^b}\mathbf{MHS}^p_{\mathbb{Q}}$ of the Abelian category $\mathbf{MHS}^p_{\mathbb{Q}}$ of graded polarizable mixed $\mathbb{Q}$-Hodge structures.

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4.2.

We now drop the assumption that Y and Z are reduced. In this subsection, we provide an alternative description of $H^n_{\mathbb{C}}$. We define

$$\Omega^p_{X|Y,Z} := \Omega^p_X(\log(Y+Z)_{\mathrm{red}}) \otimes \mathscr{O}_X(-Y+Z-Z_{\mathrm{red}}) \subset j_{Z*}\Omega^p_{X\setminus Z}.$$

In particular, we have (recall that $d = \dim X$)

$$\Omega^0_{X|Y,Z} = \mathscr{O}_X(-Y+Z-Z_{\mathrm{red}}) \quad \text{and} \quad \Omega^d_{X|Y,Z} = \Omega^d_X \otimes \mathscr{O}_X(Y_{\mathrm{red}}-Y+Z).$$

They form a subcomplex $\Omega^\bullet_{X|Y,Z}$ of $j_{Z*}\Omega^\bullet_{X\setminus Z}$. When Y and Z are reduced, this complex agrees with the one considered in the previous subsection and is consistent with the notation in Lemma 10. For another pair of effective divisors Y' and Z' on X, we have

$$\Omega^\bullet_{X|Y,Z} \subset \Omega^\bullet_{X|Y',Z'} \quad \text{if } Y \geqslant Y' \text{ and } Z \leqslant Z'. \tag{20}$$

In particular, we have a commutative diagram of complexes in which all arrows are inclusion maps:

$$\begin{array}{ccccc} & & \Omega^\bullet_{X|Y,Z} & & \\ & \nearrow & & \searrow & \\ \Omega^\bullet_{X|Y,Z_{\mathrm{red}}} & & & & \Omega^\bullet_{X|Y_{\mathrm{red}},Z} \\ & \searrow & & \nearrow & \\ & & \Omega^\bullet_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}}. & & \end{array} \tag{21}$$

The following proposition plays an important role in this work.

Proposition 12. *All maps in (21) are quasi-isomorphisms. Consequently, we have*

$$H^n_{\mathbb{C}} \cong H^n(X, \Omega^\bullet_{X|*,*'})$$

for all $* \in \{Y, Y_{\mathrm{red}}\}$, $*' \in \{Z, Z_{\mathrm{red}}\}$ *and* n. *(See Proposition 9 (1).)*

Proof. The assertion for the left lower arrow is proved in [4, Lemma 6.1] (under a weaker assumption that Y and Z have no common irreducible component). The same proof works for the other arrows without any change. However, we include a brief account here because of its importance in our work.

By induction, it suffices to show the following:

(1) For any irreducible component T of Y, $\Omega^\bullet_{X|Y,Z}/\Omega^\bullet_{X|Y+T,Z}$ is acyclic.

(2) For any irreducible component T of Z, $\Omega^\bullet_{X|Y,Z+T}/\Omega^\bullet_{X|Y,Z}$ is acyclic.

In what follows, we outline the proof of (2) by adapting that of [4, Lemma 6.2] (which is precisely (1)). The complex in question can be rewritten as

$$\Omega^0_{X|Y,Z+T} \otimes \mathscr{O}_T \xrightarrow{d^0_T} \Omega^1_{X|Y,Z+T} \otimes \mathscr{O}_T \xrightarrow{d^1_T} \Omega^2_{X|Y,Z+T} \otimes \mathscr{O}_T \xrightarrow{d^2_T} \cdots .$$

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We have an exact sequence

$$0 \to \Omega_X^p(\log((Y+Z)_{\mathrm{red}} - T)) \to \Omega_X^p(\log(Y+Z)_{\mathrm{red}}) \overset{\mathrm{Res}_T^p}{\to} \omega_T^{p-1} \to 0,$$

where $\omega_T^p := \Omega_T^p(\log(Z_{\mathrm{red}} - T)|_T)$ and Res_T^p is the residue map. By taking tensor product with $\mathcal{O}_X(-Y+Z-Z_{\mathrm{red}}+T) \otimes \mathcal{O}_T$, we obtain another exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_X(-Y+Z-Z_{\mathrm{red}}+T) \otimes \omega_T^p \to \Omega_{X|Y,Z+T}^p \otimes \mathcal{O}_T \\ &\overset{\mathrm{Res}_{Z,T}^p}{\to} \mathcal{O}_X(-Y+Z-Z_{\mathrm{red}}+T) \otimes \omega_T^{p-1} \to 0,\end{aligned}$$

by which we regard $\mathcal{O}_X(-Y+Z-Z_{\mathrm{red}}+T) \otimes \omega_T^p$ as a subsheaf of $\Omega_{X|Y,Z+T}^p \otimes \mathcal{O}_T$. Then a direct computation shows

$$d_T^{p-1} \circ \mathrm{Res}_{Z,T}^p + \mathrm{Res}_{Z,T}^{p+1} \circ d_T^p = e \cdot \mathrm{id}_{\Omega_{X|Y,Z+T}^p \otimes \mathcal{O}_T},$$

where $e = -\mathrm{ord}_T(Z)$. We get a homotopy operator that proves (2). □

4.3.

For any integers k and p, we define

$$\begin{aligned}\Omega_{X|Y,Z}^{(k)p} &:= \begin{cases} \Omega_{X|Y,Z_{\mathrm{red}}}^p & \text{if } p < k, \\ \Omega_{X|Y_{\mathrm{red}},Z}^p & \text{if } p \geqslant k \end{cases} \\ &= \begin{cases} \Omega_X^p(\log(Y+Z)_{\mathrm{red}}) \otimes \mathcal{O}_X(-Y) & \text{if } p < k, \\ \Omega_X^p(\log(Y+Z)_{\mathrm{red}}) \otimes \mathcal{O}_X(-Y_{\mathrm{red}}+Z-Z_{\mathrm{red}}) & \text{if } p \geqslant k. \end{cases}\end{aligned}$$

For each k, they form a subcomplex $\Omega_{X|Y,Z}^{(k)\bullet}$ of $j_{Z*}\Omega_{X\setminus Z}^\bullet$. We have a sequence of inclusion maps

$$\Omega_{X|Y,Z_{\mathrm{red}}}^\bullet = \Omega_{X|Y,Z}^{(d+1)\bullet} \subset \Omega_{X|Y,Z}^{(d)\bullet} \subset \cdots \subset \Omega_{X|Y,Z}^{(1)\bullet} \subset \Omega_{X|Y,Z}^{(0)\bullet} = \Omega_{X|Y_{\mathrm{red}},Z}^\bullet, \tag{22}$$

which fits the middle row of the diagram (21). We have $\Omega_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}}^{(k)\bullet} = \Omega_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}}^\bullet$ for any k. Similarly to (20), we have for another pair of effective divisors Y' and Z' on X and for any k

$$\Omega_{X|Y,Z}^{(k)\bullet} \subset \Omega_{X|Y',Z'}^{(k)\bullet} \quad \text{if } Y \geqslant Y' \text{ and } Z \leqslant Z'. \tag{23}$$

Remark 13. Some cases of the complex $\Omega_{X|Y,Z}^{(k)\bullet}$ have been used in the literature. When $d=1$, $\Omega_{X|Y,Z}^{(1)\bullet} = [\mathcal{O}_X(-Y) \to \Omega_X^1 \otimes \mathcal{O}_X(Z)]$ has been used in [14]. When $k=d$ and $Z=\emptyset$, $\Omega_{X|Y,\emptyset}^{(d)\bullet}$ agrees with the complex $S_Y^\bullet$ used in [15].

For integers n and k, we define

$$\mathcal{H}^{n,k}(X,Y,Z) := H^n(X, \Omega_{X|Y,Z}^{(k)\bullet}), \tag{24}$$

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$$\begin{aligned}
\mathcal{F}^{n,k}(X,Y,Z) &:= H^n(X, \Omega^{(k)\bullet \geqslant k}_{X|Y,Z}),\\
(\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z) &:= H^n(X, \Omega^{(k)\bullet <k}_{X|Y,Z}),\\
H^{n,k}_{\mathrm{add}}(X,Y) &:= H^{n-1}\left(X, \Omega^{\bullet<k}_X(\log Y_{\mathrm{red}}) \otimes \frac{\mathcal{O}_X(-Y_{\mathrm{red}})}{\mathcal{O}_X(-Y)}\right),\\
H^{n,k}_{\mathrm{inf}}(X,Z) &:= H^{n-1}\left(X, \Omega^{\bullet<k}_X(\log Z_{\mathrm{red}}) \otimes \frac{\mathcal{O}_X(Z-Z_{\mathrm{red}})}{\mathcal{O}_X}\right).
\end{aligned}$$

By definition, we have $\Omega^{(k)\bullet\geqslant k}_{X|Y,Z} = \Omega^{(k)\bullet\geqslant k}_{X|Y_{\mathrm{red}},Z}$, $\Omega^{(k)\bullet<k}_{X|Y,Z} = \Omega^{(k)\bullet<k}_{X|Y,Z_{\mathrm{red}}}$, and, therefore, $\mathcal{F}^{n,k}(X,Y,Z) = \mathcal{F}^{n,k}(X,Y_{\mathrm{red}},Z)$, $(\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z) = (\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z_{\mathrm{red}})$.

Theorem 14. *Let n and k be integers.*

(1) *Let a and b' be the maps induced by the inclusion maps from (23):*

$$\mathcal{H}^{n,k}(X,Y,Z_{\mathrm{red}}) \underset{b}{\overset{a}{\rightleftarrows}} \mathcal{H}^{n,k}(X,Y,Z) \underset{a'}{\overset{b'}{\rightleftarrows}} \mathcal{H}^{n,k}(X,Y_{\mathrm{red}},Z).$$

Then there are canonical maps b and a' such that $b \circ a = \mathrm{id}$ and $b' \circ a' = \mathrm{id}$.

(2) *We have canonical isomorphisms*

$$\begin{aligned}
\mathrm{Coker}(a) &\cong H^{n,k}_{\mathrm{inf}}(X,Z) \cong H^n\left(X, \Omega^{\bullet\geqslant k}_X(\log Z_{\mathrm{red}}) \otimes \frac{\mathcal{O}_X(Z-Z_{\mathrm{red}})}{\mathcal{O}_X}\right),\\
\ker(b') &\cong H^{n,k}_{\mathrm{add}}(X,Y) \cong H^n\left(X, \Omega^{\bullet\geqslant k}_X(\log Y_{\mathrm{red}}) \otimes \frac{\mathcal{O}_X(-Y_{\mathrm{red}})}{\mathcal{O}_X(-Y)}\right).
\end{aligned}$$

(3) *The sequence*

$$0 \to \mathcal{F}^{n,k}(X,Y,Z) \xrightarrow{i} \mathcal{H}^{n,k}(X,Y,Z) \xrightarrow{p} (\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z) \to 0$$

is exact. Hereafter, we regard $\mathcal{F}^{n,k}(X,Y,Z)$ as a subspace of $\mathcal{H}^{n,k}(X,Y,Z)$.

(4) *We have*

$$\begin{aligned}
a(\mathcal{F}^{n,k}(X,Y,Z_{\mathrm{red}})) &\subset \mathcal{F}^{n,k}(X,Y,Z),\\
b'(\mathcal{F}^{n,k}(X,Y,Z)) &= \mathcal{F}^{n,k}(X,Y_{\mathrm{red}},Z).
\end{aligned}$$

(Note, however, that a' and b do not preserve $\mathcal{F}^{n,k}$.) Moreover, there are commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & H^{n,k}_{\mathrm{inf}}(X,Z) & = & H^{n,k}_{\mathrm{inf}}(X,Z) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathcal{F}^{n,k}(X,Y,Z) & \to & \mathcal{H}^{n,k}(X,Y,Z) & \to & (\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z) & \to & 0\\
 & & \uparrow & & \uparrow{\scriptstyle a} & & \| & & \\
0 & \to & \mathcal{F}^{n,k}(X,Y,Z_{\mathrm{red}}) & \to & \mathcal{H}^{n,k}(X,Y,Z_{\mathrm{red}}) & \to & (\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z_{\mathrm{red}}) & \to & 0\\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

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and

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^{n,k}_{\mathrm{add}}(X,Y) & = & H^{n,k}_{\mathrm{add}}(X,Y) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}^{n,k}(X,Y,Z) & \to & \mathcal{H}^{n,k}(X,Y,Z) & \to & (\mathcal{H}/\mathcal{F})^{n,k}(X,Y,Z) & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle b'} & & \downarrow & & \\
0 & \to & \mathcal{F}^{n,k}(X,Y_{\mathrm{red}},Z) & \to & \mathcal{H}^{n,k}(X,Y_{\mathrm{red}},Z) & \to & (\mathcal{H}/\mathcal{F})^{n,k}(X,Y_{\mathrm{red}},Z) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

(5) *The inclusion map from (22) induces a map*

$$\tau^{n,k} : \mathcal{H}^{n,k}(X,Y,Z) \to \mathcal{H}^{n,k-1}(X,Y,Z)$$

and it holds that $\tau^{n,k}(\mathcal{F}^{n,k}(X,Y,Z)) \subset \mathcal{F}^{n,k-1}(X,Y,Z)$. *The same map also induces maps*

$$
\begin{aligned}
\tau^{n,k}_{\mathrm{add}} &: H^{n,k}_{\mathrm{add}}(X,Y) \to H^{n,k-1}_{\mathrm{add}}(X,Y),\\
\tau^{n,k}_{\mathrm{inf}} &: H^{n,k}_{\mathrm{inf}}(X,Z) \to H^{n,k-1}_{\mathrm{inf}}(X,Z).
\end{aligned}
$$

Proof. We introduce complexes $\Omega^{(k)'\bullet}_{X|Y,Z}$ and $\Omega^{(k)''\bullet}_{X|Y,Z}$ by setting

$$\Omega^{(k)'p}_{X|Y,Z} := \begin{cases} \Omega^p_{X|Y,Z_{\mathrm{red}}} & \text{if } p < k, \\ \Omega^p_{X|Y,Z} & \text{if } p \geqslant k, \end{cases} \qquad \Omega^{(k)''p}_{X|Y,Z} := \begin{cases} \Omega^p_{X|Y,Z} & \text{if } p < k, \\ \Omega^p_{X|Y_{\mathrm{red}},Z} & \text{if } p \geqslant k. \end{cases}$$

Altogether, they fit in a diagram extending (21) in which all arrows are inclusions:

$$
\begin{array}{ccccccccc}
 & & & & \Omega^\bullet_{X|Y,Z} & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & \Omega^{(k)'\bullet}_{X|Y,Z} & & & & \Omega^{(k)''\bullet}_{X|Y,Z} & & \\
 & \nearrow & & \searrow & & \nearrow{\scriptstyle\beta} & & \searrow & \\
\Omega^\bullet_{X|Y,Z_{\mathrm{red}}} & & & & \Omega^{(k)\bullet}_{X|Y,Z} & & & & \Omega^\bullet_{X|Y_{\mathrm{red}},Z} \\
 & \searrow & & \nearrow{\scriptstyle\alpha} & & \searrow & & \nearrow & \\
 & & \Omega^{(k)\bullet}_{X|Y,Z_{\mathrm{red}}} & & & & \Omega^{(k)\bullet}_{X|Y_{\mathrm{red}},Z} & & \\
 & & & \searrow & & \nearrow & & & \\
 & & & & \Omega^\bullet_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}}. & & & &
\end{array}
\tag{25}
$$

The map a is induced by α. The cokernel of the degree p part of $\beta \circ \alpha$ is given by

$$\Omega^p_X(\log(Y+Z)_{\mathrm{red}}) \otimes \frac{\mathcal{O}_X(-Y+Z-Z_{\mathrm{red}})}{\mathcal{O}_X(-Y)} \quad \text{for } p < k,$$

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$$\Omega^p_X(\log(Y+Z)_{\mathrm{red}}) \otimes \frac{\mathscr{O}_X(-Y_{\mathrm{red}}+Z-Z_{\mathrm{red}})}{\mathscr{O}_X(-Y_{\mathrm{red}})} \quad \text{for } p \geqslant k,$$

but since $|Y| \cap |Z| = \emptyset$, we have

$$\frac{\mathscr{O}_X(-Y+Z-Z_{\mathrm{red}})}{\mathscr{O}_X(-Y)} \cong \frac{\mathscr{O}_X(-Y_{\mathrm{red}}+Z-Z_{\mathrm{red}})}{\mathscr{O}_X(-Y_{\mathrm{red}})} \cong \frac{\mathscr{O}_X(Z-Z_{\mathrm{red}})}{\mathscr{O}_X}.$$

It follows that the cokernel of $\beta \circ \alpha$ is isomorphic to that of the lower right arrow in (21), hence acyclic by Proposition 12. We have shown that $\beta \circ \alpha$ is a quasi-isomorphism. We define b to be the composition of $H^n(\beta)$ and $H^n(\beta \circ \alpha)^{-1}$, showing the first half of (1). Since we have seen that a is injective for any n, we also get $\mathrm{Coker}(a) \cong H^n(X, \mathrm{Coker}(\alpha))$, which is nothing but the right-hand side of the first displayed formula in (2). Similarly, as we have seen that b is surjective for any n, we get $\mathrm{Coker}(a) \cong \ker(b) \cong H^{n-1}(X, \mathrm{Coker}(\beta)) = H^{n,k}_{\mathrm{inf}}(X, Z)$, proving the first half of (2). The rest of (1) and (2) is shown by a dual argument.

To prove (3), we consider a diagram

$$\begin{array}{ccccccccc}
 & \Omega^{(k)\bullet \geqslant k}_{X|Y_{\mathrm{red}},Z} & \xrightarrow{f} & \Omega^{(k)\bullet}_{X|Y_{\mathrm{red}},Z} & \xrightarrow{g} & \Omega^{(k)\bullet<k}_{X|Y_{\mathrm{red}},Z} & & & \\
 // & & \nearrow & & \nwarrow & & \backslash\backslash & & \\
\Omega^{(k)\bullet \geqslant k}_{X|Y,Z} & \xrightarrow{f'} & \Omega^{(k)\bullet}_{X|Y,Z} & & \Omega^{(k)\bullet}_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}} & & & \xrightarrow{g'} & \Omega^{(k)\bullet<k}_{X|Y_{\mathrm{red}},Z_{\mathrm{red}}}.
\end{array}$$

By Proposition 9 (2), g' induces surjections on cohomology, hence so does g. It follows that f induces injections on cohomology, hence so does f'. We have shown the injectivity of i. A dual argument proves the surjectivity of p. (3) follows.

The first half of (4) is obvious. The rest is obtained by taking cohomology of the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \dfrac{\Omega^{(k)\bullet \geqslant k}_{X|Y,Z}}{\Omega^{(k)\bullet \geqslant k}_{X|Y,Z_{\mathrm{red}}}} & = & \dfrac{\Omega^{(k)\bullet}_{X|Y,Z}}{\Omega^{(k)\bullet}_{X|Y,Z_{\mathrm{red}}}} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \Omega^{(k)\bullet \geqslant k}_{X|Y,Z} & \longrightarrow & \Omega^{(k)\bullet}_{X|Y,Z} & \longrightarrow & \Omega^{(k)\bullet<k}_{X|Y,Z} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \Omega^{(k)\bullet \geqslant k}_{X|Y,Z_{\mathrm{red}}} & \longrightarrow & \Omega^{(k)\bullet}_{X|Y,Z_{\mathrm{red}}} & \longrightarrow & \Omega^{(k)\bullet<k}_{X|Y,Z_{\mathrm{red}}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and its dual diagram. (5) is obvious. □

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Corollary 15. *We have* $H^{n,k}_{\mathrm{add}}(X, Y) = H^{n,k}_{\mathrm{inf}}(X, Z) = 0$ *if one of the following conditions is satisfied:*

(1) $k \leqslant 0$;

(2) $k < n - d + 1$;

(3) $k > d$;

(4) $k > n$.

Proof. Since the sheaves $\mathscr{O}_X(-Y_{\mathrm{red}})/\mathscr{O}_X(-Y)$ and $\mathscr{O}_X(Z - Z_{\mathrm{red}})/\mathscr{O}_X$ are supported in a closed subvariety of dimension $d-1$, (1) and (2) follow from the definition (24). Theorem 14 (2) implies the cases (3) and (4). □

4.4.

We arrive at our main definition.

Definition 16. For each integer n, we define an object

$$\mathcal{H}^n(X, Y, Z) = (H^n, H^{n,\bullet}_{\mathrm{add}}, H^{n,\bullet}_{\mathrm{inf}}, \mathcal{F}^{n,\bullet})$$

of **MHSM** as follows. Let H^n be the mixed Hodge structure considered in §4.1. We define two objects $H^{n,\bullet}_{\mathrm{inf}}$ and $H^{n,\bullet}_{\mathrm{add}}$ of $\mathbf{Vec}^{\bullet}_{\mathbb{C}}$ to be $(H^{n,k}_{\mathrm{inf}}(X, Z), \tau^{n,k}_{\mathrm{inf}})_k$ and $(H^{n,k}_{\mathrm{add}}(X, Y), \tau^{n,k}_{\mathrm{add}})_k$, respectively. For each $k \in \mathbb{Z}$, we have

$$\begin{aligned}\mathcal{H}^{n,k}(X, Y, Z) &\cong \mathcal{H}^{n,k}(X, Y_{\mathrm{red}}, Z) \oplus H^{n,k}_{\mathrm{add}}(X, Y)\\ &\cong \mathcal{H}^{n,k}(X, Y, Z_{\mathrm{red}}) \oplus H^{n,k}_{\mathrm{inf}}(X, Z)\\ &\cong \mathcal{H}^{n,k}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \oplus H^{n,k}_{\mathrm{add}}(X, Y) \oplus H^{n,k}_{\mathrm{inf}}(X, Z).\\ &\cong H^n_{\mathbb{C}} \oplus H^{n,k}_{\mathrm{add}}(X, Y) \oplus H^{n,k}_{\mathrm{inf}}(X, Z).\end{aligned}$$

Here we applied Theorem 14 (1–2) (to (X, Y, Z), (X, Y_{red}, Z), and (X, Y, Z_{red})), and for the last isomorphism, we used Proposition 9 (1). We then define $\mathcal{F}^{n,k} \subset H^n_{\mathbb{C}} \oplus H^{n,k}_{\mathrm{inf}} \oplus H^{n,k}_{\mathrm{add}}$ to be the subspace corresponding to $\mathcal{F}^{n,k}(X, Y, Z) \subset \mathcal{H}^{n,k}(X, Y, Z)$, using Theorem 14 (3). Theorem 14 (4–5) implies that they satisfy conditions **(1-a)**–**(1-d)** in Definition 1.

4.5.

Let (X, Y, Z) and (X', Y', Z') be two triples as in the beginning of § 4. Let $f : X \to X'$ be a morphism of $\mathbb{C}$-schemes such that $f(X) \not\subset |Y'| \cup |Z'|$. If f verifies the conditions

$$Y \leqslant f^*Y', \quad Z_{\mathrm{red}} \geqslant (f^*Z')_{\mathrm{red}}, \quad Z - Z_{\mathrm{red}} \geqslant f^*(Z' - Z'_{\mathrm{red}}), \tag{26}$$

then it induces a morphism $f^* : \mathcal{H}^n(X', Y', Z') \to \mathcal{H}^n(X, Y, Z)$ for any n. To see this, we first note that the first two items in (26) implies that f restricts to $X \setminus Z_{\mathrm{red}} \to X' \setminus Z'_{\mathrm{red}}$ and to $Y_{\mathrm{red}} \to Y'_{\mathrm{red}}$. Hence, we have a pull-back map $f^* : H^n(X' \setminus Z'_{\mathrm{red}}, Y'_{\mathrm{red}}, \mathbb{Z}) \to H^n(X \setminus Z_{\mathrm{red}}, Y_{\mathrm{red}}, \mathbb{Z})$ in **MHS**. We next note that (26) implies $\Omega^{(k)\bullet}_{X|f^*Y', f^*Z'} \subset \Omega^{(k)\bullet}_{X|Y,Z}$ for any $k \in \mathbb{Z}$. Hence, we have a pull-back map $f^* : \mathcal{H}^{n,k}(X', Y', Z') \to \mathcal{H}^{n,k}(X, Y, Z)$ induced by the maps of complexes $\Omega^{(k)\bullet}_{X'|Y',Z'} \to f_*\Omega^{(k)\bullet}_{X|f^*Y',f^*Z'} \to f_*\Omega^{(k)\bullet}_{X|Y,Z}$, which

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verifies $f^*(\mathcal{F}^{n,k}(X',Y',Z')) \subset \mathcal{F}^{n,k}(X,Y,Z)$. By Theorem 14 (2), it induces $f^*: H^{n,k}_{\mathrm{add}}(X',Y') \to H^{n,k}_{\mathrm{add}}(X,Y)$ and $f^*: H^{n,k}_{\mathrm{inf}}(X',Z') \to H^{n,k}_{\mathrm{inf}}(X,Z)$. These maps define the desired morphism.

Remark 17. Composition of two morphisms satisfying (26) need not satisfy (26). Here is an example: $(\operatorname{Spec}\mathbb{C},\emptyset,\emptyset) \to (\mathbb{P}^1,\emptyset,\emptyset) \to (\mathbb{P}^1,x,\emptyset)$, where the first map is the immersion to a closed point x, and the second map is given by the identity.

4.6.

As an example, we give an explicit description of $\mathcal{H} = \mathcal{H}^1(X,Y,Z)$ when $d=1$. Write $\mathcal{H} = (H, H^\bullet_{\mathrm{inf}}, H^\bullet_{\mathrm{add}}, \mathcal{F}^\bullet)$ so that $H = H^1(X\setminus Z, Y, \mathbb{Z})$. If $k \neq 1$, then we have $H^k_{\mathrm{inf}} = H^k_{\mathrm{add}} = 0$ and hence $\mathcal{H}^k = H_{\mathbb{C}}$, $\mathcal{F}^k = F^k H_{\mathbb{C}}$. We have

$$\begin{aligned}
&H_{\mathbb{C}} \cong H^1(X, [\mathcal{O}_X(-Y_{\mathrm{red}}) \to \Omega^1_X \otimes \mathcal{O}_X(Z_{\mathrm{red}})]),\\
&F^1 H_{\mathbb{C}} \cong H^0(X, \Omega^1_X \otimes \mathcal{O}_X(Z_{\mathrm{red}})), \quad F^2 H_{\mathbb{C}} = H_{\mathbb{C}}, \quad F^0 H_{\mathbb{C}} = 0,\\
&H^1_{\mathrm{add}} = H^0(X, \mathcal{O}_X(-Y_{\mathrm{red}})/\mathcal{O}_X(-Y)) \cong H^0(X, \Omega^1_X \otimes (\mathcal{O}_X/\mathcal{O}_X(Y_{\mathrm{red}}-Y))),\\
&H^1_{\mathrm{inf}} = H^0(X, \mathcal{O}_X(Z-Z_{\mathrm{red}})/\mathcal{O}_X) \cong H^0(X, \Omega^1_X \otimes (\mathcal{O}_X(Z)/\mathcal{O}_X(Z_{\mathrm{red}}))),\\
&\mathcal{H}^1 = H^1(X, [\mathcal{O}_X(Y) \to \Omega^1_X \otimes \mathcal{O}_X(Z)]) \cong H_{\mathbb{C}} \oplus H^1_{\mathrm{add}} \oplus H^1_{\mathrm{inf}},\\
&\mathcal{F}^1 = H^0(X, \Omega^1_X \otimes \mathcal{O}_X(Z)).
\end{aligned}$$

These gadgets are considered in [14, Propositions 10, 14 and Definition 13].

5. Duality

Throughout this section, let X be a connected smooth proper variety of dimension d over $\mathbb{C}$, and let Y, Z be effective divisors on X such that $|Y|\cap|Z| = \emptyset$ and such that $(Y+Z)_{\mathrm{red}}$ is a simple normal crossing divisor. The main result of this section is the following.

Theorem 18. *For every integer $n \in \mathbb{Z}$, there exists an isomorphism in* **MHSM**

$$\mathcal{H}^n(X,Y,Z)^\vee \cong \mathcal{H}^{2d-n}(X,Z,Y)(d)_{\mathrm{fr}},$$

where $(-)^\vee$ is the duality functor described in § 2.8, and $(-)(d)$ (resp. $(-)_{\mathrm{fr}}$) is the Tate twist (resp. free part) introduced in § 2.9 (resp. § 2.7).

5.1.

Let us first assume that Y, Z are reduced. For a $\mathbb{C}$-scheme V with structural map $a: V \to \operatorname{Spec} A$, let $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V,A)$ be the bounded derived category of sheaves of A-modules with (algebraically) constructible cohomology and $\mathbb{D}_V := R\mathcal{H}om(-, a^! A) : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V,A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V,A)$ be the Verdier duality functor, where A is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$. Since we have $\mathbb{D}_U(A_U) = A_U(d)[2d]$ as $U = X\setminus(|Y|\cup|Z|)$ is smooth of dimension d, we have

$$\begin{aligned}
\mathbb{D}_X(A_{X|Y,Z}) &= \mathbb{D}_X(Rj_{Z*}j'_{Y!}A_U) = j_{Z!}Rj'_{Y*}\mathbb{D}_U(A_U) \qquad (27)\\
&\overset{(*)}{=} Rj_{Y*}j'_{Z!}A_U(d)[2d] = A_{X|Z,Y}(d)[2d],
\end{aligned}$$

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where we used the notations from (12) for the maps j_Y, etc. (see (16) for $(*)$). The induced pairing $A_{X|Y,Z} \otimes A_{X|Z,Y} \to A_X$ factors through

$$A_{X|Y,Z} \otimes A_{X|Z,Y} \to (j_U)_! A_U \tag{28}$$

since $(A_{X|Y,Z})_y = 0$ for all $y \in Y$ and $(A_{X|Z,X})_z = 0$ for all $z \in Z$ (as extension by zero). Therefore, we obtain a pairing

$$H^n(X \setminus Z, Y, A) \otimes H^{2d-n}(X \setminus Y, Z, A)(d) \xrightarrow{\smile} H_c^{2d}(U, A)(d) \cong A, \tag{29}$$

where the last isomorphism is the trace map. As this pairing is perfect, up to torsion when $A = \mathbb{Z}$, we obtain an isomorphism

$$\mathrm{Hom}_A(H^n(X \setminus Z, Y, A), A) \cong H^{2d-n}(X \setminus Y, Z, A)(d)_{\mathrm{fr}}. \tag{30}$$

(Here $(-)_{\mathrm{fr}}$ makes no effect if $A = \mathbb{Q}$ or $\mathbb{C}$.)

We have a canonical pairing

$$\Omega^\bullet_{X|Y,Z} \otimes \Omega^\bullet_{X|Z,Y} \to {}^c\Omega^\bullet_U := \Omega^\bullet_X(\log(Y+Z)_{\mathrm{red}}) \otimes \mathscr{O}_X(-Y_{\mathrm{red}} - Z_{\mathrm{red}}) \tag{31}$$

defined by the wedge product. Since ${}^c\Omega^\bullet_U$ is a resolution of $(j_U)_!\mathbb{C}_U$, we obtain a pairing

$$H^n(X, \Omega^\bullet_{X|Y,Z}) \otimes H^{2d-n}(X, \Omega^\bullet_{X|Z,Y}) \xrightarrow{\smile} H_c^{2d}(U, \mathbb{C}) \cong \mathbb{C}, \tag{32}$$

where the last isomorphism is the trace map.

The two pairings (29) (with $A = \mathbb{C}$) and (32) are compatible with respect to the isomorphisms $H^n(X \setminus Z, Y, \mathbb{C}) \cong H^n(X, \Omega^\bullet_{X|Y,Z})$ and $H^{2d-n}(X \setminus Y, Z, \mathbb{C}) \cong H^{2d-n}(X, \Omega^\bullet_{X|Z,Y})$ from Proposition 9. This follows from a commutative diagram

$$\begin{array}{ccc} \mathbb{C}_{X|Y,Z} \otimes \mathbb{C}_{X|Z,Y} & \longrightarrow & (j_U)_!\mathbb{C}_U \\ \downarrow & & \downarrow \\ \Omega^\bullet_{X|Y,Z} \otimes \Omega^\bullet_{X|Z,Y} & \longrightarrow & {}^c\Omega^\bullet_U, \end{array}$$

where the horizontal maps are (28) and (31).

Lemma 19. *The morphism (30) defines an isomorphism of mixed Hodge structures*

$$H^n(X, Y, Z)^\vee \xrightarrow{\simeq} H^{2d-n}(X, Z, Y)(d)_{\mathrm{fr}}.$$

Proof. Since we already know that it is an isomorphism of underlying Abelian groups, it suffices to prove that it is a morphism of **MHS**. (This is known when $Y = \emptyset$ or $Z = \emptyset$; see [12].) This amounts to showing that under the pairing (29),

$$F^p H^n(X \setminus Z, Y, \mathbb{C}) \text{ and } F^{1-p}(H^{2d-n}(X \setminus Z, Y, \mathbb{C})(d)) = F^{d+1-p} H^{2d-n}(X \setminus Z, Y, \mathbb{C})$$

annihilate each other, and the same for

$$W_q H^n(X \setminus Z, Y, \mathbb{Q}) \text{ and } W_{-q-1}(H^{2d-n}(X \setminus Z, Y, \mathbb{Q})(d)) = W_{2d-q-1} H^{2d-n}(X \setminus Z, Y, \mathbb{Q}).$$

Both immediately follow from the description of the filtrations given in Lemma 10. □

Remark 20. Alternatively, one can prove this lemma in the same way as (27) upon replacing $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V, A)$ by the category of mixed Hodge modules $\mathbf{MHM}(V, \mathbb{Q})$ [22].

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5.2.

Let $C^\bullet, D^\bullet$ be complexes of sheaves of $\mathbb{C}$-vector spaces on X. Let $\wedge : C^\bullet \otimes D^\bullet \to {}^c\Omega^\bullet_U$ be a morphism of complexes. The map $\wedge$ together with the trace map induces a morphism

$$\mathbf{H}^i(X, C^\bullet) \otimes_{\mathbb{C}} \mathbf{H}^{2d-i}(X, D^\bullet)(d) \xrightarrow{\wedge} \mathbf{H}^{2d}(X, {}^c\Omega^\bullet_U)(d) \xrightarrow{\simeq} H^{2d}_c(U, \mathbb{C})(d) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

that defined a canonical morphism

$$\mathbf{H}^{2d-i}(X, D^\bullet)(d) \to \mathbf{H}^i(X, C^\bullet)^\vee. \tag{33}$$

Note that ${}^c\Omega^d_U = \Omega^d_X$ by definition.

Lemma 21. *Assume that C^i and D^i are locally free $\mathscr{O}_X$-modules for all i and that $C^i = D^i = 0$ if $i \notin [0, d]$. If the map $\wedge$ induces an isomorphism $C^i \xrightarrow{\simeq} \mathcal{H}om_{\mathscr{O}_X}(D^{d-i}, \Omega^d_X)$ for all i, then the morphism (33) is an isomorphism for every integer i.*

Proof. Let $n \in [0, d]$ be the greatest integer such that $C^n \neq 0$. Consider the truncated complexes $C'^\bullet = C^{\bullet<n}$ and $D'^\bullet = D^{\bullet>d-n}$ so that we have the exact sequences of complexes of sheaves of $\mathbb{C}$-vector spaces on X

$$0 \to C^n[-n] \to C^\bullet \to C'^\bullet \to 0, \quad 0 \to D'^\bullet \to D^\bullet \to D^{d-n}[n-d] \to 0.$$

These sequences, together with the pairing $\wedge$ and the trace map, induce a morphism of long exact sequences

$$\begin{array}{ccccccc} \cdots \to & \mathbf{H}^{2d-i}(X, D'^\bullet)(d) & \to & \mathbf{H}^{2d-i}(X, D^\bullet)(d) & \to & \mathbf{H}^{2d-i}(X, D^{d-n}[n-d])(d) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \mathbf{H}^i(X, C'^\bullet)^\vee & \to & \mathbf{H}^i(X, C^\bullet)^\vee & \to & \mathbf{H}^i(X, C^n[-n])^\vee & \to \cdots. \end{array}$$

Let $\mathscr{F} = D^{d-n}$ and $r = 2d - i - n$. Note that

$$\mathbf{H}^{2d-i}(X, C^n[-n]) = \mathbf{H}^r(X, \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_X)) = \mathrm{Ext}^r_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_X)$$

and $\mathbf{H}^i(X, D^{d-n}[n-d]) = \mathbf{H}^{d-r}(X, \mathscr{F})$. Hence, we are reduced by induction to showing that the map

$$\mathrm{Ext}^r_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_X) \to H^{d-r}(X, \mathscr{F})^\vee$$

induced by the canonical pairing

$$\mathrm{Ext}^r_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_X) \times H^{d-r}(X, \mathscr{F}) \to H^d(X, \Omega^d_X) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

is an isomorphism. This is Serre duality from [24] (see also [13, Theorem 7.6]). □

5.3.

Let us come back to the general assumption of Theorem 18 (that is, Y and Z may not be reduced). Let $k \in \mathbb{Z}$ be an integer. Then we have a canonical pairing

$$\wedge : \Omega^{(k)\bullet}_{X|Y,Z} \otimes_{\mathbb{C}} \Omega^{(d+1-k)\bullet}_{X|Z,Y} \to {}^c\Omega^\bullet_U$$

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that induces a morphism (for every $n \in \mathbb{Z}$)

$$\mathcal{H}^{n,k}(X, Y, Z) \otimes_{\mathbb{C}} \mathcal{H}^{2d-n,d+1-k}(X, Z, Y) \to H^{2d}(X, {}^c\Omega_U^\bullet) \cong \mathbb{C}$$

and therefore via the trace map, a morphism

$$\mathcal{H}^{2d-n,d+1-k}(X, Z, Y) \to \mathcal{H}^{n,k}(X, Y, Z)^\vee. \tag{34}$$

By Lemma 21, applied with $C^\bullet = \Omega^{(k)\bullet}_{X|Y,Z}$ and $D^\bullet = \Omega^{(d+1-k)\bullet}_{X|Z,Y}$, we see that the morphism (34) is an isomorphism for every integer $n, k \in \mathbb{Z}$. Theorem 14 (2) also implies that (34) induces isomorphisms

$$\begin{aligned} H^{n,k}_{\mathrm{add}}(X, Y)^\vee &\cong H^{2d-n,d+1-k}_{\mathrm{inf}}(X, Y), \\ H^{n,k}_{\mathrm{inf}}(X, Z)^\vee &\cong H^{2d-n,d+1-k}_{\mathrm{add}}(X, Z). \end{aligned} \tag{35}$$

To prove Theorem 18, it suffices to show the following lemma.

Lemma 22. *Under the duality of (34), $\mathcal{F}^{n,k}(X, Y, Z)$ is the exact annihilator of $\mathcal{F}^{2d-n,d+1-k}(X, Z, Y)$.*

Proof. They annihilate each other for the simple reason of degrees. Thus, it suffices to show

$$\dim \mathcal{H}^{n,k}(X, Y, Z)/\mathcal{F}^{n,k}(X, Y, Z) = \dim \mathcal{F}^{2d-n,d+1-k}(X, Z, Y).$$

By (2) and (6), the left-hand side is equal to

$$\dim H^n(X \setminus Z, Y, \mathbb{C})/F^k H^n(X \setminus Z, Y, \mathbb{C}) + \dim H^{n,k}_{\mathrm{add}}(X, Y),$$

while by (4) and (6), the right-hand side is equal to

$$\dim F^{d+1-k} H^{2d-n}(X \setminus Y, Z, \mathbb{C}) + \dim H^{2d-n,d+1-k}_{\mathrm{inf}}(X, Y).$$

The first terms coincide by Lemma 19, and the second terms also agree since we have the isomorphisms (35). □

This completes the proof of Theorem 18.

5.4.

Let (X', Y', Z') be another triple as in the beginning of this section and set $d' = \dim X'$. Let $f : X \to X'$ be a morphism of $\mathbb{C}$-schemes such that $f(X) \not\subset |Y'| \cup |Z'|$. If f verifies the conditions

$$Y_{\mathrm{red}} \geqslant (f^*Y')_{\mathrm{red}}, \quad Y - Y_{\mathrm{red}} \geqslant f^*(Y' - Y'_{\mathrm{red}}), \quad Z \leqslant f^*(Z'), \tag{36}$$

then by Theorem 18 and § 4.5, we obtain an induced map

$$f_* : \mathcal{H}^{2d-n}(X, Y, Z)(d)_{\mathrm{fr}} \to \mathcal{H}^{2d'-n}(X', Y', Z')(d')_{\mathrm{fr}},$$

which is dual to $f^* : \mathcal{H}^n(X', Z', Y') \to \mathcal{H}^n(X, Z, Y)$.

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6. Picard and Albanese 1-motives

6.1.

Let (X, Y, Z) be as in the beginning of § 4, and let us consider the objects $\mathcal{H}^1(X, Y, Z)$ and $\mathcal{H}^{2d-1}(X, Y, Z)$ in **MHSM** from Definition 16. Proposition 9 (3) and Corollary 15 show that $\mathcal{H}^1(X, Y, Z)(1)_{\mathrm{fr}}$ and $\mathcal{H}^{2d-1}(X, Y, Z)(d)_{\mathrm{fr}}$ belong to $\mathbf{MHSM}_1$ (see § 3.3), where (m) denotes the Tate twists and $(-)_{\mathrm{fr}}$ denotes the free part (see § 2.9 and § 2.7).

Definition 23. We define the Picard and Albanese 1-motives $\mathrm{Pic}(X, Y, Z)$ and $\mathrm{Alb}(X, Y, Z)$ to be the Laumon 1-motive that corresponds to $\mathcal{H}^1(X, Y, Z)(1)_{\mathrm{fr}}$ and $\mathcal{H}^{2d-1}(X, Y, Z)(d)_{\mathrm{fr}}$, respectively, under the equivalence $\mathbf{MHSM}_1 \cong \mathscr{M}_1^{\mathrm{Lau}}$ from Corollary 8.

In view of § 3.7 and (9), Theorem 18 shows that $\mathrm{Pic}(X, Y, Z)$ and $\mathrm{Alb}(X, Z, Y)$ are dual to each other under the Cartier duality.

6.2.

Let (X, Y, Z) and (X', Y', Z') be two triples as in the beginning of § 4 and put $d = \dim X$ and $d' = \dim X'$. Let $f : X \to X'$ be a morphism of $\mathbb{C}$-schemes such that $f(X) \not\subset |Y'| \cup |Z'|$. If conditions (26) are satisfied, then by § 4.5, there is an induced map $f^* : \mathcal{H}^1(X', Y', Z')(1) \to \mathcal{H}^1(X, Y, Z)(1)$ and hence we obtain

$$f^* : \mathrm{Pic}(X', Y', Z') \to \mathrm{Pic}(X, Y, Z).$$

Similarly, if conditions (36) are satisfied, then by § 5.4, there is an induced map $f_* : \mathcal{H}^{2d-1}(X, Y, Z)(d)_{\mathrm{fr}} \to \mathcal{H}^{2d'-1}(X', Y', Z')(d')_{\mathrm{fr}}$ and hence we obtain

$$f_* : \mathrm{Alb}(X, Y, Z) \to \mathrm{Alb}(X', Y', Z').$$

6.3.

Suppose $d = 1$. In this case, we have $\mathrm{Pic}(X, Y, Z) = \mathrm{Alb}(X, Y, Z)$ and we write it as $\mathrm{Jac}(X, Y, Z)$. We give its geometric description. Note that $\mathrm{Jac}(X, Y, Z)$ and $\mathrm{Jac}(X, Z, Y)$ are Cartier dual to each other.

In [14, Definition 25], we considered a Laumon 1-motive $\mathsf{LM}(X, Y, Z)$. Explicitly, this is given as follows (see [14, §5.2]). Let X_Y be a proper $\mathbb{C}$-curve that is obtained by collapsing Y into a single (usually singular) point (see [25, Chapter IV, §3–4]). Let $G(X, Y)$ be the generalized Jacobian of X with modulus Y in the sense of Rosenlicht–Serre [25] or, which amounts to the same, the Picard scheme $\underline{\mathrm{Pic}}^0(X_Y)$ of X_Y. Let $F_{\text{ét}}(X, Z) = \mathrm{Div}^0_Z(X)$ be the group of degree zero (Cartier) divisors supported on Z. Define $F(X, Z)_{\mathrm{inf}}$ by $\mathrm{Lie}\, F(X, Z)_{\mathrm{inf}} = H^0(X, \mathscr{O}_X(Z - Z_{\mathrm{red}})/\mathscr{O}_X)$. Put $F(X, Z) := F_{\text{ét}}(X, Z) \times F_{\mathrm{inf}}(X, Z)$. Let $u_{\text{ét}} : F_{\text{ét}}(X, Z) \to G(X, Y)$ be the map that associates to a divisor its isomorphism class. (We identify Z with its image in X_Y.) Let $u_{\mathrm{inf}} : F_{\mathrm{inf}}(X, Z) \to G(X, Y)$ be the map such that $\mathrm{Lie}\, u_{\mathrm{inf}}$ is given by the composition of

$$\begin{aligned}\mathrm{Lie}\, F_{\mathrm{inf}}(X, Z) &= H^0(X, \mathscr{O}_X(Z - Z_{\mathrm{red}})/\mathscr{O}_X) \\ &= H^0(X_Y, \mathscr{O}_{X_Y}(Z - Z_{\mathrm{red}})/\mathscr{O}_{X_Y}) \overset{(*)}{\to} H^1(X_Y, \mathscr{O}_{X_Y}) = \mathrm{Lie}\, G(X, Y),\end{aligned}$$

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where $(*)$ is the connecting map with respect to the short exact sequence

$$0 \to \mathscr{O}_{X_Y} \to \mathscr{O}_{X_Y}(Z - Z_{\mathrm{red}}) \to \mathscr{O}_{X_Y}(Z - Z_{\mathrm{red}})/\mathscr{O}_{X_Y} \to 0.$$

Put $u = u_{\text{ét}} \times u_{\mathrm{inf}}$, and define

$$\mathsf{LM}(X, Y, Z) = [u : F(X, Z) \to G(X, Y)].$$

By [14, eq. (29)], its Deligne part $[F_{\text{ét}}(X, Z) \to G(X, Y)_{\mathrm{sa}}]$ agrees with Degline's 1-motive $H^1_m(X_Y \setminus Z)(1)$ from [8, 10.3.4]. Here $G(X, Y)_{\mathrm{sa}}$ denotes the maximal semi-Abelian quotient of $G(X, Y)$.

Proposition 24. *We have* $\mathsf{LM}(X, Y, Z) \cong \mathrm{Jac}(X, Y, Z)$.

Proof. Let $\mathcal{L}'$ be the object of $\mathbf{MHSM}_1$ corresponding to $\mathsf{LM}(X, Y, Z)$, and set $\mathcal{L} = (L, L^\bullet_{\mathrm{add}}, L^\bullet_{\mathrm{inf}}, \mathcal{G}^\bullet) := \mathcal{L}'(-1)$. Let $\mathcal{H} := \mathcal{H}^1(X, Y, Z)$ be the object described in § 4.6 and write $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$. It suffices to show that $\mathcal{L} \cong \mathcal{H}$.

We first show that $L \cong H$ as mixed Hodge structures. Let us consider a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\;i\;} & X & \xrightarrow{\;\;p\;\;} & X_Y \\ & \searrow^{i'} & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j'} \\ & & X \setminus Z & \xrightarrow[p']{} & U := X_Y \setminus Z, \end{array}$$

in which i and i' are closed immersions, j and j' are open immersions, and p and p' are finite morphisms. By applying Rj'_* to an exact sequence

$$0 \to \mathbb{Z}_U \to p'_*\mathbb{Z}_{X \setminus Z} \to (p' \circ i')_*\mathbb{Z}_Y \to 0,$$

we obtain

$$\begin{aligned} Rj'_*\mathbb{Z}_U &\cong Rj'_*\mathrm{Cone}(p'_*\mathbb{Z}_{X\setminus Z} \to (p' \circ i')_*\mathbb{Z}_Y)[-1] \\ &= \mathrm{Cone}(p_*Rj_*\mathbb{Z}_{X\setminus Z} \to p_*i_*\mathbb{Z}_Y)[-1] \\ &= p_*\mathbb{Z}_{X|Y,Z}. \end{aligned}$$

It follows that $H^1(U, \mathbb{Z}) \cong H^1(X, \mathbb{Z}_{X|Y,Z}) = H$. On the other hand, we have $L \cong H^1(U, \mathbb{Z})$ by [8, 10.3.8], whence $L \cong H$.

By definition, we have (see § 3.6)

$$\begin{aligned} H^1_{\mathrm{inf}} &= L^1_{\mathrm{inf}} = H^0(X, \mathscr{O}_X(Z - Z_{\mathrm{red}})/\mathscr{O}_X), \\ H^1_{\mathrm{add}} &= H^0(X, \mathscr{O}_X(-Y_{\mathrm{red}})/\mathscr{O}_X(-Y)), \\ L^1_{\mathrm{add}} &= \mathrm{Lie}\, G(X, Y)_{\mathrm{add}}, \end{aligned}$$

where $G(X, Y)_{\mathrm{add}}$ denotes the additive part of $G(X, Y)$. It follows $H^1_{\mathrm{add}} \cong L^1_{\mathrm{add}}$ by [14, Lemma 24]. In particular, we get $\mathcal{L}^1 \cong \mathcal{H}^1$. Finally, we have

$$\mathcal{G}^1 = \ker(\mathcal{L}^1 \to \mathrm{Lie}\, G(X, Y)) \overset{(*)}{\cong} \ker(\mathcal{H}^1 \to H^1(X, \mathscr{O}_X(-Y))) = \mathcal{F}^1,$$

where for $(*)$, we used [25, Chapter V, §10, Proposition 5]. We are done. $\square$

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The construction of $\mathsf{LM}(X, Y, Z)$ works over any field of characteristic zero. Thus, one can ask the following question.

Question 25. Can $\mathrm{Pic}(X, Y, Z)$ and $\mathrm{Alb}(X, Y, Z)$ be constructed over any field of characteristic zero when $d > 1$?

When $Z = \emptyset$, this has been done by Kato and Russell [15, §5] (see the next subsection) and extended by Russell to arbitrary perfect base field [21].

6.4.

We now consider a smooth projective variety X of dimension d and an effective divisor D on X. Kato and Russell defined in [15, §6.1] objects $H^1(X, D_+)(1)$ and $H^{2d-1}(X, D_-)(d)$ of their category $\mathcal{H}_1$ (see § 3.2) and gave their explicit description in [15, §6.3, 6.4]. The Laumon 1-motive corresponding to $H^{2d-1}(X, D_-)(d)$ has trivial formal group part, that is, it can be written as $[0 \to \mathrm{Alb}^{KR}(X, D)]$ where $\mathrm{Alb}^{KR}(X, D)$ is a commutative algebraic group, and $\mathrm{Alb}^{KR}(X, D)$ is Kato–Russell's Albanese variety of X with modulus D.

Let us denote by $\mathcal{H}^1(X, D_+)(1)$ and $\mathcal{H}^{2d-1}(X, D_-)(d)$ the objects in $\mathbf{MHSM}_1$ that correspond to $H^1(X, D_+)(1)$ and $H^{2d-1}(X, D_-)(d)$ under the equivalence from Proposition 7. Suppose now that D_{red} is a simple normal crossing divisor in X. By comparing our construction of $\mathcal{H}^n(X, Y, Z)$ with [15, §6.3, 6.4], we obtain

$$\mathcal{H}^1(X, D_+)(1) \cong \mathcal{H}^1(X, 0, D)(1)_{\mathrm{fr}}, \quad \mathcal{H}^{2d-1}(X, D_-)(d) \cong \mathcal{H}^{2d-1}(X, D, 0)(d)_{\mathrm{fr}}.$$

Therefore, we obtain the following.

Proposition 26. *We have* $\mathrm{Alb}(X, D, 0) \cong [0 \to \mathrm{Alb}^{KR}(X, D)]$.

6.5.

Lekaus [17] has defined Laumon 1-motives $\mathrm{Pic}^+_a(U)$ and $\mathrm{Alb}^+_a(U)$ for an equidimensional quasi-projective $\mathbb{C}$-scheme U of dimension d such that its singular locus is proper over $\mathbb{C}$. Their associated Deligne 1-motives agree with the cohomological Picard and Albanese 1-motives $\mathrm{Pic}^+(U)$ and $\mathrm{Alb}^+(U)$ constructed by Barbieri-Viale and Srinivas [3]; hence, they correspond to the objects $H^1(U, \mathbb{Z})(1)$ and $H^{2d-1}(U, \mathbb{Z})(d)$ of $\mathbf{MHS}_1$ under Deligne's equivalence $\mathscr{M}_1^{\mathrm{Del}} \cong \mathbf{MHS}_1$ from [8, §10]. (Lekaus has also defined Laumon 1-motives $\mathrm{Pic}^-_a(U)$ and $\mathrm{Alb}^-_a(U)$ whose associated Deligne 1-motives correspond to the homology groups of U.)

We may define objects $\mathcal{H}^1(U), \mathcal{H}^{2d-1}(U)$ of $\mathbf{MHSM}$ as follows. Let $\mathcal{H}^1$ and $\mathcal{H}^{2d-1}$ be the objects of $\mathbf{MHSM}_1$ that correspond to $\mathrm{Pic}^+_a(U)$ and $\mathrm{Alb}^+_a(U)$ under the equivalence $\mathscr{M}_1^{\mathrm{Lau}} \cong \mathbf{MHSM}_1$ from Corollary 8. Then we define $\mathcal{H}^1(U) := \mathcal{H}^1(-1)$ and $\mathcal{H}^{2d-1}(U) := \mathcal{H}^{2d-1}(-d)$.

Question 27. Can the definition of $\mathcal{H}^n(U)$ be extended to $n \neq 1, 2d-1$?

Remark 28. The nature of $\mathrm{Pic}^+_a(U)$ and $\mathrm{Alb}^+_a(U)$ are rather different from our $\mathrm{Pic}(X, Y, Z)$ and $\mathrm{Alb}(X, Y, Z)$. For instance, suppose that U is an affine irreducible curve

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and let $\overline{U}$ be a good compactification. Then the Laumon 1-motive $\mathrm{Pic}^+_a(U) = \mathrm{Alb}^+_a(U) = [F_{\text{ét}} \times F_{\text{inf}} \to G]$ verifies

$$G = \underline{\mathrm{Pic}}^0(\overline{U}), \;\; F_{\text{ét}} = \mathrm{Div}^0_{\overline{U}\setminus U}(\overline{U}), \;\; \mathrm{Lie}\, F_{\text{inf}} = H^1(\overline{U}, \mathscr{O}_{\overline{U}}).$$

(In particular, F_{inf} depends only on $\overline{U}$ as long as U is affine.) On the other hand, let X be a smooth proper curve and let Y, Z be effective divisors with disjoint support, and let $U := X_Y \setminus Z$ be the curve considered in § 6.3. Then $\mathrm{Pic}(X, Y, Z) = \mathrm{Alb}(X, Y, Z) = \mathrm{Jac}(X, Y, Z)$ is written as $[F_{\text{ét}} \times F'_{\text{inf}} \to G]$ using the same $F_{\text{ét}}$ and G as above, but

$$\mathrm{Lie}\, F'_{\text{inf}} = H^0(X, \mathscr{O}_X(Z - Z_{\text{red}})/\mathscr{O}_X).$$

7. Relation with enriched and formal Hodge structures

7.1.

Let $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$ be the subcategory of $\mathbf{Z}^{\text{op}}\mathbf{Vec}_{\mathbb{C}}$ (see § 2.1) formed by the objects $V^\bullet$ from (1) such that V^k are trivial for all sufficiently small k and such that τ^k_V are isomorphic for all sufficiently large k. For an object $V^\bullet$ of $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$, we denote by V^∞ the projective limit of $(V^k, \tau^k_V)_{k\in\mathbb{Z}}$ and by $\tau^{\infty,k}_V : V^\infty \to V^k$ the canonical map. For a mixed Hodge structure H, we define an object of $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$ by

$$H_{\mathbb{C}}/F^\bullet := (\cdots \to H_{\mathbb{C}}/F^k H_{\mathbb{C}} \to H_{\mathbb{C}}/F^{k-1} H_{\mathbb{C}} \to \cdots),$$

where all maps are the projection maps. We have $H_{\mathbb{C}}/F^\infty = H_{\mathbb{C}}$.

Recall from [6] that an enriched Hodge structure is a tuple $E = (H, V^\bullet, v^\bullet, s)$ of a mixed Hodge structure H, an object $V^\bullet$ of $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$, a morphism $v^\bullet : V^\bullet \to H_{\mathbb{C}}/F^\bullet$ of $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$, and a $\mathbb{C}$-linear map $s : H_{\mathbb{C}} \to V^\infty$ such that $v^\infty \circ s = \mathrm{id}$. A morphism between two enriched Hodge structures is a pair of morphisms of **MHS** and of $\widetilde{\mathbf{Vec}}^\bullet_{\mathbb{C}}$ that is compatible with structural maps $(v^\bullet, s)$. The category of enriched Hodge structures is denoted by **EHS**. Let $\mathbf{EHS}_\triangle$ be the full subcategory of **EHS** consisting of objects $(H, V^\bullet, v^\bullet, s)$ such that v^k are isomorphic for all sufficiently large k (hence, $s = (v^\infty)^{-1}$). Recall from § 2.3 that we have defined a subcategory $\mathbf{MHSM}_{\text{add}}$ of **MHSM**.

Lemma 29. *The categories* $\mathbf{EHS}_\triangle$ *and* $\mathbf{MHSM}_{\text{add}}$ *are equivalent.*

Proof. Take an object $(H, V^\bullet, v^\bullet, s)$ of $\mathbf{EHS}_\triangle$. We define an object $(H, H^\bullet_{\text{add}}, 0, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}_{\text{add}}$ by setting for each integer k

$$H^k_{\text{add}} := \ker(v^k), \quad \mathcal{F}^k := \ker(H_{\mathbb{C}} \oplus \ker(v^k) \to V^k),$$

where the last map is defined by $\tau^{\infty,k}_V \circ s : H_{\mathbb{C}} \to V^k$ and the inclusion map $\ker(v^k) \hookrightarrow V^k$. This yields a functor $\mathbf{EHS}_\triangle \to \mathbf{MHSM}_{\text{add}}$. Next, take an object $(H, H^\bullet_{\text{add}}, 0, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}_{\text{add}}$. We define an object $(H, V^\bullet, v^\bullet, s)$ of $\mathbf{EHS}_\triangle$ by setting for each integer k

$$V^k := (H_{\mathbb{C}} \oplus H^k_{\text{add}})/\mathcal{F}^k, \quad v^k : V^k \to H_{\mathbb{C}}/F^k H_{\mathbb{C}}, \quad s := (v^\infty)^{-1},$$

where v^k is induced by the composition of the projection maps $H_{\mathbb{C}} \oplus H^k_{\text{add}} \to H_{\mathbb{C}} \to H_{\mathbb{C}}/F^k H_{\mathbb{C}}$. This yields a functor $\mathbf{EHS}_\triangle \to \mathbf{MHSM}_{\text{add}}$. It is easy to see that these two functors are quasi-inverse to each other. □

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Remark 30. One can construct a category that contains both **EHS** and **MHSM** as follows. Let $\widetilde{\mathbf{Vec}}^{\bullet\vee}_{\mathbb{C}}$ be the subcategory of $\mathbf{Z}^{\mathrm{op}}\mathbf{Vec}_{\mathbb{C}}$ formed by the objects $V^{\bullet}$ such that V^k are trivial for all sufficiently large k and such that τ_V^k are isomorphic for all sufficiently small k. We define $\widetilde{\mathbf{MHSM}}$ to be the category of tuples $(H, H^{\bullet}_{\mathrm{inf}}, H^{\bullet}_{\mathrm{add}}, \mathcal{F}^{\bullet})$ consisting of an object $(H, H^{\bullet}_{\mathrm{inf}}, H^{\bullet}_{\mathrm{add}})$ of $\mathbf{MHS} \times \widetilde{\mathbf{Vec}}^{\bullet}_{\mathbb{C}} \times \widetilde{\mathbf{Vec}}^{\bullet\vee}_{\mathbb{C}}$ and a linear subspace $\mathcal{F}^k$ of $\mathcal{H}^k = H_{\mathbb{C}} \oplus H^k_{\mathrm{inf}} \oplus H^k_{\mathrm{add}}$ for each $k \in \mathbb{Z}$, subject to conditions **(1-a)**–**(1-d)** in Definition 1. Then **EHS** is identified with a subcategory $\widetilde{\mathbf{MHSM}}_{\mathrm{add}}$ of $\widetilde{\mathbf{MHSM}}$ consisting of objects such that $H^{\bullet}_{\mathrm{inf}} = 0$. Note that the functor R from § 2.10 cannot be extended to $\widetilde{\mathbf{MHSM}}$. Note also that $\widetilde{\mathbf{Vec}}^{\bullet}_{\mathbb{C}}$ (resp. $\widetilde{\mathbf{Vec}}^{\bullet\vee}_{\mathbb{C}}$) is not Noetherian (resp. Artinian); hence, $\widetilde{\mathbf{MHSM}} \otimes \mathbb{Q}$ is neither Noetherian nor Artinian, while $\mathbf{Vec}^{\bullet}_{\mathbb{C}}$ and $\mathbf{MHSM} \otimes \mathbb{Q}$ (as well as $\mathbf{MHS} \otimes \mathbb{Q}$) are both Artinian and Noetherian.

7.2.

Let n be a positive integer. We write $\mathbf{MHS}^n$ for the subcategory of **MHS** consisting of mixed Hodge structures H such that $\mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H_{\mathbb{C}} = 0$ unless $p, q \in [0, n]$. Denote by $\mathbf{EHS}^n$ the full subcategory of **EHS** consisting of objects $E = (H, V^{\bullet}, v^{\bullet}, s)$ such that $H_{\mathbb{C}}$ belongs to $\mathbf{MHS}^n$, $V^k = 0$ for any $k \leqslant 0$, and $\tau_V^{\infty,k}$ are isomorphic for any $k > n$. Let $\mathbf{EHS}^n_{\triangle}$ be the intersection of $\mathbf{EHS}^n$ and $\mathbf{EHS}_{\triangle}$. We define a functor $\triangle_n : \mathbf{EHS}^n \to \mathbf{EHS}^n_{\triangle}$ by $\triangle_n(H, V^{\bullet}, v^{\bullet}, s) = (H, V^{\bullet}_{\triangle}, v^{\bullet}_{\triangle}, \mathrm{id})$, where

$$V^k_{\triangle} := \begin{cases} V^k & \text{if } k \leqslant n \\ H_{\mathbb{C}} & \text{if } k > n, \end{cases} \quad \tau^k_{V_{\triangle}} := \begin{cases} \tau^k_V & \text{if } k \leqslant n \\ \tau_V^{\infty,n} \circ s & \text{if } k = n+1 \\ \mathrm{id}_{H_{\mathbb{C}}} & \text{if } k > n+1, \end{cases} \quad v^k_{\triangle} := \begin{cases} v^k & \text{if } k \leqslant n \\ \mathrm{id}_{H_{\mathbb{C}}} & \text{if } k > n. \end{cases}$$

According to Mazzari [18], a formal Hodge structure of level $\leqslant n$ is a tuple (E, U, u) of an object $E = (H, V^{\bullet}, v^{\bullet}, (v^{\infty})^{-1})$ of $\mathbf{EHS}^n_{\triangle}$, a finite dimensional $\mathbb{C}$-vector space U, and a $\mathbb{C}$-linear map $u : U \to V^n$. The category of formal Hodge structures of level $\leqslant n$ is denoted by $\mathbf{FHS}^n$. We identify the subcategory of $\mathbf{FHS}^n$ formed by objects of the form $(E, 0, 0)$ with $\mathbf{EHS}^n_{\triangle}$.

Let $\mathbf{MHSM}^n_{\square}$ be a full subcategory of **MHSM** consisting of objects $\mathcal{H} = (H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet})$ such that H belongs to $\mathbf{MHS}^n$, $H^k_{\mathrm{add}} = 0$ unless $k \in [1, n]$, and $H^k_{\mathrm{inf}} = 0$ for all $k \neq n$. Let $\mathbf{MHSM}^n_{\diamond}$ be the full subcategory of $\mathbf{MHSM}^n_{\square}$ consisting of objects $\mathcal{H} = (H, H^{\bullet}_{\mathrm{add}}, H^{\bullet}_{\mathrm{inf}}, \mathcal{F}^{\bullet})$ such that the composition map $H^n_{\mathrm{inf}} \hookrightarrow \mathcal{H}^n_{\mathrm{inf}} \twoheadrightarrow \mathcal{H}^n_{\mathrm{inf}}/\mathcal{F}^n_{\mathrm{inf}}$ is the zero map.

Lemma 31.

(1) *The categories* $\mathbf{FHS}^n$ *and* $\mathbf{MHSM}^n_{\square}$ *are equivalent.*

(2) *The categories* $\mathbf{EHS}^n$ *and* $\mathbf{MHSM}^n_{\diamond}$ *are equivalent.*

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Proof. (1) Take an object $((H, V^\bullet, v^\bullet, s), U, u)$ of $\mathbf{FHS}^n$. We define an object $(H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}^n_\square$ by setting for each integer k

$$H^k_{\mathrm{add}} := \ker(v^k), \quad H^k_{\mathrm{inf}} := \begin{cases} U & k = n \\ 0 & k \neq n, \end{cases}$$

$$\mathcal{F}^k := \begin{cases} \ker(H_{\mathbb{C}} \oplus \ker(v^n) \oplus U \to V^n) & k = n \\ \ker(H_{\mathbb{C}} \oplus \ker(v^k) \to V^k), & k \neq n, \end{cases}$$

where the last map is defined by $\tau_V^{\infty,k} \circ s : H_{\mathbb{C}} \to V^k$, the inclusion map $\ker(v^k) \hookrightarrow V^k$, and $u : U \to V^n$. This yields a functor $\mathbf{FHS}^n \to \mathbf{MHSM}^n_\square$. Next, take an object $\mathcal{H} = (H, H^\bullet_{\mathrm{add}}, H^\bullet_{\mathrm{inf}}, \mathcal{F}^\bullet)$ of $\mathbf{MHSM}^n_\square$. Let E be an enriched Hodge structure that corresponds to $\mathcal{H}_{\mathrm{add}} = \pi_{\mathrm{add}}(\mathcal{H})$ (see (5)) under the equivalence in Lemma 29. Then $E = (H, V^\bullet, v^\bullet, s)$ belongs to $\mathbf{EHS}^n_\triangle$, and we have $V^k = \mathcal{H}^k_{\mathrm{add}}/\mathcal{F}^k_{\mathrm{add}}$. Set $U = H^n_{\mathrm{inf}}$. We define a linear map $u : H^n_{\mathrm{inf}} \to \mathcal{H}^n_{\mathrm{add}}/\mathcal{F}^n_{\mathrm{add}}$ as the composition of

$$H^n_{\mathrm{inf}} \hookrightarrow \mathcal{H}^n \twoheadrightarrow \mathcal{H}^n/\mathcal{F}^n \overset{\cong}{\leftarrow} \mathcal{H}^n_{\mathrm{add}}/\mathcal{F}^n_{\mathrm{add}},$$

where the last isomorphism is from (4). We have defined an object (E, U, u) of $\mathbf{FHS}^n$. This yields a functor $\mathbf{MHSM}^n_\square \to \mathbf{FHS}^n$. It is easy to see that these two functors are quasi-inverse to each other, proving (1).

(2) There is a full faithful functor $\sigma_n : \mathbf{EHS}^n \to \mathbf{FHS}^n$ given by

$$E = (H, V^\bullet, v^\bullet, s) \mapsto (\triangle_n(E), \ker(v^\infty), u),$$

where u is the composition of the inclusion map $\ker(v^\infty) \hookrightarrow V^\infty$ and $\tau_V^{\infty,n} : V^\infty \to V^n$. Its essential image is formed by objects $((H, V^\bullet, v^\bullet, s), U, u)$ of $\mathbf{FHS}^n$ such that $v^n \circ u = 0$. (See [2, Proposition 4.2.3] for the case $n = 1$. Formal Hodge structures satisfying the last condition are called special in [18].) Under the equivalence from the first part of the lemma, the last condition is translated to $H^n_{\mathrm{inf}} \to \mathcal{H}^n_{\mathrm{inf}}/\mathcal{F}^n_{\mathrm{inf}}$ being the zero map by (7). □

Lemma 32. *Denote by* $\iota_n : \mathbf{FHS}^n \to \mathbf{MHSM}$ *the composition of the equivalence functor* $\mathbf{FHS}^n \cong \mathbf{MHSM}^n_\square$ *from Lemma 31 and the inclusion functor* $\mathbf{MHSM}^n_\square \subset \mathbf{MHSM}^n$. *Then, for any objects* D, D' *of* $\mathbf{FHS}^n$, *we have*

$$\mathrm{Ext}^1_{\mathbf{FHS}^n}(D, D') \cong \mathrm{Ext}^1_{\mathbf{MHSM}}(\iota_n D, \iota_n D').$$

Proof. By Lemma 31, it suffices to show that $\mathbf{MHSM}^n_\square$ is a thick Abelian subcategory of $\mathbf{MHSM}$. This follows from Remark 5. □

7.3.

Let n be an integer and let X be a proper irreducible variety over $\mathbb{C}$ of dimension d. Bloch and Srinivas constructed an enriched Hodge structure $H^n_{\mathrm{BS}}(X) = (H, V^\bullet, v^\bullet, s)$ as follows. (There are variants; see [6, Corollary 2.2].) Let $H = H^n(X, \mathbb{Z})$ be Deligne's mixed Hodge structure. Take a smooth (proper) hypercovering $\pi : X_* \to X$. Then we

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have $H^n(X_*, \Omega^{\bullet<k}_{X_*}) = H_{\mathbb{C}}/F^k H_{\mathbb{C}}$, and there is a commutative diagram

$$\begin{array}{ccccccc} & & H^n(X_*, \mathbb{C}) & \xrightarrow{\cong} & H^n(X_*, \Omega^{\bullet}_{X_*}) & \longrightarrow & H^n(X_*, \Omega^{\bullet<k}_{X_*}) \\ & & \cong \Big\uparrow & & v^{\infty} \Big\uparrow & & v^k \Big\uparrow \\ H_{\mathbb{C}} & \xrightarrow{=} & H^n(X, \mathbb{C}) & \xrightarrow{s} & H^n(X, \Omega^{\bullet<d+1}_X) & \longrightarrow & H^n(X, \Omega^{\bullet<k}_X). \end{array}$$

We define an object $H^n_{\mathsf{BS}}(X)$ of $\mathbf{EHS}^n$ by setting $V^k := H^n(X, \Omega^{\bullet<k}_X)$ if $k \leqslant d$ and $V^k := H^n(X, \Omega^{\bullet<d+1}_X)$ if $k > d$. This belongs to $\mathbf{EHS}^d$ if $n \geqslant d$.

In [18, Definition 3.1], Mazzari defined a similar object $H^{n,k}_{\sharp}(X)$ of $\mathbf{EHS}^n_{\triangle}$ for each $k = 1, \dots, n$. Let us have a look at a special case $H^{2d-1,d}_{\sharp}(X)$ from [18, Example 3.2], which actually belongs to $\mathbf{EHS}^d_{\triangle}$ and the same as $\triangle_d(H^{2d-1}_{\mathsf{BS}}(X))$. We write $\mathcal{H}^{2d-1}_{\triangle}(X)$ for the corresponding object of **MHSM** under the equivalence from Lemma 31. By Lemma 32, Mazzari's result [18, Proposition 3.5] can be rewritten as follows.

Proposition 33. *Let X be an irreducible proper variety over $\mathbb{C}$ of dimension d. Denote by $A^d(X)$ the Albanese variety of X in the sense of Esnault–Srinivas–Viehweg [11]. Then there is an isomorphism*

$$A^d(X) \cong \mathrm{Ext}^1_{\mathbf{MHSM}}(\mathbb{Z}(-d), \mathcal{H}^{2d-1}_{\triangle}(X)).$$

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