Is Quantum Mechanics Pointless?

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There exist well-known conundrums, such as measure-theoretic paradoxes and problems of contact, which, within the context of classical physics, can be used to argue against the existence of points in space and space-time. I examine whether quantum mechanics provides additional reasons for supposing that there are no points in space and space-time.

1. Introduction. Our standard account of regions and their sizes, has some bizarre features. In the first place, one cannot cut a region exactly in two halves. For if one of the two regions includes its boundary (is closed), then the other does not include it (is open). One might reasonably think that this difference between open and closed regions is an artifact of our mathematical representation of regions that does not correspond to a difference in reality. Secondly, regions of finite size are composed of points, each of which have zero size. One might think it rather strange that when one gathers together countably many points one must have a region of size zero, while if one gathers together uncountably many points, one can form a region of any size. Thirdly, finite sized regions must have parts that have no well-defined size, that is, are unmeasurable. One might swallow parts that have zero size, but parts that cannot have any well-defined size could lead to gagging. Fourthly, Banach and Tarski have shown that one can break any finite sized region into finitely many parts that can then be reassembled without stretching or squeezing, to form a larger or smaller region. There are also problems about contact: physical objects that occupy closed regions can never touch, indeed they must always be a finite distance apart. Now, I do not say that problems such as these are a decisive argument against the standard account. But I do say that they form a good reason to devise a geometry that does not have these problems, and to see whether modern physics can plausibly be set in such a geometry.

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Caratheodory, and others following him, have devised such "pointless geometries" (see Caratheodory 1963, Skyrms 1993). Let me give an example of such a pointless geometry. Start by designating the collection of all open intervals on the real line as regions. Then declare that the union of any countable set of regions is a region, declare that the intersection of any two regions is a region, declare that the complement of any region is a region, and declare that these are all the regions that there are. This collection of regions is the so-called "Borel algebra" of regions. Now this collection of regions includes point-sized regions, regions that differ only in being open or closed, and more generally distinct regions whose differences have size 0. Let us get rid of all such differences by regarding as equivalent any regions such that the differences between those regions have size 0. That is, let us declare regions to be equivalence classes of regions that differ at most by (Lebesque) measure 0. This collection of regions, and their sizes, comprises an example of a pointless geometry. Since any distinct points differ by measure 0, all points will correspond to one and the same region, namely, the "null region," which is the complement of the region consisting of the entire space. Any other region has well-defined finite size (measure). Breaking up and reassembling never changes the size of a region. Regions can always be cut exactly in half. And there are no problems about contact between objects since there are no differences between open and closed regions.

This seems very pleasing. It therefore seems worthwhile to examine whether physics can be done in such a setting. In this paper I will take a look at quantum mechanics. I will argue that the formalism of quantum mechanics strongly suggests that its value spaces, including physical space and space-time, are pointless spaces.

2. Continuous Observables in Quantum Mechanics. It is well known that, strictly speaking, on the standard account of the state-space of quantum mechanics as a separable Hilbert space, continuous observables do not have eigenstates. For instance, there exists no quantum mechanical state $|x = 5\rangle$ which is an eigenstate of the position operator X corresponding to the point x = 5 in physical space. Indeed, there exists no quantum mechanical state such that a measurement of a position in that state will, with probability equal to one, yield a particular value. For if there were position eigenstates there would have to be uncountably many mutually orthogonal states, but a separable Hilbert space has only countably many dimensions.

What is not often noted is that there is a more general conclusion that can be drawn from the assumption that the quantum mechanical state-space is a separable Hilbert space, namely, that wave functions are functions on pointless spaces. To be more precise, it is a consequence of the fact that wave functions are representations of states in a separable Hilbert space that

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each wave function is not simply a square integrable function, but rather an equivalence class of square integrable functions that differ in their values at most on a set of (Lebesque) measure 0. The reason for this is pretty straightforward. One of the axioms of the theory of Hilbert spaces is that there is a unique vector whose norm (inner product with itself) is zero. In the position representation, the norm of a wave function f(x) is $\int|f(x)|^2dx$. But there are many different functions for which $\int|f(x)|^2dx = 0$. So, in order for wave functions to be representations of vectors in a Hilbert space one needs to assume that wave functions correspond to equivalence classes of (square integrable) functions that differ at most on a set of measure 0. Now one can show that mappings (homomorphisms) on pointless spaces correspond exactly to equivalence classes of functions that differ at most on a set of measure 0 (see Skyrms 1993). Thus wave functions are functions on pointless spaces. Quantum mechanics thus provides us with evidence that the value-space for any continuous observable is a pointless space. However, let me now turn to two ways in which point values for continuous observables can be reintroduced into quantum mechanics.

3. Rigged Hilbert Spaces. There is a standard way of reintroducing eigenstates of continuous observables in a rigorous way, namely, the "rigged Hilbert space" formalism. Let me outline this formalism (for more detail see Böhm 1978).

Let us use the simplest example, the harmonic oscillator. I will assume that the reader is familiar with the construction of the "ladder" of eigenstates $\phi_n = (a^+)^n\phi_0/\sqrt{n!}$ of the number operator N, which starts "at the bottom" with the state ϕ_0 which has the feature that $N\phi_0 = 0$. Let us now consider all and only the *finite* superpositions of these states, that is, the states of form $\phi = \Sigma c_n\phi_n$, where we superpose only *finitely* many ϕ_n. Let us denote this linear space of states as Ψ. Using the standard scalar product (ϕ,ψ) and norm $|\psi|^2 = (\psi,\psi)$ one can then define the standard Hilbert space topology on the space Ψ, and the accompanying standard notion of convergence: $\phi_k \rightarrow \phi$ iff $|\phi_k - \phi| \rightarrow 0$ as $k \rightarrow \infty$. Given this topology the space Ψ is not "complete," that is, there exist Cauchy sequences (converging sequences) that have no limit point in Ψ. If one now completes Ψ by adding all such limit points, one obtains the standard Hilbert space H of the harmonic oscillator. It is important to note that this has as a consequence that the Hilbert space H will contain "infinite energy" states: there will exist Cauchy sequences of states $\phi_1 = c_1E_1$, $\phi_2 = d_1E_1+d_2E_2$, $\phi_3 = e_1E_1+e_2E_2+e_3E_3$, . . . , such that as $n \rightarrow \infty$, the expectation value of Energy $= (1/\Sigma|c_i|^2)(\Sigma|c_i|^2E_i) \rightarrow \infty$ (each E_i denotes an energy eigenstate). By the completeness of the Hilbert space H there must exist a limit state corresponding to each Cauchy sequence. Hence there will exist a state that one can reasonably call an "infinite energy" state, even though this state, strictly speaking, is not in the domain of the energy operator.

Let us now define a different topology, a "nuclear" topology, on Ψ and the accompanying different, "nuclear" notion of convergence: $\phi_k \rightarrow \phi$ iff $((\phi_k - \phi),(N+1)^p(\phi_k - \phi)) \rightarrow 0$ as $k \rightarrow \infty$ for any p. Roughly speaking, the factor $(N+1)^p$ is a factor designed to weigh the higher-number eigenstates heavier than the lesser-number eigenstates, so that differences in the higher-number coefficients have to converge to 0 very rapidly if the norm $((\phi_k - \phi), (N+1)^p(\phi_k - \phi))$ is to converge to 0 as k converges to infinity. Thus, any sequence of states in Ψ that is a Cauchy sequence according to the nuclear topology, is also a Cauchy sequence according to the Hilbert space topology, but not vice versa. Now let us complete Ψ according to the nuclear sense of convergence. Of course, this will add only a proper subset of the states that get added when one completes Ψ according to the Hilbert space topology. We then obtain a "linear topological" space of states Φ.

It is interesting to note that Φ does not contain infinite energy states. The reason for this is that the coefficients of higher-number (higher-energy) states have to drop to 0 very rapidly (faster than any polynomial) in order for the sequence to be a Cauchy sequence according to the Nuclear topology. This might seem to be a rather appealing feature of space Φ.

We need just a little more machinery in order to construct such point-valued states. A so-called "antilinear functional" F on a linear space Θ is a function $F(\theta)$, often denoted as $\langle\theta|F\rangle$, from elements θ of Θ to complex numbers, such that $\langle c_1\theta_1 + c_2\theta_2|F\rangle = c_1^*\langle\theta_1|F\rangle + c_2^*\langle\theta_2|F\rangle$. (Here the c_i denotes complex numbers, and * denotes complex conjugation.) The space Θ^X of linear functionals on a linear space Θ is linear itself, and is called the space "conjugate to" Θ. It is easy to see that each vector f in a linear space Θ with a scalar product (θ,η) defines an anti-linear functional F as follows: $\langle\theta|F\rangle = (\theta,f)$. It is also fairly easy to show that for a Hilbert space H, there is a one-to-one correspondence between antilinear functionals $|\eta\rangle$ and vectors $\langle\eta|$, so that H and H^X can be taken to be the same space. This, however, is not true for the space Φ that we obtained from Ψ by completing it according to the nuclear topology. Rather, one can show that $\Phi \subset H \subset \Phi^X$. This triplet of spaces is known as a "rigged Hilbert space," or a "Gelfand Triplet." Corresponding to any continuous linear operator A on states in Φ there exists an adjoint operator A^X on states in Φ^X, which is defined by the demand that $\langle\phi|A^X|F\rangle =^{def} \langle\phi|A^XF\rangle = \langle A\phi|F\rangle$ for all $\langle\phi|$ and all $|F\rangle$. Now we can define so-called "generalized" eigenvectors of an operator A on Φ. A "generalized" eigenvector of A corresponding to the "generalized" eigenvalue λ is an antilinear functional $F \in \Phi^X$ such that: $\langle A\phi|F\rangle = \langle\phi|A^X|F\rangle = \lambda^*\langle\phi|F\rangle$ for all $\langle\phi| \in \Phi$, which may also be stated as $A^X|F\rangle = \lambda^*|F\rangle$. One can then show that, for our harmonic oscillator system, there is a continuum of generalized eigenvalues and eigenvectors of both the X and P operators. And one can show, for our harmonic oscillator system, that any state $|\phi\rangle$ in Φ^X that corresponds to a state $\langle\phi|$ in Φ, has a unique expansion in terms of a measure over the generalized eigenvectors $|x\rangle$ of the position operator X, and a unique expansion in terms of a measure over the generalized eigenvectors $|p\rangle$ of the momentum operator P. This all seems great. Let us now consider some unappealing features of rigged Hilbert spaces.

A rigged Hilbert space, that is, a Gelfand triple $\Phi \subset H \subset \Phi^X$, is not as simple and natural a state-space as a Hilbert space. Just look at the machinery that I needed above in order to explain the basics of rigged Hilbert spaces, and compare it to the simplicity and naturalness of (the axioms of) the normal (separable) Hilbert space formalism. Moreover, a rigged Hilbert space is a rather nonunified, cobbled together, state-space that consists of three quite distinct parts Φ, H and Φ^X, where states in the distinct parts have distinct properties. For instance, given any two states ϕ and ψ in H, one can take their scalar product $\langle\phi|\psi\rangle$, which is a complex number. But the scalar product $\langle f|g\rangle$ of states f and g that are in Φ^X but not in H, is not an ordinary complex number. The scalar product in Φ^X exists only in a distributional sense, that is, it is defined as the distribution that satisfies $\langle f|\phi\rangle = \int dg \langle f|g\rangle\langle g|\phi\rangle$ for all ϕ in Φ. And there is the awkward, but essential, use of two distinct topologies, the one corresponding to the usual inner product, the other being the "nuclear" topology. It's all rather messy.

A more serious problem is the following. Since one cannot spectrally decompose a position eigenstate in terms of the eigenstates of such an observable, one cannot make sense of probabilities of the results of a measurement of such an observable when the object is in a position eigenstate. More generally, in a state f one can only make sense of the ratios of expectation values $\langle f|A|f\rangle/\langle f|B|f\rangle$ of "admissible" observables A and B, where an observable A is said to be admissible iff $A|f\rangle$ belongs to the domain of $\langle f|$ (see, for example, Bogolubov, Logunov, and Todorov 1975, ch. 4).

In the specific case of the observable Energy, matters are even worse. There is a relatively clear sense in which position eigenstates are 'infinite energy' states. Consider any sequence of wave functions $\{\psi_i(x)\}$ such that each $\psi_i(x)$ has a well-defined finite expectation value for its energy, which becomes more and more concentrated around a given point in space, that is, suppose that in the limit as i goes to infinity the wave functions $\psi_i(x)$ become arbitrarily well confined to arbitrarily small regions around that point in space. One can then show that the expectation value of energy of this sequence of states must increase without bound as i goes to infinity.[1]

1. Although this is a rather suggestive fact one has to be a bit careful as to what it means. For instance, it is not true that this sequence of wave functions converges to the corre-

It seems that we have a a bit of a dilemma. Either position eigenstates are physically possible, in which case, in a rather clear sense, gross violations of energy conservation are possible. This seems implausible. Or they are not physically possible, in which case it is unclear why one would go to such lengths in order to introduce such states into the quantum mechanical state-space. This dilemma can be brought out a bit more sharply by considering the dynamics of quantum states.

What is the Hamiltonian time evolution of position eigenstates? If one initially has a probability distribution over values of observables that corresponds to a state in the ordinary Hilbert space H, then, as long as the development is a Hamiltonian development, the state will always be in the ordinary Hilbert space H. Thus, if at any time the state is in the ordinary Hilbert space, then the rest of the rigged Hilbert space is redundant. If, on the other hand, at any time the state is a position eigenstate, then it will always be in an eigenstate of a continuous observable, and never return to the ordinary Hilbert space H.

On the other hand, suppose that one believes that during measurements the dynamics is governed by the projection postulate. And suppose that exact position measurements were possible. Then one could, with certainty, create an "infinite energy" state by measuring the exact position of a particle. While this could be a great boon or a great disaster to humanity, it seems implausible that it could ever happen. However, if position eigenstates could not possibly be produced by such measurements, nor by a unitary dynamics, why introduce the mathematical artifice of position eigenstates in the first place?

In general it would seem that eigenstates of continuous observables, at best, are redundant. Since, in addition, they complicate the mathematical formalism, it seems best to not countenance them in the first place.

4. Recovering Point Values in the Algebraic Approach. Hans Halvorson (Halvorson 2001a and 2001b) has recently proposed a different way, set within the algebraic approach to quantum mechanics, to introduce quantum mechanical states corresponding to point values for continuous observables. Let me sketch the basic idea behind his reintroduction of points.

Suppose that physical space is pointless. And suppose that in order to completely specify the locational state of an object one has to specify for each region whether the object is entirely confined to that region. It would then seem that, despite the fact that no point-sized regions exist, there nonetheless can be point-sized objects with point-like locational properties. For instance, suppose that the locational state of an object is wholly

sponding position eigenstate in the sense that the inproduct of this sequence with that position eigenstate converges to 1.

confined to each of the following regions: (−1,1), (−1/2,1/2), (−1/4,1/4)…. The only possible understanding of that collection of locational properties is surely that it is a point particle that is located exactly at point x = 0. Of course there is no region that corresponds to this point. But it seems impossible to understand the locational properties of the object in any other way: it is smaller than any region, so it cannot have finite size, and it is located in each of a set of regions that "converge" to point x = 0. Thus it appears that the fact that space is pointless does not rule out states of objects that correspond to the occupation of a point-sized location. This, in essence, is the way in which Halvorson reintroduces point values in the algebraic approach to quantum mechanics.

In the algebraic approach one identifies a quantum mechanical state, of a system characterized by an operator algebra *A*, with a linear map from operators to complex numbers such that any observable O (self-adjoint operator O) gets mapped onto a positive real number, the expectation value of O. In particular, states will assign expectation values to projection operators. The expectation value of a projection operator is just the probability that the value of that projection operator is 1, since projection operators only have 1 and 0 as possible values. Given a continuous observable Q one can form a Boolean algebra $\{Q_s\}$ of projection operators Q_s where S is a range of values on the real line R, and Q_s corresponds to the claim that the value of Q lies in range S. When one does this, regions S that differ by measure 0 will all correspond to one and the same projection operator. Thus, for example, all measure 0 regions correspond to one and the same (null) operator. Indeed this algebra is isomorphic to the Borel algebra of equivalence classes of regions on the real line R which differ at most by (Lebesque) measure 0, which I previously called a "pointless geometry."

Nonetheless, as I explained with my analogy, on the algebraic account of quantum states there can be states, so-called "singular states" that correspond to point values for continuous observables. For consider a state that assigns a probability of 1 or 0 to every projection operator in the algebra $\{Q_S\}$. Such a singular state determines for any region of possible values of Q whether the value of Q is inside that region or not. In particular there will be a set of regions that converge to a point value for Q such that the value of Q is, with probability 1, in each of these regions. Thus on the algebraic approach one can introduce states corresponding to point values for continuous observables, and this is exactly what Halvorson suggests doing. Indeed, one can even fit all of these algebraic states into a single *nonseparable* Hilbert space. Now let me quickly evaluate the merits of Halvorson's proposal.

Let me begin by noting that Halvorson's singular states will violate countable additivity, that is, "singular" algebraic states will correspond to probability distributions that violate countable additivity. My own view is that violations of countable additivity are perfectly acceptable in this case.[2] However, this is a somewhat involved issue that I cannot satisfactorily address in a couple of paragraphs. Other than a brief indication of my view in footnote 2, I will therefore set this issue aside.

In other respects, the problems with Halvorson's approach are very similar to the problems with the rigged Hilbert space approach. A nonseparable Hilbert space that includes all the eigenstates of continuous observables does not appear to be as mathematically attractive as the standard separable Hilbert space. For instance, the fact that it is a nonseparable Hilbert space means that standard forms of reasoning in terms of finite or countable superpositions do not go through. Also, as in the case of the rigged Hilbert space, the nonseparable Hilbert space decomposes into two quite distinct parts: the part that corresponds to the standard separable Hilbert space (that is, the eigenstates of discrete observables plus their countable superpositions), and the part that corresponds to the "singular" states (the eigenstates of continuous observables). Moreover, as before, a unitary Hamiltonian dynamics cannot take one into, or out of, the standard separable Hilbert space. Finally, position eigenstates do not have well-defined expectation values for momenta and energies. And one cannot make sense of probabilities of results of measurements of observables that have a complete set of eigenvectors in the standard Hilbert space (the Schrodinger representation). All of this suggests that we should stick with the standard Hilbert space.

5. Pointless Spaces and Finite Energies in Quantum Mechanics. Let me now suggest a modification of the standard Hilbert space approach. As I noted before infinite energy states occur in the standard Hilbert space H. Should we not get rid of all infinite energy states from the standard Hilbert space? A natural way in which to remove all infinite energy states is to go back to the rigged Hilbert space construction, and to let the state-space be the space Φ that is the completion, with regard to the nuclear topology, of the space Ψ of finite superpositions of energy eigenstates. As I previously noted this space Φ contains only states with finite expectation values for energy. It also has some other attractive features. One can show that there

2. Here is a very brief indication of why I think violations of countable additivity are acceptable in this case. The sense in which countable additivity is violated in Halvorson's theory is that the probability of a countable Boolean disjunction can be 1 even though the probability of each of the disjuncts is 0. Normally countable additivity violations imply that there exists a countable Dutch book. However, that is not so in this case. The reason for this is that in this case truth need not "distribute over countable Boolean disjunction," that is, one can have it that each of countably many disjuncts is false, while the countable disjunction is true (which is not normally the case).

exists a large algebra of operators such that the expectation value of every Hermitian operator in this algebra is finite for every state, and that every operator in this algebra is everywhere defined. In the case of the harmonic oscillator the relevant operator algebra consists of all finite polynomials in the position and momentum operators. So in space Φ one does not have the problems that one has when one has unbounded operators in a Hilbert space, namely "infinite expectation values" and operators that do not have the entire Hilbert space as their domain. At the same time, it has to be admitted that Φ, in other ways, is not as natural as the standard Hilbert space H: Φ makes essential the use of two different topologies, and it does not contain all countable superpositions that have norm equal to one. So, as yet, it is not obvious which state-space is the better candidate. Now let us shift the discussion from quantum mechanics to quantum field theory.

6. Pointless Space-Time in Quantum Field Theory. In quantum field theory, the fundamental observables from which all other observables are built, are field observables, such as field strengths, rather than particle observables, such as position, or region, occupation. It would then seem that no conclusions about the existence or nonexistence of points in space, or space-time, can be drawn from the existence or nonexistence of point values for continuous observables. In fact, one might think that since the fundamental observables are field-strengths *at points* in space-time, therefore quantum field theory actually *presupposes the existence of points in space-time*. However, this is not so.

In quantum field theory there are no well-defined field operators associated with points in space-time. Rather than field operators defined at points, there are "smeared" field operators associated with weighted regions. Let me explain how this is done in more detail in order to make clear that the procedure whereby such smeared field operators are defined does not presuppose the existence of space-time points.

A quantum field Φ(f) is defined as a linear map from "test functions" f(x) to operators. The test functions are functions on space-time, and the operators are operators on a Hilbert space (or on a rigged Hilbert space). If one takes, for example, a test function f(x) which is 0 everywhere except in some space-time region R, and is 1 everywhere in region R, then the operator Φ(g) corresponds to the average value of the field in region R. However, one does not usually use such a test function since it is not continuous. If one instead uses a function f(x) that varies smoothly, then one obtains a field operator corresponding to the weighted average of the field values, where the weight is given by the value of the function. Each such linear map from test functions f to operators Φ(f) is usually represented as an integral $\Phi(f) = \int\Phi(x)f(x)dx$, where the integration is over all of space-time. One might think that this construction presupposes the existence of points in space-time, since the smeared field operators are defined in terms of integrations of $\Phi(x)$ and $f(x)$, where $\Phi(x)$ and $f(x)$ are supposed to have well-defined values at *points* x in space-time. If that were correct then the existence of points in space-time would be, after all, presupposed in quantum field theory.

In order to disarm this argument I need to explain why one can represent linear maps from test functions to operators as integrations. Let us start by assuming that $\Phi(x)$ and $f(x)$ are ordinary functions from space-time to the real numbers, and let us take for granted that all the functions that we are dealing with are suitably integrable. In that case any function $\Phi(x)$ will indeed induce a linear map from test functions $f(x)$ to the real numbers, via the formula $\Phi(f) = \int\Phi(x)f(x)dx$. However, even then, the formula $\Phi(f) = \int\Phi(x)f(x)dx$ does not generate a one-to-one correspondence between functions $\Phi(x)$, and functionals $\Phi(f)$. There are two reasons for this.

In the first place, there exist linear maps $\Phi(f)$ that are not generated by functions $\Phi(x)$, but instead are generated by sequences of functions $\Phi_n(x)$. For instance, consider a sequence of functions $\delta^5{}_n(x)$ such that, as n increases, the functions $\delta^5{}_n$ become more and more peaked around $x = 5$, while, for each n, satisfying $\int\delta^5{}_n(x)dx = 1$. Then, for any test function $f(x)$ that is continuous at $x = 5$, the integral $\int\delta^5{}_n(x)f(x)dx$ will approach $f(5)$ as n goes to infinity, and thus the sequence of integrals can be said to map $f(x)$ to $f(5)$. This map from $f(x)$ to $f(5)$ is a linear map from functions to numbers that is generated, not by a single function, but by a sequence of functions. The reason why this map cannot be generated by a single function is that there exists no limit function to which the functions $\delta^5{}_n$ converge as n goes to infinity. One can however introduce the notion of a "distribution," and define the "distribution" $\delta(5)$ to be such that $\int\delta(5)f(x)dx = f(5)$ for functions f that are continuous at $x = 5$, while being careful not to use the distribution $\delta(5)$ in contexts other than such an "integration." This will allow us to represent all linear maps from test functions to reals as integrations.

Secondly, note that if $\Phi(x)$ and $\Phi'(x)$ are functions, which differ at most on a set of points of (Lebesque) measure 0, then the map $\Phi(f)$ and $\Phi'(f)$ will be the same. Similarly if test functions $f(x)$ and $f'(x)$ differ at most on a set of points of (Lebesque) measure 0, they will be mapped onto the same number by any Φ. So, rather than taking functions $f(x)$ and $\Phi(x)$ as the objects that we use to construct smeared fields, we should take as our objects equivalence classes of functions $[f(x)]$ and $[\Phi(x)]$ that differ at most on (Lebesque) measure 0. Indeed, we must do so, in order to maintain that $\Phi([f]) = \int[\Phi(x)][f(x)]dx$ generates a one-to-one correspondence between $[\Phi(x)]$ and $\Phi([f])$.

To summarize, one can indeed think of smeared field operators as being generated by *integrations* of two types of underlying quantities. But the underlying quantities are *equivalence classes of functions that differ by at most (Lebesque) measure 0*. Rather than presuppose that space-time contains points, this procedure instead strongly suggests that space-time contains only extended regions, that is, that space-time is pointless, since that is the natural habitat of such equivalence classes of functions.

7. Conclusion. There are well-known conceptual oddities, such as measure theoretic paradoxes and problems of contact, associated with the existence of points in space and space-time. In quantum particle mechanics there are additional reasons to reject states that correspond to point values for continuous observables, including positions. In the first place, such states cannot exist in the standard separable Hilbert space formulation. They can be introduced, but only at the expense of a prima facie less natural formulation of quantum particle mechanics. Moreover, exact value states for one observable imply undefined expectation values for many other observables. Indeed, it seems hard to make sense of the probabilities of the results of measurements of perfectly ordinary observables when one starts out, for example, in a position eigenstate.

There exist (at least) two fairly natural quantum particle state-spaces that avoid such problems: the standard (separable) Hilbert space H, and the "nuclear" space Φ. Whichever of those two options one prefers, the spaces consisting of all possible values of continuous observables, including positions, are then pointless spaces. Furthermore, quantum field theory supplies an independent argument that space, and space-time, are pointless. For in quantum field theory there are no operators defined at points in space-time. There are only smeared operators, and these "live" in a pointless space-time.

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