

Stability of scalar delay differential equations with dominant delayed terms

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This paper deals with scalar delay differential equations with dominant delayed terms. Sufficient conditions are obtained for uniform stability, uniformly asymptotic stability and globally asymptotic stability of the equations. The criteria extend and improve some existing ones. The main results are applied to two physiological models. Some counterexamples are also given to show the invalidity of some existing results.

1. Introduction

For $\tau > 0$, let C^τ be the space of continuous functions on $[-\tau, 0]$. Define the norm $\|\cdot\|$ in C^τ by

$$\|\phi\| = \sup_{s \in [-\tau, 0]} |\phi(s)|$$

for $\phi \in C^\tau$. For $H > 0$, let

$$C^\tau(H) = \{\phi \in C^\tau : \|\phi\| < H\}.$$

Consider the scalar delay differential equation

$$x'(t) = -\lambda x(t) + F(t, x_t), \quad t \geq 0, \quad (1.1)$$

where $\lambda \in \mathbf{R}$, $F : [0, \infty) \times C^\tau(H) \rightarrow \mathbf{R}$ is continuous, $F(t, 0) \equiv 0$ and $x'(t)$ denotes the left-hand derivative of $x(t)$. Our main concern is the uniform stability of the trivial solution of (1.1).

Equation (1.1) includes many model equations (directly or after some transformation) arising from various fields, among which are the model for the survival of red blood cells in an animal,

$$N'(t) = -\lambda N(t) + pe^{-rN(t-\tau)}, \quad t \geq 0, \quad (1.2)$$

which has been studied in [3, 8, 9, 11], the models of hematopoiesis (blood cell production) proposed by Mackey and Glass [10],

$$N'(t) = -rN(t) + \frac{\beta\theta^n}{\theta^n + N^n(t - \tau)}, \quad t \geq 0, \tag{1.3}$$

$$N'(t) = -rN(t) + \frac{\beta\theta^n N(t - \tau)}{\theta^n + N^n(t - \tau)}, \quad t \geq 0, \tag{1.4}$$

the model of Nicholson’s blowflies proposed by Gurney *et al.* [5],

$$N'(t) = -\delta N(t) + pN(t - \tau)e^{-aN(t-\tau)}, \quad t \geq 0, \tag{1.5}$$

and the model of hematopoiesis proposed by Gopalsamy and Weng [4]

$$N'(t) = -rN(t) + \alpha \int_0^\infty \frac{k(s)}{1 + N^n(t - s)} ds, \quad t \geq 0. \tag{1.6}$$

Note that (1.2)–(1.5) are all autonomous, and thus the local stability of an equilibrium for each of these equations is determined by the stability of its linearization of the equation at the equilibrium, which is of the form

$$x'(t) = -\lambda x(t) - \alpha x(t - \tau), \quad t \geq 0. \tag{1.7}$$

Equation (1.2) has been studied from early times in the development of stability theory of delay differential equations. From the theory of characteristic equations it is known (see, for example, [6, 7]) that the zero solution of (1.2) is uniformly stable if and only if λ and α satisfy one of the following conditions:

- (C₁) $\lambda \geq |\alpha|$;
- (C₂) $\lambda = \alpha \sin \eta, 0 \leq \alpha\tau \leq (\eta + \frac{1}{2}\pi) / \cos \eta, -\frac{1}{2}\pi < \eta < \frac{1}{2}\pi$;
- (C₃) $-\lambda = \alpha, 0 \leq \alpha\tau < 1$.

Note that (C₂) implies that $\alpha > |\lambda|$, which means the delayed term dominates the instantaneous term.

When it comes to a non-autonomous equation, the theory of characteristic equations is not applicable. In such a case, Lyapunov’s method is most frequently employed to attack the stability problem. However, constructing a working Lyapunov function/functional is never an easy job, and the resulting conditions on a given equation for its stability heavily depend on the skills (and luck) an author has in constructing the Lyapunov function/functional.

In the case $\lambda = 0$ in (1.1), Yorke [17] developed a method, which is different from Lyapunov’s, to show that if

(H₁) there exists a constant $\alpha > 0$ such that

$$-\alpha M(\phi) \leq F(t, \phi) \leq \alpha M(-\phi) \quad \text{for } t \geq 0, \quad \phi \in C^\tau(H),$$

where $M(\phi) = \max\{0, \sup_{s \in [-\tau, 0]} \phi(s)\}$; and

(H₂) $\alpha\tau \leq \frac{3}{2}$,

then the zero solution of the equation

$$x'(t) = F(t, x_t), \quad t \geq 0, \tag{1.8}$$

is uniformly stable. For related results, see [1, 2, 6, 12]. Later, Yoneyama [14] and Yoneyama and Sugie [16] extended (H₁) and (H₂) to more general conditions:

(A₁) there exists a continuous function $a : [0, \infty) \rightarrow [0, \infty)$ such that

$$-a(t)M(\phi) \leq F(t, \phi) \leq a(t)M(-\phi) \quad \text{for } t \geq 0, \quad \phi \in C^\tau(H),$$

where $M(\phi)$ is the same as in (H₁); and

$$(A_2) \int_{t-\tau}^t a(s) ds \leq \frac{3}{2}, \quad t \geq \tau.$$

In a recent paper, Yu [18] made an attempt to extend Yoneyama's result to (1.1) with $\lambda \geq 0$, and showed that if $\lambda \geq 0$ and (A₁) holds, and

$$\int_{t-\tau}^t a(s)e^{\lambda(s-t)} ds \leq 1 + \frac{1}{2}e^{-\lambda\tau}, \quad t \geq \tau, \tag{1.9}$$

then the zero solution of (1.1) is uniformly stable.

This paper is a continuation of the aforementioned work. In §2, we will improve (1.9) to

$$\int_{t-\tau}^t a(s)e^{\lambda(s-t)} ds \leq 1 + \frac{1}{2} \left(1 + \frac{\lambda^2}{\alpha^2} \right) e^{-\lambda\tau}, \quad t \geq \tau, \tag{1.10}$$

where

$$a(t) \leq \alpha \quad \text{for all } t \geq 0, \tag{1.11}$$

and α is allowed to be ∞ . Section 3 is dedicated to the case $\lambda < 0$, and there we will establish some convenient criteria for uniform stability of the trivial solution of (1.1) in this case. In the last section, we will give two counterexamples to some of the main theorems in [15], which deals with the stability of the zero solution of an equation related to, but more general than, equation (1.1).

2. Case $\lambda > 0$

LEMMA 2.1. *Assume that (A₁) holds and $\lambda > 0$. For some $t_1 \geq 0$, let $x(t)$ be a solution (1.1) on $[t_1 - 2\tau, t_1]$ such that $|x(t)| > 0$ for all $t \in (t_1 - \tau, t_1]$. Then*

$$x(t_1)x'(t_1) < 0.$$

Proof. We may assume, without loss of generality, that $x(t) > 0$ for all $t \in (t_1 - \tau, t_1]$. Then, by (1.1) and (A₁), we have

$$\begin{aligned} x(t_1)x'(t_1) &= -\lambda x^2(t_1) + x(t_1)F(t_1, x_{t_1}) \\ &\leq -\lambda x^2(t_1) + a(t_1)x(t_1) \max \left\{ 0, \sup_{s \in [t_1 - \tau, t_1]} (-x(s)) \right\} \\ &\leq -\lambda x^2(t_1) < 0. \end{aligned}$$

The proof is complete. □

LEMMA 2.2. Assume that (1.11) and (A_1) hold, $0 < \lambda < \alpha$, and that

$$\int_{t-\tau}^t a(s)e^{\lambda(s-t)} ds \leq D \tag{2.1}$$

and

$$\theta \equiv \frac{D - \frac{1}{2}(1 - \lambda/\alpha)^2 e^{-\lambda\tau}}{1 + (\lambda/\alpha)e^{-\lambda\tau}} \leq 1. \tag{2.2}$$

Let $x(t)$ be a solution (1.1) on $[t_1 - 2\tau, T]$ for some $T > t_1 \geq 0$ such that

$$x(t_1) = 0 \quad \text{and} \quad x(t) > 0 \quad \text{for all } t \in (t_1, T]. \tag{2.3}$$

Then

$$|x(t)| \leq \theta \sup_{s \in [t_1 - 2\tau, t_1]} |x(s)| \quad \text{for all } t \in (t_1, T]. \tag{2.4}$$

Proof. We may assume, without loss of generality that $D \geq 1$. Let

$$r = \sup_{s \in [t_1 - 2\tau, t_1]} |x(s)|.$$

In view of (2.3), it suffices to show that, for each $\epsilon > 0$,

$$x(t) < \theta(r + \epsilon) \quad \text{for all } t \in (t_1, T]. \tag{2.5}$$

Suppose that there exist $\epsilon > 0$ and $t_3 \in (t_1, T]$ such that

$$x(t_3) = \theta(r + \epsilon) \quad \text{and} \quad x(t) < \theta(r + \epsilon) \quad \text{for all } t \in (t_1, t_3). \tag{2.6}$$

Then, from (1.1) (1.11), (A_1) and (2.6), we have

$$0 \leq x'(t_3) = -\lambda x(t_3) + F(t_3, x_{t_3}) \leq -\lambda\theta(r + \epsilon) + \alpha \max \left\{ 0, \sup_{s \in [-\tau, 0]} (-x(t_3 + s)) \right\}.$$

It follows that $t_3 < t_1 + \tau$ and there exists $t_2 \in [t_3 - \tau, t_1)$ such that

$$x(t_2) = -\frac{\lambda}{\alpha}\theta(r + \epsilon). \tag{2.7}$$

From (1.1), (A_1) and (2.6), we have

$$(x(t)e^{\lambda t})' = e^{\lambda t} F(t, x_t) \leq r a(t)e^{\lambda t}, \quad t_1 - \tau \leq t \leq t_1, \tag{2.8}$$

and

$$(x(t)e^{\lambda t})' = e^{\lambda t} F(t, x_t) \leq (r + \epsilon)a(t)e^{\lambda t}, \quad t_1 \leq t \leq t_3. \tag{2.9}$$

For $t \in [t_1, t_3]$ and $s \in [-\tau, t_1 - t]$, integrating (2.8) from $t + s$ to t_1 ,

$$-x(t + s) \leq r \int_{t+s}^{t_1} a(u)e^{\lambda(u-t-s)} du \leq r e^{\lambda\tau} \int_{t-\tau}^{t_1} a(u)e^{\lambda(u-t)} du.$$

Substituting this into (1.1) and using (A₁) and (2.3), we have

$$\begin{aligned} (x(t)e^{\lambda t})' &= e^{\lambda t} a(t) \max \left\{ 0, \sup_{s \in [-\tau, t_1 - t]} (-x(t + s)) \right\} \\ &\leq r e^{\lambda \tau} a(t) \int_{t-\tau}^{t_1} a(s) e^{\lambda s} ds, \quad t_1 \leq t \leq t_3. \end{aligned} \tag{2.10}$$

There are now two possible cases to consider.

CASE 1. $\int_{t_1}^{t_3} a(s) e^{\lambda(s-t_3)} ds \leq (1 - \lambda/\alpha) e^{-\lambda \tau}$.

In this case, by (2.1) and (2.10),

$$\begin{aligned} x(t_3) &\leq r e^{\lambda \tau} \int_{t_1}^{t_3} a(t) e^{\lambda(t-t_3)} \int_{t-\tau}^{t_1} a(s) e^{\lambda(s-t)} ds dt \\ &\leq r e^{\lambda \tau} \int_{t_1}^{t_3} a(t) e^{\lambda(t-t_3)} \left(\int_{t-\tau}^t a(s) e^{\lambda(s-t)} ds - \int_{t_1}^t a(s) e^{\lambda(s-t_3)} ds \right) dt \\ &\leq r e^{\lambda \tau} \left[D \int_{t_1}^{t_3} a(s) e^{\lambda(s-t_3)} ds - \frac{1}{2} \left(\int_{t_1}^{t_3} a(s) e^{\lambda(s-t_3)} ds \right)^2 \right] \\ &\leq r e^{\lambda \tau} \left[D \left(1 - \frac{\lambda}{\alpha} \right) e^{-\lambda \tau} - \frac{1}{2} \left(1 - \frac{\lambda}{\alpha} \right)^2 e^{-2\lambda \tau} \right] \\ &= r \left[D - \frac{1}{2} \left(1 - \frac{\lambda}{\alpha} \right)^2 e^{-\lambda \tau} - \frac{\lambda}{\alpha} D \right] \\ &= r \left[\theta \left(1 + \frac{\lambda}{\alpha} e^{-\lambda \tau} \right) - \frac{\lambda}{\alpha} D \right] \\ &< \theta(r + \epsilon). \end{aligned}$$

CASE 2. $\int_{t_1}^{t_3} a(s) e^{\lambda(s-t_3)} ds > (1 - \lambda/\alpha) e^{-\lambda \tau}$.

In this case, there exists $\eta \in (t_1, t_3)$ such that

$$\int_{\eta}^{t_3} a(s) e^{\lambda(s-t_3)} ds = \left(1 - \frac{\lambda}{\alpha} \right) e^{-\lambda \tau}.$$

If

$$\int_{t_2}^{t_1} a(s) e^{\lambda(s-t_3)} ds \leq \frac{\lambda}{\alpha} e^{-\lambda \tau},$$

then, by (2.1), (2.8), (2.9) and (2.10),

$$\begin{aligned} x(t_3) &\leq x(t_2) e^{\lambda(t_2-t_3)} + r \int_{t_2}^{t_1} a(s) e^{\lambda(s-t_3)} ds + (r + \epsilon) \int_{t_1}^{\eta} a(s) e^{\lambda(s-t_3)} ds \\ &\quad + r e^{\lambda \tau} \int_{\eta}^{t_3} a(t) e^{\lambda(t-t_3)} \int_{t-\tau}^{t_1} a(s) e^{\lambda(s-t)} ds dt \end{aligned}$$

$$\begin{aligned}
 &< (r + \epsilon) \left[-\frac{\lambda}{\alpha} \theta e^{-\lambda\tau} + \int_{t_2}^{t_1} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\quad + \left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds \\
 &\quad \left. + e^{\lambda\tau} \int_{\eta}^{t_3} a(t)e^{\lambda(t-t_3)} \int_{t-\tau}^{\eta} a(s)e^{\lambda(s-t)} ds dt \right] \\
 &\leq (r + \epsilon) \left\{ -\frac{\lambda}{\alpha} \theta e^{-\lambda\tau} + \int_{t_2}^{t_1} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\quad + \left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds \\
 &\quad \left. + e^{\lambda\tau} \left[D \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds - \frac{1}{2} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \right\} \\
 &\leq (r + \epsilon) \left\{ -\frac{\lambda}{\alpha} \theta e^{-\lambda\tau} + e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \int_{t_2}^{t_1} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\quad + \left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \left(D - \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \\
 &\quad \left. + e^{\lambda\tau} \left[D \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds - \frac{1}{2} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \right\} \\
 &\leq (r + \epsilon) \left[-\frac{\lambda}{\alpha} \theta e^{-\lambda\tau} + D - \left(1 - \frac{\lambda}{\alpha} \right) \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\quad \left. + \frac{1}{2} e^{\lambda\tau} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \\
 &= (r + \epsilon) \left[D - \frac{1}{2} \left(1 - \frac{\lambda}{\alpha} \right)^2 e^{-\lambda\tau} \right] - \frac{\lambda}{\alpha} e^{-\lambda\tau} x(t_3).
 \end{aligned}$$

It follows that $x(t_3) < \theta(r + \epsilon)$.

If

$$\int_{t_2}^{t_1} a(s)e^{\lambda(s-t_3)} ds \geq \frac{\lambda}{\alpha} e^{-\lambda\tau},$$

then

$$\int_{t_1}^{t_3} a(s)e^{\lambda(s-t_3)} ds \leq D - \frac{\lambda}{\alpha} e^{-\lambda\tau}.$$

Therefore, by (2.1), (2.9) and (2.10),

$$\begin{aligned}
 x(t_3) &\leq (r + \epsilon) \int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds + re^{\lambda\tau} \int_{\eta}^{t_3} a(t)e^{\lambda(t-t_3)} \int_{t-\tau}^{t_1} a(s)e^{\lambda(s-t)} ds dt \\
 &< (r + \epsilon) \left[\int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds + e^{\lambda\tau} \int_{\eta}^{t_3} a(t)e^{\lambda(t-t_3)} \int_{t-\tau}^{t_1} a(s)e^{\lambda(s-t)} ds dt \right]
 \end{aligned}$$

$$\begin{aligned}
 &\leq (r + \epsilon) \left[\left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\qquad \qquad \qquad \left. + e^{\lambda\tau} \int_{\eta}^{t_3} a(t)e^{\lambda(t-t_3)} \int_{t-\tau}^{\eta} a(s)e^{\lambda(s-t)} ds dt \right] \\
 &\leq (r + \epsilon) \left\{ \left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \int_{t_1}^{\eta} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\qquad \qquad \qquad \left. + e^{\lambda\tau} \left[D \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds - \frac{1}{2} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \right\} \\
 &\leq (r + \epsilon) \left\{ \left(1 - e^{\lambda\tau} \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \right. \\
 &\qquad \qquad \times \left(D - \frac{\lambda}{\alpha} e^{-\lambda\tau} - \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right) \\
 &\qquad \qquad \left. + e^{\lambda\tau} \left[D \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds - \frac{1}{2} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \right\} \\
 &\leq (r + \epsilon) \left[-\frac{\lambda}{\alpha} e^{-\lambda\tau} + D - \left(1 - \frac{\lambda}{\alpha} \right) \int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right. \\
 &\qquad \qquad \qquad \left. + \frac{1}{2} e^{\lambda\tau} \left(\int_{\eta}^{t_3} a(s)e^{\lambda(s-t_3)} ds \right)^2 \right] \\
 &= (r + \epsilon) \left[D - \frac{1}{2} \left(1 + \frac{\lambda^2}{\alpha^2} \right) e^{-\lambda\tau} \right] \\
 &\leq \theta(r + \epsilon).
 \end{aligned}$$

Cases 1 and 2 imply that we always have $x(t_3) < \theta(r + \epsilon)$, which contradicts (2.6). Thus (2.4) is true and the proof is complete. □

In a similar way, we can show the following lemma.

LEMMA 2.3. *Assume that (1.11), (A₁), (2.1) and (2.2) hold, $0 < \lambda < \alpha$, and let $x(t)$ be a solution (1.1) on $[t_1 - 2\tau, T]$ for some $T > t_1 \geq 0$ such that*

$$x(t_1) = 0 \quad \text{and} \quad x(t) < 0 \quad \text{for all } t \in (t_1, T]. \tag{2.11}$$

Then (2.4) holds.

From lemmas 2.2 and 2.3, we immediately have the following result.

COROLLARY 2.4. *Assume that (1.11), (A₁), (2.1) and (2.2) hold, $0 < \lambda < \alpha$, and let $x(t)$ be a solution (1.1) on $[t_1 - 2\tau, T]$ for some $T > t_1 \geq 0$ such that $x(t_1) = 0$. Then (2.4) holds.*

We now state a theorem on the uniform stability of the zero solution of (1.1).

THEOREM 2.5. *Assume that (1.11) and (A₁) hold, $\lambda > 0$, and that*

$$\int_{t-\tau}^t a(s)e^{\lambda(s-t)} ds \leq 1 + \frac{1}{2} \left(1 + \frac{\lambda^2}{\alpha^2} \right) e^{-\lambda\tau}, \quad t \geq \tau. \tag{2.12}$$

Then the zero solution of (1.1) is uniformly stable.

Proof. Let $\epsilon \in (0, H)$ be given, and let $\delta = \frac{1}{2}\epsilon \exp(-2 - 2e^{\lambda\tau})$. For $t_0 \geq 0$ and $\phi \in C^\tau(\delta)$, we shall prove that

$$|x(t)| = |x(t; t_0, \phi)| < \epsilon \quad \text{for } t \geq t_0. \tag{2.13}$$

If $\lambda \geq \alpha$, then (1.9) holds and the conclusion follows from Yu [18, theorem 1]. In the sequel, we only consider the case where $0 < \lambda < \alpha$. Let $D = 1 + \frac{1}{2}(1 + \lambda^2/\alpha^2)e^{-\lambda\tau}$. Then, by (2.2),

$$\theta \equiv \frac{D - \frac{1}{2}(1 - \lambda/\alpha)^2 e^{-\lambda\tau}}{1 + (\lambda/\alpha)e^{-\lambda\tau}} = 1.$$

From (1.1) and (A₁),

$$x(t)x'(t) = -\lambda x^2(t) + x(t)F(t, x_t) \leq a(t)\|x_t\|^2, \quad t \geq t_0,$$

which yields

$$\|x_t\|^2 \leq \|\phi\|^2 + 2 \int_{t_0}^t a(s)\|x_s\|^2 ds, \quad t \geq t_0.$$

In view of Gronwall’s inequality, we have

$$\|x_t\| \leq \|\phi\| \exp\left(\int_{t_0}^t a(s) ds\right), \quad t \geq t_0. \tag{2.14}$$

Thus, for $t_0 \leq t \leq t_0 + 2\tau$, we have

$$\begin{aligned} |x(t)| &\leq \max\{\|x_{t_0+\tau}\|, \|x_{t_0+2\tau}\|\} \\ &\leq \|\phi\| \exp\left(\int_{t_0}^{t_0+2\tau} a(s) ds\right) \\ &\leq \delta \exp(2 + 2e^{\lambda\tau}) \\ &< \epsilon. \end{aligned}$$

Next we prove that

$$|x(t)| < \epsilon \quad \text{for } t \geq t_0 + 2\tau. \tag{2.15}$$

Assume that (2.15) is not true. Then there must be some $T > t_0 + 2\tau$ such that $|x(T)| = \epsilon$ and $|x(t)| < \epsilon$ for $t_0 \leq t < T$. Then $x(T)x'(T) \geq 0$. Thus, by lemma 2.1, there exists $t_1 \in [T - \tau, T)$ such that $x(t_1) = 0$. Then it follows from corollary 2.4 that

$$|x(t)| \leq \theta \sup_{s \in [t_1 - 2\tau, t_1]} |x(s)| < \epsilon \quad \text{for } t \in (t_1, T],$$

which contradicts $|x(T)| = \epsilon$. The proof is now complete. □

THEOREM 2.6. *Assume that (1.11) and (A₁) hold, $\lambda > 0$, and that*

$$\int_{t-\tau}^t a(s)e^{\lambda(s-t)} ds \leq D < 1 + \frac{1}{2}\left(1 + \frac{\lambda^2}{\alpha^2}\right)e^{-\lambda\tau}, \quad t \geq \tau. \tag{2.16}$$

Then the zero solution of (1.1) is uniformly asymptotically stable.

Proof. In view of theorem 2.5, the zero solution of (1.1) is uniformly stable and so, for any $t_0 \geq 0$, there exists $\delta > 0$, which is independent of t_0 , such that $\phi \in C^\tau(\delta)$ implies

$$|x(t)| = |x(t; t_0, \phi)| < \frac{1}{2}H, \quad t \geq t_0. \tag{2.17}$$

Next we prove that

$$\lim_{t \rightarrow \infty} x(t) = 0. \tag{2.18}$$

If $\lambda \geq \alpha$, then the inequality in (1.9) becomes strict and the conclusion follows from [18, theorem 2]. In the sequel, we only consider the case where $0 < \lambda < \alpha$. Set

$$\theta \equiv \frac{D - \frac{1}{2}(1 - \lambda/\alpha)^2 e^{-\lambda\tau}}{1 + (\lambda/\alpha)e^{-\lambda\tau}}.$$

Then $\theta < 1$. If $x(t)$ is non-oscillatory, then, by (1.1) and (A₁), we eventually have

$$\frac{d}{dt}|x(t)| + \lambda|x(t)| \leq 0,$$

which implies that (2.18) holds. If $x(t)$ is oscillatory. Set $v = \limsup_{t \rightarrow \infty} |x(t)|$. Then $0 \leq v \leq \frac{1}{2}H$. It suffices to show that $v = 0$. For any $0 < \epsilon < \frac{1}{2}H$, there exists $T > t_0 + 2\tau$ such that

$$|x(t - \tau)| < v + \epsilon \quad \text{for } t \geq T. \tag{2.19}$$

Let $\{l_n\}$ be an increasing sequence such that $l_n \rightarrow \infty$ as $n \rightarrow \infty$, $l_n > T$, $x(l_n)\dot{x}(l_n) \geq 0$ and $\lim_{n \rightarrow \infty} |x(l_n)| = v$. By lemma 2.1, there exists $\xi_n \in [l_n - \tau, l_n]$ such that $x(\xi_n) = 0$. In view of corollary 2.4, we have

$$|x(l_n)| \leq \theta \sup_{s \in [\xi_n - 2\tau, \xi_n]} |x(s)| < \theta(v + \epsilon), \quad n = 1, 2, \dots$$

Letting $n \rightarrow \infty$ and $\epsilon \rightarrow 0$, we have $v \leq \theta v$. Note that $\theta < 1$. It follows that $v = 0$. The proof is complete. □

REMARK 2.7. From the proof of theorem 2.6, we can see (by replacing $\frac{1}{2}H$ with M in the proof, where $|x(t)| < M$ for $t \geq t_0$) that, under the conditions of this theorem, any bounded solution $x(t)$ of (1.1) actually converges to 0 as $t \rightarrow \infty$.

REMARK 2.8. When equation (1.1) is in the following simpler form,

$$x'(t) = -\lambda x(t) - a(t)f(x(t - \tau)), \quad t \geq 0, \tag{2.20}$$

it is easily seen that (A₁) is implied by

$$xf(x) > 0 \quad \text{and} \quad |f(x)| \leq |x| \quad \text{for } -M_1 < x < M_2, \tag{2.21}$$

where M_1 and M_2 can be ∞ .

To conclude this section, we apply the main results in this section to (1.2) and (1.3) to obtain globally asymptotic stability of the positive equilibrium. Due to the physiological or biological backgrounds of the model equations (1.2) and (1.3), we assume that the initial conditions for (1.2) and (1.3) are of the type

$$N(s) = \phi(s), \quad s \in [-\tau, 0], \quad \phi(0) > 0, \quad \phi \in C([-\tau, 0], [0, \infty)). \tag{2.22}$$

THEOREM 2.9. Assume that $\lambda, p, r, \tau > 0$ and

$$\frac{rp}{\lambda}(1 - e^{-\lambda\tau}) < 1 + \frac{1}{2}\left(1 + \frac{\lambda^2}{r^2p^2}\right)e^{-\lambda\tau}. \tag{2.23}$$

Then every solution of (1.2) and (2.22) tends to the positive equilibrium N^* of (1.2) as $t \rightarrow \infty$.

Proof. It is easy to show that the solution $N(t)$ of (1.2) and (2.22) exists on $[0, \infty)$ and satisfies $0 < N(t) < M$ for some $M > 0$ for all $t \geq 0$. Let $x(t) = N(t) - N^*$. Then $-N^* < x(t) < M - N^*$ and (1.2) is transformed to (2.20), with $a(t) = rp$ and

$$f(x) = \frac{1}{rp}\lambda N^*(1 - e^{-rx}).$$

By a simple calculation, we can verify that

$$xf(x) > 0 \quad \text{and} \quad |f(x)| \leq |x| \quad \text{for} \quad -N^* < x < \infty.$$

The conclusion follows from remarks 2.7 and 2.8, and the proof is complete. □

For (1.3), we have the following result.

THEOREM 2.10. Assume that $\beta, \theta, r, \tau > 0, n > 1$ and

$$\begin{aligned} &\frac{1}{4rn\theta}\beta(n+1)^{(n+1)/n}(n-1)^{(n-1)/n}(1 - e^{-r\tau}) \\ &< 1 + \frac{1}{2}\left[1 + \frac{16(rn\theta)^2}{\beta^2}(n+1)^{-2(n+1)/n}(n-1)^{-2(n-1)/n}\right]e^{-r\tau}. \end{aligned} \tag{2.24}$$

Then every solution $N(t)$ of (1.3) and (2.22) tends to the positive equilibrium N^* of (1.3) as $t \rightarrow \infty$.

Proof. It is easily seen that a solution $N(t)$ of (1.3) and (2.22) exists on $[0, \infty)$ and satisfies $0 < N(t) < M$ for some constant $M > 0$, and all $t \geq 0$. Let $x(t) = N(t) - N^*$, and

$$f(x) = \frac{4n\theta}{\beta}(n+1)^{-(n+1)/n}(n-1)^{-(n-1)/n}\left[rN^* - \frac{\beta\theta^n}{\theta^n + (x + N^*)^n}\right].$$

Then $x(t)$ satisfies $-N^* < x(t) < M - N^*$ and the following equation:

$$x'(t) = -rx(t) - \frac{1}{4n\theta}\beta(n+1)^{(n+1)/n}(n-1)^{(n-1)/n}f(x(t - \tau)), \quad t \geq 0. \tag{2.25}$$

Note that

$$f'(x) = \frac{4n^2\theta^{n+1}}{[\theta^n + (x + N^*)^n]^2}(x + N^*)^{n-1}(n+1)^{-(n+1)/n}(n-1)^{-(n-1)/n}$$

and

$$f''(x) = \frac{n^2\theta^{n+1}[(n-1)\theta^n - (n+1)(x + N^*)^n](x + N^*)^{n-2}}{[\theta^n + (x + N^*)^n]^3(n+1)^{(n+1)/n}(n-1)^{(n-1)/n}}.$$

Since $n > 1$, $f(x)$ has a unique inflection point at $x_0 = ((n - 1)/(n + 1))^{1/n}\theta - N^*$. It follows that

$$xf(x) > 0 \quad \text{and} \quad |f(x)| \leq |x| \quad \text{for} \quad -N^* < x < M - N^*.$$

Hence the conclusion follows from remarks 2.7 and 2.8, and the proof is complete. □

REMARK 2.11. Kuang [9] also considered the globally asymptotic stability of (1.3) and obtained (see [9, corollary 8.2, p. 156]) the following criterion:

$$\frac{\beta\tau}{4n\theta}(n + 1)^{(n+1)/n}(n - 1)^{(n-1)/n} < 1 - r\tau. \tag{2.26}$$

Noting that

$$\frac{1 - e^{-r\tau}}{r} < \tau,$$

we see that (2.24) improves (2.26).

REMARK 2.12. It is interesting to compare, in the case $\lambda < \alpha$, condition (2.12) in theorem 2.5 with condition (C₂) for the linear autonomous equation (1.7) (and hence for the local stability of the corresponding nonlinear equation). In this case, $a(t) = \alpha$ and $c := \lambda/\alpha < 1$. Then (2.12) reduces to

$$\frac{1}{c}(1 - e^{-\lambda\tau}) \leq 1 + \frac{1}{2}(1 + c^2)e^{-\lambda\tau}, \tag{2.27}$$

which is equivalent to

$$\begin{aligned} \alpha\tau &\leq \frac{1}{c} \ln \frac{2 + c + c^3}{2(1 - c)} \\ &= \frac{1}{c} [\ln(1 + \frac{1}{2}(c + c^3)) - \ln(1 - c)] \\ &= \frac{1}{c} [(\frac{1}{2}(c + c^3) - \frac{1}{4}((c + c^3)^2) + \frac{1}{8}((c + c^3)^3) - \dots) \\ &\qquad\qquad\qquad + (c + \frac{1}{2}c^2 + \frac{1}{3}c^3 + \frac{1}{4}c^4 + \dots)] \\ &= \frac{3}{2} + \frac{3}{8}c + \frac{7}{8}c^2 - \frac{1}{64}c^3 + \dots. \end{aligned} \tag{2.28}$$

On the other hand, condition (C₂) can be rewritten as

$$\begin{aligned} \alpha\tau &\leq (\arcsin c + \frac{1}{2}\pi)(1 - c^2)^{-1/2} \\ &= (\frac{1}{2}\pi + c + \frac{1}{6}c^3 + \dots)(1 + \frac{1}{2}c^2 + \frac{3}{8}c^4 + \dots) \\ &= \frac{1}{2}\pi + c + \frac{1}{4}\pi c^2 + \frac{2}{3}c^3 + \frac{3}{16}\pi c^4 + \dots. \end{aligned} \tag{2.29}$$

When $c = 0$ (corresponding to $\lambda = 0$), equation (2.28) reduces to the well-known criterion $\alpha\tau \leq \frac{3}{2}$ for the *globally* asymptotic stability of the positive equilibrium of the delayed logistic equation

$$x'(t) = \alpha x(t)[1 - x(t - \tau)], \tag{2.30}$$

while (2.29) reduces to $\alpha\tau \leq \frac{1}{2}\pi$, which is exactly the necessary and sufficient condition for the *local* stability of the positive equilibrium of (2.30). We point out that

the range of $\alpha\tau$ for global stability of (2.30) can be further extended beyond $\frac{3}{2}$, say, at least to $\frac{37}{24}$ (see, for example, [13] or [9, p. 125]). Note that $\frac{37}{24} = 1.541\dots$ is very close to $\frac{1}{2}\pi = 1.571\dots$. Whether or not the range can be extended exactly to $\frac{1}{2}\pi$ has been a long-lasting open problem.

3. The case $\lambda < 0$

In this section, we consider the case $\lambda < 0$. We further assume that

$$|\lambda| \leq a(t) \leq \alpha \quad \text{for all } t \geq 0 \tag{3.1}$$

and

$$\lambda < 0 \quad \text{and} \quad -\lambda\tau < 1. \tag{3.2}$$

See [15] for a justification for conditions (3.1) and (3.2). Moreover, as a replacement of (A_1) , we need the following condition.

(A_1^*) There exists a continuous function $a : [0, \infty) \rightarrow [0, \infty)$ such that

$$-a(t) \sup_{s \in [-\tau, 0]} \phi(s) \leq F(t, \phi) \leq a(t) \sup_{s \in [-\tau, 0]} (-\phi(s)) \quad \text{for } t \geq 0, \quad \phi \in C^\tau(H).$$

For $\lambda < 0$ and $\alpha > 0$, equation (1.2) is an example of (1.1), and if either $-\lambda \geq \alpha$ or $-\lambda\tau > 1$, then the zero solution of (1.1) is unstable. For a continuous function $w : [0, \infty) \rightarrow \mathbf{R}$, we let $w_+(t) = \max\{0, w(t)\}$ and $w_-(t) = \max\{0, -w(t)\}$. The first three lemmas are taken from [15, lemmas 4.1–4.3].

LEMMA 3.1. Assume that (A_1^*) , (3.1) and (3.2) hold. For some $t_1 \geq 0$ and $T > t_1 + \tau$, let $x(t)$ be a solution of (1.1) on $[t_1, T]$ such that $x(t) > 0$ for all $t \in (t_1, T]$. Then

$$\int_{t_1+\tau}^t x'_+(s) \, ds \leq \frac{-\lambda\tau}{1 + \lambda\tau} \int_{t_1}^{t_1+\tau} x'_+(s) \, ds \quad \text{for all } t \in [t_1 + \tau, T]. \tag{3.3}$$

LEMMA 3.2. Assume that (A_1^*) , (3.1) and (3.2) hold. For some $t_1 \geq 0$ and $T > t_1 + \tau$, let $x(t)$ be a solution of (1.1) on $[t_1, T]$ such that $x(t) < 0$ for all $t \in (t_1, T]$. Then

$$\int_{t_1+\tau}^t x'_-(s) \, ds \leq \frac{-\lambda\tau}{1 + \lambda\tau} \int_{t_1}^{t_1+\tau} x'_-(s) \, ds \quad \text{for all } t \in [t_1 + \tau, T]. \tag{3.4}$$

LEMMA 3.3. Assume that (A_1^*) , (3.1) and (3.2) hold. For some $t_1 \geq 0$ and $T > t_1$, let $x(t)$ be a solution of (1.1) on $[t_1 - 2\tau, T]$ such that $|x(t)| > 0$ for all $t \in (t_1 - \tau, T]$. Then

$$|x(t)| \leq \left[1 + \frac{-\lambda\tau^2}{1 + \lambda\tau} (\alpha - \lambda) \right] \sup_{s \in [t_1 - 2\tau, t_1]} |x(s)| \quad \text{for all } t \in [t_1, T]. \tag{3.5}$$

As replacements of [15, lemmas 4.4, 4.5], which may not be correct (see our counterexamples and discussion in §4), we can establish the following lemmas.

LEMMA 3.4. Assume that (A_1^*) , (3.1) and (3.2) hold. For some $t_1 \geq 0$ and $T > t_1 + \tau$, let $x(t)$ be a solution of (1.1) on $[t_1 - 2\tau, T]$ such that $x(t_1) = 0$ and $x(t) > 0$ for all $t \in (t_1, T]$, and let $r = \sup_{s \in [t_1 - 2\tau, t_1]} |x(s)|$. Suppose that there exists $\eta \geq 0$ such that

$$\frac{\alpha}{-\lambda} \left[(e^{-\lambda\tau} - 1 + \lambda\tau) + \frac{\alpha}{-\lambda} (e^{\lambda\tau} - 1 - \lambda\tau) \right] \leq 1 - \eta + \lambda\tau \tag{3.6}$$

or

$$\frac{\alpha + \lambda}{-\lambda} (e^{-\lambda\tau} - 1) > 1 \quad \text{and} \quad \frac{\alpha}{-\lambda} \left(e^{-\lambda\tau} + \frac{\alpha + \lambda}{-\lambda} \ln \frac{\alpha}{\alpha + \lambda} - 2 \right) \leq 1 - \eta + \lambda\tau. \tag{3.7}$$

Then

$$x(t) \leq \int_{t_1}^t x'_+(s) ds \leq (1 - \eta + \lambda\tau)r \quad \text{for all } t \in [t_1, t_1 + \tau]. \tag{3.8}$$

Proof. Suppose that there exists $t_4 \in (t_1, t_1 + \tau]$ such that $x(t_4) > (1 - \eta + \lambda\tau)r$. Then we can choose $t_3 < t_4$ so that $x(t) < (1 - \eta + \lambda\tau)r$ for all $t \in (t_1, t_3]$ and

$$x(t_3) = (1 - \eta + \lambda\tau)r. \tag{3.9}$$

By (1.1), (3.1) and (A_1^*) ,

$$(x(t)e^{\lambda t})' = e^{\lambda t} F(t, x_t) \leq \alpha r e^{\lambda t}, \quad t_1 - \tau \leq t \leq t_3. \tag{3.10}$$

For $t \in [t_1, t_3]$, first integrating (3.10) from t_1 to t ,

$$x(t) \leq \frac{\alpha r}{-\lambda} (e^{-\lambda(t-t_1)} - 1),$$

and then integrating (3.10) from $t + s$ to t_1 with $s \in [-\tau, 0]$,

$$-x(t + s) \leq \frac{\alpha r}{-\lambda} (1 - e^{-\lambda(t-t_1-\tau)}).$$

Substituting these into (1.1) and using (3.1) and (A_1^*) , we have

$$x'_+(t) \leq \alpha r (e^{-\lambda(t-t_1)} - 1) + \min \left\{ \alpha r, \frac{\alpha^2 r}{-\lambda} (1 - e^{-\lambda(t-t_1-\tau)}) \right\}, \quad t_1 \leq t \leq t_3. \tag{3.11}$$

If (3.6) holds, it then follows from $t_3 < t_1 + \tau$ that

$$\begin{aligned} x(t) &\leq \int_{t_1}^t x'_+(s) ds \\ &\leq \int_{t_1}^{t_3} \left[\alpha r (e^{-\lambda(s-t_1)} - 1) + \frac{\alpha^2 r}{-\lambda} (1 - e^{-\lambda(s-t_1-\tau)}) \right] ds \\ &< \int_{t_1}^{t_1+\tau} \left[\alpha r (e^{-\lambda(s-t_1)} - 1) + \frac{\alpha^2 r}{-\lambda} (1 - e^{-\lambda(s-t_1-\tau)}) \right] ds \\ &= \frac{\alpha r}{-\lambda} \left[(e^{-\lambda\tau} - 1 + \lambda\tau) + \frac{\alpha}{-\lambda} (e^{\lambda\tau} - 1 - \lambda\tau) \right] \\ &\leq (1 - \eta + \lambda\tau)r. \end{aligned}$$

If (3.7) holds, then choose $t_2 \in [t_1, t_1 + \tau)$ so that

$$e^{\lambda(t_1+\tau-t_2)} = \frac{\alpha + \lambda}{\alpha}.$$

Hence, by (3.11), we have

$$\begin{aligned} x(t) &\leq \int_{t_1}^t x'_+(s) \, ds \\ &< \alpha r \int_{t_1}^{t_1+\tau} (e^{-\lambda(s-t_1)} - 1) \, ds + \int_{t_1}^{t_2} \alpha r \, ds + \frac{\alpha^2 r}{-\lambda} \int_{t_2}^{t_1+\tau} (1 - e^{-\lambda(s-t_1-\tau)}) \, ds \\ &= \frac{\alpha r}{-\lambda} \left(e^{-\lambda\tau} + \frac{\alpha + \lambda}{-\lambda} \ln \frac{\alpha}{\alpha + \lambda} - 2 \right) \\ &\leq (1 - \eta + \lambda\tau)r. \end{aligned}$$

In either case, we have a contradiction to (3.9) at $t = t_3$, and at the same we have

$$\int_{t_1}^t x'_+(s) \, ds \leq (1 - \eta + \lambda\tau)r \quad \text{for all } t \in [t_1, t_1 + \tau].$$

The proof is complete. □

LEMMA 3.5. Assume that (A_1^*) , (3.1) and (3.2) hold. For some $t_1 \geq 0$ and $T > t_1 + \tau$, let $x(t)$ be a solution of (1.1) on $[t_1 - 2\tau, T]$ such that $x(t_1) = 0$ and $x(t) < 0$ for all $t \in (t_1, T]$, and let $r = \sup_{[s \in t_1 - 2\tau, t_1]} |x(s)|$. Suppose that there exists $\eta \geq 0$ such that (3.6) or (3.7) holds. Then

$$|x(t)| \leq \int_{t_1}^t x'_-(s) \, ds \leq (1 - \eta + \lambda\tau)r \quad \text{for all } t \in [t_1 + \tau, T].$$

Proof. The proof is similar to that of lemma 3.4, and is omitted here. □

With [15, lemmas 4.4, 4.5] being replaced by the above lemmas 3.4, 3.5, respectively, we now can easily follow the same line in the proofs of [15, theorems 4.1, 4.2], but using lemmas 3.1–3.5 now, to prove the following theorem on the uniform stability of the zero solution of (1.1).

THEOREM 3.6. Assume that (A_1^*) , (3.1) and (3.2) hold, and that

$$\frac{\alpha}{-\lambda} \left[(e^{-\lambda\tau} - 1 + \lambda\tau) + \frac{\alpha}{-\lambda} (e^{\lambda\tau} - 1 - \lambda\tau) \right] \leq 1 + \lambda\tau \tag{3.12}$$

or

$$\frac{\alpha + \lambda}{-\lambda} (e^{-\lambda\tau} - 1) > 1 \quad \text{and} \quad \frac{\alpha}{-\lambda} \left(e^{-\lambda\tau} + \frac{\alpha + \lambda}{-\lambda} \ln \frac{\alpha}{\alpha + \lambda} - 2 \right) \leq 1 + \lambda\tau. \tag{3.13}$$

Then the zero solution of (1.1) is uniformly stable.

REMARK 3.7. When $\lambda \rightarrow 0$, equation (3.13) deduces to $1 < \alpha\tau \leq \frac{3}{2}$.

THEOREM 3.8. Assume that (A_1^*) , (3.1) and (3.2) hold, and that

$$\int_0^\infty (a(s) + \lambda) ds = \infty \tag{3.14}$$

and

$$\frac{\alpha}{-\lambda} \left[(e^{-\lambda\tau} - 1 + \lambda\tau) + \frac{\alpha}{-\lambda} (e^{\lambda\tau} - 1 - \lambda\tau) \right] < 1 + \lambda\tau \tag{3.15}$$

or

$$\frac{\alpha + \lambda}{-\lambda} (e^{-\lambda\tau} - 1) > 1 \quad \text{and} \quad \frac{\alpha}{-\lambda} \left(e^{-\lambda\tau} + \frac{\alpha + \lambda}{-\lambda} \ln \frac{\alpha}{\alpha + \lambda} - 2 \right) < 1 + \lambda\tau. \tag{3.16}$$

Then the zero solution of (1.1) is uniformly asymptotically stable.

Proof. In view of theorem 3.6, the zero solution of (1.1) is uniformly stable and so, for any $t_0 \geq 0$, there exists $\delta > 0$, which is independent of t_0 , such that $\phi \in C^\tau(\delta)$ implies

$$|x(t)| = |x(t; t_0, \phi)| < \frac{1}{2}H, \quad t \geq t_0. \tag{3.17}$$

Next we prove that

$$\lim_{t \rightarrow \infty} x(t) = 0. \tag{3.18}$$

First, assume that $x(t)$ is eventually positive solution. Then there exists $T_1 > 0$ such that $x(t) > 0$ for all $t \geq T_1$. By lemma 3.1,

$$\int_{T_1}^\infty x'_+(s) ds < \infty. \tag{3.19}$$

From (1.1), (A_1^*) and (3.1), we have

$$\begin{aligned} x'(t) &= -\lambda x(t) + F(t, x_t) \\ &\leq -\lambda x(t) + a(t) \sup_{s \in [-\tau, 0]} (-x(t+s)) \\ &= -\lambda x(t) - a(t) \inf_{s \in [-\tau, 0]} x(t+s) \\ &\leq -\lambda \int_{t-\tau}^t x'_+(s) ds - [a(t) + \lambda] \inf_{s \in [-\tau, 0]} x(t+s). \end{aligned}$$

Integrating the above from $T_1 + \tau$ to $T > T_1 + 4\tau$, we obtain

$$\begin{aligned} x(T) - x(T_1 + \tau) &\leq -\lambda \int_{T_1 + \tau}^T \int_{t-\tau}^t x'_+(s) ds dt - \int_{T_1 + \tau}^T [a(t) + \lambda] \inf_{s \in [-\tau, 0]} x(t+s) dt \\ &\leq -\lambda\tau \int_{T_1}^T x'_+(s) ds - \int_{T_1 + \tau}^T [a(t) + \lambda] \inf_{s \in [-\tau, 0]} x(t+s) dt. \end{aligned}$$

Suppose that $\liminf_{t \rightarrow \infty} x(t) > 0$. Then it follows from (3.14) and (3.19) that

$$x(T) \rightarrow -\infty \quad \text{as } T \rightarrow \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t \geq T_1$. Thus we have

$$\liminf_{t \rightarrow \infty} x(t) = 0. \tag{3.20}$$

Suppose that $\limsup_{t \rightarrow \infty} x(t) > 0$. Then, by (3.20), there exist $\epsilon > 0$ and two sequences $\{s_n\}$ and $\{t_n\}$ tending to ∞ such that $s_n < t_n < s_{n+1}$, $x(s_n) = \frac{1}{2}\epsilon$, $\frac{1}{2}\epsilon < x(t) < \epsilon$ for all $t \in (s_n, t_n)$ and $x(t_n) = \epsilon$. Hence, by (3.19),

$$\frac{1}{2}\epsilon = x(t_n) - x(s_n) \leq \int_{s_n}^{t_n} x'_+(s) \, ds \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which yields a contradiction. Thus (3.18) holds. In a similar way, we can show that (3.18) holds for any eventually negative solution $x(t)$ of (1.1). Finally, we show (3.18) for any oscillatory solution $x(t)$ of (1.1). In view of (3.15) or (3.16), there exists $\eta > 0$ such that (3.6) or (3.7) holds. Choose a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$ and $x(t) \neq 0$ for $t \neq t_n$. Let $r_n = \sup_{s \in [t_n - 2\tau, t_n]} |x(s)|$. In order to prove (3.18), it suffices to show that, for each n ,

$$|x(t)| \leq (1 - \eta)r_n \quad \text{for all } t \in [t_n, t_{n+1}].$$

We may assume that $x(t) > 0$ for all $t \in (t_n, t_{n+1})$, since the proof in the other case is similar. If $t_{n+1} \leq t_n + \tau$, then, by lemma 3.4,

$$x(t) \leq (1 - \eta + \lambda\tau)r_n \leq (1 - \eta)r_n \quad \text{for all } t \in [t_n, t_{n+1}].$$

If $t_{n+1} > t_n + \tau$, then, by lemma 3.4,

$$x(t) \leq \int_{t_n}^t x'_+(s) \, ds \leq (1 - \eta + \lambda\tau)r_n \quad \text{for all } t \in [t_n, t_n + \tau],$$

and, by lemma 3.1,

$$\begin{aligned} x(t) &\leq x(t_n + \tau) + \int_{t_n + \tau}^t x'_+(s) \, ds \\ &\leq (1 - \eta + \lambda\tau)r_n + \frac{-\lambda\tau}{1 + \lambda\tau}(1 + \lambda\tau)r_n \\ &= (1 - \eta)r_n \end{aligned}$$

for all $t \in (t_n, t_{n+1})$. Thus the proof is complete. □

4. Counterexamples

In this last section, we will give two counterexamples to show that some of the main theorems in [15] are not true. For convenience, we first state [15, theorem 3.1].

THEOREM 4.1 (cf. theorem 3.1 of [15]). *Suppose that $\lambda > 0$ and there exist $\alpha > 0$ such that (H_1) holds and that $\lambda \geq \alpha$ or*

$$\frac{\alpha^2}{\lambda^2} \left\{ 1 - \frac{\alpha - \lambda}{\alpha} \left[1 + \frac{2\lambda\alpha}{(\alpha - \lambda)^2} e^{-\lambda\tau} \right]^{1/2} \right\} \leq 1. \tag{4.1}$$

Then the zero solution of (1.1) is uniformly stable.

EXAMPLE 4.2. Consider the equation

$$x'(t) = -c \sin\left(\frac{1}{12}\pi\right)x(t) - cx(t - \tau), \quad t \geq 0, \tag{4.2}$$

where $c > 0, \tau > 0$. In view of (C_2) , the zero solution of (4.2) is uniformly stable if and only if

$$c\tau \leq \frac{\eta + \frac{1}{2}\pi}{\cos \eta} = \frac{\frac{1}{12}\pi + \frac{1}{2}\pi}{\cos(\frac{1}{12}\pi)} = 1.8972.$$

But, when $1.8972 < c\tau \leq 2.1870$, by a simple calculation, we see that

$$\begin{aligned} & \frac{\alpha^2}{\lambda^2} \left\{ 1 - \frac{\alpha - \lambda}{\alpha} \left[1 + \frac{2\lambda\alpha}{(\alpha - \lambda)^2} e^{-\lambda\tau} \right]^{1/2} \right\} \\ &= \frac{1}{\sin^2(\frac{1}{12}\pi)} \left\{ 1 - (1 - \sin(\frac{1}{12}\pi)) \left[1 + \frac{2\sin(\frac{1}{12}\pi)}{(1 - \sin(\frac{1}{12}\pi))^2} e^{-c\tau \sin(\pi/12)} \right]^{1/2} \right\} \\ &< 1. \end{aligned}$$

Thus [15, theorem 3.1] implies that the zero solution of (4.2) is uniformly stable. This is a contradiction, which shows that [15, theorem 3.1] is not true.

Let us take a further look into the source that leads to the invalidity of [15, theorem 3.1]. In the proof of [15, theorem 3.1], the following key lemma (cf. [15, lemma 2.1]) is employed.

LEMMA 4.3. *Let $x(t)$ be a continuously differentiable function on $[T_1, T_2]$ such that $x(t_1) = 0$ for some $t_1 \in [T_1, T_2]$ and*

$$\frac{d}{dt}|x(t)| \leq \lambda|x(t)| + c \quad \text{for all } t \in [T_1, T_2], \tag{4.3}$$

where $\lambda \neq 0$ and $c \geq 0$. Then

$$|x(t)| \leq \frac{c}{\lambda}(e^{\lambda|t-t_1|} - 1) \quad \text{for all } t \in [T_1, T_2]. \tag{4.4}$$

However, the above lemma is false. A counterexample is as follows.

EXAMPLE 4.4. Consider the function

$$x(t) = \begin{cases} \frac{c}{\lambda}(e^{\lambda(t-t_1)} - 1), & t \geq t_1, \\ \frac{c}{\lambda_1}(1 - e^{\lambda_1(t_1-t)}), & t < t_1, \end{cases}$$

where $\lambda_1 > \lambda > 0$. It is easy to verify that the above function $x(t)$ satisfies (4.3) for all $t \in [t_1 - 1, t_1 + 1]$. But when $t \in [t_1 - 1, t_1]$,

$$|x(t)| = \frac{c}{\lambda_1}(e^{\lambda_1(t_1-t)} - 1) > \frac{c}{\lambda}(e^{\lambda|t-t_1|} - 1),$$

which contradicts (4.4), and so lemma 4.3 is false.

In addition to [15, theorem 3.1], the proofs of lemmas 4.4, 4.5 and theorems 3.2, 4.1, 4.2 in [15] also all make use of [15, lemma 2.1], and thus these lemmas and theorems may not be correct as well. Actually, the above observation is one of the motivations of this paper (the others are stated in § 1).

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