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$\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-MANIFOLDS

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Abstract

We show that closed $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifolds are Seifert fibred, with general fibre the torus, and base one of the flat 2-orbifolds $T, Kb, \mathbb{A}, \mathbb{M}b, S(2,2,2,2), P(2,2)$ or $\mathbb{D}(2,2)$, and outline how such manifolds may be classified.

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1. Introduction

A closed 4-manifold M is homeomorphic to an infrasolvmanifold if and only if $\chi(M) = 0$ and $\pi_1(M)$ is torsion-free and virtually poly-Z, of Hirsch length 4. Every such group is realised in this way, and M is determined up to homeomorphism by π. Such manifolds are either mapping tori of self-homeomorphisms of three-dimensional infrasolvmanifolds or are unions of two twisted I-bundles over such 3-manifolds. (See [3, Ch. 8].)

There are six families of four-dimensional infrasolvmanifolds, corresponding to the geometries $\mathbb{E}^4$, $\mathbb{N}\mathrm{il}^3 \times \mathbb{E}^1$, $\mathbb{N}\mathrm{il}^4$, $\mathbb{S}\mathrm{ol}^4_0$, $\mathbb{S}\mathrm{ol}^4_1$ and $\mathbb{S}\mathrm{ol}^4_{m,n}$ of solvable Lie type. The 74 flat 4-manifolds can be listed, while $\mathbb{N}\mathrm{il}^3 \times \mathbb{E}^1$- and $\mathbb{N}\mathrm{il}^4$-manifolds (infranilmanifolds of dimension 4) were classified in [2]. Every torsion-free, virtually poly-Z group of Hirsch length 4 which is not virtually nilpotent is the fundamental group of a 4-manifold with one of the remaining geometries [4]. Manifolds with geometry $\mathbb{S}\mathrm{ol}^4_{m,n}$ (with $m \neq n$) or $\mathbb{S}\mathrm{ol}^4_0$ are mapping tori of self-homeomorphisms of the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$, and so may be classified in terms of conjugacy classes of matrices in $GL(3,\mathbb{Z})$. The relationship between the various $\mathbb{S}\mathrm{ol}^4_{m,n}$ geometries is not obvious (see [3, page 137].) However, when $m = n$ all agree with the product geometry $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$. Partial classifications of $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$- and $\mathbb{S}\mathrm{ol}^4_1$-manifolds were given in [1]. A complete classification of $\mathbb{S}\mathrm{ol}^4_1$-manifolds has recently appeared [6].

The next section is on notation and terminology, and Section 3 gives some simple observations on subgroups of $GL(2, \mathbb{Z})$ with two ends. In Section 4 we show that $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifolds have canonical Seifert fibrations, with general fibre the torus, and base one of the seven flat 2-orbifolds T, Kb, $\mathbb{A}$, $\mathbb{M}b$, $S(2,2,2,2)$, $P(2,2)$ or $\mathbb{D}(2,2)$. The fibration is unique, and so this suggests a route to the classification of such manifolds, in which the key elements are the base B, the action α of $\pi_1^{\mathrm{orb}}(B)$ on $N = \pi_1(F)$, where F is the general fibre, and an 'Euler class' in $H^2(\beta; N^\alpha)$. The manifolds which are mapping tori may also be classified in terms of conjugacy classes of automorphisms. In Section 5 we consider the interaction of the Seifert fibrations, mapping torus structure and orientability for such manifolds, but shall not otherwise classify them explicitly. The others all have base orbifold either $S(2,2,2,2)$, $P(2,2)$ or $\mathbb{D}(2,2)$. The orbifold fundamental groups all admit epimorphisms to the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, and so the $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-group has a corresponding decomposition as a generalised free product with amalgamation. We use this structure in Section 6 to give examples of each of these three types. In Section 7 we sketch why the Seifert approach should suffice to classify the $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifolds with such bases, but we do not pursue such a classification in detail.

2. Notation and terminology

If G is a group let G' be its commutator subgroup and let $\beta_1(G)$ be the rank of G/G'. Let $I(G)$ be the preimage of the torsion of G/G' in G, and let $\sqrt{G}$ be the Hirsch–Plotkin radical of G. (In the cases of interest below, $\sqrt{G}$ is the unique maximal nilpotent normal subgroup of G.) Let $X^2(G)$ be the subgroup generated by squares. If $x \in G$ let c_x be the automorphism induced by conjugation by x. If $H \leq G$ let $C_G(H)$ and $N_G(H)$ be the centraliser and normaliser of H in G, respectively. In particular, $\zeta G = C_G(G)$ is the centre of G. If G is virtually solvable let $h(G)$ be its Hirsch length.

The symbols $G_1, \ldots, G_6$ and $B_1, \ldots, B_4$ denote the six orientable and four nonorientable flat 3-manifold groups, respectively. (See [3, Ch. 8].)

Our notation for flat 2-orbifolds is taken from [7, Appendix A], embellished with 'blackboard bold' font for the initial letters of the symbols for such orbifolds with reflector curves. Similarly, $\mathbb{I}$ denotes the reflector interval (the quotient of S^1 by complex conjugation). (This font is otherwise used for the integers, rationals and real numbers, and for the initial letters of names of geometries. We use italics for the symbols for the associated model spaces, in this paper the Lie group $Sol^3 \times \mathbb{R}$.)

3. Some lemmas on subgroups of GL(2, $\mathbb{Z}$)

Let $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ be the infinite dihedral group, with presentation $\langle u, v \mid u^2 = v^2 = 1\rangle$. Recall that a group G has two ends if and only if G has an infinite cyclic subgroup of finite index if and only if G has a maximal finite normal subgroup with quotient $\mathbb{Z}$ or D_∞.

Lemma 3.1. *Let F be a finite subgroup of $G = GL(2, \mathbb{Z})$. If $N_G(F)$ is infinite then $F \leq \{\pm I\}$.*

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Proof. If $P \in F \setminus \{\pm I\}$ then it is conjugate to one of $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)$. (These have orders 2, 2, 3, 4 and 6, respectively.) In each case $C_G(P)$ is finite, and so $C_G(F)$ is finite. Since $\mathrm{Aut}(F)$ is finite the lemma follows. □

Since we may assume without loss of generality that $-I \in F$, this lemma also follows from the fact that $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

Lemma 3.2. *Let $H < \mathrm{GL}(2,\mathbb{Z})$ have two ends. Then either*

(1) $H \cong \mathbb{Z}$; *or*
(2) $H \cong \mathbb{Z} \oplus \langle -I\rangle$; *or*
(3) $H = \langle A, B\rangle$ *where* $A^2 = B^2 = I$; *or*
(4) $H = \langle A, B, -I\rangle$ *where* $A^2 = B^2 = I$; *or*
(5) $H = \langle A, B\rangle$ *where* $A^2 = -I$ *and* $B^2 = I$; *or*
(6) $H = \langle A, B\rangle$ *where* $A^2 = B^2 = -I$.

In each case neither A nor B is $-I$.

Proof. Let F be the maximal finite normal subgroup of H. Then $F \le \{\pm I\}$, by Lemma 3.1. If $H/F \cong \mathbb{Z}$ then (1) or (2) holds.

Suppose that $H/F \cong D_\infty$, and let $A, B \in H$ represent generators of the free factors of D_∞. Then A and B each have order dividing 4. Since AB has infinite order, neither A nor B is $-I$. If A has order 2 then $\det(A) = -1$, while if A has order 4 then $\det(A) = +1$ and $A^2 = -I$, and similarly for B. Thus if $F = 1$ then (3) holds, while if $F = \{\pm I\}$ then (4), (5) or (6) holds. □

Let $\widetilde{D}_\infty = \langle a, b \mid a^2 = b^4 = 1,\ ab^2 = b^2 a\rangle$ be the central extension of D_∞ arising in case (5) of Lemma 2.

4. Seifert fibrations

The component of the identity in $\mathrm{Isom}(\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1)$ is the solvable Lie group $\mathcal{S} = \mathrm{Sol}^3 \times \mathbb{R} \cong \mathbb{R}^3 \rtimes_\Theta \mathbb{R}$, where $\Theta(t) = \mathrm{diag}[e^t, 1, e^{-t}]$. The nilradical of $\mathcal{S}$ is $\sqrt{\mathcal{S}} \cong \mathbb{R}^3$, and $\mathcal{S}/\sqrt{\mathcal{S}} \cong \mathbb{R}$. (See [10], or [3, page 137].) The group of path components of $\mathrm{Isom}(\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1)$ is $D_8 \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.1. *Let π be a discrete cocompact subgroup of* $\mathrm{Isom}(\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1)$. *Then* $\sqrt{\pi} \cong \mathbb{Z}^3$, *and* $\pi \cap \mathcal{S}' \cong \mathbb{Z}^2$.

Proof. Let $T(\pi) = \pi \cap \mathcal{S}$. Then $[\pi : T(\pi)] \le |D_8 \times \mathbb{Z}/2\mathbb{Z}| = 16$. The intersection $T(\pi) \cap \sqrt{\mathcal{S}} = \pi \cap \sqrt{\mathcal{S}}$ is a lattice in $\sqrt{\mathcal{S}} \cong \mathbb{R}^3$, and so $T(\pi)/\pi \cap \sqrt{\mathcal{S}}$ is a discrete subgroup of $\mathcal{S}/\sqrt{\mathcal{S}} \cong \mathbb{R}$. (See [8, Ch. 2].) Therefore $T(\pi) \cong \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, where $A \in \mathrm{GL}(3,\mathbb{Z})$ has eigenvalues $\xi, 1, \xi^{-1}$, for some $\xi > 1$. Hence $T(\pi)' \cong \mathbb{Z}^2$ and $\sqrt{T(\pi)} = \pi \cap \sqrt{\mathcal{S}} \cong \mathbb{Z}^3$. Since $\pi \cap \sqrt{\mathcal{S}}$ is normal in π, it is a subgroup of $\sqrt{\pi}$. Since $[\pi : T(\pi)]$ is finite, $h(\sqrt{\pi}) = 3$, and since $\sqrt{\pi}$ is torsion-free nilpotent it follows that $\sqrt{\pi} \cong \mathbb{Z}^3$.

The intersection $\pi \cap \mathcal{S}'$ is a discrete subgroup of $\mathcal{S}' \cong \mathbb{R}^2$, and so has rank at most 2. Since $T(\pi)' \le \pi \cap \mathcal{S}'$, we see that $\pi \cap \mathcal{S}' \cong \mathbb{Z}^2$. □

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The intersection $T(\pi) = \pi \cap \mathcal{S}$ is the *translation subgroup* of π.

Theorem 4.2. *Let M be a $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifold. Then M has an essentially unique Seifert fibration, with general fibre T and base $B = T$, Kb, $\mathbb{A}$, $\mathbb{M}b$, $S(2,2,2,2)$, $P(2,2)$ or $\mathbb{D}(2,2)$. In particular, $\beta_1(\pi) \le 2$.*

Proof. The manifold M is a quotient of the Lie group $\mathcal{S}$ by a lattice $\pi = \pi_1(M)$ in $\mathrm{Isom}(\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1)$. The foliation of $\mathcal{S}$ by translates of its commutator subgroup is preserved by the isometry group, and so it induces a canonical foliation on M. Since $\pi \cap \mathcal{S}' \cong \mathbb{Z}^2$, by Lemma 4.1, the leaf map for this foliation is a Seifert fibration $p : M \to B$, with base B a flat 2-orbifold and general fibre F a flat 2-manifold. Hence π has a normal subgroup N such that $\pi/N \cong \beta = \pi_1^{\mathrm{orb}}(B)$ is a flat 2-orbifold group.

Conjugation in π determines an 'action' homomorphism $\alpha : \beta \to \mathrm{Out}(N)$. Since π is not virtually nilpotent, $\mathrm{Im}(\alpha)$ is infinite. Therefore $N \cong \mathbb{Z}^2$ and $F \cong T$, since $\mathrm{Out}(\pi_1(Kb))$ is finite. Since $\mathrm{Im}(\alpha)$ is an infinite solvable subgroup of $\mathrm{Out}(N) \cong GL(2,\mathbb{Z})$, it must be virtually $\mathbb{Z}$. Hence B fibres over S^1 or $\mathbb{I}$.

Let $\widehat{M}$ be the finite covering space induced from a torus $\widehat{B}$ covering B, and let $\widehat{\pi} = \pi_1(\widehat{M})$. Then $\beta_1(\widehat{\pi}) = \beta_1(\widehat{\pi}/N) = 2$, and so N and $\widehat{\pi}'$ are commensurable. Hence N has finite image in π/π'. In particular, $\beta_1(\pi) = \beta_1(\beta) \le 2$.

Suppose that $q : M \to \overline{B}$ is another Seifert fibration and $\overline{N}$ is the fundamental group of the general fibre. The base $\overline{B}$ is a flat 2-orbifold, since π is solvable, and again must itself fibre over S^1 or $\mathbb{I}$. After passing to a subgroup $\widehat{\pi}$ of finite index, if necessary, we may assume that $\widehat{\pi}/N \cong \mathbb{Z}^2$. Since N and $\overline{N} \cap \widehat{\pi}$ have finite image in $\widehat{\pi}/\widehat{\pi}'$, N and $\overline{N}$ must each be commensurable with $\widehat{\pi}' \cong \mathbb{Z}^2$. Thus N and $\overline{N}$ each have finite index in $N\overline{N}$. Since the groups of flat 2-orbifolds do not have nontrivial finite normal subgroups it follows that $N = \overline{N}$. Thus the fibration is unique (up to automorphisms of the base).

Let $\tilde{\alpha}$ be the composition of the projection of π onto β with the action α, and let $\nu = \tilde{\alpha}^{-1}(F)$, where F is the maximal finite normal subgroup of $\mathrm{Im}(\alpha)$. Since $N < \sqrt{\pi} \cong \mathbb{Z}^3$, we see that $\sqrt{\pi} \le \mathrm{Ker}(\tilde{\alpha})$. Since N is central in $\mathrm{Ker}(\tilde{\alpha})$, which is a torsion-free virtually poly-Z group of Hirsch length 3, it follows that $\mathrm{Ker}(\tilde{\alpha}) = \sqrt{\pi}$, and $\nu = \sqrt{\pi}$ or $\nu \cong G_2$. Since $\sqrt{\pi}/N$ is an abelian normal subgroup of $\beta = \pi/N$ and β has no nontrivial finite normal subgroup $\sqrt{\pi}/N \cong \mathbb{Z}$, and so $\sqrt{\pi} \cong N \times \mathbb{Z}$.

If $\nu = \sqrt{\pi}$ then $\nu/N \cong \mathbb{Z}$. If $\nu \cong G_2$ then $N = I(\nu)$, and so we again have $\nu/N \cong \mathbb{Z}$. In each case B must fibre over S^1 or $\mathbb{I}$ with general fibre S^1, and so $B = T$, Kb, $\mathbb{A}$, $\mathbb{M}b$, $S(2,2,2,2)$, $P(2,2)$, or $\mathbb{D}(2,2)$. □

The existence of such a Seifert fibration is discussed briefly in [3, pages 146 and 176].

The torus only fibres over S^1, the next three have fibrations of both kinds, while the remaining three only fibre over $\mathbb{I}$. If π/π' is finite then $B = S(2,2,2,2)$, $P(2,2)$ or $\mathbb{D}(2,2)$, and in each case the epimorphism from β to $\pi_1^{\mathrm{orb}}(\mathbb{I}) \cong D_\infty$ is unique up to composition with an automorphism of β. (This is easily verified by considering the infinite cyclic normal subgroups of β.) If $B = S(2,2,2,2)$ or $\mathbb{D}(2,2)$ there is also an essentially unique epimorphism to $D_\infty \times \mathbb{Z}/2\mathbb{Z}$, but none to $\widetilde{D}_\infty$. If $B = P(2,2)$ there

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is no epimorphism to $D_\infty \times \mathbb{Z}/2\mathbb{Z}$, but there is one to $\widetilde{D}_\infty$. There are no actions with image as in case (6) of Lemma 3.2.

We shall show below that each of these seven flat 2-orbifolds is the base of the Seifert fibration of some $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifold. There are two further flat 2-orbifolds which fibre over 1-orbifolds, namely $\mathbb{D}(2,\overline{2},\overline{2})$ and $\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2})$, but these do not arise here, by Theorem 4.2.

We may derive some of the algebraic consequences of Theorem 4.2 as follows. Let τ be a characteristic subgroup of finite index in π, and such that $\tau \leq \mathcal{S}$. (For instance, we could let τ be be the intersection of all normal subgroups of index 8 in π.) Then $\tau' \cong \mathbb{Z}^2$, and π/τ' is virtually $\mathbb{Z}^2$. Let N be the preimage in π of the maximal finite normal subgroup of π/τ'. Then N is a characteristic subgroup and $h(N) = 2$. Hence $N \cong \mathbb{Z}^2$ and $h(\pi/N) = 2$. Since π/N is virtually $\mathbb{Z}^2$ and has no nontrivial finite normal subgroup, it is a flat 2-orbifold group.

Closed $\mathbb{N}\mathrm{il}^3 \times \mathbb{E}^1$- and $\mathbb{N}\mathrm{il}^4$-manifolds also have canonical Seifert fibrations. For these, the images of the fundamental group of the general fibre in π are $\zeta\sqrt{\pi}$ and $\zeta_2\sqrt{\pi}$ (the second stage of the upper central series), respectively. In general, $\mathbb{N}\mathrm{il}^3 \times \mathbb{E}^1$-manifolds may have many Seifert fibrations, but in the $\mathbb{N}\mathrm{il}^4$ case the fibration is unique.

5. Mapping tori

Let π be the fundamental group of a $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifold M. If π/π' is infinite then π is a semidirect product $\kappa \rtimes \mathbb{Z}$, where κ is a torsion-free virtually poly-Z group of Hirsch length 3. Either κ is the group of a $\mathbb{S}\mathrm{ol}^3$-manifold or $\sqrt{\pi} \leq \kappa$, and then $[\kappa : \sqrt{\pi}] \leq 2$, by [3, Theorem 8.4]. Such semidirect products may be classified in terms of conjugacy classes in $\mathrm{Out}(\kappa)$. In this section we shall consider the interactions between Seifert fibrations, mapping tori and orientability for these groups. We shall consider the groups with π/π' finite in later sections.

Lemma 5.1. *If $\pi \cong \kappa \rtimes \mathbb{Z}$, where $\kappa = \mathbb{Z}^3$ or G_2 or is a $\mathbb{S}\mathrm{ol}^3$-group such that $\kappa/\sqrt{\kappa} \cong \mathbb{Z}$, then $B = T$ or Kb. Conversely, if $B = T$ or Kb and $\sqrt{\pi} \leq \kappa$ then $\pi \cong \sqrt{\pi} \rtimes \mathbb{Z}$ or $G_2 \rtimes \mathbb{Z}$.*

Proof. In each case $N < \kappa$ and $\kappa/N \cong \mathbb{Z}$. Hence $\beta \cong \mathbb{Z}^2$ or $\mathbb{Z} \rtimes_{-I} \mathbb{Z}$ and so $B = T$ or Kb.

If $B = T$ or Kb and $\sqrt{\pi} \leq \kappa$ then $\mathrm{Im}(\alpha)$ is $\mathbb{Z}$ or $\mathbb{Z} \oplus \langle -I \rangle$, by Lemma 3.2, and so $\pi \cong \sqrt{\pi} \rtimes \mathbb{Z}$ or $G_2 \rtimes \mathbb{Z}$. □

In Theorem 4.2 it was shown that $\beta_1(\pi) = \beta_1(\beta) \leq 2$, and clearly $\beta_1(\pi) = 2$ if and only if $B = T$. If so, then $\pi \cong \sqrt{\pi} \rtimes \mathbb{Z}$ or $G_2 \rtimes \mathbb{Z}$, since $\mathrm{Im}(\alpha)$ is abelian. The group π is also a semidirect product $\sigma \rtimes \mathbb{Z}$, where σ is a $\mathbb{S}\mathrm{ol}^3$-group (with $\sigma/\sqrt{\sigma} \cong \mathbb{Z}$), in infinitely many ways. (However, π need not be a direct product $\sigma \times \mathbb{Z}$.) There are orientable examples and nonorientable examples. (All T-bundles over T have been classified, in terms of extension data [9]. However [9, Proposition 3] appears to overlook some cases.)

If $\beta(\pi) = 1$ then $B = Kb$, $\mathbb{A}$ or $\mathbb{M}b$, and the splitting of π as a semidirect product is unique. If $B = Kb$ there are orientable and nonorientable examples with $\pi \cong \sqrt{\pi} \rtimes \mathbb{Z}$ and with $\sigma \rtimes \mathbb{Z}$, where σ is a $\mathbb{S}\mathrm{ol}^3$-group such that $\sigma/\sqrt{\sigma} \cong \mathbb{Z}$. (See [3, Section 8 of Ch. 8].) Conversely, if $\pi \cong \sigma \rtimes \mathbb{Z}$, where $\sigma/\sqrt{\sigma} \cong \mathbb{Z}$, then $B = T$ or Kb.

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Lemma 5.2. *Let σ be a $\mathbb{S}\mathrm{ol}^3$-group such that $\sigma/\sqrt{\sigma} \cong D_\infty$. Then σ is orientable, and automorphisms of σ are orientation preserving.*

Proof. The hypotheses imply that σ/σ' is finite. Thus $H_1(\sigma; \mathbb{Q}) = 0$. Since σ is a PD_3-group, $\chi(\sigma) = 0$, Therefore $H_3(\sigma; \mathbb{Q}) \neq 0$, and so σ is orientable. (This can also be deduced from the fact that if $N \in GL(2, \mathbb{C})$ is conjugate to N^{-1} then either $\det(N) = 1$ or $N^2 = 1$.) Let $[\sigma] \in H_3(\sigma; \mathbb{Z})$ be a generator.

Let u and $v \in \sigma$ represent generating involutions of D_∞, and let $t = uv$. Let f be an automorphism of σ. Then f restricts to an automorphism of $\sqrt{\sigma}$, and induces an automorphism of $\sigma/\sqrt{\sigma}$. After composition with an inner automorphism of σ, if necessary, we may assume that either $f(u) \equiv u$ and $f(v) \equiv v$, or $f(u) \equiv v$ and $f(v) \equiv u \bmod \sqrt{\sigma}$. Let $P = f|_{\sqrt{\sigma}}$, and suppose that $f(t) \equiv t^\epsilon \bmod \sqrt{\sigma}$. Then $f_*[\sigma] = \epsilon \det(P)[\sigma]$.

In the first case, $f(t) \equiv t \bmod \sqrt{\sigma}$, while $Pc_u|_{\sqrt{\sigma}} = c_u|_{\sqrt{\sigma}}P$ and $Pc_v|_{\sqrt{\sigma}} = c_v|_{\sqrt{\sigma}}P$, and so $P = I$. In the second case, $f(t) \equiv t^{-1} \bmod \sqrt{\sigma}$, while $Pc_u|_{\sqrt{\sigma}}P^{-1} = c_v|_{\sqrt{\sigma}}$ and $Pc_v|_{\sqrt{\sigma}}P^{-1} = c_u|_{\sqrt{\sigma}}$. Hence $P^2 = I$. Since $c_t = c_u c_v$ and $c_t|_{\sqrt{\sigma}}$ has infinite order, $P \neq \pm I$. Therefore $\det P = -1$. In each case, f is orientation preserving. □

If σ is a $\mathbb{S}\mathrm{ol}^3$-group then $\mathrm{Out}(\sigma)$ is finite, by [3, Theorem 8.10].

Theorem 5.3. *Suppose that $B = \mathbb{A}$ or $\mathbb{M}b$. Then M is orientable if and only if $\pi \cong \sigma \rtimes \mathbb{Z}$, where σ is a $\mathbb{S}\mathrm{ol}^3$-group such that $\sigma/\sqrt{\sigma} \cong D_\infty$. If M is nonorientable then $\pi \cong B_1 \rtimes \mathbb{Z}$.*

Proof. If M is Seifert fibred over $B = \mathbb{A}$ or $\mathbb{M}b$ then $\beta_1(M) = \beta_1(\beta) = 1$. Hence there is a unique splitting $\pi = \kappa \rtimes_\theta \mathbb{Z}$. Moreover, $N < \kappa$ and $\kappa/N \cong D_\infty$, since $B = \mathbb{A}$ or $\mathbb{M}b$. If κ is a $\mathbb{S}\mathrm{ol}^3$-group then $N = \sqrt{\kappa}$, since $\sqrt{\kappa}$ is characteristic and $\nu/\sqrt{\kappa}$ has no nontrivial finite normal subgroup. Since κ is orientable and θ is orientation preserving, by Lemma 5.2, M is orientable.

Conversely, if $\pi \cong \sigma \rtimes \mathbb{Z}$, where σ is a $\mathbb{S}\mathrm{ol}^3$-group such that $\sigma/\sqrt{\sigma} \cong D_\infty$, then $N = \sqrt{\sigma}$ and so $\pi/N \cong D_\infty \rtimes \mathbb{Z}$. Hence $B = \mathbb{A}$ or $\mathbb{M}b$.

If κ maps onto D_∞ and is virtually abelian then $[\kappa : \sqrt{\pi}] = 2$, by [3, Theorem 8.4]. Since κ is not G_2, by Lemma 5.1, it must be B_1 or B_2, and since B_2 does not map onto D_∞, we must have $\kappa = B_1$. Hence $N = \zeta B_1 \cong \mathbb{Z}^2$, since $\zeta D_\infty = 1$ and $B_1/\zeta B_1 \cong D_\infty$. Since B_1 is nonorientable, M is nonorientable. □

There are examples of each type allowed by Theorem 5.3. For instance, let σ be the $\mathbb{S}\mathrm{ol}^3$-group with presentation

$$\langle x, y, u, v \mid xy = yx,\ u^2 = x,\ uyu^{-1} = y,\ v^2 = x^3y^{-2}, vxv^{-1} = x^{17}y^{-12}, vyv^{-1} = x^{24}y^{-17}\rangle.$$

Then $\sqrt{\sigma} = \langle x, y\rangle$ and $\sigma/\sqrt{\sigma} \cong D_\infty$. We may define an involution f of σ by $f(u) = v$, $f(y) = x^4y^{-3}$ and $f(v) = u$. The groups $\sigma \times \mathbb{Z}$ and $\sigma \rtimes_f \mathbb{Z}$ are groups of orientable $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifolds which are Seifert fibred over $\mathbb{A}$ and $\mathbb{M}b$, respectively.

The flat 3-manifold group B_1 has a presentation

$$\langle X, y, z \mid Xz = zX,\ XyX^{-1} = y^{-1},\ yz = zy\rangle.$$

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Let θ and ψ be the automorphisms defined by $\theta(X) = X^3z^4$, $\theta(y) = y$ and $\theta(z) = X^2z^3$, and $\psi(X) = Xyz$, $\psi(y) = y^{-1}$ and $\psi(z) = X^2z^3$, respectively. Then $B_1 \rtimes_\theta \mathbb{Z}$ and $B_1 \rtimes_\psi \mathbb{Z}$ are the groups of nonorientable $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifolds which are Seifert fibred over $\mathbb{A}$ and $\mathbb{M}b$, respectively.

6. Examples with π/π' finite

Suppose now that $\beta_1(\pi) = 0$. Then $\pi/\nu \cong D_\infty$, and so $\pi \cong G *_\nu H$, where ν is the preimage of the maximal finite normal subgroup of $\mathrm{Im}(\alpha)$ and $[G : \nu] = [H : \nu] = 2$. Moreover, either $\nu = \sqrt{\pi}$ and $\mathrm{Im}(\alpha) \cong D_\infty$ or $\nu \cong G_2 \cong \mathbb{Z} \rtimes_{-I} \mathbb{Z}$ and $\mathrm{Im}(\alpha) \cong \widetilde{D}_\infty$ or $D_\infty \times \mathbb{Z}/2\mathbb{Z}$. Since ν/N is a normal subgroup of β it has no nontrivial finite normal subgroup. Therefore if $\nu \cong \mathbb{Z}^3$ then $\nu/N \cong \mathbb{Z}$ and N is a direct summand of ν, while if $\nu \cong G_2$ then either $\nu/N \cong \mathbb{Z}$ and $N = I(\nu)$ or $\nu/N \cong D_\infty$. (However, N need not be a canonical subgroup of ν.)

In order to describe our examples clearly, we should be more precise about our definitions of such amalgamated free products. We shall assume that ν is given as a subgroup of G and that $\phi : \nu \to H$ is a monomorphism. Then we shall write

$$G *_\phi H = G * H/\langle\langle \phi(n) = n \ \forall\, n \in \nu \rangle\rangle.$$

Since ν, G and H are each finite extensions of $\sqrt{\pi}$, they are flat 3-manifold groups. If G and H were both nonorientable then π would be orientable, and so $\beta_1(\pi) = 1 + \frac{1}{2}(\beta_2(\pi) - \chi(\pi)) > 0$, contrary to the assumption. Hence we may assume that H is orientable. A Mayer–Vietoris argument shows that $\beta_1(G) + \beta_1(H) \leq \beta_1(\nu)$.

If $\nu = \sqrt{\pi}$ then G and H each have holonomy of order 2 or less, and so $\beta(G)$ and $\beta(H)$ are each greater than 0. We may then assume that $H = G_2$ and $G = G_2, B_1$ or B_2. In each case $\nu = \sqrt{G} = \sqrt{H}$.

If $\nu \cong G_2$ then we may assume that $H \cong G_6$ and $G \cong G_2, G_4, G_6$, B_3 or B_4. If $G \cong G_4, B_3$ or B_4 it has a unique subgroup of index 2 which is isomorphic to G_2, while if $G \cong G_2$ or G_6 there are three such subgroups, which are equivalent under automorphisms of G.

We shall use the following presentations for these groups:

$$\begin{aligned}
\mathbb{Z}^3 &= \langle x, y, z \mid xy = yx,\ xz = zx,\ yz = zy\rangle,\\
G_2 &= \langle r, y, z \mid ryr^{-1} = y^{-1},\ rzr^{-1} = z^{-1},\ yz = zy\rangle,\\
G_4 &= \langle t, y, z \mid tyt^{-1} = z,\ tzt^{-1} = y^{-1},\ yz = zy\rangle,\\
G_6 &= \langle r, s \mid rs^2r^{-1} = s^{-2},\ sr^2s^{-1} = r^{-2}\rangle,\\
B_1 &= \langle X, y, z \mid Xz = zX,\ XyX^{-1} = y^{-1},\ yz = zy\rangle,\\
B_2 &= \langle X, y, z \mid XyX^{-1} = y^{-1},\ XzX^{-1} = yz,\ yz = zy\rangle,\\
B_3 &= \langle t, Y, z \mid tYt^{-1} = Y^{-1},\ tz = zt,\ YzY^{-1} = z^{-1}\rangle,\\
B_4 &= \langle t, Y, z \mid tYt^{-1} = Y^{-1}z,\ tz = zt,\ YzY^{-1} = z^{-1}\rangle.
\end{aligned}$$

(Here $r^2 \in G_2$, $t^4 \in G_4$, $r^2 \in G_6$, $X^2 \in B_1$, $X^2 \in B_2$, $t^2 \in B_3$ and $t^2 \in B_4$ correspond to $x \in \mathbb{Z}^3$, while $s^2 \in G_6$, $Y^2 \in B_3$ and $Y^2 \in B_4$ correspond to $y \in \mathbb{Z}^3$, and $(rs)^2 \in G_6$

corresponds to $z \in \mathbb{Z}^3$.) In each case, let $A(G)$ be the maximal abelian normal subgroup of the flat 3-manifold group G.

In order to realise the remaining bases $S(2,2,2,2)$, $P(2,2)$ and $\mathbb{D}(2,2)$, it shall suffice to consider the case $\nu \cong \mathbb{Z}^3$. We shall assume that $A(\nu)$, $A(G)$ and $A(H)$ have bases $\{x, y, z\}$, as above. Clearly $N < A(\nu) \leq A(G) \cap A(H)$. Since G/N has $\nu/N \cong \mathbb{Z}$ as an index-2 subgroup, $G/N \cong \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or D_∞. If $G \cong G_2$ then either $N = I(G) = \langle y, z\rangle$ or $N \cong \langle x, w\rangle$ for some $w \in I(G)$, since N is normal in G. However, if $N = I(G)$ then G would act on N via $-I$, and so $\mathrm{Im}(\alpha)$ would be finite. Hence, we may assume that $N \cong \langle x, y\rangle$, and so $G/N \cong D_\infty$.

If $G \cong B_1$ then either $N \cong \langle x^a z^b, y\rangle$, with $(2a, b) = 1$ and $G/N \cong \mathbb{Z}$, or $N \cong \langle x^a z^{2c}, y\rangle$, with $(2a, c) = 1$ and $G/N \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, or $N \cong \langle x, z\rangle$ and $G/N \cong D_\infty$. But in the latter case G would act on N via $-I$, and so $\mathrm{Im}(\alpha)$ would be finite.

If $G \cong B_2$ we find that G/N can be either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However, B_2 does not admit any epimorphisms to D_∞.

Let $G, H \cong G_2$, and let $\phi : A(G) \to A(H)$ be the isomorphism given in terms of standard bases by the bordered 3×3 matrix $C \oplus [1] = \left(\begin{smallmatrix} C & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $C = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, and let $\pi = G *_\phi H$. Then $N = \langle x, y\rangle < G$ is normal in π, and $\beta = \pi/N \cong D_\infty *_\mathbb{Z} D_\infty$. Let $u \in G$ and $v \in H$ correspond to $r \in G_2$. Then the action of uv on ν by conjugation has matrix $\left(\begin{smallmatrix} 3 & -2 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. The corresponding semidirect product is the fundamental group of a mapping torus which is a $\mathbb{S}\mathrm{ol}^3 \times E^1$-manifold. Hence the overgroup π is the fundamental group of a $\mathbb{S}\mathrm{ol}^3 \times E^1$-manifold which is Seifert fibred over $B = S(2,2,2,2)$. If we set $G = B_1$ instead, and use the same matrices, we get an example with $B = \mathbb{D}(2,2)$ instead, since $\pi/N \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) *_\mathbb{Z} D_\infty$. Modifying the 3×3 matrix so that each entry in its third column is 1 gives an example with $B = P(2,2)$, since $\pi/N \cong \mathbb{Z} *_\mathbb{Z} D_\infty$.

Similar examples can be constructed when $\nu \cong G_2$. We then have $G/N \cong \mathbb{Z}$ (and $N = I(G)$) if $G \cong G_4$, $G/N \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $G = B_3$ or B_4, and $G/N \cong D_\infty$ if $G \cong G_6$. Restriction from G_2 to $I(G_2)$ induces an epimorphism from $\mathrm{Aut}(G_2)$ to $\mathrm{Aut}(I(G_2))$. Thus, given $C \in GL(2, \mathbb{Z})$ and $G = G_4, G_6, B_3$ or B_4, there is an embedding of G_2 in G whose restriction to $I(G_2)$ has matrix C with respect to the standard bases, and which fixes t. As before, the corresponding groups $\pi = G *_\phi H$ are $\mathbb{S}\mathrm{ol}^3 \times E^1$-groups, with $\beta = \pi/N \cong \mathbb{Z} *_\mathbb{Z} D_\infty$, $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) *_\mathbb{Z} D_\infty$ or $D_\infty *_\mathbb{Z} D_\infty$, respectively.

It is easy to see that every flat 3-manifold group or $\mathbb{S}\mathrm{ol}^3$-group can be generated by at most three elements [5], and hence that every $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-group requires at most four generators. This is best possible in general. If $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$ then $\sigma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a $\mathbb{S}\mathrm{ol}^3$-group such that $H_1(\sigma; \mathbb{F}_2) \cong \mathbb{F}_2^3$, and so $\sigma \times \mathbb{Z}$ is a $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-group that cannot be generated by three elements. Similarly, if $\phi = \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL(3, \mathbb{Z})$ then $\pi = G_2 *_\phi G_2$ needs four generators.

7. Outline of the classification in terms of Seifert data

The subgroup N is characteristic in π. Therefore any isomorphism $f : \pi \to \tilde{\pi}$ of such groups induces isomorphisms $f|_N : N \to \tilde{N}$ and $\bar{f} : \pi/N \to \tilde{\pi}/\tilde{N}$. Hence the

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classification of such groups may be derived from the classification of the extensions

$$\xi : 1 \to N \to \pi(\xi) \to \beta \to 1.$$

The ingredients of such a classification are the quotient group β, the action $\alpha : \beta \to \mathrm{Out}(N) \cong GL(2,\mathbb{Z})$ and the cohomology class $e(\xi) \in H^2(\beta; N^\alpha)$, where N^α is N considered as a $\mathbb{Z}[\beta]$-module with module structure determined by α. Given β, N and α, the groups $\pi(\xi)$ and $\pi(\xi')$ are isomorphic if and only if $e(\xi') = g_\# e(\xi)$, where g is a β-linear automorphism of N.

The group $\pi(\xi)$ determined by such an extension is the fundamental group of a $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-manifold if and only if it is torsion-free and $\mathrm{Im}(\alpha)$ contains a matrix with trace greater than 2. The torsion condition can be checked by restricting the extension to the finite cyclic subgroups of β. In all cases of interest to us, β is either torsion-free or a semidirect product $\gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ where γ is torsion-free. The 2-torsion must act via I or $\pm U$, where $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (and not via $-I$ or $\pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

Lemma 7.1. *If $\beta \cong \gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ with γ torsion-free then $\pi(\xi)$ is torsion-free if and only if $e(\xi|_{\mathbb{Z}/2\mathbb{Z}}) \neq 0$ in $H^2(\mathbb{Z}/2\mathbb{Z}; (\mathbb{Z}^2)^\alpha)$.*

Proof. Since the nontrivial finite subgroups of β have order 2, and are all conjugate, π is torsion-free if and only if any one of these subgroups has torsion-free preimage in π. This is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}^2$, with action trivial or via U, and the claim follows easily. □

The identification of N with $\mathbb{Z}^2$ and π/N with β is only well defined up to compositions with automorphisms, and so the same is true for the action α.

Let $D(P)$ be the subgroup of $GL(2,\mathbb{Z})$ generated by U and $V = PUP^{-1}$, where $P \in GL(2,\mathbb{Z})$ is such that UV has infinite order. Then $D(P) \cong D_\infty$. If $\mathrm{Im}(\alpha) \cong D_\infty$ then it is generated by elements conjugate to U, and so is conjugate to some $D(P)$. There is a matrix with trace greater than 2 in $\mathrm{Im}(\alpha)$ if and only if UV has an eigenvalue not equal to ± 1. The pair of involutions u, v generating D_∞ is unique up to interchange and (simultaneous) conjugation. Therefore $D(P)$ is conjugate to $D(\widetilde{P})$ if and only if $\widetilde{P} = U^\delta P^\epsilon U^\delta$, where $\delta = 0$ or 1 and $\epsilon = \pm 1$.

We may find the epimorphisms from β to D_∞ by considering the possible kernels, which are normal subgroups of Hirsch length 1.

If $B = S(2,2,2,2)$ then $\beta = \mathbb{Z}^2 \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$, so every subgroup of $\sqrt{\beta} = \mathbb{Z}^2$ is normal, while normal subgroups with nontrivial torsion have finite index. In this case the normal subgroups of Hirsch length 1 are infinite cyclic, and all epimorphisms from β to D_∞ are equivalent up to composition with an automorphism of β. Similarly, all epimorphisms to $D_\infty \times \mathbb{Z}/2\mathbb{Z}$ are equivalent. On the other hand, this group has no epimorphisms to $\widetilde{D}_\infty$.

If $B = \mathbb{D}(2,2)$ then $\beta = \langle j, u, v \mid ujv = jvu,\ j^2 = u^2 = v^2 = 1\rangle$, and there are just two maximal normal subgroups of Hirsch length 1, namely $\langle jv\rangle$ and $\langle u, (ju)^2\rangle$. The quotients by $\langle (jv)^2\rangle$ and $\langle (ju)^2\rangle$ are each $D_\infty \times \mathbb{Z}/2\mathbb{Z}$. In each case, the epimorphisms are inequivalent. On the other hand, this group has no epimorphisms to $\widetilde{D}_\infty$.

If $B = P(2,2)$ then $\beta = \langle s, u \mid (us^2)^2 = u^2 = 1\rangle$, and there are again just two maximal normal subgroups of Hirsch length 1, namely $\langle s^2\rangle$ and $\langle (us)^2\rangle$. There is an automorphism that fixes u and swaps s with us. The quotients by $\langle s^4\rangle$ and $\langle (us)^4\rangle$ are each $\widetilde{D}_\infty$. In each case, the epimorphisms are equivalent. On the other hand, since this group has a 2-generator presentation it has no epimorphism to $D_\infty \times \mathbb{Z}/2\mathbb{Z}$.

The possible actions α with $\mathrm{Im}(\alpha) \cong D_\infty$ and with given kernel may be parametrised by matrices $P \in GL(2,\mathbb{Z})$ such that $UPUP^{-1}$ has an eigenvalue not equal to ± 1, modulo inversion and conjugation by U. Similarly for epimorphisms to $D_\infty \times \mathbb{Z}/2\mathbb{Z}$, since the $\mathbb{Z}/2\mathbb{Z}$ direct factor must be generated by $\pm I$. Let $W = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then actions with $\mathrm{Im}(\alpha) \cong \widetilde{D}_\infty$ are conjugate to actions generated by U and PWP^{-1}, such that $UPWP^{-1}$ has an eigenvalue not equal to ± 1, modulo conjugation by U.

The final stage of the classification is the determination of the possible extensions with given base and action, modulo automorphisms of the coefficients. We shall settle for a slightly weaker result.

Theorem 7.2. *There are only finitely many $\mathbb{S}\mathrm{ol}^3 \times \mathbb{E}^1$-groups with given base group β and action α.*

Proof. The cohomology groups $H^2(\beta; N^\alpha)$ may be estimated by using the Lyndon–Hochschild–Serre spectral sequence

$$H^p(\beta/\sqrt{\beta}; H^q(\sqrt{\beta}; N^\alpha)) \Rightarrow H^{p+q}(\beta; N^\alpha)$$

for β as a extension of the finite group $\beta/\sqrt{\beta}$ by $\sqrt{\beta} \cong \mathbb{Z}^2$. Now $H^0(\sqrt{\beta}; N^\alpha) = 0$, since $\alpha(\sqrt{\beta})$ contains matrices with no eigenvalue 1, and $H^q(\sqrt{\beta}; N^\alpha) = 0$, for $q > 2$. Hence this spectral sequence has just two nonzero columns, and so there is an exact sequence

$$H^1(\beta/\sqrt{\beta}; H^1(\sqrt{\beta}; N^\alpha)) \to H^2(\beta; N^\alpha) \to H^0(\beta/\sqrt{\beta}; H^2(\sqrt{\beta}; N^\alpha)).$$

Since $H^q(\sqrt{\beta}; N^\alpha)$ is finitely generated, for all q, the first term is finite. Poincaré duality for $\sqrt{\beta}$ gives $H^2(\sqrt{\beta}; N^\alpha) \cong H_0(\sqrt{\beta}; N^\alpha)$. This is again finite, since $\alpha(\sqrt{\beta})$ contains matrices with no eigenvalue 1, and so the result follows. □

References

[1] R. Cobb, 'Infrasolvmanifolds of dimension four', PhD Thesis, School of Mathematics and Statistics, University of Sydney, 1999.

[2] K. Dekimpe, *Almost Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics, 1639 (Springer, Berlin, 1996).

[3] J. A. Hillman, *Four-Manifolds, Geometries and Knots*, GT Monographs, 5 (Geometry and Topology Publications, Coventry, 2002), revised in 2007 and 2014.

[4] J. A. Hillman, 'Geometries and infrasolvmanifolds in dimension 4', *Geom. Dedicata* (2007), 57–72.

[5] J. A. Hillman, 'The $\mathbb{F}_2$-cohomology rings of $\mathbb{S}\mathrm{ol}^3$-manifolds', *Bull. Aust. Math. Soc.* **89** (2014), 191–201.

[6] K. B. Lee and S. Thuong, 'Infra-solvmanifolds of Sol_1^4', *J. Korean Math. Soc.* **52** (2015), 1209–1251.

[7] J. M. Montesinos, *Classical Tessellations and Three-Manifolds*, Universitext (Springer, Berlin, 1987).

https://doi.org/10.1017/S1446788717000258 Published online by Cambridge University Press

[8] M. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 2, Bd 68 (Springer, Berlin, 1972).

[9] K. Sakamoto and S. Fukuhara, 'Classification of T^2-bundles over T^2', *Tokyo J. Math.* **6** (1983), 311–327.

[10] C. T. C. Wall, 'Geometric structures on compact complex analytic surfaces', *Topology* **25** (1986), 119–153.

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