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# DIFFERENCE-MAKING CONDITIONALS AND THE RELEVANT RAMSEY TEST

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**Abstract.** This article explores conditionals expressing that the antecedent makes a difference for the consequent. A 'relevantised' version of the Ramsey Test for conditionals is employed in the context of the classical theory of belief revision. The idea of this test is that the antecedent is relevant to the consequent in the following sense: a conditional is accepted just in case (i) the consequent is accepted if the belief state is revised by the antecedent *and* (ii) the consequent fails to be accepted if the belief state is revised by the antecedent's negation. The connective thus defined violates almost all of the traditional principles of conditional logic, but it obeys an interesting logic of its own. The article also gives the logic of an alternative version, the 'Dependent Ramsey Test,' according to which a conditional is accepted just in case (i) the consequent is accepted if the belief state is revised by the antecedent *and* (ii) the consequent is rejected (e.g., its negation is accepted) if the belief state is revised by the antecedent's negation. This conditional is closely related to David Lewis's counterfactual analysis of causation.

## §1. Introduction.

One of the two prime sources of the classical AGM theory of belief change (after Alchourrón, Gärdenfors, & Makinson, 1985) was Gärdenfors's (1978) analysis of conditionals in terms of belief revisions. It was Gärdenfors himself who later showed that there are serious difficulties with this project if conditionals are treated as beliefs just like sentences containing only Boolean connectives (Gärdenfors, 1986). But the main idea, due to Ramsey (1931) and made popular by Stalnaker (1968), remains attractive:[1] 'If $A$ then $B$' is accepted in a belief state just in case $B$ is an element of the belief set $Bel * A$ that results from a revision of the belief set $Bel$ by the sentence (information that, supposition that) $A$.

However, the Ramsey Test does not take into account a fundamental feature of conditionals as used in natural language: typically, conditionals also express that the antecedent

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[1] See, for instance, Bradley (2007) and the long debate this article sparked in the journal *Mind*.—Many researchers have suggested variations of the Ramsey Test in order to get around Gärdenfors's (1986) famous impossibility theorem. Gärdenfors (1987) showed that the Strong Ramsey Test (see §3) does not help to solve the problem raised by this theorem. Giordano, Gliozzi, & Olivetti (2005) introduce another 'Strong Ramsey Test' that simultaneously refers to conditionals and negations of conditionals. Ismail (2010) presents a reason maintenance system based on relevance logic and rejects one half of Gärdenfors's formulation of the Ramsey Test. An early overview of variations of the Ramsey Test is given by Lindström & Rabinowicz (1998). My concern in this article is not the Gärdenfors triviality theorem. I avoid the problems here simply by not treating conditionals as expressing propositions (as beliefs) in my renderings of the Ramsey Test and its variations.—A different solution to the problem indicated by Gärdenfors's theorem that is true to 'the spirit of AGM' is proposed in Rott (2011).



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is relevant to the consequent. The idea then is this: 'If $A$ then $B$' is accepted in a belief state just in case $B$ is a belief in the belief set $Bel * A$ that results from a revision of the belief set $Bel$ by the sentence (information that, supposition that) $A$, but $B$ is not a belief in the revision of $Bel$ by the negation $\neg A$ of $A$. I will call conditionals that are evaluated by this principle *difference-making conditionals*, because roughly speaking, revising by the antecedent makes a difference to the doxastic status of the consequent. This idea was the basis of Rott (1986). More recently, there has been an increasing number of voices arguing that relevance is part of the meaning of some conditionals, among them Douven (2008; 2016), Spohn (2015), Skovgaard-Olsen (2016), Krzyżanowska, Collins, & Hahn (2017), van Rooij & Schulz (2019), Crupi & Iacona (2018), and Skovgaard-Olsen, Collins, Krzyżanowska, Hahn, & Klauer (2019).[2]

There are suggestions how to encode relevance in conditionals within a probabilistic (Douven; Crupi & Iacona) or a ranking-theoretic (Spohn) framework. In contrast to this, the present article will explore the logic of difference-making conditionals in an entirely qualitative framework.

The purpose of this article is not to investigate whether and when it is part of the message of natural-language conditionals that the antecedent is relevant to the consequent (but see §2). Furthermore, I will not take a stand on the important question whether this part of the message, if it exists, is a matter of pragmatics or semantics. Though I consider it legitimate to say that belief revision models provide a semantics for conditionals, it would not harm if it turned out that difference-making conditionals represent 'only' what is conveyed pragmatically by the utterance of conditionals in natural language.[3] The main aim of the article is to specify the logic of such conditionals, or more precisely, of conditionals the acceptance conditions of which are defined by what I will call the *Relevant Ramsey Test*.

**§2. Almost all traditional principles of conditional logic fail for difference-making conditionals.** The conditional logic corresponding to AGM revision is commonly taken to be very close to System VC of David Lewis (1973b) (with embedded conditionals) or System R of Lehmann & Magidor (1992) (without embedded conditionals).[4] But let us rely on a weaker standard. In this section the corner '$>$' is used as a generic 'if-then' connective, but this will change from the next section on. The following principles have often been considered to be the conservative core of reasoning with conditionals (Adams, 1975; Burgess, 1981; Veltman, 1985; Pearl, 1989; Kraus, Lehmann, & Magidor, 1990):

(**Ref**) $A > A$. (Reflexivity)

(**LLE**) If $Cn(A) = Cn(B)$, then $A > C$ iff $B > C$. (Left Logical Equivalence)

(**RW**) If $A > B$ and $C \in Cn(B)$, then $A > C$. (Right Weakening)

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2 According to Douven and Crupi & Iacona, this applies only to 'evidential conditionals', which are contrasted with 'suppositional conditionals'. The relevance criterion does not apply to all kinds of conditionals: certainly not to those that have an expression indicating that the relevance connection is suspended ('even if', 'still', ...), and not to some conditionals without such a linguistic marker, like, e.g., so-called biscuit conditionals.—Also compare Evans & Over (2004) on the Ramsey Test and relevance.

3 One of the main conclusions of Skovgaard-Olsen et al. (2019, p. 60) is that 'the reason-relation reading [i.e., the positive relevance reading, H.R.] of indicative conditionals is a conventional aspect of their meaning, which cannot be cancelled without contradiction'.

4 It was the professed aim of Gärdenfors (1978) to reproduce Lewis's VC using a belief revision semantics.

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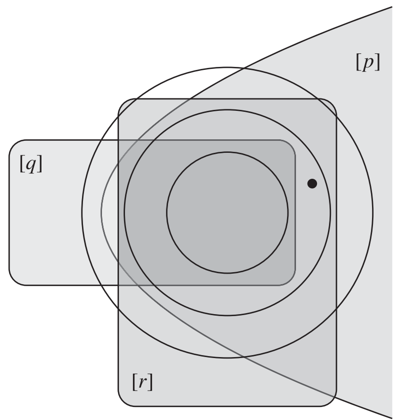


Fig. 1. Cautious Monotony is invalid for the difference-making conditionals: $p > q$ and $p > r$, but not $p \wedge q > r$. (All counterexamples against Cautious Monotony have $p, q, r \in Bel$.)

(**And**) If $A > B$ and $A > C$, then $A > B \wedge C$.
(**CMon**) If $A > B$ and $A > C$, then $A \wedge B > C$. (Cautious Monotony)
(**Cut**) If $A > B$ and $A \wedge B > C$, then $A > C$.
(**Or**) If $A > C$ and $B > C$, then $A \vee B > C$.

This collection is often referred to as System **P** (Kraus, Lehmann, & Magidor, 1990).[5]

It turns out that difference-making and relevance considerations wreak havoc on traditional principles of conditional logic. Only two of these seven principles remain valid in our modelling of difference-making conditionals, viz., LLE and And.[6] I will now discuss a few examples that illustrate how some of the other principles can come to fail.

*Against Cautious Monotony.* A research project with two postdoc positions is about to start. I believe that Pam and Quinn will work on the project ($p$ and $q$), and that the project will be successful ($r$). I know that Pam is an excellent and dedicated researcher, and if she is missing, the project might fail. On the other hand, I know that Quinn is neither the greatest researcher and nor terribly interested in the topic of the project. But Quinn likes Pam a lot, and if Pam is not in, it is not sure that Quinn will be in. So I think 'If Pam works on the project, Quinn will work on it, too,' and I also think 'If Pam works on the project, the project will be successful.' It sounds strange, however, to say 'If Pam and Quinn work on the project, the project will be successful,' because should one of them not be in the project, it will most likely be Quinn who is missing—remember he is not keen on the topic—and the project will be a success anyway because of Pam's work. Figure 1 gives a diagram representing this situation by a systems of spheres in the style of Grove (1988).

5 The Cut rule is redundant in this collection. I keep it because it is important for defining 'cumulative reasoning' (Makinson, 1989; Kraus, Lehmann, & Magidor, 1990) which is System P without Or and with Cut.—The 'factual', non-conditional language is assumed to be governed by some reflexive, monotonic and idempotent consequence operation ('background logic') $Cn$ which is supraclassical and compact and satisfies the deduction theorem: $\Gamma \subseteq Cn(\Gamma)$; if $\Gamma \subseteq \Delta$ then $Cn(\Gamma) \subseteq Cn(\Delta)$; $Cn(Cn(\Gamma)) = Cn(\Gamma)$; if $Cn_0$ is classical tautological implication then $Cn_0(\Gamma) \subseteq Cn(\Gamma)$; if $A \in Cn(\Gamma)$ then $A \in Cn(\Gamma_0)$ for some finite subset $\Gamma_0$ of $\Gamma$; and finally, $B \in Cn(\Gamma \cup \{A\})$ iff $A \supset B \in Cn(\Gamma)$. We write $Cn(A)$ for $Cn(\{A\})$. $Cn$ is used in LLE and RW.

6 The reason for the failure of Reflexivity is not only the relevance condition alone, but also our not endorsing the fifth AGM axiom. See §4 below.

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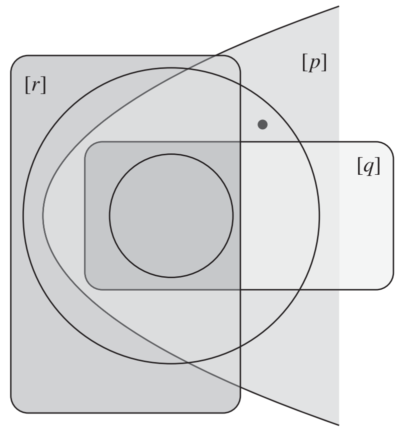



Fig. 2. Cut is invalid for difference-making conditionals: $p > q$ and $p \wedge q > r$, but not $p > r$. (All counterexamples against Cut have $p, q, r \in Bel$.)

The innermost circle contains the possible worlds deemed most plausible by the agent; they define her beliefs. The ring around it contains the second most plausible worlds, the next ring the third most plausible worlds. Propositions, i.e., sets of possible worlds at which a certain sentence is true, are coloured.

*Against Cut.* Another research project with two postdoc positions is about to start. There have been many highly qualified applicants. I believe that Peter and Quiana will work on this project ($p$ and $q$), and that the project will be successful ($r$). I know that Peter is not the greatest researcher but an exceptionally nice person, and that Quiana is brilliant but the topic of the project is not her favourite one. However, Quiana likes Peter a lot, and if Peter is not in, it is very unlikely that Quina will be in. Peter and Quiana form a very good team, but if one of them is missing, this will be Quiana and the project is likely to fail (it is Quiana's contribution that is crucial for the success of the project). So I think 'If Peter works on the project, Quiana will work on it, too,' and I also think 'If Peter and Quiana work on the project, the project will be successful.' It sounds strange, however, to say 'If Peter works on the project, the project will be successful,' because should Peter not be in the project, it will be successful anyway, since there are many competent applicants for this project. Figure 2 presents a diagram illustrating this situation.

*Against Or.* Pam and Quinn live in a village with two pubs. They both prefer the Irish pub to the Spanish pub, but they don't avoid the latter altogether. I know that they will go out tonight and they want to meet in a pub, but it is not quite clear in which pub. I believe that Pam will go to the Irish pub ($p$), that Quinn will go to the Irish pub ($q$), and that they will meet each other ($r$). It makes sense to say 'If Pam goes to the Irish pub, they will meet,' because if Pam does not go to the Irish pub (and go to the Spanish pub instead), they will most likely miss each other. Similarly for 'If Quinn goes to the Irish pub, they will meet.' But it sounds odd to say 'If Pam goes to the Irish pub or Quinn goes to the Irish, they will meet,' because if neither of them goes to the Irish pub, they will meet each other anyway in the Spanish pub.[7] This situation is depicted in Figure 3.

[7] While the last conditional does sound awkward, it may be for a different reason. Intuitively, it is tempting to interpret the antecedent as short for 'If one of them goes to the Irish pub *and the other doesn't*', and then of course they won't meet each other. On this interpretation, no relevance considerations are required to explain the oddness of the conditional.

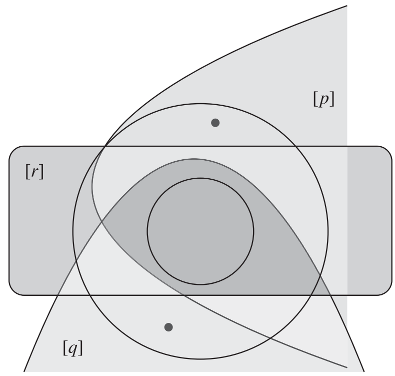



Fig. 3. Or is invalid for the difference-making conditionals: $p > r$ and $q > r$, but not $p \vee q > r$. (All counterexamples against Or have $p, q, r \in Bel$.)

That difference-making conditionals—as analysed by the Relevant Ramsey Test against the background of the full AGM theory—violate these three properties, Cautious Monotony, Cut and Or, is striking. It should be noted, however, that counterexamples against them can only be constructed with cases in which the antecedent of the relevant conditionals are all believed to be true.[8] So the counterexamples all involve 'factual conditionals' (Goodman, 1947, p. 114), they rely on somewhat untypical cases in which 'because' or 'since' may appear to be better words than 'if.' This is different for Right Weakening, as we are now going to show.

*Against Right Weakening.* It makes perfect sense to say 'If you pay an extra fee ($p$), your letter will be delivered ($q$) by express ($r$),' because the fee will buy you a special service. But it sounds odd to say 'If you pay an extra fee, your letter will be delivered,' because the letter would be delivered anyway, even if you did not pay the extra fee. No special belief about whether or not you actually pay the extra fee is required here. See Figure 4.

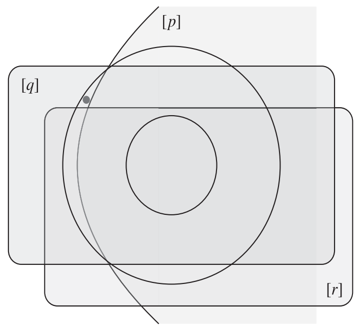
It is a truism about natural language conditionals that they are quite different from material conditionals. The conditionals modelled by traditional conditional logics deviate from material conditionals most importantly by not generally allowing to strengthen the antecedent. 'Left Strengthening' is invalid.

Difference-making conditionals characteristically violate a second basic principle of monotonicity, one that has not been questioned by standard conditional logics. It is in general not allowed to weaken the consequent: 'Right Weakening' is invalid.

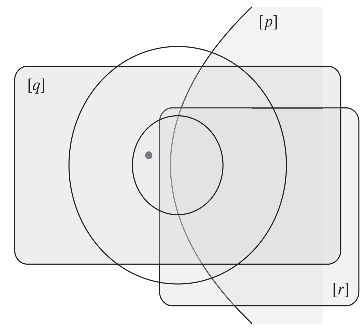
We have seen that the case against Right Weakening is easier to make than that against any of Cautious Monotony, Cut and Or. In my view, it is *the hallmark of difference-making conditionals* that they do not satisfy Right Weakening.[9] The conditionals modelled by conditional logics in the wake of Stalnaker and Lewis don't require the antecedent to be relevant for the consequent. Just as it is *the* most striking feature of such conditionals that they don't validate 'Left Strengthening' ('Strengthening the Antecedent'), it is *the* most striking feature of difference-making conditionals that the invalidate Right Weakening ('Weakening the Consequent').

8 This claim will be substantiated later, after Observation 5.4.

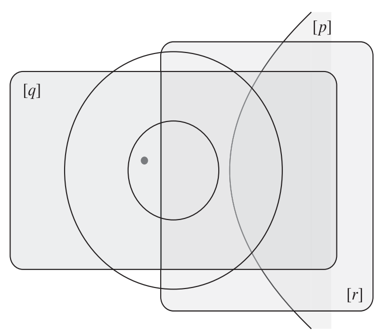
9 This is related to the phenomenon of 'transmission failure' of evidential support studied by Chandler (2013).




*(a)* Counterexample with $p, q, r \in Bel$.




*(b)* Counterexample with $p, \neg p \notin Bel$.




*(c)* Counterexample with $\neg p \in Bel$.

Fig. 4. RW is invalid for difference-making conditionals: $p > q \wedge r$, but not $p > q$.

**§3. The Ramsey Test, relevance and dependence.** We denote a belief state by $\mathfrak{B}$ and a belief set by *Bel*. The nature of $\mathfrak{B}$ is left completely open, except that we assume that one can always determine, with the help of a uniform method *Bel*, the agent's set of beliefs in belief state $\mathfrak{B}$. These beliefs are expressed in a certain language. Our object languages in this article features the logical constants $\top$ (*verum*) and $\bot$ (*falsum*), the usual truth-functional propositional operators $\neg$, $\wedge$, $\vee$, and $\supset$, as well as a conditional connective (mainly $\gg$, but briefly also $>$ and $\rightsquigarrow$). Conditionals formed with the help of this connective are not embedded in more complex sentences. I do not think of conditionals as expressing propositions (rather as expressing 'bi-propositional' attitudes, to use a term of Spohn, 2015, p. 5). The 'logic' of conditionals as studied in this article takes place in the meta-language—just as the logic of belief revision does within the AGM approach.

In the following, *Bel* is short for $Bel(\mathfrak{B})$, and $Bel * A$ is short for $Bel(\mathfrak{B} * A)$, which denotes the belief set obtained after revising one's belief state $\mathfrak{B}$ by a new piece of information or by a hypothetical assumption $A$.[10] We always presuppose in this article that the initial, unrevised belief set $Bel = Bel(\mathfrak{B})$ is consistent.

[10] Caution: The very notation '$Bel * A$' is slightly misleading. The correct notation would be '$Bel(\mathfrak{B} * A)$', but I hope the use of simpler expressions will not cause any confusion in this article.—I don't want to commit myself to the thesis that revising by new information and revising by a hypothetical assumption lead to the same result. They don't, as the scenarios underlying Thomason conditionals show.

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We will base this article essentially on the classical AGM theory of belief revision which will be recapitulated in §4. One of the standard readings of a conditional 'If $A$ then $C$' is given by the Ramsey Test:

$$A > C \mathbin{@} \mathfrak{B} \quad \text{iff} \quad C \in Bel * A\,. \tag{RT}$$

'$A > C$' may be read as 'If $A$ then plainly $C$', and '@ $\mathfrak{B}$' as 'is accepted in belief state $\mathfrak{B}$'. The Ramsey Test RT with '$A > C \in Bel$' substituted for '$A > C \mathbin{@} \mathfrak{B}$' was pioneered by Gärdenfors (1978) and recognised to be problematic by him a few years later. The notation using '@' is borrowed from Fuhrmann & Levi (1994);[11] it signals that conditionals are refused the status of beliefs (membership in *Bel*), thus blocking Gärdenfors's triviality result. In conjunction with the AGM theory, the Ramsey Test yields that whenever $A$ and $C$ are any two beliefs in *Bel*, the conditional 'If $A$ then $C$' is invariably accepted in $\mathfrak{B}$, regardless of whether $A$ is in any way (considered) relevant for $C$ and of whether there is any substantive connection between $A$ and $C$. This inference is counterintuitive and should be blocked.[12]

The idea now is to express the relevance of the antecedent $A$ for the consequent $C$ in the conditional, or—to change the perspective—to encode the idea of relevance in a single conditional connective. Rott (1986) proposed a reading of the conditional 'If $A$ then $C$' and recommended to use what he called the Strong Ramsey Test.[13] Now I prefer to use a more descriptive label and call it the *Relevant Ramsey Test*:

$$A \gg C \mathbin{@} \mathfrak{B} \quad \text{iff} \quad C \in Bel * A \ \text{ and } \ C \notin Bel * \neg A\,. \tag{RRT}$$

'$A \gg C$' may be read as 'If $A$ then relevantly $C$.' Let us call the conditional $\gg$ evaluated through $*$ by means of Definition RRT *difference-making conditional*.[14] If $A \vee C$ is not a belief, then $A \gg C$ has the same acceptability conditions as $A > C$. But crucially, the Relevant Ramsey Test makes good, non-trivial sense even for conditionals with true antecedents. It does not force us to accept the conditional 'If $A$ then $C$' for any two arbitrary beliefs $A$ and $C$ in *Bel*. I don't conceive of $A \gg C$ as a compound of two object-language sentences like $(A > C) \wedge \neg(\neg A > C)$. '$\gg$' is rather an intrinsically contrastive connective, much like the natural-language connectives 'because' and 'since'. It was in fact one of the main motivating ideas of Rott (1986) to bring out the similarities of 'if' and 'because'.[15]

[11] However, Fuhrmann and Levi apply '@' to theories rather than belief states.

[12] This view is not universally shared. Walters & Williams (2013) defend 'conjunction conditionalization' (also known as the 'and-to-if inference') within a possible-worlds framework against attacks from authors endorsing 'the Connection Hypothesis'.

[13] Chandler (2013) uses another variant of the Ramsey Test suggested in Rott (1986) for his analysis of the relation '$A$ is evidence for $C$'. Building on the same variant, Andreas & Günther (2019) explore the consequences of yet another variant for the analysis of 'Because $A$, $C$'. Since both Ramsey Test variants employ iterated changes of belief which are much more complex and controversial than one-shot belief changes, it will be much more difficult to carry through a project like the present one for them.

[14] Another good name would be 'explanatory conditional'. Even better might be the label 'relevance conditional', but unfortunately this term is quite commonly used in linguistics for something entirely different, namely for conditionals that are also known as (Austinian) 'biscuit conditionals' or 'speech-act conditionals'.

[15] In the terminology of Spohn (2012, pp. 107–108), $A \gg C$ expresses that $A$ is a sufficient reason for $C$. This was essentially pointed out by Raidl (2018, p. 230).

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Besides Gärdenfors's (1978), the work closest to the project of the present article is due to Fariñas & Herzig (1996). This is what they understand by the phrase '$C$ depends on $A$':

$$A \rightsquigarrow C \,@\, \mathfrak{B} \quad \text{iff} \quad C \in Bel \text{ and } C \notin Bel \dot{-} A\,. \tag{FHD}$$

Here $Bel\dot{-}A$ denotes that result of withdrawing the belief $A$ from the belief set $Bel$ (a 'belief contraction' in the terminology of AGM). Given the Harper identity that is widely endorsed within the AGM theory, FHD can also be written thus:

$$\begin{aligned} A \rightsquigarrow C \,@\, \mathfrak{B} \quad &\text{iff} \quad C \in Bel \text{ and } C \notin (Bel \cap Bel * \neg A) \\ &\text{iff} \quad C \in Bel \text{ and } C \notin Bel * \neg A\,. \end{aligned}$$

$A \rightsquigarrow C$ can only be accepted in a belief state if $A$ is a belief, due to the vacuity principle of AGM ('don't revise your belief set if the input is already believed, don't contract it if the input isn't believed'). If, on the other hand, $A$ is a belief, then $A \rightsquigarrow C$ has the same acceptability conditions as $A \gg C$.

It is easy to see that for all belief states $\mathfrak{B}$,

$$A \rightsquigarrow C \,@\, \mathfrak{B} \quad \Rightarrow \quad A \gg C \,@\, \mathfrak{B} \quad \Rightarrow \quad A > C \,@\, \mathfrak{B}\,.$$

The article by Fariñas and Herzig is particularly interesting because it demonstrates, in a way, that the theory of belief revision can in principle be based just upon the notion of dependence. Their dependency relation is restricted because its domain is only the agent's current belief set: $A \rightsquigarrow C \,@\, \mathfrak{B}$ implies that $A, C \in Bel$. Intuitively, however, we acknowledge many dependencies between non-beliefs, i.e., propositions that we either believe to be false or suspend judgement on. Counterfactuals, for instance, typically express dependencies between non-beliefs (in fact, even disbeliefs). This is why we will work with RRT rather than FHD.[16]

**§4. Classical belief revision.** We will base our discussion upon the following classical postulates of AGM belief revision:

(∗**1**) $Bel * A = Cn(Bel * A)$. (Closure)

(∗**2**) $A \in Bel * A$. (Success)

(∗**3**) $Bel * A \subseteq Cn(Bel \cup \{A\})$. (Inclusion)

(∗**4**) If $\neg A \notin Bel$, then $Bel \subseteq Bel * A$. (Preservation)

(∗**5a**) If $\bot \in Bel * A$, then $\bot \in Bel * A \wedge B$.

(∗**5b**) If $\bot \in Bel * A$ and $\bot \in Bel * B$, then $\bot \in Bel * A \vee B$.

(∗**6**) If $Cn(A) = Cn(B)$, then $Bel * A = Bel * B$. (Intensionality)

(∗**7**) $Bel * (A \wedge B) \subseteq Cn((Bel * A) \cup \{B\})$. (Conditionalisation)

(∗**8**) If $\neg B \notin Bel * A$, then $Bel * A \subseteq Bel * (A \wedge B)$. (Rational Monotony)

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16 It is interesting to compare the situation here with that of 'evidential conditionals' that are based on positive probabilistic relevance *and* high conditional probability. See Douven's (2008; 2016, chap. 4, p. 108) 'Evidential support thesis'. In the probabilistic context, there is no difference analogous to that between RRT and FHD, because $\Pr(C|A) > \Pr(C)$ if and only if $\Pr(C|A) > \Pr(C|\neg A)$. In this respect, ranking functions are closer to qualitative belief revision than to probabilities; see Spohn (2012, pp. 106–107).—Comments similar to those on (FHD) apply to the conditional defined by the clause $C \in Bel * A$ and $C \notin Bel$. It can only be accepted in a belief state if $A$ is *not* a belief. I will not further discuss this conditional.

Postulates (∗1)–(∗6) are called *basic* by AGM and are usually considered to be weak. Postulates (∗7) and (∗8)—also known as 'Conditionalisation' and 'Rational Monotony'—are called *supplementary* by AGM; they are very strong and make the AGM theory interesting in the first place. (∗1) is equivalent to the conjunction of the following two postulates:

(∗**1a**) If $B \in Bel * A$ and $C \in Bel * A$, then $B \wedge C \in Bel * A$. (And)

(∗**1b**) If $B \in Bel * A$ and $C \in Cn(B)$, then $C \in Bel * A$. (Singleton closure)

I have replaced the traditional fourth AGM postulate by (∗4) which is also known as 'Preservation'. The former is somewhat stronger than the latter, but its additional strength can be gained from (∗1) and (∗2). A similar simplification has been applied to (∗8). I do not endorse the traditional fifth AGM postulate stating that only revisions by contradictions will result in inconsistent belief sets, with the idea that non-contradictory doxastic impossibilities should be allowed, and use the much weaker postulates (∗5a) and (∗5b).[17]

We look at two prominent weakenings of the supplementary postulates that are well known in the literature of defeasible reasoning as characterising 'Cumulative Reasoning'.

(∗**7c**) If $B \in Bel * A$, then $Bel * (A \wedge B) \subseteq Bel * A$. (Cut)

(∗**8c**) If $B \in Bel * A$, then $Bel * A \subseteq Bel * (A \wedge B)$. (Cumulative Monotony)

Within the AGM theory, useful equivalents of (∗7c), (∗7), (∗8c), and (∗8) are

(∗**7c′**) If $A \in Bel * A \vee B$, then $Bel * A \subseteq Bel * A \vee B$.

(∗**7′**) $Bel * A \cap Bel * B \subseteq Bel * A \vee B$. (Or)

(∗**8c′**) If $A \in Bel * A \vee B$, then $Bel * A \vee B \subseteq Bel * A$.

(∗**8′**) If $\neg A \notin Bel * A \vee B$, then $Bel * A \vee B \subseteq Bel * A$.

For motivation and discussion of all these conditions, as well as proofs of equivalences,[18] the reader is referred to Gärdenfors (1988, chap. 3) or Hansson (1999).

Finally, it is good to be aware of the following fact about case-based reasoning that follows from (∗1) and (∗3) (and the deduction theorem):

(∗**Cas**) If $B \in Bel * A$ and $B \in Bel * \neg A$, then $B \in Bel$.

We suppose throughout this article that *Bel* is consistent; this is our zeroth postulate as it were. So by (∗3) and (∗4), $Bel = Bel * \top$.[19] By (∗Cas), $\bot \in Bel * A$ implies $\bot \notin Bel * \neg A$.

We call a sentence $A$ a (*doxastic*) *necessity* iff $\bot \in Bel * \neg A$, a (*doxastic*) *possibility* iff not $\bot \in Bel * A$, and a (*doxastic*) *impossibility* iff $\bot \in Bel * A$. We call $A$ a *contingent sentence* iff it is not a necessity.[20]

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17 It would also be interesting to see what happens if AGM is further changed by using; (∗2′) $A \in Bel * A$ or $Bel * A = Bel$ [or (∗2″) $A \in Bel * A$ or $\bot \in Cn(A)$] (Weak success); and (∗5′) $Bel * A$ is consistent (Strong consistency). The results would be quite different.

18 Notice that (∗8c) does not follow from (∗8) if we don't have AGM's fifth postulate, i.e., if we allow for doxastic impossibilities that are not logical contradictions. Notice also that (∗8c) implies (∗5a), (∗7′) implies (∗5b).

19 If *Bel* is inconsistent, this identity ceases to hold in the original AGM theory due to their fifth postulate.

20 By slight abuse of terminology, then, impossibilities and even contradictions are counted as contingent sentences here.

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| | | |
|---|---|---|
| $\bot \gg \bot$ | $\bot \notin Bel * \top$ | $Bel = Bel * \top$ is consistent. |
| $A \gg \bot$ | $\bot \in Bel * A$ | $A$ is a doxastic impossibility. |
| $\neg A \gg \bot$ | $\bot \in Bel * \neg A$ | $A$ is a doxastic necessity. |
| $\bot \gg A$ | $A \notin Bel * \top$ | $A$ is a non-belief. |
| $A \gg A$ | $A \notin Bel * \neg A$ | $A$ is contingent. |
| $A \gg A \wedge C$ | $C \in Bel * A$ and $\bot \notin Bel * \neg A$ | $C$ is in the revision $Bel * A$ and $A$ is contingent. |
| $A \gg A \vee C$ | $C \notin Bel * \neg A$ | $C$ is not in the revision $Bel * \neg A$. |
| not $\neg A \gg \neg A \vee C$ | $C \in Bel * A$ | $C$ is in the revision $Bel * A$. |

Table 1. *The meanings of some basic difference-making conditionals. We suppose that Bel is consistent*

**§5. Principles for difference-making conditionals.** Remember that we suppose throughout this article that the belief set *Bel* is consistent (though $Bel * A$ may well be inconsistent—this just means that $A$ is an impossibility). As the Relevant Ramsey Test makes reference only to revised belief sets, it does not say anything about the original belief set *Bel*. It is very natural to identify *Bel* with $Bel * \top$, and this identification is in fact a consequence of the consistency of *Bel* together with ($*3$) and ($*4$). The condition expressing the consistency of *Bel* is '$\bot \gg \bot$' (or: '$\bot \gg A$ for some $A$'). The condition expressing that $A$ is a belief is 'not $\bot \gg A$'. The 'meanings' of some basic difference-making conditionals are collected in Table 1.[21]

A conditional of the form $A \gg A \wedge C$ is a de-relevantised conditional; it expresses 'If $A$ then plainly $C$' (plus a side condition) using the difference-making conditional $\gg$. It is almost equivalent to the standard Ramsey conditional $A > C$ to which it only adds that $A$ is a contingent sentence. Since $A \gg A \wedge C$ is strictly weaker than $A \gg C$, the difference-making conditionals do not satisfy what Keenan & Stavi (1986, p. 275) and van Benthem (1986, pp. 8, 77) called 'Conservativity'. This comes as no surprise, because these conditionals, being contrastive, care about what happens when the antecedent is false.

***5.1. Basic principles.*** If a conditional of the form 'If $A$ then $C$' is used in everyday discourse and meant to convey that $A$ is relevant for $C$, then $A$ and $C$ are hardly ever logically related. What is common usage in a seminar on propositional logic is apt to cause bewilderment in practical contexts. In the sense of 'relevance' intended here, it sounds odd to say that a sentence is relevant for some of its subsentences or for some Boolean compounds containing it. Still odder does it sound to say that a sentence is relevant for itself. Yet, if one wants to present a conditional logic, it is just one's principal task to deal with such statements. We should not expect that the principles valid for difference-making conditionals are generally intuitively appealing. The most important thing to bear in mind is that the Relevant Ramsey Test RRT provides a clear and simple doxastic semantics that is applicable to arbitrary compounds of propositional sentences. We are now going to explore the logic of conditionals governed by RRT.

We list the *basic principles of difference-making conditionals*. All these principles are to be read as quantified over all belief states $\mathfrak{B}$, but the clause '@ $\mathfrak{B}$' after each conditional

[21] We can also express that $A$ is at most as entrenched as $B$ ($A \leq_{ent} B$), i.e., that $A \notin Bel * \neg(A \wedge B)$ or $\bot \in Bel * \neg B$, by '$A \wedge B \gg A$ or $\neg B \gg \bot$'.

is left implicit throughout: '$A \gg C$' is short for '$A \gg C$ @ $\mathfrak{B}$' and 'not $A \gg C$' is short for 'not $A \gg C$ @ $\mathfrak{B}$'. I trust that this somewhat sloppy notation will not cause confusion. A principle of the form 'If $\Phi$, then $\Psi$' formulates a *validity* in the sense that *for every belief state* $\mathfrak{B}$, if the conditionals mentioned in $\Phi$ are all accepted in $\mathfrak{B}$, then the conditionals mentioned in $\Psi$ are accepted in $\mathfrak{B}$. The variables $A$, $B$, and $C$ range over propositional sentences without any occurrences of the conditional connective.

(**≫0**) $\bot \gg \bot$.
(**≫1**) If $A \gg B \wedge C$, then $A \gg B$ or $A \gg C$.
(**≫2a**) $A \gg C$ iff ($A \gg A \wedge C$ and $A \gg A \vee C$).
(**≫2b**) $A \gg A \wedge C$ iff (not $\neg A \gg \neg A \vee C$ and $A \gg A$).
(**≫3–4**) $\bot \gg A \vee C$ iff ($\bot \gg A$ and $A \gg A \vee C$).
(**≫5**) $A \vee B \gg \bot$ iff ($A \gg \bot$ and $B \gg \bot$).
(**≫6**) If $Cn(A) = Cn(B)$ and $Cn(C) = Cn(D)$, then: $A \gg C$ iff $B \gg D$.

Condition (≫0) expresses that *Bel* is consistent. This is not part of the AGM theory, but a (harmless) general presumption made in this article. Somewhat surprisingly, it turns out that it makes sense to place contradictions to the left or to the right of difference-making conditionals.[22] As we have already seen, they can be used for characterising beliefs and non-beliefs, as well as doxastic necessities and possibilities. Conditions (≫3–4) and (≫5) refer to contradictions, too.

Condition (≫1) is our weaker replacement of a condition that is often called 'Right Weakening'. Read contrapositively, it says that when one denies 'If $A$ then relevantly $B$' and one denies 'If $A$ then relevantly $C$', then one denies 'If $A$ then relevantly $B$ and $C$'.

Condition (≫2a) is valid but close to trivial in ordinary conditional reasoning. It says that if one accepts 'If $A$ then relevantly $C$', i.e., if $A$ is a relevant antecedent for $C$, then $A$ is also a relevant antecedent for both the conjunction $A \wedge C$ and the disjunction $A \vee C$—and vice versa. Condition (≫2b) says that if $A$ is a relevant antecedent for $A \wedge C$, then $A$ is also a relevant antecedent for itself, but $\neg A$ is not a relevant antecedent for $\neg A \vee C$—and vice versa. By chaining (≫2a) and (≫2b), we can see that only contingent sentences can be relevant antecedents: If $A \gg C$, then $A \gg A$. We will later see that only contingent sentences can be consequents, too.[23]

Condition (≫3–4) says that $A \vee C$ is not believed if and only if $A$ is not believed and $A$ is a relevant antecedent for $A \vee C$. A part of the left-to-right direction of (≫3–4), namely

(**≫3**) If $\bot \gg A \vee C$, then $A \gg A \vee C$.

corresponds to (∗3), while the right-to-left direction of (≫3–4)

(**≫4**) If $\bot \gg A$ and $A \gg A \vee C$, then $\bot \gg A \vee C$.

corresponds to (∗4). Notice that (≫3–4) is actually more than the conjunction of (≫3) and (≫4). The surplus content, viz. the condition 'If $\bot \gg A \vee C$, then $\bot \gg A$', expresses the closure of *Bel* under singleton entailment.

[22] Things are very different for tautologies. We will soon derive principles stating that there is no role for tautologies on either side of a difference-making conditional: they can never be relevant antecedents or relevant consequents. This is what one would expect intuitively.

[23] In our non-standard sense of 'contingent'! See footnote 20 above.

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Condition ($\gg$5) says that impossibilities are closed under converse singleton entailment and disjunction. Strengthening an impossibility or forming the disjunction of two impossibilities will not lead to a possibility.

($\gg$6) is an intensionality principle that corresponds to AGM's sixth and part of their first postulates. Relevant antecedents and consequents may be replaced by logically equivalent sentences.

The fact that principles ($\gg$0), ($\gg$1), ($\gg$2a), ($\gg$2b), ($\gg$3–4), ($\gg$5), and ($\gg$6) are all valid on the basis of the Relevant Ramsey Test (RTT) and the basic AGM theory as presented in §4 will be formulated in Theorem 6.1 below.

The following lemma offers a rather long list of derived principles. Some of them are interesting in their own right, some will be useful in later proofs. (All proofs are collected in the Appendix.)

LEMMA 5.1 (Derived conditions). *Let principles* ($\gg$*0*), ($\gg$*1*), ($\gg$*2a*), ($\gg$*2b*), ($\gg$*3–4*), ($\gg$*5*), *and* ($\gg$*6*) *be given. Then*

(d1) *Not* $\top \gg A$.

(d2) *If* $A \gg A \wedge C$, *then* $A \gg A$. *Hence, if* $A \gg C$, *then* $A \gg A$.

(d3) $A \gg A$ *iff* ($A \gg A \vee C$ *or* $A \gg A \vee \neg C$).

(d4) $\bot \gg C$ *or* $\bot \gg \neg C$.

(d5) *If* $\neg A \gg \neg A$, *then*: *not both* $A \gg A \wedge C$ *and* $A \gg A \wedge \neg C$.

(d6) *If* $A \gg C$, *then not* $\bot \gg A \supset C$.

(d7) *If* $\bot \gg A$, *then* $A \gg A$.

(d8) $\bot \gg A$ *iff* ($\bot \gg A \wedge C$ *and* $A \wedge C \gg A$).

(d9) $\bot \gg A \wedge C$ *iff* ($\bot \gg A$ *or* $\bot \gg C$).

(d10) *If* $\bot \gg A$ *and* $A \gg C$, *then* $\bot \gg C$.

(d11) $A \gg A$ *iff not* $\neg A \gg \bot$.

(d12) *If* $A \gg \bot$, *then not* $\bot \gg \neg A$.

(d13) *If* $A \gg \bot$, *then* $\bot \gg A$.

(d14) *Not* $A \gg \top$.

(d15) *If* $A \gg A \vee C$, *then* $\bot \gg \neg A$ *or* $\bot \gg C$.

(d16) *If* $\bot \gg C$ *and not* $\bot \gg \neg A$, *then* $A \gg A \vee C$.

(d17) *If* $A \gg C$, *then* ($\bot \gg A$ *iff* $\bot \gg C$).

(d18) *If* $A \gg A \wedge C$ *and* $\neg A \gg \neg A \wedge C$, *then not* $\bot \gg C$.

(d19) *If* $\bot \gg C$, *then* $A \gg C$ *or* $\neg A \gg \neg A \vee C$.

(d20) *Not both* $A \gg \bot$ *and* $\neg A \gg \bot$.

(d21) $A \wedge B \gg A \wedge B$ *iff* ($A \gg A$ *or* $B \gg B$).

(d22) *Not* $A \gg \neg A \vee C$.

(d23) *Not* $A \vee C \gg \neg A$.

(d24) *If* $A \gg A \wedge B \wedge C$, *then* $A \gg A \wedge C$.

(d25) *If* $A \gg A \vee B \vee C$, *then* $A \gg A \vee C$.

(d26) *If* $A \gg B$ *and* $A \gg C$, *then* $A \gg B \wedge C$.

(d27) *If* $A \gg \bot$, *then* $A \gg A \wedge C$.

(d28) *If* $A \gg B \wedge C$ *and* $\bot \gg A \vee B$, *then* $A \gg C$.

(d29) *If* $A \gg C$ *and* $\bot \gg A$ *and* $\bot \gg \neg A$, *then* $\neg C \gg \neg A$.

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A few brief comments on some of these conditions are in order. The reason for principle (d1) is that $A \notin Bel * \bot$ is impossible, due to ($*$1) and ($*$2). The meaning of (d4) is that *Bel* is consistent. (d7) says that negations of non-beliefs are possibilities. (d9) says that the beliefs in *Bel* are closed under singleton entailment and under conjunction. (d11) identifies two equivalent ways of saying that $A$ is not a doxastic necessity. (d12) says that negations of impossibilities are beliefs; it is an important bridge between bottom-right and bottom-left principles. (d13) says that impossibilities are non-beliefs. The reason for principle (d14) is that $\top \notin Bel * \neg A$ is impossible, due to ($*$1). As it turns out, tautologies, and indeed doxastic necessities in general, can never be relevant antecedents or relevant consequents. (d16) corresponds to a vacuity condition for revisions stating that $Bel * A = Bel$ whenever $A \in Bel$. (d17) is an interesting condition equivalent to the conjunction of Modus Ponens and Affirming the Consequent. (d18) corresponds to the case-based reasoning condition ($*$CAS). As we pointed out above, Right Weakening in general is invalid for difference-making conditionals. Like ($\gg$1), the principles (d24), (d27), and (d28) are weakened forms of Right Weakening. Condition (d26) says that from 'If $A$ then relevantly $B$' and 'If $A$ then relevantly $C$', one can infer 'If $A$ then relevantly $B$ and $C$'. This principle is often called 'And' in the literature, and it is the only pattern of ordinary conditional reasoning that remains valid for the difference-making conditional $\gg$![24]

A number of important elementary implications are represented in Figures 5 and 6.

Recalling that 'not $\bot \gg A$' expresses that $A$ is a belief, the following principles are reminiscent of much studied valid and invalid patterns of inference:

**Observation 5.2.** *The difference-making conditional* $\gg$ *satisfies the following principles*:

($\gg$**MP**) *If* $A \gg C$ *and not* $\bot \gg A$*, then not* $\bot \gg C$. (*Modus Ponens*)

($\gg$**MT**) *If* $A \gg C$ *and not* $\bot \gg \neg C$*, then not* $\bot \gg \neg A$. (*Modus Tollens*)

($\gg$**AC**) *If* $A \gg C$ *and not* $\bot \gg C$*, then not* $\bot \gg A$. (*Affirming the Consequent*)

($\gg$**DA**$^w$) *If* $A \gg C$ *and not* $\bot \gg \neg A$*, then* $\bot \gg C$. (*quite a weak form of Denying the Antecedent*)

($\gg$MP) and ($\gg$MT) are forms of *Modus Ponens* and *Modus Tollens*. If $A \gg C$ is accepted and $A$ is a belief, then $C$ is a belief, too. If $A \gg C$ is accepted and $\neg C$ is a belief, then $\neg A$ is a belief, too. Moreover, due to the fact that $\gg$ embodies an idea of relevance, it also satisfies a form of the 'fallacy' of *Affirming the Consequent*, ($\gg$AC). However, the dual 'fallacy' of *Denying the Antecedent* is not satisfied by $\gg$; only a much weaker form holds: If $A \gg C$ and $\neg A$ is believed, it does not follow that $\neg C$ is believed, but only that $\neg C$ is consistent with the agent's beliefs.[25] Alleged fallacies become sound inferences when conditionals are understood in a different, difference-making way.

**Observation 5.3.** *The difference-making conditional* $\gg$ *satisfies the following connexivity principles*:

($\gg$**Arist1**) *Not* $A \gg \neg A$.

[24] The observation that And is not needed as a separate basic postulate for difference-making conditionals is due to Eric Raidl.

[25] Denying the Antecedent is fully valid for the dependence conditional $\succcurlyeq$ of §7. For general discussions concerning Affirming the Consequent and Denying the Antecedent, see Oaksford & Chater (2007, chap. 5) and Godden & Zenker (2015).

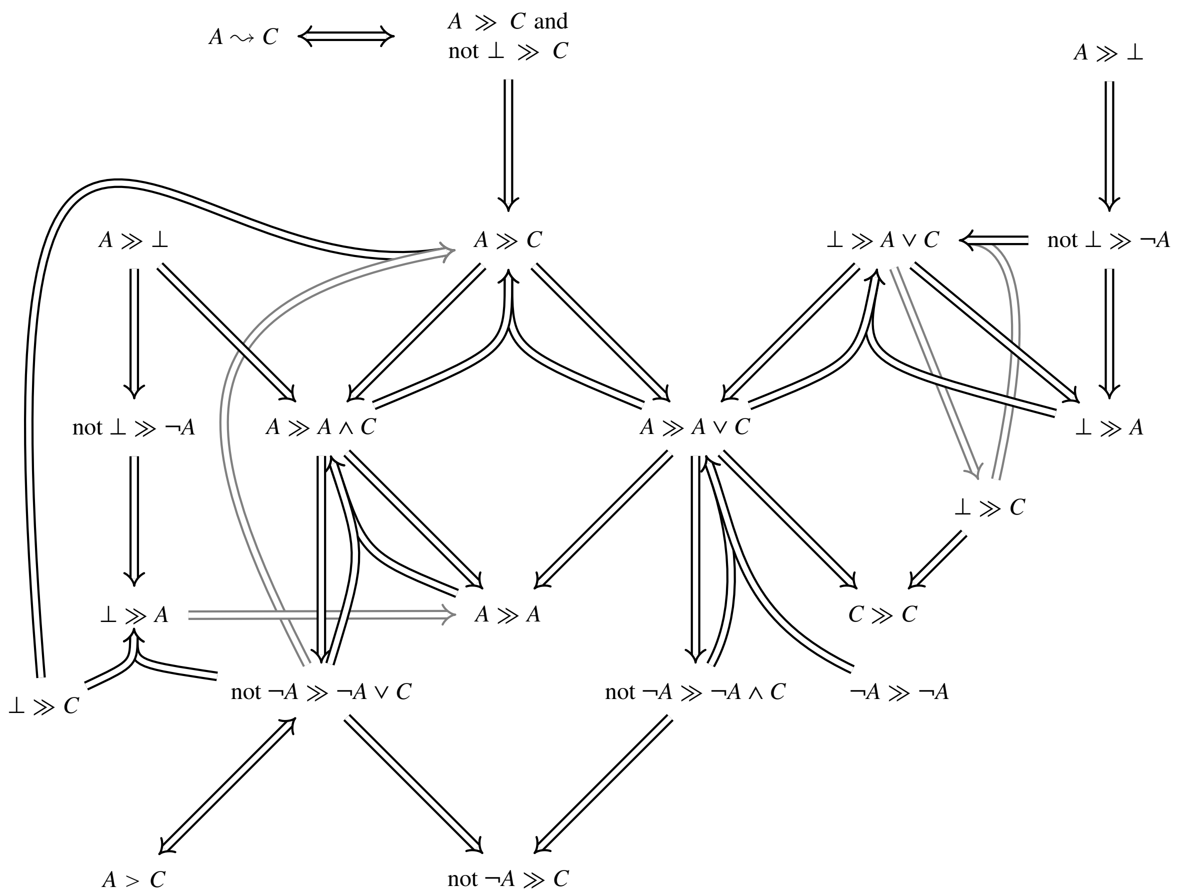



Fig. 5. Some useful implications and non-implications. The group of conditions $A \gg \bot$, not $\bot \gg \neg A$, $\bot \gg A$ and $\bot \gg C$ is represented twice, in order to avoid too many crossing arrows. Recall that $A \gg A$ is equivalent to not $\neg A \gg \bot$.

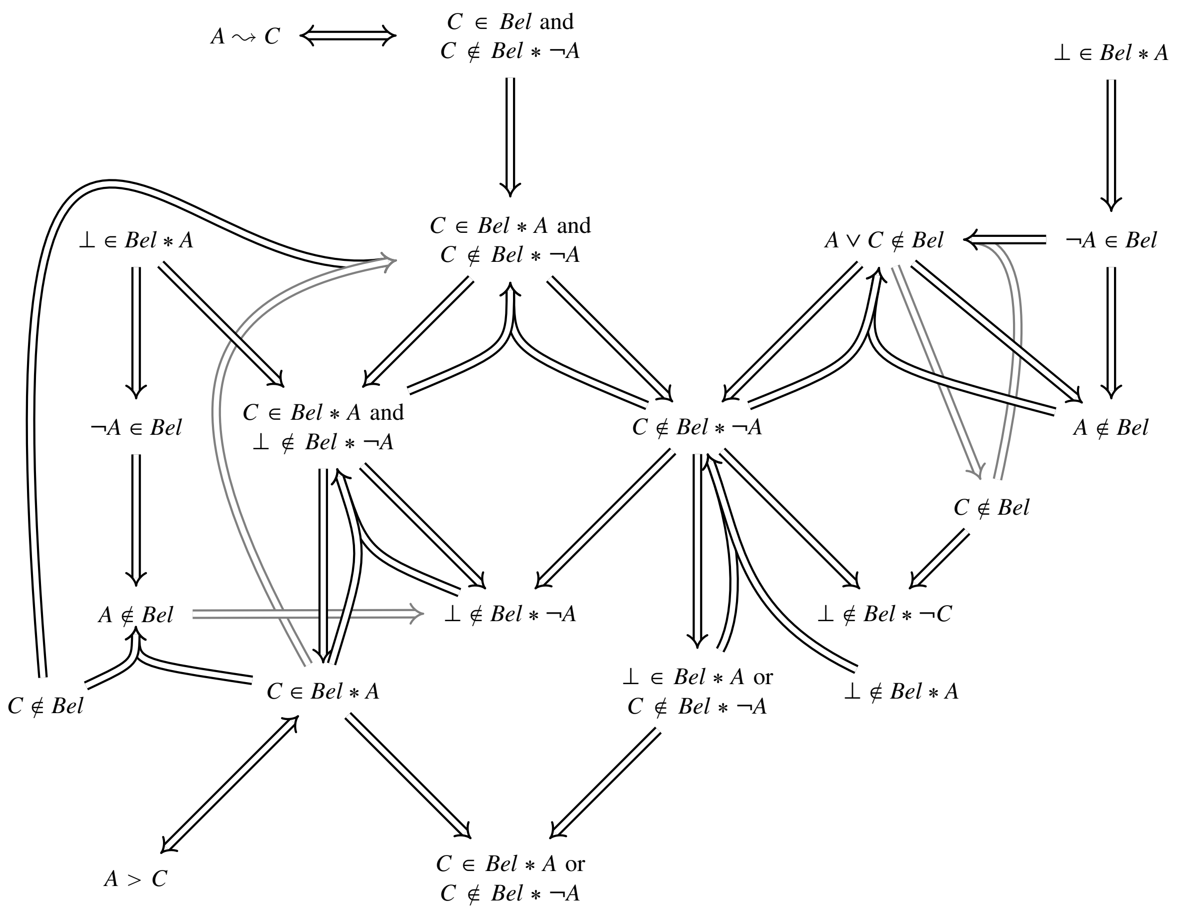
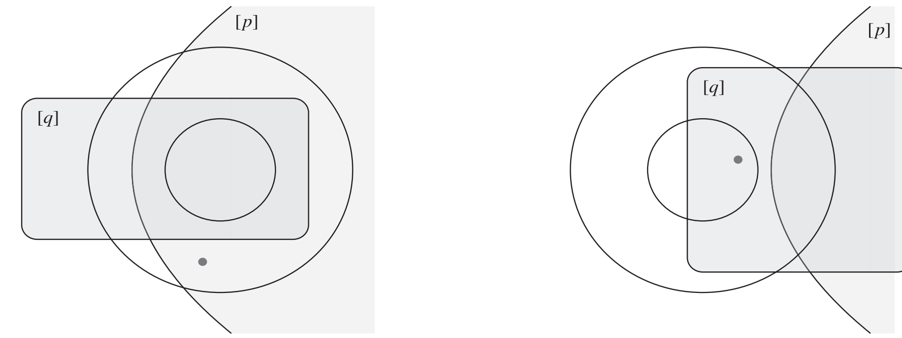



Fig. 6. The meanings of the conditions of Fig. 5.




*(a)* Counterexample with $p \in Bel$. *(b)* Counterexample with $\neg p \in Bel$.

Fig. 7. Contraposition is invalid for difference-making conditionals: $p \gg q$, but not $\neg q \gg \neg p$. (All counterexamples against Contraposition have either $p \in Bel$ or $\neg p \in Bel$.)

($\gg$**Arist2**) *Not both* $A \gg C$ *and* $\neg A \gg C$.

($\gg$**Ablrd**$^r$) *Not both* $A \gg C$ *and* $A \gg \neg C$*, unless* ($A \gg \bot$ *and* $\bot \gg C$ *and* $\bot \gg \neg C$).

($\gg$Arist1) is a variant of Aristotle's Thesis, ($\gg$Arist2) is Aristotle's Second Thesis.[26] ($\gg$Ablrd$^r$) is a substantial restriction of Abelard's First Principle. Thus the difference-making conditional $\gg$ has clear features of connexivity. In particular, it is excluded that 'If $A$ then relevantly $C$' and 'If $\neg A$ then relevantly $C$' at the same time. It can, however, happen that 'If $A$ then relevantly $C$' and 'If $A$ then relevantly $\neg C$' at the same time. In fact, this holds just in case $A$ is a doxastic impossibility and neither $C$ nor $\neg C$ is a belief. Thus $\gg$ satisfies only a restricted form of Abelard's Thesis and fails to be fully connexive.[27]

Finally, a word about Contraposition. Suppose that $A \gg C$. Contraposition into $\neg C \gg \neg A$ is illegitimate if either $A \in Bel$ (because then not in general $\neg A \in Bel * \neg C$) or $\neg A \in Bel$ (because then not in general $\neg A \notin Bel * C$); see Figure 7. As a counterexample, we can use a simplified version of our counterexample against Cautious Monotony. I believe that Pam works on the project ($p$) and that the project will be successful ($q$), because Pam is an excellent and dedicated researcher, and if she is missing, the project might fail. So I think 'If Pam works on the project, the project will be successful'. On the other hand, there is a (slightly remote) possibility that Pam will not perform well, and this is why I reject 'If the project is not successful, Pam has not worked on the project'. So contraposition in general fails for difference-making conditionals. However, we have a weaker form of Contraposition that is restricted to antecedents that the agent suspends judgment on (neither $A$ nor $\neg A$ is a belief): this is Condition (d29).[28]

***5.2. Compound-antecedent principles.*** We now turn to a systematic discussion of the effects of AGM's 'supplementary' belief revision postulates. As we pointed out above,

26 Since McCall (2012, p. 416), these names have become fairly common. Martin (1987, pp. 379–381) who introduced the labels had the numbering the other way round.

27 Abelard's First Principle could be fully validated by postulating that *all* revisions $Bel * A$ are consistent. Such an adaptation of AGM is feasible—one would then have to restrict the success postulate ($*2$) to possibilities—, but this is not the route taken in this article.

28 In the probabilistic context (cf. footnote 16), Contraposition is valid for positive-relevance conditionals: $\Pr(C|A) > \Pr(C|\neg A)$ if and only if $\Pr(\neg A|\neg C) > \Pr(\neg A|C)$. Contraposition is also valid in the system of Crupi and Iacona.

almost all of the characteristic axioms of the usual conditional logics fail for difference-making conditionals, the only exception being And, also known as Conjunction in the Consequent. Because they in a way ‘say too much’—their acceptability conditions all have two quite different internal parts—, difference-making conditionals behave quite oddly.

Is there any further logic left? Yes, there is. It is actually not difficult to ‘translate’ the belief revision postulates and their counterparts for the non-relevant conditional $>$ into the language of difference-making conditionals. This is possible by making extensive use of de-relevantised conditionals of the form ‘$A \gg A \wedge C$’. These are *supplementary principles of difference-making conditionals:*

(**≫7c**) If $A \gg A \wedge B$ and $A \wedge B \gg A \wedge B \wedge C$, then $A \gg A \wedge C$.

(**≫7**) If $A \wedge B \gg A \wedge B \wedge C$ and $A \gg A$, then $A \gg A \wedge (B \supset C)$.

(**≫7′**) If $A \gg A \wedge C$ and $B \gg B \wedge C$ and $A \vee B \gg A \vee B$, then $A \vee B \gg (A \vee B) \wedge C$.

(**≫8c**) If $A \gg A \wedge B$ and $A \gg A \wedge C$, then $A \wedge B \gg A \wedge B \wedge C$.

(**≫8**) If not $A \gg A \wedge \neg B$ and $A \gg A \wedge C$, then $A \wedge B \gg A \wedge B \wedge C$.

Given the basic principles, (≫7) is interderivable with (≫7′). There are counterparts to the supplementary AGM postulates that are simpler, but restricted to cases where the antecedent of a crucial conditional is not believed (and perhaps some other sentence is disbelieved):

**Observation 5.4.** *The conditions* (≫7c)*,* (≫7′)*,* (≫8c)*, and* (≫8)*, respectively, imply the following restricted simpler versions*:

(**≫7c$^r$**) *If $A \wedge B \gg C$ and $A \gg B$ and $\bot \gg A$, then $A \gg C$.*

(**≫7$'^r$**) *If $A \gg C$ and $B \gg C$ and $\bot \gg A \vee B$, then $A \vee B \gg C$.*

(**≫8c$^r$**) *If $A \gg C$ and $A \gg B$ and $\bot \gg A$, then $A \wedge B \gg C$.*

(**≫8$^r$**) *If $A \gg C$ and not $A \gg \neg B$ and $\bot \gg A$, then $A \wedge B \gg C$.*

Observation 5.4 explains why counterexamples against Cautious Monotony, Cut and Or for $\gg$ have to be cases in which $A$, $B$, and $C$ are beliefs (cf. §2). More precisely, (≫7c$^r$) and (≫8c$^r$) show that counterexamples against Cautious Monotony and Cut require not $\bot \gg A$, i.e., that $A$ is a belief. Given the antecedents of (≫7c$^r$) and (≫8c$^r$) and (d17), this entails that $B$ and $C$ are beliefs, too. For Or, we look at (≫7$'^r$) which tells us that counterexamples have to have $A \vee B$ as a belief. Applying (d6) to the premises of (≫7$'^r$) and then using (d9) shows that $C$ is a belief, which in turn entails, by (d17), that $A$ and $B$ are beliefs, too.

**§6. Constructing revisions from difference-making conditionals.** So far we have taken an AGM revision function in the sense of §4 as given, and analysed the conditional connective $\gg$ as obtained from $*$ by RRT. In this section, we also take the converse perspective. Given the set of difference-making conditionals accepted by the agent in her belief state $\mathfrak{B}$, can we determine the result of the revision of her beliefs by a new sentence $A$? It turns out that we can.

$$C \in Bel * A \quad \text{iff} \quad A \gg A \wedge C \,@\, \mathfrak{B} \text{ or } (\neg A \gg \bot \,@\, \mathfrak{B} \text{ and not } \bot \gg C \,@\, \mathfrak{B}). \quad (\text{Def}*)$$

According to this definition, $C$ is in the revised belief set $Bel * A$ just in case $A \gg A \wedge C$ is accepted in belief state $\mathfrak{B}$ (principal case) *or* $A$ is a doxastic necessity and $C$ a belief

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(limiting case). There is an equivalent and more compact alternative definition:

$$C \in Bel * A \quad \text{iff} \quad \text{not } \neg A \gg \neg A \vee C \mathbin{@} \mathfrak{B}\,. \qquad \text{(Def2}*\text{)}$$

Although this second definition is more elegant than the first one, we will work with the first which is much more intuitive. This is because (Def∗) centres on elementhood in a revision, while (Def2∗) centres on non-elementhood in a revision (and is thus closer to the idea of a belief *contraction* than to a belief revision).[29]

What is the belief set, given the set of difference-making conditionals? We use the identity $Bel = Bel * \top$ and note that this set is consistent by hypothesis (in the original AGM theory: by (∗5)). We can thus get from $\gg$ to *Bel* by either (Def∗) or (Def2∗) and obtain:

$$A \in Bel \quad \text{iff} \quad \text{not } \bot \gg A \mathbin{@} \mathfrak{B}\,. \qquad \text{(Def-Bel)}$$

We are finally in a position to list the collection of validities of difference-making conditionals.

THEOREM 6.1. *Let ∗ be a basic AGM revision function in the sense of §4, and let $\gg$ be obtained from ∗ by RRT. Then $\gg$ satisfies the principles* ($\gg$*0*), ($\gg$*1*), ($\gg$*2a*), ($\gg$*2b*), ($\gg$*3–4*), ($\gg$*5*), *and* ($\gg$*6*), *and* (*Def∗*) *is satisfied, too.*
*If ∗ in addition satisfies* (∗*7c*), (∗*7*), (∗*7′*), (∗*8c*) *or* (∗*8*), *then $\gg$ satisfies* ($\gg$*7c*), ($\gg$*7*), ($\gg$*7′*), ($\gg$*8c*) *or* ($\gg$*8*), *respectively.*

In a converse to Theorem 6.1, it can be shown that any conditional satisfying the basic principles characterising difference-making conditionals can be represented as based on an AGM revision function and the Relevant Ramsey Test.

THEOREM 6.2 (Representation theorem). *Let $\gg$ be a conditional satisfying the principles* ($\gg$*0*), ($\gg$*1*), ($\gg$*2a*), ($\gg$*2b*), ($\gg$*3–4*), ($\gg$*5*), *and* ($\gg$*6*), *and let ∗ be obtained from $\gg$ by* (*Def∗*). *Then ∗ is a basic AGM revision function in the sense of §4, and RRT is satisfied.*
*If $\gg$ in addition satisfies* ($\gg$*7c*), ($\gg$*7*), ($\gg$*7′*), ($\gg$*8c*) *or* ($\gg$*8*), *then ∗ satisfies* (∗*7c*), (∗*7*), (∗*7′*), (∗*8c*) *or* (∗*8*), *respectively.*

**§7. The Dependent Ramsey Test.** The Relevant Ramsey Test RRT seems to be the most principled way of implementing the idea of relevance in conditionals. But there is an alternative to it that is attractive, too. Let us call it the *Dependent Ramsey Test*:

$$A \succcurlyeq C \mathbin{@} \mathfrak{B} \quad \text{iff} \quad C \in Bel * A \ \text{ and } \ \neg C \in Bel * \neg A\,. \qquad \text{(DRT)}$$

We will read '$A \succcurlyeq C$' as 'If *A* then dependently *C*.' The term is not supposed to refer to the article by Fariñas and Herzig, but it is chosen in analogy to Lewis (1973a, p. 563) who famously proposed reducing causal dependence to counterfactual dependence in the following way: If *A* and *C* describe the occurrences of two events *a* and *c*, then *c* causally depends on *a* if and only if the counterfactuals $A \mathbin{\Box\!\!\rightarrow} C$ and $\neg A \mathbin{\Box\!\!\rightarrow} \neg C$ are both true. The clause of DRT is stronger than that of RRT unless $Bel * \neg A$ is inconsistent (i.e., unless *A* is a necessity).

[29] Another alternative definition would be $C \in Bel * A$ iff $A \gg A \wedge C$ or ($\bot \gg \neg A$ and not $\bot \gg A \supset C$). This would only use bottom-left conditions, but it is still more complex and has a much wider 'limiting-case clause' than (Def∗). (Actually, '$\bot \gg \neg A$ and not $\bot \gg A \supset C$' does not really specify a limiting case any more.)

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Let us suppose that $*$ is a basic AGM revision function in the sense of §4. Then the most conspicuous property of the dependence conditional $\succ$ is that $A \succ C$ is equivalent to $\neg A \succ \neg C$. This follows immediately from Definition DRT together with ($*$1) and ($*$6), and it shows that the connective $\succ$ embodies something like 'conditional perfection' (van der Auwera, 1997). But it does not imply that $\succ$ is a biconditional. The meaning of $A \succ C$ is still different from that of $C \succ A$. As with the difference-making conditional $\gg$, contraposition is invalid for the dependence conditional $\succ$.

We can express that $A$ is a belief by $\top \succ A$ (or by $\bot \succ \neg A$), and that $A$ is a necessity by $\neg A \succ \bot$ (or by $A \succ \top$). Our general assumption that *Bel* is consistent can be captured by 'not $\top \succ \bot$'. It is not hard to verify the claims of the following observation.

**Observation 7.1.** *Let $*$ be a basic AGM revision function in the sense of §4. Then the dependence conditional $\succ$ satisfies the following principles*:

($\succ$**0**) *Not* $\top \succ \bot$.
($\succ$**1a**) *If* $A \succ B$ *and* $A \succ C$, *then* $A \succ B \wedge C$.
($\succ$**1b**) *If* $A \succ A \wedge B \wedge C$, *then* $A \succ A \wedge C$.
($\succ$**1c**) $A \succ C$ *iff* $\neg A \succ \neg C$.
($\succ$**2a**) $A \succ C$ *iff* ($A \succ A \wedge C$ *and* $A \succ A \vee C$).
($\succ$**2b**) $A \succ A$.
($\succ$**3–4**) $\top \succ A \vee C$ *iff* ($\top \succ A$ *or* $A \succ A \vee \neg C$).
($\succ$**5**) $A \vee B \succ \bot$ *iff* ($A \succ \bot$ *and* $B \succ \bot$).
($\succ$**6**) *If* $Cn(A) = Cn(B)$ *and* $Cn(C) = Cn(D)$, *then*: $A \succ C$ *iff* $B \succ D$.
($\succ$**MP**) *If* $A \succ C$ *and* $\top \succ A$, *then* $\top \succ C$. (*Modus Ponens*)
($\succ$**MT**) *If* $A \succ C$ *and* $\top \succ \neg C$, *then* $\top \succ \neg A$. (*Modus Tollens*)
($\succ$**AC**) *If* $A \succ C$ *and* $\top \succ C$, *then* $\top \succ A$. (*Affirming the Consequent*)
($\succ$**DA**) *If* $A \succ C$ *and* $\top \succ \neg A$, *then* $\top \succ \neg C$. (*Denying the Antecedent*)
($\succ$**Arist1**) *Not* $A \succ \neg A$. (*Aristotle's first thesis*)
($\succ$**Arist2**) *Not both* $A \succ C$ *and* $\neg A \succ C$. (*Aristotle's second thesis*)
($\succ$**Ablrd**) *Not both* $A \succ C$ *and* $A \succ \neg C$. (*Abelard's first principle*)

Thus the dependence conditional satisfies quite a few basic principles of difference-making conditionals, and also some variants of them.[30] It validates not only Modus Ponens and Modus Tollens, but also the supposed fallacies of Denying the Antecedent and Affirming the Consequent. It also validates all the analogues to the basic principles of connexive logic. The supplementary AGM postulates give rise to principles of dependence conditionals similar to—and slightly more straightforward than—the corresponding principles of difference-making conditionals:

($\succ$**7c**) If $A \succ A \wedge B$ and $A \wedge B \succ A \wedge B \wedge C$, then $A \succ A \wedge C$.
($\succ$**7**) If $A \wedge B \succ A \wedge B \wedge C$, then $A \succ A \wedge (B \supset C)$.
($\succ$**7′**) If $A \succ A \wedge C$ and $B \succ B \wedge C$, then $A \vee B \succ (A \vee B) \wedge C$.
($\succ$**8c**) If $A \succ A \wedge B$ and $A \succ A \wedge C$, then $A \wedge B \succ A \wedge B \wedge C$.
($\succ$**8**) If not $A \succ A \wedge \neg B$ and $A \succ A \wedge C$, then $A \wedge B \succ A \wedge B \wedge C$.

30 Notice, in particular, that ($\succ$1a) and ($\succ$1b) correspond to (d26) and (d24) above, respectively.

The content of a plain conditional $A > C$ can easily be expressed by the independenised conditional $A \ggcurly A \wedge C$. AGM revisions can be constructed from dependence conditionals by putting

$$C \in Bel * A \quad \text{iff} \quad A \ggcurly A \wedge C \ @ \ \mathfrak{B}. \qquad \text{(Ddef}*\text{)}$$

It is then possible to derive an analogue to Theorem 6.2 for $\ggcurly$.

THEOREM 7.2 (Representation theorem for dependence conditionals). *Let $\ggcurly$ be a conditional satisfying the principles ($\ggcurly$0), ($\ggcurly$1a), ($\ggcurly$1b), ($\ggcurly$1c), ($\ggcurly$2a), ($\ggcurly$2b), ($\ggcurly$3–4), ($\ggcurly$5), and ($\ggcurly$6), and let $*$ be obtained from $\ggcurly$ by (Ddef$*$). Then $*$ is a basic AGM revision function in the sense of §4, and DRT is satisfied.*
*If $\ggcurly$ in addition satisfies ($\ggcurly$7c), ($\ggcurly$7), ($\ggcurly$7′), ($\ggcurly$8c) or ($\ggcurly$8), then $*$ satisfies ($*$7c), ($*$7), ($*$7′), ($*$8c) or ($*$8), respectively.*

Overall, even though dependence conditionals are considerably easier to handle than difference-making conditionals, they are connectives that are as interesting as the latter.

**§8. Conclusion.** Different kinds of conditionals validate different sets of inference patterns. For natural-language conditionals as opposed to material (or strict) conditionals, strengthening the antecedent is invalid. This has been one of the most important messages of conditional logic ever since the times of Goodman, Adams, Stalnaker, and Lewis. For difference-making conditionals as opposed to the conditionals commonly studied in conditional logic, the dual pattern of weakening the consequent is invalid, too. What raises the doxastic status of $C$ does not necessarily raise the doxastic status of $B \vee C$. Many other well-known patterns get lost and conditionals appear to behave rather irregularly if the relevance idea is heeded. Still there is a logic of difference-making conditionals as captured by the Relevant Ramsey Test. I have presented this logic in this article, and I have shown how it relates to the classical AGM logic of belief revision.

However, the idea pursued in this article is not tied to belief-revision semantics. The reader uncomfortable with this framework may choose her favourite semantics for a conditional $>$ that doesn't encode relevance and then introduce the difference-making conditional $\gg$ by defining that $A \gg C$ is true/accepted/acceptable/assertable if and only if $A > C$ is, but $\neg A > C$ is not, true/accepted/acceptable/assertable. The axioms of standard conditional logics like system **P** (that have counterparts in the belief revision principles like those discussed in §5.2) should then map to axioms of difference-making conditionals in a way similar to the above.

It has been the ambition of this article to investigate a qualitative idea for the meaning of conditionals and a notion of validity that are both natural and simple. Thus it may seem surprising that the resulting logic is rather complex and violates most of the standard principles for conditional logic. But there is little correlation between the former and the latter. This can also be seen from an analogy with the situation in probabilistic frameworks. Hawthorne & Makinson (2007) couple a simple meaning (conditional probability over a given threshold) with a simple notion of validity ('probabilistic soundness', acceptance in all probability functions) and find that And, CMon, Cut and Or are all violated. Hawthorne and Makinson's account does not even involve relevance. The study of a conditional logic incorporating relevance in a probabilistic framework has, to the best of my knowledge, only begun very recently, with the analysis of 'evidential conditionals' by Douven (2016, chap. 5) and Crupi & Iacona (2018). Crupi and Iacona's conditionals have a more complex meaning ('degree of evidential support') and a more complex notion of validity (Adams's

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uncertainty sum soundness), and the logic they identify does violate RW—the hallmark of relevance!—, but it validates And, CMon, Or and even Contraposition.[31]

It is a pity that the axioms for difference-making conditionals are not easily accessible intuitively. But we should not expect them to be. First, the relevance relation encoded by the conditional connective $\gg$ *is* rather complex, and thus grasping its precise content is cognitively demanding. And there is a second reason that may be even more important. While the idea of relevance in conditionals is very natural, the antecedents and consequents of ordinary conditionals as used in natural language are rarely—if ever—logically related. Yet this is exactly what almost all of the valid principles of the logic of difference-making conditionals are about. So we should not be surprised to see that many of them come across as rather weird.

## §9. Appendix: Proofs.

*Proof of Lemma 5.1.* Let principles ($\gg$0), ($\gg$1), ($\gg$2a), ($\gg$2b), ($\gg$3–4), ($\gg$5), and ($\gg$6) be given.

(d1) Suppose for reductio that $\top \gg A$. Then, by ($\gg$2a), $\top \gg \top \vee A$, so, by ($\gg$2b), not $\neg\top \gg \neg\top \wedge A$. By ($\gg$6), this means that not $\bot \gg \bot$, contradicting ($\gg$0).

(d2) Suppose that $A \gg A \wedge C$. Then, by ($\gg$2a), $A \gg A \vee (A \wedge C)$. So $A \gg A$, by ($\gg$6). For the second part, suppose that $A \gg C$. Then, by ($\gg$2a), $A \gg A \wedge C$, and we can apply the first part.

(d3) For the left-to-right direction, note that $A$ is equivalent to $(A \vee C) \wedge (A \vee \neg C)$ and apply ($\gg$1). For the right-to-left direction, note that $A \gg A \vee C$ implies $A \gg A \wedge (A \vee C)$, by ($\gg$2a), which is equivalent to $A \gg A$, by ($\gg$6). Similarly for $A \gg A \vee \neg C$.

(d4) follows from ($\gg$0) and (d3); use $A = \bot$.

(d5) follows from ($\gg$2b) and (d3).

(d6) Suppose that $A \gg C$. Then, by ($\gg$2a), $A \gg A \wedge C$, so, by ($\gg$2b), not $\neg A \gg \neg A \vee C$. By ($\gg$3), then, not $\bot \gg \neg A \vee C$, and, by ($\gg$6), not $\bot \gg A \supset C$.

(d7) follows from ($\gg$3) and ($\gg$6); use $C = A$.

(d8) is equivalent to ($\gg$3–4).

(d9) The direction from left to right follows immediately from ($\gg$1); just substitute $\bot$ for $A$ in ($\gg$1). For the converse direction, first note that, by ($\gg$6), $\bot \gg A$ is equivalent to $\bot \gg (A \wedge C) \vee (A \wedge \neg C)$. Using ($\gg$3–4), we conclude that $\bot \gg A \wedge C$.

(d10) Suppose that $\bot \gg A$ and $A \gg C$. From the latter, by ($\gg$2a), $A \gg A \vee C$. From this and $\bot \gg A$, by ($\gg$4), $\bot \gg A \vee C$. By ($\gg$3–4), $\bot \gg C$.

(d11) For the direction from left to right, suppose that $A \gg A$, i.e., $A \gg A \vee A$. Then, by ($\gg$2b), not $\neg A \gg \neg A \wedge A$, i.e., not $\neg A \gg \bot$. For the other direction, we first note that ($\gg$0), ($\gg$6), and (d9) together give us that either $\bot \gg A$ or $\bot \gg \neg A$. Applying (d7) twice, we get that $A \gg A$ or $\neg A \gg \neg A$. Second, by substituting $\neg A$ for $C$ in ($\gg$2b), we get that not $\neg A \gg \neg A$ and $A \gg A$ together imply $A \gg \bot$. Using $A \gg A$ or $\neg A \gg \neg A$, this reduces to the fact that not $\neg A \gg \neg A$ implies $A \gg \bot$. Equivalently, not $\neg A \gg \bot$ implies $A \gg A$, as desired.

(d12) follows from (d11) and (d7).

(d13) follows from (d4) and (d12).

(d14) Suppose for reductio that $A \gg \top$. Then, by ($\gg$2a), $A \gg A \vee \top$, and, by ($\gg$2b), not $\neg A \gg \neg A \wedge \top$, i.e., not $\neg A \gg \neg A$. This means, by (d11), $A \gg \bot$, and, by (d13),

[31] Douven's logic violates RW, And, CMon, Cut, Or and Contraposition.

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$\bot \gg A$. On the other hand, if we substitute $\neg A$ (or $\top$) for $C$ in ($\gg$4), we get that $\bot \gg A$ and $A \gg \top$ imply $\bot \gg \top$. Now we know from (d1) that not $\top \gg \top$, so, by (d7), not $\bot \gg \top$. We conclude that if $A \gg \top$, then not $\bot \gg A$. By our supposition that $A \gg \top$, we get that not $\bot \gg A$, contradicting what we inferred before. So not $A \gg \top$.

(d15) follows from (d4) and ($\gg$4).

(d16) Suppose that $\bot \gg C$ and not $\bot \gg \neg A$. Because $\neg A \wedge (A \vee C)$ implies $C$, we get $\bot \gg \neg A \wedge (A \vee C)$, by ($\gg$6) and (d9). By the other direction of (d9), then $\bot \gg A \vee C$. But thus $A \gg A \vee C$, by ($\gg$3).

(d17) follows from ($\gg$2a), ($\gg$4), and (d16).

(d18) Suppose that $A \gg A \wedge C$ and $\neg A \gg \neg A \wedge C$. Then, by (d6) and ($\gg$6), not $\bot \gg A \supset C$ and not $\bot \gg \neg A \supset C$. So by (d9) and ($\gg$6), not $\bot \gg C$.

(d19) Suppose that $\bot \gg C$ and not $\neg A \gg \neg A \vee C$. Then, by (d16), $\bot \gg A$ and, by (d7), $A \gg A$, from which we get, by ($\gg$2b), $A \gg A \wedge C$. On the other hand, by (d9), $\bot \gg A \vee C$ and, by ($\gg$3), $A \gg A \vee C$. So, by ($\gg$2a), $A \gg C$.

(d20) follows from (d1) and ($\gg$5). (This is a dual to (d4).)

(d21) is equivalent to ($\gg$5), given (d11).

(d22) follows from ($\gg$2a) and (d14).

(d23) follows from ($\gg$2a) and (d14).

(d24) Suppose $A \gg A \wedge B \wedge C$. Then, by ($\gg$6), $A \gg A \wedge C \wedge (A \supset B)$. So by ($\gg$1), either $A \gg A \wedge C$ or $A \gg A \supset B$. But the latter is impossible, by (d22) and ($\gg$6).

(d25) Suppose that $A \gg A \vee B \vee C$. Case 1: $\neg A \gg \neg A$. By ($\gg$2b), not $\neg A \gg \neg A \wedge (B \vee C)$. Hence, by (d24), not $\neg A \gg \neg A \wedge (B \vee C) \wedge (\neg B \vee C)$, and, by ($\gg$6), not $\neg A \gg \neg A \wedge C$. So, by ($\gg$2b), $A \gg A \vee C$. Case 2: not $\neg A \gg \neg A$. Then, by (d11), $A \gg \bot$. So, by (d13), $\bot \gg A$. Suppose that $A \gg A \vee B \vee C$. Then, by ($\gg$4), $\bot \gg A \vee B \vee C$. By (d9), $\bot \gg A \vee C$, and, by ($\gg$3), $A \gg A \vee C$.

(d26) Suppose that $A \gg B$ and $A \gg C$. By the left-to-right directions of ($\gg$2a) and ($\gg$2b), we get $A \gg A \vee B$ and $A \gg A \vee C$, and $A \gg A$, not $\neg A \gg \neg A \vee B$ and not $\neg A \gg \neg A \vee C$. From the latter two, we can deduce with ($\gg$1) that not $\neg A \gg (\neg A \vee B) \wedge (\neg A \vee C)$. Thus by ($\gg$6), not $\neg A \gg \neg A \vee (B \wedge C)$. Since $A \gg A$, we can apply ($\gg$2b) in the right-to-left direction and get $A \gg A \wedge (B \wedge C)$. On the other hand, we know that $A \gg A \vee B$, which is equivalent to $A \gg A \vee (B \wedge C) \vee (B \wedge \neg C)$, by ($\gg$6). So, by (d25), $A \gg A \vee (B \wedge C)$. Putting this and $A \gg A \wedge (B \wedge C)$ together with the help of the right-to-left direction of ($\gg$2a), we get $A \gg B \wedge C$.

(d27) follows from (d24) and ($\gg$6).

(d28) From $A \gg B \wedge C$, we get $A \gg A \wedge B \wedge C$, by ($\gg$2a). Thus, $A \gg A \wedge C$, by (d24), from which we conclude not $\neg A \gg \neg A \vee C$, by ($\gg$2b). From $\bot \gg A \vee B$, we get $\bot \gg A$, by (d9), from which and $A \gg A \wedge C$ we conclude that $\bot \gg C$, by ($\gg$4). From this and not $\neg A \gg \neg A \vee C$, we conclude that $A \gg C$, by (d19), and this is just what we desired.

(d29) Suppose that $A \gg C$, $\bot \gg A$ and $\bot \gg \neg A$. Then by (d6), not $\bot \gg A \supset C$. From $\bot \gg A$ and $A \gg C$, we get $\bot \gg C$, by (d17). From this and not $\bot \gg A \supset C$, we get not $C \gg \neg A \vee C$, by ($\gg$4). On the other hand, we have $\bot \gg \neg A$, i.e., $\bot \gg ((A \supset C) \wedge (\neg A \vee \neg C))$, and not $\bot \gg A \supset C$, for which we infer, with the help of (d9) that $\bot \gg \neg A \vee \neg C$, and thus, by ($\gg$3), $\neg C \gg \neg A \vee \neg C$. This entails $\neg C \gg \neg C$, by (d2). This, together with not $C \gg \neg A \vee C$, gives us $\neg C \gg \neg A \wedge \neg C$, by ($\gg$2b). This together with $\neg C \gg \neg A \vee \neg C$ at last yields $A \gg C$, by ($\gg$2a), as desired. □

*Proof of Observation 5.2.* ($\gg$MP) follows from ($\gg$2a) and (d16).

For ($\gg$MT), suppose that $A \gg C$ and $\bot \gg \neg A$. We need to show that $\bot \gg \neg C$. From $A \gg C$ we get, by ($\gg$2a) and ($\gg$2b), not $\neg A \gg \neg A \vee C$. By ($\gg$3), not $\bot \gg \neg A \vee C$. From

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$\bot \gg \neg A$, we get with (d9) that $\bot \gg (\neg A \vee C) \wedge \neg C$. From this and not $\bot \gg \neg A \vee C$, we conclude that $\bot \gg \neg C$, by (d9).

($\gg$AC) follows from ($\gg$2a) and ($\gg$4).

($\gg$DA$^w$) follows from ($\gg$2a) and (d15). □

*Proof of Observation 5.3.* For ($\gg$Arist1), suppose that $A \gg \neg A$. By ($\gg$2a), we get $A \gg A \vee \neg A$. But this is impossible, by (d14).

For ($\gg$Arist2), suppose that $A \gg C$. By ($\gg$2a), we get that $A \gg A \wedge C$. Thus, by ($\gg$2b), not $\neg A \gg \neg A \vee C$. So, by ($\gg$2a) again, not $\neg A \gg C$.

For ($\gg$Boet$^r$), suppose that both $A \gg C$ and $A \gg \neg C$. By (d26) and ($\gg$6), we get that $A \gg \bot$. By (d11), we get not $\neg A \gg \neg A$, so by (d7), not $\bot \gg \neg A$. So, by (d4), $\bot \gg A$. On the other hand, we have $A \gg A \vee C$ and $A \gg A \vee \neg C$, by ($\gg$2a). This, together with $\bot \gg A$, gives us $\bot \gg A \vee C$ and $\bot \gg A \vee \neg C$, by ($\gg$3–4). Using ($\gg$3–4) in the other direction, we finally get $\bot \gg C$ and $\bot \gg \neg C$. □

*Proof of Observation 5.4.* For ($\gg$7c$^r$), suppose that $A \wedge B \gg C$, $A \gg B$ and $\bot \gg A$. By ($\gg$2a), we get $A \wedge B \gg A \wedge B \wedge C$ and $A \gg A \wedge B$, and thus, by ($\gg$7c), $A \gg A \wedge C$. Bearing in mind ($\gg$2a), we realise that we are ready if we show that $A \gg A \vee C$. From $\bot \gg A$, we derive $\bot \gg A \wedge B$, by (d9). We apply ($\gg$2a) to $A \wedge B \gg C$ again and obtain $A \wedge B \gg (A \wedge B) \vee C$. Taken together with $\bot \gg A \wedge B$, this implies $\bot \gg (A \wedge B) \vee C$, by ($\gg$4). Since $(A \vee C) \wedge (A \supset B)$ implies $(A \wedge B) \vee C$, (d9) gives us $\bot \gg (A \vee C) \wedge (A \supset B)$. Thus either $\bot \gg A \vee C$ or $\bot \gg A \supset B$, by (d9). By $A \gg B$ and (d6), we get that not $\bot \gg A \supset B$, so $\bot \gg A \vee C$. By ($\gg$3), then $A \gg A \vee C$, as desired.

For ($\gg$7$'^r$), suppose that $A \gg C$, $B \gg C$ and $\bot \gg A \vee B$. By ($\gg$2a), we get $A \gg A \wedge C$ and $B \gg B \wedge C$. From $\bot \gg A \vee B$ we can derive $A \vee B \gg A \vee B$, by (d7). So we can apply ($\gg$7$'$) and get $A \vee B \gg (A \vee B) \wedge C$. Bearing in mind ($\gg$2a), we realise that we are ready if we can show that $A \vee B \gg A \vee B \vee C$. In order to show this, we first note that our supposition gives us $A \gg A \vee C$, by ($\gg$2a), and also $\bot \gg A$, by ($\gg$3–4). Thus, by ($\gg$4), we get $\bot \gg A \vee C$, and thus $\bot \gg C$, by ($\gg$3–4). Moreover, we can conclude from $A \gg C$ and $B \gg C$ that not $\bot \gg A \supset C$ and not $\bot \gg B \supset C$, by (d6). Thus not $\bot \gg (A \vee B) \supset C$, by (d9). Combining this with $\bot \gg C$, we get that $\bot \gg A \vee B \vee C$, by (d9) again. Applying ($\gg$3), we conclude that $A \vee B \gg A \vee B \vee C$, as desired.

For ($\gg$8c$^r$), suppose that $A \gg B$, $A \gg C$ and $\bot \gg A$. By ($\gg$2a), we get $A \gg A \wedge B$ and $A \gg A \wedge C$, and thus, by ($\gg$8c), $A \wedge B \gg A \wedge B \wedge C$. Bearing in mind ($\gg$2a), we realise that we are ready if we show that $A \wedge B \gg (A \wedge B) \vee C$. By ($\gg$2a), we get $A \gg A \vee B$ and $A \gg A \vee C$. Since $\bot \gg A$, we can use ($\gg$4) and conclude that $\bot \gg A \vee B$ and $\bot \gg A \vee C$. From the latter, we get $\bot \gg (A \wedge B) \vee C$, by (d9). But now we can apply ($\gg$3) and find that $A \wedge B \gg (A \wedge B) \vee C$, as desired.

For ($\gg$8$^r$), suppose that $A \gg C$, not $A \gg A \wedge \neg B$ and $\bot \gg A$. By ($\gg$2a), we get $A \gg A \wedge C$, so we can apply ($\gg$8) and get $A \wedge B \gg A \wedge B \wedge C$. Bearing in mind ($\gg$2a), we realise that we are ready if we show that $A \wedge B \gg (A \wedge B) \vee C$. By ($\gg$2a), we get $A \gg A \vee C$. Since $\bot \gg A$, we can use ($\gg$4) and conclude that $\bot \gg A \vee C$. From this, we get $\bot \gg (A \wedge B) \vee C$, by (d9). But now we can apply ($\gg$3) and find that $A \wedge B \gg (A \wedge B) \vee C$, as desired. □

*Proof of the equivalence of (Def∗) and (Def∗2).* Assume the RHS of (Def∗). If $A \gg A \wedge C$, then not $\neg A \gg \neg A \vee C$, by the left-to-right direction of ($\gg$2b). So suppose that $\neg A \gg \bot$ and not $\bot \gg C$. From the former we get, by (d12), not $\bot \gg \neg\neg A$, i.e., not $\bot \gg A$. From this and not $\bot \gg C$, we get that not $\neg A \gg \neg A \vee C$, by (d15).

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For the converse assume the RHS of (Def$*$2), i.e., not $\neg A \gg \neg A \vee C$. Then the right-to-left direction of ($\gg$2b) gives us $A \gg A \wedge C$ or not $A \gg A$. If the former, we are done. So assume that not $A \gg A$. By (d11), we get $\neg A \gg \bot$. It remains to prove that not $\bot \gg C$. From not $\neg A \gg \neg A \vee C$, we get, by (d16), that not $\bot \gg C$ or $\bot \gg A$. On the other hand, not $A \gg A$ implies that not $\bot \gg A$, by (d7). Thus we conclude that not $\bot \gg C$, as desired. □

*Proof of Theorem 6.1.* First we prove the properties of $\gg$.

For ($\gg$0), we show that $\bot \gg \bot$, i.e., $\bot \in Bel * \bot$ and $\bot \notin Bel * \top$. The former is true by ($*$2), the latter is just our general supposition that $Bel = Bel * \top$ is consistent.

For ($\gg$1), we show that $A \gg B \wedge C$ implies that $A \gg B$ or $A \gg C$. Suppose that $B \wedge C \in Bel * A$ and $B \wedge C \notin Bel * \neg A$. But then again, with the help of ($*$1), it follows immediately that either $B \in Bel*A$ and $B \notin Bel*\neg A$ or that $C \in Bel*A$ and $C \notin Bel*\neg A$.

For ($\gg$2a), we show that $A \gg C$ if and only if $A \gg A \wedge C$ and $A \gg A \vee C$. By RRT, this means that

$$\begin{array}{l} C \in Bel * A \text{ and} \\ C \notin Bel * \neg A \end{array} \quad \text{iff} \quad \left( \begin{array}{l} A \wedge C \in Bel * A \text{ and} \\ A \wedge C \notin Bel * \neg A \end{array} \quad \text{and} \quad \begin{array}{l} A \vee C \in Bel * A \text{ and} \\ A \vee C \notin Bel * \neg A \end{array} \right).$$

For the left-to-right direction, we first infer from $C \in Bel * A$ that $A \wedge C \in Bel * A$ and $A \vee C \in Bel * A$, by ($*$1) and ($*$2). Since $C \notin Bel * \neg A$, $A \wedge C \notin Bel * \neg A$, by ($*$1). Suppose $A \vee C \in Bel * \neg A$. Then by ($*$1) and ($*$2), $C \in Bel * \neg A$, contradicting the supposition.

For the right-to-left direction, we get $C \in Bel*A$ from $A \wedge C \in Bel*A$, and $C \notin Bel*\neg A$ from $A \vee C \notin Bel * \neg A$, both by ($*$1).

For ($\gg$2b), we show that $A \gg A \wedge C$ if and only if not $\neg A \gg \neg A \vee C$ and $A \gg A$. By RRT, this means that

$$\begin{array}{l} A \wedge C \in Bel * A \text{ and} \\ A \wedge C \notin Bel * \neg A \end{array} \quad \text{iff} \quad \left( \begin{array}{l} \neg A \vee C \notin Bel * \neg A \text{ or} \\ \neg A \vee C \in Bel * A \end{array} \quad \text{and} \quad \begin{array}{l} A \in Bel * A \text{ and} \\ A \notin Bel * \neg A \end{array} \right).$$

For the left-to-right direction, we first note that $A \in Bel * A$, by ($*$2). Since $A \wedge C \notin Bel * \neg A$, $Bel * \neg A$ is consistent, and thus $A \notin Bel * \neg A$, by ($*$1) and ($*$2). Since $A \wedge C \in Bel * A$, $\neg A \vee C \in Bel * A$, by ($*$1).

For the right-to-left direction, we first note that $\neg A \vee C \notin Bel * \neg A$ is impossible, due to ($*$1) and ($*$2). So $\neg A \vee C \in Bel * A$, from which we get $A \wedge C \in Bel * A$, by ($*$1) and ($*$2). Finally, we get $A \wedge C \notin Bel * \neg A$ from $A \notin Bel * \neg A$, by ($*$1).

For ($\gg$3–4), we show that $\bot \gg A \vee C$ if and only if $\bot \gg A$ and $A \gg A \vee C$. By RRT, this means that

$$\begin{array}{l} A \vee C \in Bel * \bot \text{ and} \\ A \vee C \notin Bel * \top \end{array} \quad \text{iff} \quad \left( \begin{array}{l} A \in Bel * \bot \text{ and} \\ A \notin Bel * \top \end{array} \quad \text{and} \quad \begin{array}{l} A \vee C \in Bel * A \text{ and} \\ A \vee C \notin Bel * \neg A \end{array} \right).$$

For the left-to-right direction, we first note that $A \in Bel * \bot$ and $A \vee C \in Bel * A$, by ($*$1) and ($*$2) alone. Then we get $A \notin Bel * \top$ from $A \vee C \notin Bel * \top$, by ($*$1). Suppose that $A \vee C \in Bel * \neg A$. Then, by ($*$3), $A \vee C \in Cn(Bel \cup \{\neg A\})$, which means $\neg A \supset (A \vee C) \in Bel$, by the deduction theorem and ($*$1). But this means, by ($*$1), $A \vee C \in Bel = Bel * \top$, contradicting the supposition.

For the right-to-left direction, we first note that $A \vee C \in Bel * \bot$, by ($*$1) and ($*$2) alone. We infer from $A \notin Bel * \top = Bel$ that $Bel \subseteq Bel * \neg A$. Since $A \vee C \notin Bel * \neg A$, also $A \vee C \notin Bel = Bel * \top$.

For ($\gg$5), we show that $A \vee C \gg \bot$ if and only if $A \gg \bot$ and $B \gg \bot$. By RRT, this means that

$$\begin{array}{l}\bot \in Bel * A \vee B \text{ and} \\ \bot \notin Bel * \neg A \wedge \neg B\end{array} \quad \text{iff} \quad \left(\begin{array}{l}\bot \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array} \quad \text{and} \quad \begin{array}{l}\bot \in Bel * B \text{ and} \\ \bot \notin Bel * \neg B\end{array}\right).$$

We first note that, given the upper lines, the lower lines are all impossible, due to the assumed consistency of *Bel* and ($*$Cas). Given this, we get the left-to-right direction from ($*$5a) and ($*$6), and the right-to-left direction from ($*$5b).

($\gg$6) follows straightforwardly from ($*$1) and ($*$6).

For ($\gg$7c), we show that $A \gg A \wedge B$ and $A \wedge B \gg A \wedge B \wedge C$ imply $A \gg A \wedge C$. By RRT, this means, after a few routine simplifications using ($*$1) and ($*$2), that

$$\text{If} \quad \begin{array}{l}B \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array} \quad \text{and} \quad \begin{array}{l}C \in Bel * A \wedge B \text{ and} \\ \bot \notin Bel * \neg A \vee \neg B\end{array}, \text{ then} \quad \begin{array}{l}C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array}.$$

It follows from $B \in Bel * A$ and $C \in Bel * A \wedge B$ by ($*$7c) that $C \in Bel * A$. Since $\bot \notin Bel * \neg A$ is in the antecedent, too, we are done.

For ($\gg$7), we show that $A \wedge B \gg A \wedge B \wedge C$ and $A \gg A$ imply $A \gg A \wedge (B \supset C)$. By RRT, this means, after a few routine simplifications using ($*$1) and ($*$2), that

$$\text{If} \quad \begin{array}{l}C \in Bel * A \wedge B \text{ and} \\ \bot \notin Bel * \neg A \vee \neg B\end{array} \quad \text{and } \bot \notin Bel * \neg A \text{, then} \quad \begin{array}{l}B \supset C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array}.$$

It follows from $C \in Bel * A \wedge B$ by ($*$7) that $C \in Cn((Bel * A) \cup \{B\}$. By the deduction theorem and ($*$1), this entails $B \supset C \in Bel * A$. Since also $\bot \notin Bel * \neg A$, we are done.

For ($\gg$7′), we show that $A \gg A \wedge C$, $B \gg B \wedge C$ and $A \vee B \gg A \vee B$ imply $A \vee B \gg (A \vee B) \wedge C$. By RRT, this means, after a few routine simplifications using ($*$1) and ($*$2), that

$$\text{If} \quad \begin{array}{l}C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array} \quad \text{and} \quad \begin{array}{l}C \in Bel * B \text{ and} \\ \bot \notin Bel * \neg B\end{array} \quad \text{and } \bot \notin Bel * \neg A \wedge \neg B,$$

$$\text{then} \quad \begin{array}{l}C \in Bel * A \vee B \text{ and} \\ \bot \notin Bel * \neg A \wedge \neg B\end{array}.$$

It follows from $C \in Bel * A$ and $C \in Bel * B$ by ($*$7′) that $C \in Bel * A \vee B$. Since $\bot \notin Bel * \neg A \wedge \neg B$ is in the antecedent, too, we are done.

For ($\gg$8c), we show that $A \gg A \wedge B$ and $A \gg A \wedge C$ imply $A \wedge B \gg A \wedge B \wedge C$. By RRT, this means, after a few routine simplifications using ($*$1) and ($*$2), that

$$\text{If} \quad \begin{array}{l}B \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array} \quad \text{and} \quad \begin{array}{l}C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array}, \text{ then} \quad \begin{array}{l}C \in Bel * A \wedge B \text{ and} \\ \bot \notin Bel * \neg A \vee \neg B\end{array}.$$

It follows from $B \in Bel * A$ and $C \in Bel * A$ by ($*$8c) that $C \in Bel * A \wedge B$. Since $\bot \notin Bel * \neg A$, we infer with the help of ($*$5a) that $\bot \notin Bel * \neg A \vee \neg B$.

For ($\gg$8), we show that not $A \gg A \wedge \neg B$ and $A \gg A \wedge C$ imply $A \wedge B \gg A \wedge B \wedge C$. By RRT, this means, after a few routine simplifications using ($*$1) and ($*$2), that

$$\text{If} \quad \begin{array}{l}\neg B \notin Bel * A \text{ or} \\ \bot \in Bel * \neg A\end{array} \quad \text{and} \quad \begin{array}{l}C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A\end{array}, \text{ then} \quad \begin{array}{l}C \in Bel * A \wedge B \text{ and} \\ \bot \notin Bel * \neg A \vee \neg B\end{array}.$$

It follows from $\neg B \notin Bel * A$ and $C \in Bel * A$ by ($*$8) that $C \in Bel * A \wedge B$. Since $\bot \notin Bel * \neg A$, we infer with the help of ($*$5a) that $\bot \notin Bel * \neg A \vee \neg B$.

Finally, let us check (Def$*$). (For this, we need to prove $Rev(RCond(*)) = *$.) We need to show that $C \in K * A$ if and only if

$$A \gg A \wedge C \text{ or } (\neg A \gg \bot \text{ and not } \bot \gg C).$$

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To analyse the latter, we use RRT and get

$$\begin{array}{l} A \wedge C \in Bel * A \\ A \wedge C \notin Bel * \neg A \end{array} \quad \text{or} \quad \left( \begin{array}{l} \bot \in Bel * \neg A \\ \bot \notin Bel * A \end{array} \quad \text{and} \quad \begin{array}{l} C \notin Bel * \bot \text{ or} \\ C \in Bel * \top \end{array} \right).$$

Using the assumption that $Bel = Bel * \top$ is consistent and ($*$Cas) as well as the basic AGM axioms, this reduces to

$$\begin{array}{l} C \in Bel * A \text{ and} \\ \bot \notin Bel * \neg A \end{array} \quad \text{or} \quad \begin{array}{l} \bot \in Bel * \neg A \text{ and} \\ C \in Bel * \top \end{array}$$

We leave the first disjunct and transform only the second. Notice that $\bot \in Bel * \neg A$ implies $A \in Bel = Bel * \top$, by ($*$3), which in turn implies $Bel * A = Bel = Bel * \top$, by ($*$3) and ($*$4) and the consistency of *Bel*. This, however, means that the second disjunct is equivalent to $\bot \in Bel * \neg A$ and $C \in Bel * A$. So the whole condition reduces to $C \in K * A$. This is what we needed to show. □

*Proof of Theorem 6.2.* First we prove the properties of $*$.

For ($*$1a), we show that $B \in Bel * A$ and $C \in Bel * A$ implies $B \wedge C \in Bel * A$. There are four cases that make the hypothesis true. Case 1. Suppose $A \gg A \wedge B$ and $A \gg A \wedge C$. Then $A \gg A \wedge B \wedge C$, by (d26) and ($\gg$6), which entails $B \wedge C \in Bel * A$. Case 2. Suppose $A \gg A \wedge B$ and ($\neg A \gg \bot$ and not $\bot \gg C$). But the former implies $A \gg A$, by ($\gg$2b), which is incompatible with $\neg A \gg \bot$, by (d11). So this case is impossible. Case 3. Analogous, with $B$ and $C$ switching roles. Case 4. Suppose ($\neg A \gg \bot$ and not $\bot \gg B$) and ($\neg A \gg \bot$ and not $\bot \gg C$). Then by (d9), not $\bot \gg B \wedge C$, which gives us, together with $\neg A \gg \bot$, $B \wedge C \in Bel * A$.

For ($*$1b), we show that $C \in Bel * A$ and $B \in Cn(C)$ implies $B \in Bel * A$. Suppose the hypothesis, i.e., that $A \gg A \wedge C$ or ($\neg A \gg \bot$ and not $\bot \gg C$). If $A \gg A \wedge C$, then we get $A \gg A \wedge B$, by (d24) and ($\gg$6), which entails $B \in Bel * A$. If, on the other hand, $\neg A \gg \bot$ and not $\bot \gg C$, then not $\bot \gg B$, by (d9) and ($\gg$6), which together with $\neg A \gg \bot$ also entails $B \in Bel * A$.

For ($*$2), we show that $A \in Bel * A$, i.e., that $A \gg A \wedge A$ or ($\neg A \gg \bot$ and not $\bot \gg A$). But by (d11), we have $A \gg A \wedge A$ or $\neg A \gg \bot$, and the latter implies not $\bot \gg A$, by (d7).

For ($*$3), we show that $C \in Bel * A$ implies $A \supset C \in Bel * \top$. Suppose the hypothesis, i.e., that $A \gg A \wedge C$ or ($\neg A \gg \bot$ and not $\bot \gg C$). We need to show that not $\bot \gg A \supset C$. If $A \gg A \wedge C$, then not $\neg A \gg \neg A \vee C$, by ($\gg$2b), and thus, by ($\gg$3), not $\bot \gg \neg A \vee C$, which is what we want, by ($\gg$6). If, on the other hand, $\neg A \gg \bot$ and not $\bot \gg C$, the latter implies, by (d9), that not $\bot \gg \neg A \vee C$, which again gives us $A \supset C \in Bel * \top$.

For ($*$4), we show that $C \in Bel * \top$ and $\neg A \notin Bel * \top$ implies $C \in Bel * A$. Suppose the hypothesis, i.e., that not $\bot \gg C$ and $\bot \gg \neg A$. From this we get, by ($\gg$4), that not $\neg A \gg \neg A \vee C$. So, by ($\gg$2b), $A \gg A \wedge C$ or not $A \gg A$. If $A \gg A \wedge C$, we get $C \in Bel * A$ immediately. If, on the other hand, not $A \gg A$, then by (d11), $\neg A \gg \bot$, which together with $\bot \gg C$ implies $C \in Bel * A$, too.

For ($*$5a), we show that $\bot \in Bel * A$ and $A \in Cn(B)$ implies $\bot \in Bel * B$. Suppose the hypothesis, i.e., that $A \gg \bot$ and $A \in Cn(B)$. Then $B \gg \bot$, by ($\gg$5) and ($\gg$6).

For ($*$5b), we show that $\bot \in Bel * A$ and $\bot \in Bel * B$ implies $\bot \in Bel * (A \vee B)$. Suppose the hypothesis, i.e., that $A \gg \bot$ and $B \gg \bot$. Then $A \vee B \gg \bot$, by ($\gg$5).

For ($*$7c), we show that $B \in Bel * A$ and $C \in Bel * (A \wedge B)$ implies $C \in Bel * A$. Suppose the hypothesis, i.e., that

$$\begin{array}{l} A \gg A \wedge B \text{ or} \\ (\neg A \gg \bot \text{ and not } \bot \gg B) \end{array} \quad \text{and} \quad \begin{array}{l} A \wedge B \gg A \wedge B \wedge C \text{ or} \\ (\neg A \vee \neg B \gg \bot \text{ and not } \bot \gg C) \end{array}.$$

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We need to show that $A \gg A \wedge C$ or ($\neg A \gg \bot$ and not $\bot \gg C$).

Case 1. Suppose the two upper lines are true. Then we get $A \gg A \wedge C$, by ($\gg$7c) straight away.

Case 2. Suppose the left upper and the right lower line is true. But $\neg A \vee \neg B \gg \bot$ entails $\neg A \gg \bot$, by ($\gg$5) and ($\gg$6), and this is incompatible with $A \gg A \wedge B$, by (d11) and ($\gg$2b).

Case 3. Suppose the left lower and the right upper line is true. $\neg A \gg \bot$ and not $\bot \gg B$ taken together imply that not $\bot \gg A \wedge B$, by (d7) and (d9). But from this and $A \wedge B \gg A \wedge B \wedge C$, we conclude that not $\bot \gg C$, by (d16). So we have both $\neg A \gg \bot$ and not $\bot \gg C$.

Case 4. Suppose the two lower lines are true. Then we have $\neg A \gg \bot$ and not $\bot \gg C$.

For ($*$7), we show that $C \in Bel * (A \wedge B)$ implies $B \supset C \in Bel * A$. Suppose the former, i.e., that $A \wedge B \gg A \wedge B \wedge C$ or ($\neg A \vee \neg B \gg \bot$ and not $\bot \gg C$). We need to show that $A \gg A \wedge (B \supset C)$ or ($\neg A \gg \bot$ and not $\bot \gg B \supset C$).

Case 1. Suppose that $A \wedge B \gg A \wedge B \wedge C$ and $A \gg A$. Then we get $A \gg A \wedge (B \wedge C)$, by ($\gg$7) straight away.

Case 2. Suppose that $A \wedge B \gg A \wedge B \wedge C$ and not $A \gg A$. From the latter, we get $\neg A \gg \bot$ by (d11), and thus also not $\bot \gg A$ by (d12). Moreover, by (d6) and ($\gg$6), $A \wedge B \gg A \wedge B \wedge C$ implies that not $A \wedge B \gg A \wedge B \wedge C$. This together with not $\bot \gg A$ gives us, by (d9) and ($\gg$6), that not $\bot \gg A \wedge (B \supset C)$. So by (d9) again, not $\bot \gg B \supset C$.

Case 3. Suppose that $\neg A \vee \neg B \gg \bot$ and not $\bot \gg C$. Then $\neg A \gg \bot$ by ($\gg$5), and not $\bot \gg B \supset C$ by (d6) and ($\gg$6).

For ($*$7′), we show that $C \in Bel * A$ and $C \in Bel * B$ implies $C \in Bel * (A \vee B)$. Suppose the hypothesis, i.e., that

$$\begin{array}{l} A \gg A \wedge C \text{ or} \\ (\neg A \gg \bot \text{ and not } \bot \gg C) \end{array} \quad \text{and} \quad \begin{array}{l} B \gg B \wedge C \text{ or} \\ (\neg B \gg \bot \text{ and not } \bot \gg C) \end{array}.$$

We need to show that $A \vee B \gg (A \vee B) \wedge C$ or ($\neg A \wedge \neg B \gg \bot$ and not $\bot \gg C$).

Case 1. Suppose the two upper lines are true. If $A \vee B \gg A \vee B$, then we can use ($\gg$7) and get $A \vee B \gg (A \vee B) \wedge C$ right away. If not $A \vee B \gg A \vee B$, on the other hand, we get $\neg A \wedge \neg B \gg \bot$, by (d11), and we have not $\bot \gg C$ anyway.

Cases 2–4. Suppose any of the lower lines is true. Then we get, by ($\gg$5), $\neg A \wedge \neg B \gg \bot$ and not $\bot \gg C$.[32]

For ($*$8c), we show that $B \in Bel * A$ and $C \in Bel * A$ implies $C \in Bel * (A \wedge B)$. Suppose the hypothesis, i.e., that

$$\begin{array}{l} A \gg A \wedge B \text{ or} \\ (\neg A \gg \bot \text{ and not } \bot \gg B) \end{array} \quad \text{and} \quad \begin{array}{l} A \gg A \wedge C \text{ or} \\ (\neg A \gg \bot \text{ and not } \bot \gg C) \end{array}.$$

We need to show that $A \wedge B \gg A \wedge B \wedge C$ or ($\neg A \vee \neg B \gg \bot$ and not $\bot \gg C$).

Case 1. Suppose the two upper lines are true. Then we get $A \wedge B \gg A \wedge B \wedge C$, by ($\gg$8c), straight away.

[32] Let us give a similar proof for AGM's original postulate ($*$7). Suppose that $C \in Bel * (A \wedge B)$, i.e., that $A \wedge B \gg A \wedge B \wedge C$ or $\neg A \vee \neg B \gg \bot$ and not $\bot \gg C$. We need to show that $C \in Cn((Bel * A) \cup \{B\})$, which means $B \supset C \in Bel * A$, i.e., that $A \gg A \wedge (B \supset C)$ or $\neg A \gg \bot$ and not $\bot \gg (B \supset C)$. But this follows quickly, if we apply ($\gg$7), ($\gg$5) and (d9) to the three respective terms of the supposition.

Cases 2–3. Suppose one of the upper and one of the lower lines are true. But this is impossible, since each of $A \gg A \wedge B$ and $A \gg A \wedge C$ entails that not $\neg A \gg \bot$, by ($\gg$2b) and (d11).

Case 4. Suppose the two lower lines are true. Then we have $\neg A \gg \bot$, not $\bot \gg B$ and not $\bot \gg C$. If in addition $\neg A \vee \neg B \gg \bot$, we are ready. So suppose in addition that not $\neg A \vee \neg B \gg \bot$. Thus $A \wedge B \gg A \wedge B$, by (d11). $\neg A \gg \bot$ and not $\bot \gg B$ taken together imply that not $\bot \gg A \wedge B$, by (d7) and (d9). Now we can employ (d15) and get that not $\neg A \vee \neg B \gg (\neg A \vee \neg B) \vee C$. Finally, we apply ($\gg$2b) and get that $A \wedge B \gg A \wedge B \wedge C$.

For ($*$8), we show that $\neg B \notin Bel * A$ and $C \in Bel * A$ implies $C \in Bel * (A \wedge B)$. Suppose the hypothesis, i.e., that

$$\begin{array}{lcl} \text{not } A \gg A \wedge \neg B \text{ and} & & A \gg A \wedge C \text{ or} \\ (\text{not } \neg A \gg \bot \text{ or } \bot \gg \neg B) & \text{and} & (\neg A \gg \bot \text{ and not } \bot \gg C) \end{array}.$$

We need to show that $A \wedge B \gg A \wedge B \wedge C$ or ($\neg A \vee \neg B \gg \bot$ and not $\bot \gg C$).

Case 1. Suppose the right upper line is true. Then we use not $A \gg A \wedge \neg B$ and get $A \wedge B \gg A \wedge B \wedge C$, by ($\gg$8), straight away.

Case 2. Suppose the right lower line is true. Then we have $\bot \gg \neg B$ and not $\bot \gg A$, by (d12). From the former, we get $\bot \gg A \wedge \neg B$, by (d9), and $\bot \gg A \wedge (\neg A \vee \neg B)$, by ($\gg$6). Together with not $\bot \gg A$, this allows us to conclude that $\bot \gg \neg A \vee \neg B$, by (d9). From the latter and not $\bot \gg C$, we get, by ($\gg$4), not $(\neg A \vee \neg B) \gg (\neg A \vee \neg B) \vee C$. Case 2a. Suppose that not $\neg A \vee \neg B \gg \bot$. By (d11), we then get $A \wedge B \gg A \wedge B$, and from this and not $(\neg A \vee \neg B) \gg (\neg A \vee \neg B) \vee C$, we may, using ($\gg$2b), infer that $A \wedge B \gg A \wedge B \wedge C$, as desired. Case 2b. Suppose that $\neg A \vee \neg B \gg \bot$. Since we have not $\bot \gg C$, we are done here immediately.

Finally, let us check RRT. We need to show that $A \gg C$ if and only if

$$C \in Bel * A \quad \text{and} \quad C \notin Bel * \neg A.$$

To analyse the latter, we use (Def$*$) and get

$$\begin{array}{lcl} A \gg A \wedge C \text{ or} & & \text{not } \neg A \gg \neg A \wedge C \text{ and} \\ (\neg A \gg \bot \text{ and not } \bot \gg C) & \text{and} & (\text{not } A \gg \bot \text{ or } \bot \gg C) \end{array}.$$

First we show that this is implied by $A \gg C$. From $A \gg C$, we get $A \gg A \wedge C$ and $A \gg A \vee C$, by ($\gg$2a), and from the latter not $\neg A \gg \neg A \wedge C$, by ($\gg$2b). Finally suppose that $A \gg \bot$; we need to show that $\bot \gg C$. From $A \gg \bot$, we get that not $\bot \gg \neg A$, by (d12). From this and $A \gg C$, we get $\bot \gg C$ (by ......), as desired.

Second, we show that the condition displayed last implies $A \gg C$. We do this by distinguishing cases.

Case 1: Suppose $A \gg A \wedge C$ and not $\neg A \gg \neg A \wedge C$ and not $A \gg \bot$. By (d11), the last fact is equivalent to $\neg A \gg \neg A$. By ($\gg$2b), $\neg A \gg \neg A$ and not $\neg A \gg \neg A \wedge C$ imply $A \gg A \vee C$. But this together with $A \gg A \wedge C$ implies $A \gg C$, by ($\gg$2a).

Case 2: Suppose $A \gg A \wedge C$ [and not $\neg A \gg \neg A \wedge C$] and $\bot \gg C$. Then not $\neg A \gg \neg A \vee C$, by ($\gg$2b), and thus $A \gg C$ by (d19).

Case 3: Suppose not $\neg A \gg \neg A \wedge C$, ($\neg A \gg \bot$ and not $\bot \gg C$) [and (not $A \gg \bot$ or $\bot \gg C$)]. (Since the expression within the first parentheses implies the expression within the second, the latter is actually not needed.) But this is not possible, since the first two terms are inconsistent, by (d27).

Thus we have checked all cases and shown what we needed to show. □

*Proof of Theorem 7.2.* First we prove the properties of $*$.

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For (∗1a), we show that $B \in Bel * A$ and $C \in Bel * A$ implies $B \wedge C \in Bel * A$. Suppose $A \gg A \wedge B$ and $A \gg A \wedge C$. Then $A \gg A \wedge B \wedge C$, by (≫1a) and (≫6), which entails $B \wedge C \in Bel * A$.

For (∗1b), we show that $C \in Bel * A$ and $B \in Cn(C)$ implies $B \in Bel * A$. Suppose that $A \gg A \wedge C$. Then $A \gg A \wedge C \wedge B$, (≫6), and $A \gg A \wedge B$, by (≫1b), which entails $B \in Bel * A$.

For (∗2), we show that $A \in Bel * A$, i.e., that $A \gg A \wedge A$. But this follows immediately from (≫6) and (≫2b).

For (∗3), we show that $C \in Bel * A$ implies $A \supset C \in Bel * \top$. Suppose that $A \gg A \wedge C$. We need to show that $\top \gg A \supset C$. From $A \gg A \wedge C$, we get $\neg A \gg \neg A \vee \neg C$, by (≫1c) and (≫6), and thus, by (≫3–4), $\top \gg \neg A \vee C$, which is what we want, by (≫6). (The following condition might be called **(≫3)**: If $A \gg A \vee \neg C$, then $\top \gg A \vee C$.)

For (∗4), we show that $C \in Bel * \top$ and $\neg A \notin Bel * \top$ implies $C \in Bel * A$. Suppose that $\top \gg C$ and not $\top \gg \neg A$. From the former, we get, by (≫3–4), $\top \gg \top \wedge (A \vee C) \wedge (\neg A \vee C)$, so by (≫1b) and (≫6), $\top \gg \neg A \vee C$. From this and not $\top \gg \neg A$ we get, by (≫3–4), that $\neg A \gg \neg A \vee \neg C$. By (≫1c), $A \gg A \wedge C$, which is what we want. (The following condition might be called **(≫4)**: If $\top \gg A \vee C$, then $\top \gg A$ or $A \gg A \vee \neg C$.—Notice that (≫3–4) is more than the conjunction of (≫3) and (≫4). The extra information is this: If $\top \gg A$, then $\top \gg A \vee C$. This is already covered by (≫1b) together with (≫6).)

For (∗5a), we show that $\bot \in Bel * A$ and $A \in Cn(B)$ implies $\bot \in Bel * B$. Suppose that $A \gg \bot$ and $A \in Cn(B)$. Then $B \gg \bot$, by (≫5) and (≫6).

For (∗5b), we show that $\bot \in Bel * A$ and $\bot \in Bel * B$ implies $\bot \in Bel * (A \vee B)$. Suppose that $A \gg \bot$ and $B \gg \bot$. Then $A \vee B \gg \bot$, by (≫5).

For (∗7c), we show that $B \in Bel * A$ and $C \in Bel * (A \wedge B)$ implies $C \in Bel * A$. Suppose that $A \gg A \wedge B$ and $A \wedge B \gg A \wedge B \wedge C$. We need to show that $A \gg A \wedge C$. But this we get from (≫7c) straight away.

For (∗7), we show that $C \in Bel * (A \wedge B)$ implies $B \supset C \in Bel * A$. Suppose that $A \wedge B \gg A \wedge B \wedge C$. We need to show that $A \gg A \wedge (B \supset C)$. But this we get from (≫7) straight away.

For (∗7′), we show that $C \in Bel * A$ and $C \in Bel * B$ implies $C \in Bel * (A \vee B)$. Suppose that $A \gg A \wedge C$ and $B \gg B \wedge C$. We need to show that $A \vee B \gg (A \vee B) \wedge C$. But this we get from (≫7) right away.

For (∗8c), we show that $B \in Bel * A$ and $C \in Bel * A$ implies $C \in Bel * (A \wedge B)$. Suppose that $A \gg A \wedge B$ and $A \gg A \wedge C$. We need to show that $A \wedge B \gg A \wedge B \wedge C$. But this we get from (≫8c) straight away.

For (∗8), we show that $\neg B \notin Bel * A$ and $C \in Bel * A$ implies $C \in Bel * (A \wedge B)$. Suppose that not $A \gg A \wedge \neg B$ and $A \gg A \wedge C$. We need to show that $A \wedge B \gg A \wedge B \wedge C$. But this we get from (≫8) straight away.

Finally, let us check DRT. We need to show that $A \gg C$ if and only if

$$C \in Bel * A \quad \text{and} \quad \neg C \in Bel * \neg A.$$

To analyse the latter, we use (Ddef∗) and get

$$A \gg A \wedge C \qquad \text{and} \qquad \neg A \gg \neg A \wedge \neg C.$$

First we show that this is implied by $A \gg C$. From $A \gg C$, we get $A \gg A \wedge C$, by (≫2a). Moreover we also get $\neg A \gg \neg C$ from $A \gg C$, by (≫1c). Applying (≫2a) again, we find that $\neg A \gg \neg A \wedge \neg C$, as desired.

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Second, we show that $A \gg A \wedge C$ and $\neg A \gg \neg A \wedge \neg C$ implies $A \gg C$. From $\neg A \gg \neg A \wedge \neg C$, we get $A \gg A \vee C$, by ($\gg$1c) and ($\gg$6). Thus, by $A \gg A \wedge C$ and ($\gg$2b), $A \gg C$, as desired. □

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