The Analytic Revolution

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Abstract

Analytic philosophy, as we recognize it today, has its origins in the work of Gottlob
Frege and Bertrand Russell around the turn of the twentieth century. Both were
trained as mathematicians and became interested in the foundations of mathematics.
In seeking to demonstrate that arithmetic could be derived from logic, they revolu-
tionized logical theory and in the process developed powerful new forms of logical
analysis, which they employed in seeking to resolve certain traditional philosophical
problems. There were important differences in their approaches, however, and these
approaches are still pursued, adapted, and debated today. In this paper I shall eluci-
date the origins of analytic philosophy in the work of Frege and Russell and explain
the revolutionary significance of their methods of logical analysis.

1. Introduction

Analytic philosophy, as it is generally regarded today, is a complex
tradition made up of various strands, some mutually reinforcing,
some in creative tension with one another. As analytic philosophy
has evolved over the last century or so, it has incorporated new
ideas, methods and arguments, ramified into all areas of philosophy,
and extended its influence right across the world. In an earlier period
of its history, it was seen as having originated in the rebellion by
Bertrand Russell (1872–1970) and G. E. Moore (1873–1958)
against British idealism around the turn of the twentieth century.
As it developed, however, especially in the work of Ludwig
Wittgenstein (1889–1951) and the logical positivists (with their
heyday in Vienna in the 1930s), the influence of Gottlob Frege
(1848–1925) manifested itself to an ever increasing extent and
Frege became recognized as one of the founders – together with
Russell, Moore and Wittgenstein – of the analytic tradition.

Frege's significance is based on his creation of modern logic and
the use that he made of this logic in analysing arithmetic. His life's
project was to demonstrate that arithmetic is reducible to logic.
After his rebellion against British idealism, this was a project to
which Russell, too, dedicated himself in the first decade of the twen-
tieth century. Russell also contributed to the development of logic
itself and offered new logical analyses of his own, most famously, in

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his theory of descriptions, which became a paradigm of analytic philosophy. In this chapter I will explain the new logic that Frege created and the kind of analyses that this logic made possible. I shall focus on Frege's and Russell's concern with the foundations of arithmetic, but I shall avoid undue technicality in order to bring out as clearly as I can the philosophical significance of their logical analyses.

In focusing on the origins of analytic philosophy in the work of Frege and Russell, I do not want to suggest that Moore did not also play an important role.[1] Nor do I want to deny that Wittgenstein was enormously influential in the subsequent development of analytic philosophy.[2] But I do think that Frege's creation of modern logic and the use that he and Russell put it to in the logical analysis of arithmetic lay at the heart of what can justifiably be regarded as the 'analytic revolution' that took place in philosophy in the decades around the turn of the twentieth century. This analytic revolution continues to inspire philosophers today, and although its significance and implications are as much debated as anything else in philosophy, its achievements and fecundity have changed the intellectual landscape irreversibly.

2. Frege's logical revolution

The analytic revolution has its origins in a logical revolution that can be given a precise date of origin: 1879. It was in this year that Frege published his first book, *Begriffsschrift*. The term 'Begriffsschrift' literally means 'concept-script' and was the name that Frege gave to his new logical system. This system was the first system of what we now call quantificational logic, which proved to be far more powerful than any system that had hitherto been developed. It opened up the semantic machinery, as we might put it, of a whole host of complex sentences that had resisted effective analysis up to that point.

To appreciate how Frege revolutionized logic we need to understand how he went beyond traditional, essentially Aristotelian logic. Crucial here was his use of function–argument analysis, which he extended from mathematics to logic. Analytic geometry provides us

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[1]   On Moore's contribution to analytic philosophy, see T. Baldwin, *G.E. Moore* (London: Routledge, 1990) and 'G.E. Moore and the Cambridge School of Analysis' in *The Oxford Handbook of the History of Analytic Philosophy* (ed.) M. Beaney (Oxford University Press, 2013), 430–50.

[2]   For an account of Wittgenstein's influence, see P.M.S. Hacker, *Wittgenstein's Place in Twentieth-Century Analytic Philosophy* (Oxford: Blackwell, 1996).

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with a simple example to illustrate the idea. In writing the equation for a line as $y = ax + b$, we exhibit y as a function of x, where a is the gradient of the line and b the point where the line cuts the y-axis on a graph. Let $a = 3$ and $b = 4$. If $x = 2$, then $y = 10$: we say that 10 is the value of the function $3x + 4$ for argument 2. Inserting different numerical values for x yields different numerical values for y, allowing us to draw the relevant line. Frege generalized this idea: not just mathematical equations but all sentences – and indeed, what those sentences express or represent – can be analysed in the same way, in function–argument terms.

Let us see how this works in the case of simple sentences such as 'Gottlob is human'. In traditional logic, such sentences were analysed as having subject–predicate form, symbolized by 'S is P', with 'S' representing the subject ('Gottlob') and 'P' the predicate ('human'), connected by the copula ('is'). Frege, however, analysed them as having function–argument form, symbolized by 'Fa', with 'a' representing the argument ('Gottlob') and 'Fx' the function ('x is human'), the variable x indicating where the argument term goes to complete the sentence. The sentence 'Gottlob is human' is taken to be the value of the functional expression 'x is human' for the argument term 'Gottlob'. At this simple level, though, the two analyses may not seem to differ much, beyond the incorporation of the copula ('is') into the functional expression ('x is human').

If we turn to the case of relational sentences, however, then the advantages of function–argument analysis start to become clear. Relational sentences, on Frege's account, are analysed as functions of two or more arguments. In 'Gottlob is shorter than Bertrand', for example, 'Gottlob' and 'Bertrand' are taken as the argument terms and 'x is shorter than y' as the relational expression, symbolized as 'Rxy' or 'xRy'. This form of analysis can be readily extended to more complex relational sentences, such as 'York is between London and Edinburgh', which can be symbolized as '$Rabc$'. This enables a unified account of relational sentences to be provided, something which is much harder to do using only subject–predicate analysis.

The greater power of function–argument analysis, however, is only fully revealed in the case of sentences involving quantifier terms such as 'all' and 'some'. Consider the sentence 'All logicians are human'. In traditional logic, this was analysed in the same way as 'Gottlob is human', the only difference being that the subject term was taken as 'all logicians' rather than 'Gottlob' and the copula as the plural 'are' rather than singular 'is'. On Frege's view, on the other hand, 'All logicians are human' has a very different and more

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complex form, symbolized in modern notation as '$(\forall x)\,(Lx \rightarrow Hx)$', read as 'For all x, if x is a logician, then x is human'. Here what we have are two functional expressions, 'x is a logician' and 'x is human', joined by the propositional connective 'if … then …' (symbolized by '$\rightarrow$') and bound by the universal quantifier ('for all x', represented using an inverted 'A').

'Some logicians are human' is also analysed by Frege as having a more complex quantificational form, symbolized in modern notation as '$(\exists x)\,(Lx \,\&\, Hx)$', read as 'There is some x such that x is a logician and x is human'. Again what we have here are two functional expressions joined in this case by the propositional connective 'and' (symbolized by '$\&$') and bound by the existential quantifier ('there is some x', represented using a backwards 'E'). In both cases there is nothing that directly corresponds to the subject term 'all logicians' or 'some logicians': these terms are 'analysed away', to use a phrase that Russell was to make famous in his theory of descriptions (as we will see).

Introducing a notation for quantification – in particular, to represent 'all' and 'some' – was Frege's key innovation in creating his logical system.[3] This enabled him to formalize sentences not just with one quantifier term but with multiple quantifier terms. Sentences involving multiple quantification had proved especially difficult to analyse within traditional logic. Since what inferences can be drawn from them depends on their quantificational structure, it is only when we can represent this structure that we can properly exhibit the relevant logical relations.

As an example involving two quantifier terms, consider the sentence 'Every philosopher respects some logician', which is actually ambiguous. Paraphrasing it out a little, it can either mean 'Take any philosopher you like, then there is some (at least one) logician whom they respect (not necessarily the same one)'; or it can mean 'There is some (at least one) logician (the same one or more) whom every philosopher respects'. Quantificational logic provides a neat way of exhibiting this ambiguity:

(1) $(\forall x)\,(Px \rightarrow (\exists y)\,(Ly \,\&\, Rxy))$.
(2) $(\exists y)\,(Ly \,\&\, (\forall x)\,(Px \rightarrow Rxy))$.

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[3] In fact, Frege only introduced a notation for the universal quantifier, relying on the equivalence between 'Something is F' and 'It is not the case that everything is not F' to represent the existential quantifier. For an account of Frege's logical notation, see App. 2 of *The Frege Reader* (ed.) M. Beaney (Oxford: Blackwell, 1997).

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The first can be read as 'For all x, if x is a philosopher, then there is some y such that y is a logician and x respects y'; the second can be read as 'There is some y such that y is a logician and for all x, if x is a philosopher, then x respects y'. The difference in meaning is reflected in the order of the quantifiers – either $\forall\exists$ or $\exists\forall$. Furthermore, there is an asymmetry in their logical relation: while the first can be inferred from the second, the second cannot be inferred from the first. Mistakenly thinking that the first implies the second is known as the quantifier shift fallacy. Quantificational logic allows us to expose the error and helps us to avoid it in our own reasoning.

We see here an excellent example of the power of Frege's logic. While the ambiguity can be clarified in ordinary language, the use of quantificational notation sharpens the expression and, more importantly, makes clear the relevant logical relations. It can then be proved, for example, that (2) implies (1) but that (1) does not imply (2). It was the revolution in logic that Frege effected in creating quantificational logic that made possible the analytic revolution.

3. Frege's use of logical analysis in logic

Frege's use of function–argument analysis in developing his logical system yielded new forms of logical analysis. To bring out the significance of the difference between function–argument analysis and subject–predicate analysis, let us return to the examples of 'Gottlob is human' and 'All logicians are human'. Traditional logic had treated them as essentially the same, the only difference being between what is taken as the subject: 'Gottlob' in the first case, 'all logicians' in the second. Frege, on the other hand, is insistent that they involve *different* logical relations: subsumption and subordination, respectively. To say that Gottlob is human is to say that a certain object, namely, Gottlob, is subsumed under – i.e., falls under – a certain concept, the concept *human*. To say that all logicians are human is to say that anything that falls under the concept *logician* also falls under the concept *human*, in other words, that the concept *logician* is subordinate to the concept *human*. The first involves a relation between an object and a concept, the second a relation between two concepts.

Another way to express the contrast is to say that while 'Gottlob is human' and 'All logicians are human' have a similar grammatical form, they have quite different logical forms. The task of logical analysis can then be described – in a way that became typical of analytic

philosophy – as the project of revealing the logical form of sentences. The point of this project, though, was not to reveal the logical form of sentences for its own sake, but to do so in solving philosophical problems. To illustrate this, let us take the problem of negative existential statements, which has puzzled philosophers from ancient times. Consider the (true) statement made by using the following sentence:

(U$_1$)   Unicorns do not exist.

If we analyse this in traditional subject–predicate terms, then we would take 'unicorns' as the subject and 'non-existent' as the predicate. If we wanted to make explicit how (U$_1$) has the form 'S is P', then we could regiment it as:

(U$_2$)   Unicorns are non-existent.

But if this is the analysis, then we might find ourselves asking what these unicorns are that have the property of non-existence. Must not unicorns exist somehow for them to be attributed *any* property? (U$_1$) and (U$_2$), after all, are true (i.e., would be understood as typically used to make a true statement). But how can this be if the subject term does not refer? Alexius Meinong (1853–1920) is one philosopher who thought that we should grant unicorns some kind of being – *subsistence* rather than *existence* – to account for how such sentences can be true.

   In quantificational logic, on the other hand, (U$_1$) would be formalized using the existential quantifier as follows:

(U$_3$) $\neg(\exists x)\ Ux$.

This can be read as 'It is not the case that there is some x which is a U', where '$\neg$' is the sign for negation and 'Ux' abbreviates 'x is a U', with 'U' representing the concept *unicorn*. Here, as in the cases of sentences involving quantifier terms considered in the previous section, there is nothing that directly corresponds to the subject term in (U$_1$): it is 'analysed away' when we make use of function–argument analysis. (U$_3$) makes clear that what we are really saying when we say that unicorns do not exist is that nothing falls under the concept *unicorn*. This suggests that what our statement is really about is not unicorns (since there aren't any!) but about the *concept* of being a unicorn. What (U$_3$) makes clear could thus be expressed in ordinary language as:

(U$_4$)   The concept *unicorn* is not instantiated.

This in turn can be clarified by making use of another important distinction that Frege draws – between first-level and second-level

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concepts. First-level concepts are concepts under which objects fall; second-level concepts are concepts within which first-level concepts fall.[4] So (U$_4$) is to be understood as saying that the first-level concept *unicorn* falls within the second-level concept *is not instantiated*.

An implication of this distinction between first-level and second-level concepts is that the quantifiers themselves are to be construed as second-level concepts. To say that something is *F* is to say that the (first-level) concept *F* is instantiated (i.e., falls within the second-level concept *is instantiated*). To say that nothing is *F* is to say that the concept *F* is not instantiated. To say that everything is *F* is to say that the concept *F* is universally instantiated. When we talk of 'some *F*s' or 'no *F*s' or 'all *F*s', in other words, we are saying something about the concept *F*.

How, then, does this resolve the philosophical problem of negative existential statements? On Frege's analysis, the statement that unicorns do not exist turns out not to involve the attributing of a first-level property (non-existence) to an object or objects (unicorns) but the attributing of a second-level property (non-instantiation) to a first-level property (being a unicorn). We do not therefore need to suppose that unicorns must have some kind of existence ('subsistence') in order for us to say something true in using (U$_1$). To deny that something exists is just to say that the relevant *concept* has no instances: there is no need to posit any mysterious *object*. Negative existential statements only commit us to there being concepts (of the relevant kind), not to there being objects (to which the subject term somehow refers).[5]

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[4]   To make clear that there are two different relations here, between object and concept and between first-level and second-level concept, Frege distinguishes between *falling under* (subsumption) and *falling within*. But the two relations are analogous. See Frege 'Über Begriff und Gegenstand', *Vierteljahrsschrift für wissenschaftliche Philosophie* 16 (1892), 192–205; tr. as 'On Concept and Object' in *The Frege Reader*, 181–93, 189. Both relations are different from subordination (as explained in the previous section), which is a relation between concepts of the same level.

[5]   As Frege himself makes clear (*Die Grundlagen der Arithmetik*, (Breslau: W. Koebner, 1884), §53/*The Frege Reader*, 103), his analysis of existential statements also offers a diagnosis of what is wrong with the traditional ontological argument for the existence of God. In its most succinct form, this may be set out as follows: (1) God has every perfection; (2) existence is a perfection; therefore (3) God exists. In (1) we are taking 'perfections' to be first-level properties, but on Frege's view, 'existence' is not to be understood as a first-level property, so the argument fails.

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We can see here how logical analysis can do genuine philosophical work: it can elucidate the logical structure of our thinking and reasoning and help clear up the confusions that may arise from misinterpreting statements we make. Before illustrating this further in considering how Frege used logical analysis in his logicist project, let me say something more to help clarify how logical analysis itself should be understood. It is common today to think of 'analysis' as primarily meaning 'decomposition'. Traditional subject–predicate analysis encourages this conception. When we analyse 'Gottlob is human' in subject–predicate terms, i.e., as having the form 'S is P', we 'decompose' it into 'Gottlob', 'is' and 'human': these are quite literally the constituents. In the case of such a simple sentence, function–argument analysis works in a similar way: it yields the constituents 'Gottlob' (the argument) and 'is human' (the function), the copula being absorbed into the functional expression to constitute one 'unit' (of logical significance). But when we consider more complex sentences, function–argument analysis yields constituents that are different from what their surface grammatical form might indicate. As we have seen, 'All logicians are human' is analysed as 'For all x, if x is a logician, then x is human', formalized as '$(\forall x)\,(Lx \rightarrow Hx)$'. Here we have a quantifier, two functional expressions and a propositional connective.

This suggests that we should distinguish two conceptions of analysis: decompositional analysis and what I have called 'interpretive analysis'.[6] On the first conception analysis is indeed seen as decomposing something into its constituents. In the case of logical analysis, however, there is a first step that needs to be taken before the relevant constituents can be identified: the sentence to be analysed must be *interpreted* by rephrasing in some appropriate way. 'All logicians are human', for example, must be interpreted (in Frege's logic) as 'For all x, if x is a logician, then x is human' or (in more ordinary language) as 'If anything is a logician, then it is human'. This is interpretive analysis.

We can think of logical analysis, then, as proceeding in two steps. We first engage in interpretive analysis, rephrasing a sentence to reveal its logical (as opposed to merely grammatical) form, and only then do we apply decompositional analysis to identify its supposed (logically significant) constituents. Of course, our sense of

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[6]   For fuller discussion of conceptions of analysis in the history of philosophy, and of the interpretive conception, which is what I think especially characterizes analytic philosophy, see M. Beaney, 'Analysis' (2009) in *The Stanford Encyclopedia of Philosophy*, online at: plato.stanford.edu/entries/analysis.

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what these constituents should be may guide us in the interpretive analysis we offer, but we should nevertheless distinguish the two steps and not underestimate the importance of the first step. In logical analysis there is no decomposition without interpretation. We shall return to the significance of interpretive analysis later.

4. Frege's logicist project

Frege published three books in his lifetime, all of them directed at demonstrating that arithmetic is reducible to logic. In the *Begriffsschrift* of 1879 he developed the logical theory by means of which he could carry out the demonstration. In the third part of this work he also gave a logical analysis of mathematical induction, an important form of reasoning in mathematics. In *The Foundations of Arithmetic* of 1884 he offered an informal sketch of his logicist project, criticizing earlier accounts of arithmetic and elucidating his main ideas. In the *Basic Laws of Arithmetic*, the first volume of which appeared in 1893 and the second in 1903, he sought to provide the necessary formal proofs, with some modifications to his earlier logical theory. I shall concentrate here on the main ideas of the *Foundations*.

    The central claim of the *Foundations* is that number statements are assertions about concepts (*Die Grundlagen der Arithmetik*, §§ 46ff.). We are already in a position to explain this claim. According to Frege, as we have seen, existential statements are assertions about concepts. But existential statements are just one type of number statement. When we say that unicorns do not exist, we mean that the concept *unicorn* is not instantiated, in other words, has 0 instances. When we say that horses do exist, we mean that the concept *horse* is instantiated, in other words, that it has at least 1 instance. When we say that there are two horses, we mean that the concept *horse* (in the relevant context) has exactly 2 instances, and so on. A statement about how many of something there are is an assertion about a concept.

    Let us then consider one of Frege's own examples of the kind of number statement that we might make in everyday life (*Die Grundlagen der Arithmetik*, §57):

(J$_1$)   Jupiter has four moons.

It might be natural to interpret this, in accord with a subject–predicate analysis, as saying something about Jupiter, namely, that it has the property of possessing four moons. But this is a complex

property, which is itself in need of further analysis. What Frege argues is that (J_1) should be interpreted, instead, as saying something about the concept *moon of Jupiter*:

(J_2)   The concept *moon of Jupiter* has four instances.

More precisely, what this says is that the first-level concept *moon of Jupiter* falls within the second-level concept *has four instances*. So (J_1) is to be construed not as about the object Jupiter, as its surface grammatical form might suggest, but about the concept *moon of Jupiter*. The number statement is an assertion about a concept.

But how does this get us any further? Is the concept *has four instances* not just as much in need of further analysis as the supposed concept *has four moons*? It is indeed, but in this case we have the logical resources to provide the analysis. We have already seen how to define the second-level concept *is instantiated*, i.e., the concept *has at least 1 instance*. This is simply the existential quantifier: '($\exists x$) Fx' means that the (first-level) concept F is instantiated. So we just need to build on this. To say that a (first-level) concept has four instances, i.e., is instantiated four-fold, is to say that there are exactly four objects that fall under it. So (J_1) can be formalized logically as follows, with 'M' representing the concept *moon of Jupiter*:

(J_3) $(\exists v, w, x, y) (Mv \ \& \ Mw \ \& \ Mx \ \& \ My \ \& \ v \neq w \neq x \neq y \ \& \ (\forall z)$
           $(Mz \rightarrow z = v \lor z = w \lor z = x \lor z = y))$.

This can be read as 'There is some v, w, x and y such that v is M and w is M and x is M and y is M and v, w, x and y are all distinct from one other, and for all z, if z is M, then z is identical with either v or w or x or y'.

(J_3) is Frege's logical analysis of (J_1).[7] (J_1) thus has a more complex (quantificational) logical form than its surface (subject–predicate) grammatical form might suggest. Revealing such logical forms is precisely what logical analysis is all about, and demonstrating how *all* arithmetical statements can be analysed purely logically is precisely what the logicist project is all about.

This is not the place to give even a sketch of how Frege attempted to carry this through. Let us confine ourselves here to seeing how Frege defined the natural numbers themselves. For we do not just use number terms adjectivally, as in 'Jupiter has four moons', but also substantivally, as in '(The number) 2 is the successor of (the

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[7]   Frege does not, in fact, provide a logical analysis of precisely this example, and I also use here modern notation; but the analysis is in the spirit of his account in the *Foundations*.

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number) 1'. So how do we define, purely logically, 0, 1, 2, 3, and so on? Here we need to introduce the idea of an *extension of a concept*, which is the class or set of things that fall under the concept. Under the concept *human*, for example, fall Frege, Russell, you, me, etc. All of these objects (all of us) are members of the class of humans – the extension of the concept *human*. This class or extension, according to Frege, is itself a kind of object, not a 'concrete' object (existing in the empirical, spatio-temporal world) but an 'abstract' object (an object of our rational thought), in this case a logical object, since the idea of a class has traditionally been seen as logical.

Leaving aside here the problem of what abstract objects are, let us accept that classes (extensions of concepts) are abstract, logical objects. Traditionally, numbers have also been regarded as abstract objects. Frege himself stressed that we talk of 'the number one', for example, indicating that it refers to an object (rather than a concept). So can numbers be regarded as *logical* objects? If so, then the obvious suggestion is to find appropriate classes with which to identify them, and this is just what Frege did.

If we are going to define the natural numbers as classes, understood as logical objects, then we need to find appropriate logical concepts. Two of the most fundamental concepts of logic are the concepts of identity and of negation. Take the concept of identity, or more precisely, the concept of being identical with itself. Every object is identical with itself, in other words, every object falls under the concept *identical with itself*. (It might be a strange thing to say, but seems to be trivially true.) So the corresponding class has as its members all objects. Now let us add the concept of negation to form the concept *not identical with itself*. Nothing is not identical with itself. (If every object is identical with itself, then no object is not identical with itself. Again, this might be a strange thing to say, but seems to be trivially true.) So the corresponding class here has no members at all. This is what logicians call the 'null class' (or 'null set'), and in this case, it has been defined purely by means of logical concepts, as the class of things that are not self-identical.

The obvious suggestion is then to identify the first of the natural numbers, namely, the number 0, with the null class. This is what is done in modern set theory and is the simplest definition. Frege, in fact, offers a more complicated definition, identifying the number 0 not directly with the null class but with the class of classes that have the same number of members as the null class; but we can ignore this complication here. For present purposes, let us accept that this gives us our first natural number, the number 0, defined

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as the null class. We can then form the concept *is identical with 0* (i.e., the concept *is identical with the null class*). Here the corresponding class has just one member, namely, 0 (the null class itself). This class (the class of things that are identical with 0) is distinct from its sole member (0, i.e., the null class), since the former has one member and the latter has no members, so we can identify the number 1 with this class (the class of things that are identical with 0). We now have two objects, and can then form the concept *is identical with 0 or 1* (using, in this case, the additional logical concept of disjunction). This gives us a corresponding class which we can identify with the number 2, and so on. Starting with the null class, then, and using only logical concepts, we can define all the natural numbers.

The two cases we have just considered – Frege's analysis of 'Jupiter has four moons' and his definition of the natural numbers – should be enough to give a sense of the feasibility of the logicist project.[8] The key point here is to highlight the role that Frege's new logic – and the accompanying philosophical understanding of it – played in this project, without which it would scarcely have been thinkable. Unfortunately, however, as we will now see, there was nevertheless a fundamental problem in Frege's conception of his project, which is where Russell enters the story.

5. Russell's paradox

Like Frege, Russell was trained as a mathematician and became interested in the foundations of mathematics. After initially being attracted to British idealism, the philosophical tradition that was then dominant in Britain, he rejected it on the grounds that it could not do justice to mathematics,[9] and he then devoted himself, like

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[8] I give a fuller sketch in Beaney *Frege: Making Sense* (London: Duckworth, 1996), chs. 3–4; and in comparing Frege's and Russell's logicist projects, in Beaney 'Russell and Frege', in *The Cambridge Companion to Bertrand Russell* (ed.) N. Griffin (Cambridge: Cambridge University Press, 2003), 128–70; Beaney 'Frege, Russell and Logicism' in (eds) M. Beaney and E.H. Reck *Gottlob Frege: Critical Assessments* (London: Routledge).

[9] For a detailed account of this, see Griffin *Russell's Idealist Apprenticeship* (Oxford: Clarendon Press, 1991); Griffin 'Russell and Moore's Revolt against British Idealism', in *The Oxford Handbook of the History of Analytic Philosophy* (ed.) M. Beaney (Oxford: Oxford University Press, 2013), 383–406.

Frege, to showing that arithmetic (and geometry, too, in Russell's case) could be reduced to logic. Russell's logicist views were first presented in *The Principles of Mathematics*, published in 1903, and those views were revised and a detailed formal demonstration offered in his main work, *Principia Mathematica*, published in three volumes between 1910 and 1913. This work was written with his former mathematics teacher at Cambridge, A. N. Whitehead (1861–1947), who was to become a significant philosopher in his own right.

Like Frege, too, Russell defined the natural numbers as classes, using only logical concepts. Unlike Frege, however, he came to believe that classes should not be taken as objects, whether logical or not. Rather, he argued, they are 'logical fictions'. In making sense of how we could nevertheless talk about such fictions in saying true things (not least in mathematics), he developed his most famous theory, the theory of descriptions, to which we will turn in the next section. To understand what motivated his views, however, we need to go back to Frege's conception of classes (extensions of concepts). Central to this conception was the principle that for every concept, there is a class of things that fall under it. (If nothing falls under it, then there is still a class – the null class.) Furthermore, if classes are objects (in any sense at all, it seems), then they can themselves be members of classes. All we need is a relevant concept under which these classes can be taken to fall. The concept *class* is obviously one such concept. This in turn means that it is possible for a class to be a member of itself, as indeed, the class of classes would be.

These ideas, however, lead to a contradiction, now known as Russell's paradox. Consider the class of horses. This class is not itself a horse, so the class is not a member of itself. Consider the class of non-horses. This class is not a horse, so the class *is* a member of itself. So classes divide into those that are members of themselves and those that are not members of themselves. Consider now the class of all classes that are not members of themselves. Is this a member of itself or not? If it is, then since it is the class of all classes that are not members of themselves, it is not. If it is not, then since this is the defining property of the classes it contains, it is. We have a contradiction.

Why should this contradiction trouble us? Why should we not just deny that there can be any such class as the class of classes that are not members of themselves? The problem is that the defining condition for such a class seems perfectly logical. If we allow the concepts of a class and of class-membership, then we can legitimately form the concepts of a class being a member of itself and of a class not being

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a member of itself. The concept of a class being a member of itself seems to determine a legitimate class – the class of classes that are members of themselves. (Is this class a member of itself or not? If it is, then it is; and if is not, then it is not; so no contradiction arises here.) So the concept of a class not being a member of itself ought also to determine a legitimate class – the class of classes that are not members of themselves. Yet it is the idea of this class that generates a contradiction.

Given that both Frege and Russell wanted to define numbers in terms of classes (and indeed, classes of classes), determined by logically legitimate concepts, Russell's paradox is potentially devastating. Russell discovered the contradiction in 1902 and wrote to Frege in June that year informing him of it. Frege immediately recognized its significance, replying that it threatened the very foundations that he had hoped to establish for arithmetic. At the time that Frege received Russell's letter, the second volume of his *Basic Laws of Arithmetic* was in press. He attempted to respond to the paradox in a hastily-written appendix, but he soon realized that his response was inadequate, and ended up abandoning his logicist project, focusing instead on the clarification of his logical ideas. Russell did not give up so easily, however, and devoted the next ten years of his life to solving the paradox and attempting to show how the logicist project could nevertheless be carried out.

Again, this is not the place for even a brief sketch of Russell's own logicist project. Let us simply highlight here the main idea behind Russell's response to the paradox before considering its implications for our concern with the nature of analysis. Essentially Russell denied that classes could be members of themselves, but he embedded this response in a theory – his so-called theory of types – that was intended to provide a philosophical justification of his solution to the paradox. On this theory, there is a *hierarchy* of objects and classes. At the most basic level, there are 'genuine' objects – objects such as horses, tables, chairs, and so on. At the next level, there are classes of objects – such as the class of horses and the class of non-horses (which contains all those genuine objects, such as tables and chairs, that are not horses). Then there are classes of classes of objects, and so on up the hierarchy. The key point is that something at any given level can only be a member of a class at a higher level. This automatically rules out any class being a member of itself; so no contradiction can be generated.

According to Russell, then, classes are not genuine objects. But what are they, and how can we apparently say true things about them (as we must do if we are to define numbers as classes)?

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Russell came to argue that classes are 'logical fictions' or 'logical constructions': they do not 'exist' in any proper sense, but we can give a satisfactory logical analysis of our talk about them. A simple example (not Russell's own) can be used to illustrate the basic idea. Let us imagine making the following true claim:

(A$_1$)   The average British woman has 1.9 children.

Here there is no such person as the average British woman, and even if there were, she could hardly have 1.9 children! So how could any such claim be true (or indeed false, as the case may be)? It is clear what we mean here, which might be unpacked by expressing it as follows:

(A$_2$)   The total number of children of British women divided by the total number of British women equals 1.9.

(A$_1$), then, is really just a disguised claim about all British women. It offers us a useful abbreviation of (A$_2$), enabling us to compare more easily the situation in different countries, for example. We can say such things as 'While the average British woman has 1.9 children, the average Chinese woman has 1.5 children'. 'The average British woman' and 'the average Chinese woman' are *logical fictions*. No such women exist, but the terms provide a convenient way of talking, enabling us to make true statements more simply.

Talk of classes can be analysed in a similar way. Consider, for example, the following true claim:

(C$_1$)   The class of horses is a subclass of the class of animals.

Do we need to suppose that such classes 'exist' in order for this statement to be true? Not at all, on Russell's view. (C$_1$) can be analysed as follows:

(C$_2$)   Anything that falls under the concept *horse* falls under the concept *animal*.

This is a claim about concepts, not classes, readily formalized in logic as:

(C$_3$) $(\forall x)\,(Hx \rightarrow Ax)$.

As we have seen, this says that one concept (the concept *horse*) is subordinate to another (the concept *animal*). We need only to suppose that concepts 'exist', therefore, not classes as well.

Given the close connection between classes and concepts, as captured in the principle that for every (legitimate) concept there is a class determined by it, talk of classes can be translated into talk of their corresponding concepts. Concepts can thus be regarded as 'ontologically prior' to the classes they determine. It is this idea

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that lies behind Russell's claim that classes are logical fictions or logical constructions. Talk of classes is 'constructed' out of our talk of concepts, in a similar way to how talk of 'the average woman' is constructed out of our talk of actual women.

6. Russell's theory of descriptions

Russell's concern with solving the paradox that bears his name, in pursuing his logicist project, is the background against which to understand his theory of descriptions.[10] For what the paradox raises is the problem of how definite descriptions, i.e., terms of the form 'the F', can contribute to the meaning and truth-value of sentences in which they appear even when they lack a referent. 'The class of all classes that are not members of themselves' seems meaningful and yet there can be no such class. If all classes are logical fictions, as Russell came to believe, then all class terms lack referents, yet we can say true things using such terms. So analysis of talk of classes is clearly called for.

Let us consider Russell's own famous example of a sentence involving a definite description:

(K_1)   The present King of France is bald.

If we were to treat this sentence as having the form 'S is P', in accord with traditional subject–predicate analysis, then we would regard 'the present King of France' as the subject term. But if there is no King of France, then what is the sentence about? A non-existent – or 'subsistent' – object? How can we understand such a sentence if the subject term lacks a referent? Can it have a truth-value in such a case? Traditional subject–predicate analysis seems to raise many questions when the subject term fails to refer.

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[10]   There has been a huge amount written both on the theory of descriptions itself and on its history, and I can do no justice to any of this here. A full understanding would have to recognize, for example, how the theory improved on Russell's own earlier theory of denoting (as presented in *The Principles of Mathematics* of 1903). For discussion, see e.g. Hylton *Russell, Idealism, and the Emergence of Analytic Philosophy* (Oxford: Clarendon Press, 1990); Hylton, 'The Theory of Descriptions', in *The Cambridge Companion to Bertrand Russell* (ed.) Griffin, N. (Cambridge: Cambridge University Press, 2003); Linsky 'Russell's Theory of Descriptions and the Idea of Logical Construction', in *The Oxford Handbook of the History of Analytic Philosophy* (ed.) M. Beaney (Oxford: Oxford University Press, 2013).

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In 'On Denoting', published in *Mind* in 1905, Russell introduced his theory of descriptions to answer these questions.[11] On the account he offered, (K_1) is analysed into a conjunction of the following three sentences, each of which can be readily formalized in quantificational logic (given in square brackets afterwards), with '*K*' representing the concept *King of France* and '*B*' the concept *bald*:

(K_a) There is at least one King of France. $[(\exists x)\ Kx]$
(K_b) There is at most one King of France. $[(\forall x)\ (\forall y)\ (Kx\ \&\ Ky \rightarrow y = x)]$
(K_c) Whatever is King of France is bald. $[(\forall x)\ (Kx \rightarrow Bx)]$

Each of these constituent sentences has a quantificational structure, and can be interpreted as saying something about a concept, not an object. The first says that the concept *King of France* is instantiated (by at least one object), and the second says that the concept *King of France* is instantiated by at most one object.[12] Taken together they say that the concept *King of France* is uniquely instantiated. The third says that whatever instantiates the concept *King of France* also instantiates the concept *bald*.

Putting all three together, we have:

(K_2) There is one and only one King of France and whatever is King of France is bald.

Formalizing this (and simplifying) yields:

(K_3) $(\exists x)\ (Kx\ \&\ (\forall y)\ (Ky \rightarrow y = x)\ \&\ Bx)$.

Reading this as saying something about a concept gives us the following interpretive analysis of the original sentence (K_1):

(K_4) The concept *King of France* is uniquely instantiated and whatever instantiates this concept also instantiates the concept *bald*.

With this analysis we can now answer our earlier questions. The surface grammatical form of (K_1) is misleading: it has a much more complex logical form. The sentence is not about a non-existent (or subsistent) object, but about a concept; and all we need to grasp to

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[11] Russell had first tried to answer these questions in his earlier theory of denoting (see the previous note). But for various reasons which we cannot address here, he soon became dissatisfied with his answer.
[12] Very roughly, it could be read as saying that were it to seem as if two objects fell under the concept *King of France*, then they would actually be one and the same.

understand the sentence are the two relevant concepts, the concept *King of France* and the concept *bald*, as well as the relevant logical ideas (conjunction, implication, identity, and existential and universal quantification). Where there is no King of France, i.e., where the subject term has no referent, the first conjunct of the analysis – (K_a) – is false, thereby making the original sentence – (K_1) – false. So (K_1) still comes out as having a truth-value.

Russell's theory of descriptions has often – and rightly – been regarded as a paradigm of analysis. But we should recognize that all the materials for his analysis were already present in Frege's work. As we have seen, Frege construed existential statements as assertions about concepts and interpreted sentences such as (K_c) as involving the subordination of concepts. To make this clearer, we could thus rephrase (K_1) further as follows:

(K_5) The concept *King of France* is uniquely instantiated and subordinate to the concept *bald*.

This might seem to be saying something rather different from what we thought was being said by (K_1), but that it does have this interpretation is precisely what is implied by its formalization into quantificational logic within Russell's theory of descriptions.

7. Interpretive analysis

As we have seen, both Frege and Russell use interpretive analysis, drawing on the new resources of quantificational logic. It is the role played by interpretive analysis that I think is especially distinctive of analytic philosophy, and it was the logical revolution that Frege inaugurated that made possible the analytic revolution that took place in philosophy around the turn of the twentieth century.

This is not to say, however, that Frege and Russell use interpretive analysis to do the same kind of philosophical work. Certainly, they were both concerned to demonstrate logicism, and interpretive analysis played an essential role in this. But their philosophical conceptions of logicism were rather different. By defining numbers as classes (extensions of concepts), Frege saw his analyses as showing that numbers are logical objects. Russell, on the other hand, in responding to the contradiction he discovered in Frege's work, came to reject the view that classes are objects, arguing instead that they are logical fictions. Such fictions may be useful in demonstrating logicism but they must ultimately be recognized for what they are.

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What is characteristic of Russell's use of interpretive analysis, then, is its role in a philosophical project that is not just *reductivist* (like Frege's) but *eliminativist*. Numbers are not just 'reduced' to classes but 'eliminated' as mere logical fictions. Talk of numbers is nevertheless shown to be logically legitimate by interpreting or rephrasing sentences involving number terms: this is also what is meant when Russell describes numbers as logical constructions. It was Russell's theory of descriptions that gave him the confidence to take this philosophical line. As Russell himself put it in explaining that theory, definite descriptions are 'analysed away'. When a sentence involving a definite description – such as (K_1) – is interpreted in accord with the theory, the definite description disappears. In (K_2), for example, the term 'the present King of France' is not used, only the concept word 'King of France'. In itself, according to Russell, the definite description is meaningless, although it may nevertheless contribute to the meaning of a sentence in which it appears.[13]

Whose approach is right: Frege's or Russell's? Philosophers today still debate the issue and take sides in their own work. To explore the issue a little further, let us return to one of our earlier examples:

(J_1) Jupiter has four moons.

We saw that this can be analysed into:

(J_3) $(\exists v, w, x, y)\,(Mv \ \& \ Mw \ \& \ Mx \ \& \ My \ \& \ v \neq w \neq x \neq y \ \& \ (\forall z)$
$(Mz \rightarrow z = v \lor z = w \lor z = x \lor z = y))$.

Here the number term 'four' is analysed away, so this might seem to support Russell's approach. We have no need to suppose that 'four' denotes an object; indeed, it hardly seems to do so when used adjectivally as in (J_1).

Frege,[14] on the other hand, noted that (J_1) could also be taken to express an identity statement:

(J_4) The number of Jupiter's moons is (the number) four.

For him, the possibility of such rephrasal or 'interpretation' – and the perceived equivalence between (J_1) and (J_4) – showed that numbers should indeed be seen as objects. His thinking was very simple. Assuming that (J_1) is true and that it is equivalent to (J_4), then (J_4) is true. But (J_4) can only be true if the terms flanking the identity sign, i.e., 'the number of Jupiter's moons' and 'the number four' have

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[13]   Russell, 'On Denoting', 488; *My Philosophical Development* (London: George Allen and Unwin, 1959), 64.
[14]   *Die Grundlagen der Arithmetik*, §57.

meaning (*Bedeutung*). But such terms, i.e., terms of the form 'the *F*' only have meaning, according to Frege, if they stand for objects.

For Frege, then, interpretive analysis was not part of an eliminativist project; on the contrary, it was employed to support a form of Platonism: numbers must be conceived as existing in a realm of abstract objects. However, from what we have already seen, Frege's actual use of interpretive analysis nevertheless has an implicit eliminativist dimension. His analysis of 'Unicorns do not exist', for example, readily suggests that we do not need to posit any objects for this sentence to be true. All that we need be ontologically committed to is the existence of the relevant concept – the concept *unicorn* (which might in turn, though, be analysable into the concepts of a horse and of a horn), together with the logical concept of negation and second-level concept *is instantiated*.

It was left to Russell, however, to properly appreciate the eliminativist potential of interpretive analysis. But does this mean that Russell is right to claim that numbers, as classes, are logical fictions? If calling something a 'fiction' implies that it does not exist, then this is misleading. For it suggests that numbers lack something that they could have. But numbers are not the kind of thing that could exist (in the empirical, spatio-temporal world). Denying that they exist, though, makes them seem more mysterious than they actually are. What we want to understand is our use of number terms, and trying to decide whether or not numbers 'exist' is to become distracted by the real issue. It is better, then, to use Russell's other term and talk of numbers as logical constructions. What is it to claim, for example, that Jupiter has four moons? It is indeed to claim that the concept *moon of Jupiter* is instantiated by one object, another object distinct from the first, another object distinct from either of the first two, a further object distinct from any of the other three, and by no other object. This is exactly what (J$_3$) captures. We can abbreviate this by saying that the number of Jupiter's moons is four, helping us to compare more easily the numbers of other types of things, such as the number of seasons in a year – in just the same way as talk of 'the average woman' may help us make comparisons across different countries.

However, the main point here is not to take sides on the dispute between Frege and Russell but just to illustrate the different uses that interpretive analysis can be put. The logical revolution may have made possible Frege's and Russell's logicist projects, but it also opened up the use of interpretive analysis for a whole range of other philosophical projects, as the subsequent history of analytic philosophy has shown.

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8. The paradox of analysis

Frege's attempt to reduce arithmetic to logic was undermined by Russell's paradox, and a natural response is to reconceive interpretive analysis as playing more of an eliminativist role. But there is also a paradox that threatens to undermine the very possibility of interpretive analysis. This is the paradox of analysis, which was first given this name in discussion of Moore's philosophy in the 1940s, but which in fact was formulated by Frege himself in 1894, in responding to criticisms that Husserl had made to the logicist analyses he had offered in *The Foundations of Arithmetic*.[15]

The paradox can be stated very simply. Call what we want to analyse (the *analysandum*) '*A*' and what is offered as the analysis (the *analysans*) '*B*'. Then either '*A*' and '*B*' have the same meaning, in which case the analysis expresses a trivial identity and is uninformative; or else they do not, in which case the analysis is incorrect, however informative it might seem. So no analysis can be both correct and informative. Let us illustrate the problem by returning to one of our earlier examples:

(L$_1$)   All logicians are human.

In quantificational logic, this is formalized as follows:

(L$_2$) $(\forall x)\,(Lx \rightarrow Hx)$.

In explaining this formalization, we might offer various interpretive analyses, including, for example:

(L$_3$)   If anything is a logician, then it is human.
(L$_4$)   For all x, if x is a logician, then x is human.
(L$_5$)   The (first-level) concept *logician* is subordinate to the (first-level) concept *human*.

All of these, we want to claim, are equivalent. But if we take, say, (L$_5$), can we really maintain that this has the same meaning as (L$_1$)? Surely someone can understand (L$_1$) without understanding (L$_5$)? They may never have come across the idea of one concept being subordinate to another (or appreciate the distinction between first-level and second-level concepts).

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[15]   For details, see Beaney *Frege: Making Sense* (London: Duckworth, 1996), ch. 5; Beaney 'Sinn, Bedeutung and the Paradox of Analysis' in *Gottlob Frege: Critical Assessments* (eds) M. Beaney and E.H. Reck (London: Routledge, 2005).

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Clearly, on some conceptions of meaning, (L_1) and (L_5) – or any of the other analyses – do not have the same meaning. But there must be something they have in common if the analysis is indeed to be taken as correct. A minimum requirement is that they are are logically equivalent, in the sense that one implies the other, and vice versa. Now without trying to specify an appropriate criterion for sameness of meaning here, on which there has been great controversy,[16] let me defend the legitimacy of analysis and respond to the paradox by stressing the *dynamic* nature of the process of analysis. Of course, someone can understand (L_1) without understanding (L_5), but once they are brought to appreciate what (L_5) means, they thereby come to recognize that (L_5) captures what is going on, conceptually, in (L_1). An analysis is informative by being transformative.

In offering an analysis we provide richer conceptual tools to understand something. This is exactly what Frege and Russell did in drawing on the powerful resources of the new logic. In coming to appreciate – or being convinced by – an analysis, we learn to use these conceptual tools ourselves in deepening our own understanding. Learning what is meant in talking of the subordination of concepts, for example, gives us a deeper insight into the logical relations between concepts and the statements we make. Consider once again the claim that Jupiter has four moons – (J_1). What underlies our understanding of (J_1) is our abilities to count and to apply concepts to objects. This is what is made explicit in (J_4): that one object and a second object and a third object and a fourth object, and no further objects, fall under the concept *moon of Jupiter*. (J_4) may have a much more complex logical form, but it is precisely this that reflects the complexity of the logical and arithmetical abilities that underpin our use of sentences such as (J_1).

In giving and understanding analyses, then, we typically utilise richer conceptual tools. In the case of logical analysis, we invoke concepts such as those of subsumption, subordination, instantiation, first-level and second-level concepts, and so on. In thinking about – or indeed, analysing – these analyses themselves, we invoke further concepts, such as those of meaning, reference, equivalence, and so on. All these logical and semantic concepts and relations might themselves be seen as logical constructions, which emerge in our activities of analysis. Logical construction permeates all of our conceptual and logical practices.

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[16]   I discuss the issue in Beaney *Frege: Making Sense* (London: Duckworth, 1996), ch. 8.

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9. Conclusion

What does 'analytic philosophy', as it is generally used today, mean? The obvious answer is that it is philosophy that accords a central role to analysis. But 'analysis', in one form or another, has always been part of Western philosophy, from ancient Greek thought onwards.[17] So this answer says little. I have suggested in this lecture that what is especially distinctive of analytic philosophy, at least of that central strand that originates in the work of Frege and Russell, is the role played by interpretive analysis, drawing on the powerful resources that the new quantificational logic provided.

This brought with it – or crystallized – a new set of concepts, which opened up a new set of questions concerning meaning, reference, and so on, that Frege and Russell began to explore but were especially taken up by the next generation of analytic philosophers, including Wittgenstein and the logical positivists. This gave rise to the so-called linguistic turn, heralded in Wittgenstein's *Tractatus*, published in 1921. Here, too, though, the roots of the linguistic turn lie in the analytic revolution that Frege and Russell effected. And my main concern here has just been to shed some light on this analytic revolution.

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[17]   I offer an account of the different – but related – conceptions and practices of analysis in the history of philosophy in Beaney 'Analysis', in *The Stanford Encyclopedia of Philosophy* (2009), online at: plato.stanford. edu/entries/analysis.