Math. Proc. Camb. Phil. Soc. (2013), **154**, 97–118 © *Cambridge Philosophical Society* 2012
doi:10.1017/S0305004112000448
First published online 1 October 2012

Hausdorff dimension of the set of elliptic functions with critical values approaching infinity

BY PIOTR GAŁĄZKA†
Faculty of Mathematics and Information Science,
Warsaw University of Technology, Warsaw 00-661*, Poland.*
e-mail: P.Galazka@mini.pw.edu.pl

(*Received* 16 *May* 2011; *revised* 16 *July* 2012)

Abstract

Let $\wp_\Lambda$ denote the Weierstrass function with a period lattice Λ. We consider *escaping parameters* in the family $\beta\wp_\Lambda$, i.e. the parameters β for which the orbits of all critical values of $\beta\wp_\Lambda$ approach infinity under iteration. Unlike the exponential family, the functions considered here are ergodic and admit a non-atomic, σ-finite, ergodic, conservative and invariant measure μ absolutely continuous with respect to the Lebesgue measure. Under additional assumptions on $\wp_\Lambda$, we estimate the Hausdorff dimension of the set of escaping parameters in the family $\beta\wp_\Lambda$ from below, and compare it with the Hausdorff dimension of the escaping set in the dynamical space, proving a similarity between the parameter plane and the dynamical space.

1. *Introduction*

In a series of papers, J. Hawkins and L. Koss [**5**, **6**, **7**] described the dynamics of Weierstrass functions. The ergodic theory of non-recurrent elliptic functions was developed by J. Kotus and M. Urbański in [**12**, **13**, **14**]. Recently, in [**8**], examples have been given of all possible types of behaviour of non-recurrent elliptic functions (in that paper, referred to as critically tame functions). This class includes maps with critical values approaching infinity. The aim of this paper is to show that the escaping parameters form a rather 'big' set.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. For $n \in \mathbb{N}$, denote by f^n the nth iterate of f. The *Fatou set* $F(f)$ of f is the set of points $z \in \mathbb{C}$ such that all iterates $f^n(z)$ are well-defined and $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of z. The complement $J(f)$ of $F(f)$ in $\overline{\mathbb{C}}$ is called the *Julia set* of f. P. Domínguez in [**4**] proved that for transcendental meromorphic functions with poles *the escaping set*

$$I(f) = \left\{ z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty \right\}$$

is non-empty and $J(f) = \partial I(f)$. Later, P. Rippon and G. Stallard [**17**] showed that if additionally f is in the Eremenko–Lyubich class $\mathcal{B}$, then $I(f) \subset J(f)$, which follows that

† Partially supported by PW - grant No 504G/1120/0077/000 and MNiSW - grant No NN 201 607640.

$\mathrm{Int}\, I(f) = \varnothing$. Recently, several authors [**1**, **2**, **3**, **18**, **19**] have studied properties of the escaping set for entire and meromorphic functions. The Hausdorff dimension $\dim_H(I(f))$ of the escaping set for some class of meromorphic functions was estimated from below by J. Kotus in [**10**]. Applying her result to elliptic functions of the form $g_\beta = \beta\wp_\Lambda$, $\beta \in \mathbb{C}\setminus\{0\}$, where $\wp_\Lambda$ is the Weierstrass elliptic function, we have $\dim_H(I(g_\beta)) \geqslant 4/3$. This estimate together with the fact proved by Bergweiler, Kotus and Urbański in [**2**, **12**] that the upper bound on $\dim_H(I(g_\beta))$ is the same as the lower bound gives

$$\dim_H(I(g_\beta)) = \frac{4}{3}.$$

In this paper, we additionally assume that the lattice of $\wp_\Lambda$ is triangular and the critical values of $\wp_\Lambda$ are poles. As a counterpart of the escaping set $I(g_\beta)$ we consider the set of escaping parameters in the family g_β, i.e.

$$\mathcal{E} = \left\{\beta \in \mathbb{C}\setminus\{0\} : \lim_{n\to\infty} g_\beta^n(c_i) = \infty,\ i = 1, 2, 3\right\},$$

where c_i is a critical point of $\wp_\Lambda$. For these maps the Julia set is the Riemann sphere $\overline{\mathbb{C}}$. In this paper, we construct a collection of Cantor subsets of $\mathcal{E}$ with a prescribed growth rate and estimate their Hausdorff dimension from below. The main result is the following theorem.

THEOREM. *For any one-parameter family of functions $g_\beta(z) = \beta\wp_\Lambda(z)$, where $\beta \in \mathbb{C}\setminus\{0\}$, $\Lambda = [\lambda_1, e^{2\pi i/3}\lambda_1]$ is a triangular lattice such that all critical values of $\wp_\Lambda$ are poles, the Hausdorff dimension of the set of escaping parameters $\mathcal{E}$ is greater or equal to* 4/3.

The paper is organised as follows. In Section 2, we give background definitions and results for studying elliptic functions, in particular the Weierstrass $\wp_\Lambda$-function. We also summarise metric properties of maps in $\mathcal{E}$. In Sections 3 and 4, we show how one can find escaping parameters. In the final section, we estimate $\dim_H(\mathcal{E})$ from below.

2. *General preliminaries*

We begin with the definition and basic properties of elliptic functions. For $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\{0\}$ such that $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$, a lattice $\Lambda \subset \mathbb{C}$ is defined as

$$\Lambda = [\lambda_1, \lambda_2] = \{l\lambda_1 + m\lambda_2,\ l, m \in \mathbb{Z}\}.$$

Definition 2·1. An elliptic function is a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ which is periodic with respect to a lattice Λ, i.e. $f(z) = f(z + l\lambda_1 + m\lambda_2)$ for all $z \in \mathbb{C}$ and $l, m \in \mathbb{Z}$.

We denote by $b_{l,m} = l\lambda_1 + m\lambda_2$, $l, m \in \mathbb{Z}$, lattice points of Λ and by

$$\mathcal{R} = \{t_1\lambda_1 + t_2\lambda_2;\ 0 \leqslant t_1, t_2 < 1\}$$

the fundamental parallelogram of Λ. For a non-constant elliptic function and a given $w \in \overline{\mathbb{C}}$ the number of solutions to the equation $f(z) = w$ in $\mathcal{R}$ equals the sum of multiplicities of poles in the fundamental parallelogram. Since the derivative of an elliptic function is also an elliptic function which is periodic with respect to the same lattice, then each elliptic function has infinitely many critical points but only finitely many critical values. Due to periodicity, elliptic functions do not have asymptotic values. Thus, they belong to the class $\mathcal{S}$.

A special case of an elliptic function is the Weierstrass elliptic function defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$$

for all $z \in \mathbb{C}$ and every lattice Λ. It is well known that $\wp_\Lambda$ is periodic with respect to Λ and has order 2. The derivative of the Weierstrass function is also an elliptic function which is periodic with respect to Λ and is defined by

$$\wp'_\Lambda(z) = -2\sum_{w\in\Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\left(\wp'_\Lambda(z)\right)^2 = 4\left(\wp_\Lambda(z)\right)^3 - g_2\wp_\Lambda(z) - g_3, \tag{2·1}$$

where $g_2 = g_2(\Lambda) = 60\sum_{w\in\Lambda\setminus\{0\}}(1/w^4)$, $g_3 = g_3(\Lambda) = 140\sum_{w\in\Lambda\setminus\{0\}}(1/w^6)$. The numbers $g_2(\Lambda), g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_i(\Lambda) = g_i(\Lambda')$, $i = 2, 3$, then $\Lambda = \Lambda'$. Moreover, for any g_2, g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there is a lattice Λ with invariants g_2, g_3. For any lattice Λ the Weierstrass function $\wp_\Lambda$ satisfies the property of homogeneity, i.e.

$$\wp_{\alpha\Lambda}(\alpha z) = \frac{1}{\alpha^2}\wp_\Lambda(z) \tag{2·2}$$

for every $\alpha \in \mathbb{C}\setminus\{0\}$. The Weierstrass function has poles of order 2 at lattice points and its derivative has poles of order 3. In the fundamental parallelogram the map $\wp_\Lambda$ has three critical points which we denote by

$$c_1 = \frac{\lambda_1}{2},\ c_2 = \frac{\lambda_2}{2},\ c_3 = \frac{\lambda_1+\lambda_2}{2}.$$

We use the symbols $e_i = \wp_\Lambda(c_i)$, $i = 1, 2, 3$ to denote the critical values of $\wp_\Lambda$. They are related to each other with the equations

$$e_1 + e_2 + e_3 = 0,\ e_1e_3 + e_2e_3 + e_1e_2 = -\frac{g_2}{4},\ e_1e_2e_3 = \frac{g_3}{4}. \tag{2·3}$$

We consider only Weierstrass functions which are periodic with respect to triangular lattices, i.e. lattices $\Lambda = [\lambda_1, \lambda_2]$ such that $\lambda_2 = e^{2\pi i/3}\lambda_1$. In other words a lattice is triangular if $\Lambda = e^{2\pi i/3}\Lambda$. For triangular lattices $g_2 = 0$ and the critical values of $\wp_\Lambda$ are the cube roots of $g_3/4$. Moreover, (2·1) and (2·3) imply that the critical value e_3 is a non-zero real number and e_1, e_2 are given by the formulas $e_1 = e^{4\pi i/3}e_3$, $e_2 = e^{2\pi i/3}e_3$. The iterates of the critical values turn out to have the same property, i.e. $\wp^n_\Lambda(e_1) = e^{4\pi i/3}\wp^n_\Lambda(e_3)$, $\wp^n_\Lambda(e_2) = e^{2\pi i/3}\wp^n_\Lambda(e_3)$, $n \geqslant 1$. It is a consequence of invariance of the triangular lattice with respect to the rotation $z \mapsto e^{2\pi i/3}z$ and the homogeneity of $\wp_\Lambda$ given in (2·2) (see [**6**] for details).

We additionally assume that all critical values of the Weierstrass function $\wp_\Lambda$ are poles. An example of a family of such lattices was given by Hawkins and Koss in [**6**].

Example 2·2. Let $\Omega = [\omega_1, \omega_2]$ be a lattice with invariants $g_2 = 0$, $g_3 = 4$. It is a triangular lattice for which $e_1 = e^{4\pi i/3}$, $e_2 = e^{2\pi i/3}$, $e_3 = 1$. Let $\gamma_1 = \sqrt[3]{e^{4\pi i/3}\omega_1^2/m}$, where m is an odd negative number and $\gamma_2 = \gamma_1\omega_2/\omega_1$. Then, the lattice $\Gamma = [\gamma_1, \gamma_2]$ is triangular and all critical values of $\wp_\Lambda$ are poles.

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

Now, we describe ergodic properties of the so-called critically tame elliptic functions studied by Kotus and Urbański in [**14**]. We start with some definitions and notations.

Definition 2·3. Let $f\colon \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function and $z \in \mathbb{C}$ such that all iterates $f^n(z)$, $n \in \mathbb{N}$ are well-defined. A point $w \in \overline{\mathbb{C}}$ is called an $\omega-$*limit* point of z for f, if there is a sequence of natural numbers $n_k \to \infty$ such that

$$\lim_{k\to\infty} \mathrm{dist}_s(f^{n_k}(z), w) = 0,$$

where dist_s denotes the spherical metric in $\overline{\mathbb{C}}$. The $\omega-$*limit* set of z is a set of all $\omega-$*limit* points of z and we denote it by $\omega(z)$.

Definition 2·4. Suppose that:

(i) $g\colon D \to \mathbb{C}$ is an analytic map where $D \subset \mathbb{C}$ is a domain;
(ii) $U(z, g^{-1}, r)$ is the connected component of $g^{-1}(B(g(z), r))$ containing z for given $z \in \mathbb{C}$ and $r > 0$;
(iii) $c \in Crit(g)$.

Then, there exist $r = r(g, c) > 0$ and $K = K(g, c) \geqslant 1$ such that

$$\frac{1}{K}|z-c|^p \leqslant |g(z) - g(c)| \leqslant K|z-c|^p$$

and

$$\frac{1}{K}|z-c|^{p-1} \leqslant |g'(z)| \leqslant K|z-c|^{p-1}$$

for all $z \in U(c, g^{-1}, r)$ and some natural $p = p(g, c)$, and also such that

$$g(U(c, g^{-1}, r)) = B(g(c), r).$$

The number p is called the order of g at the critical point c and is denoted by p_c. The number $p_c - 1$ is the multiplicity of the zero of g' at c.

Denote by $\mathcal{P}_n(f)$, $n \geqslant 1$, the set of prepoles of order n of f, i.e.

$$\mathcal{P}_n(f) = \{z \in \mathbb{C}\colon\ f^n(z) = \infty\}.$$

In particular, $\mathcal{P}_1(f)$ is the set of poles of f.

Definition 2·5. Suppose that $f\colon \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function and $b \in \mathcal{P}_1(f)$. Let η_b denote the multiplicity of the pole b. We define

$$q := \sup\{\eta_b\colon b \in \mathcal{P}_1(f)\} = \max\{\eta_b\colon b \in \mathcal{P}_1(f) \cap \mathcal{R}\}.$$

Denote by $\mathrm{Crit}(f)$ the set of critical points of f, i.e.

$$\mathrm{Crit}(f) = \{z \in \mathbb{C}\colon\ f'(z) = 0\}.$$

Let $\mathrm{Crit}_b(f)$ be the set of all prepole critical points, i.e.

$$\mathrm{Crit}_b(f) = \mathrm{Crit}(f) \cap \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(f).$$

Moreover, we define the set of all critical points of f whose trajectories approach infinity, i.e.

$$\mathrm{Crit}_\infty(f) = \left\{c \in \mathrm{Crit}(f)\colon\ \lim_{n\to\infty} f^n(c) = \infty\right\}.$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

Note that $\mathcal{P}_n(f) = f^{-1}(\mathcal{P}_{n-1}(f))$ for all $n \geqslant 2$ and $\mathcal{P}_n(f) \subset J(f)$. For every $c \in \text{Crit}_b(f)$ there is a unique $n \in \mathbb{N}$ such that $c \in \mathcal{P}_n(f)$. For all $c \in \text{Crit}_\infty(f)$ and every $R > 0$ there exists a natural N such that for all $n \geqslant N : |f^{n+1}(c)| > R$. This inequality is equivalent to the fact that $f^n(c)$ lies close to a unique pole b_n. That implies that for all $c \in \text{Crit}_\infty(f)$ one can define a sequence of poles b_n close to the iterates of f.

Definition 2·6. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function. For $c \in \text{Crit}_\infty(f)$ we define

$$q_c := \limsup_{n\to\infty} \eta_{b_n},$$

where the sequence $\{b_n\}_{n\geqslant 1}$ was defined above. Moreover, let

$$l_\infty = \max\{p_c q_c : c \in \text{Crit}_\infty(f)\},$$

where p_c is as in Definition 2·4.

Definition 2·7. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function and $c \in \text{Crit}(f)$. We say that f is critically tame if the following conditions are satisfied:

(a) if $c \in F(f)$, then there exists an attracting or parabolic cycle of period p, $S = \{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ such that $\omega(c) = S$.
(b) if $c \in J(f)$, then one of the following holds:
  (i) $\omega(c)$ is a compact subset of $\mathbb{C}$ such that $c \notin \omega(c)$;
  (ii) $c \in \text{Crit}_b(f)$;
  (iii) $c \in \text{Crit}_\infty(f)$ and

$$\dim_H(J(f)) > \frac{2l_\infty}{l_\infty + 1}.$$

Denote by $\text{Tr}(f) \subset J(f)$ the set of all transitive points of f, that is the set of points in $J(f)$ such that their forward trajectories are dense in $J(f)$.

We quote two results from [**14**], which became an inspiration for studying the escaping parameters $\mathcal{E}$. Below, a conformal measure m is defined by means of the spherical metric.

PROPOSITION 2·8. *Suppose that f is a critically tame elliptic function, denote $h = \dim_H(J(f))$. Then there exist*:

(a) *a unique atomless h-conformal measure m for $f : J(f) \setminus \{\infty\} \to J(f)$ where m is ergodic, conservative and* $m(\text{Tr}(f)) = 1$;
(b) *a non-atomic, σ-finite, ergodic, conservative and invariant measure μ for f, equivalent to the measure m. Additionally, μ is unique up to a multiplicative constant and is supported on $J(f)$.*

The next proposition gives sufficient conditions for an elliptic function f to satisfy the conditions given in Definition 2·7.

PROPOSITION 2·9. *If every critical point c of f is such that $c \in$ $\text{Crit}_b(f)$ or $c \in$ $\text{Crit}_\infty(f)$, then $J(f) = \overline{\mathbb{C}}$ and f is critically tame.*

Proposition 2·8 and Proposition 2·9 imply that the elliptic functions considered in subsequent sections are ergodic with respect to the Riemann measure m. This shows a contrast with Lyubich's result [**15**] which says that e^z is not ergodic with respect to the Lebesgue measure. The escaping parameters in the exponential family $f_\lambda(z) = \lambda e^z$, $\lambda \in \mathbb{C} \setminus \{0\}$, were also studied by Urbański and Zdunik in [20]. Under the assumption that absolute values of points in the forward trajectory of 0 grow exponentially fast (this includes the case $\lambda > 1/e$), they showed that $\omega(z) = \{f_\lambda^n(0) : n \geqslant 0\} \cup \{\infty\}$ for a.e. $z \in J(f_\lambda) = \overline{\mathbb{C}}$. Later, Hemke [9] proved that these maps are non-recurrent. His results cover fast escaping parameters in the tangent family $f_\lambda(z) = \lambda \tan(z)$, $\lambda \in \mathbb{C} \setminus \{0\}$, for which again he proved that $\omega(z) = \{f_\lambda^n(\pm\lambda i) : n \geqslant 0\} \cup \{\infty\}$ for a.e. $z \in J(f_\lambda) = \overline{\mathbb{C}}$. In all cases the existence of a non-atomic, σ-finite, ergodic, conservative and invariant measure μ for f, absolutely continuous with respect to the Lebesgue measure, follows from [11] or Proposition 2·8.

At the end of this section we recall the definition of distortion. Let U be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ be a conformal map, then its distortion is defined as

$$L(f, U) := \frac{\sup_{z\in U} |f'(z)|}{\inf_{z\in U} |f'(z)|}.$$

For conformal maps we have

$$L(f, U) = L(f^{-1}, f(U)). \tag{2·4}$$

In order to prove the lower bound on $\dim_H(\mathcal{E})$, we use the following theorem proved by C. McMullen in [16].

PROPOSITION 2·10. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a finite collection of disjoint compact subsets of $\mathbb{R}^d$, each of which has positive d-dimensional Lebesgue measure. Define*

$$\mathcal{U}_n = \bigcup_{A_n \in \mathcal{A}_n} A_n, \quad A = \bigcap_{n=1}^{\infty} \mathcal{U}_n.$$

Suppose that for each $A_n \in \mathcal{A}_n$ there is $A_{n+1} \in \mathcal{A}_{n+1}$ and a unique $A_{n-1} \in \mathcal{A}_{n-1}$ such that $A_{n+1} \subset A_n \subset A_{n-1}$. If Δ_n, d_n are such that, for each $A_n \in \mathcal{A}_n$,

$$\begin{aligned} \frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} &\geqslant \Delta_n > 0, \\ \mathrm{diam}(A_n) &\leqslant d_n < 1, \\ d_n &\overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

then

$$\dim_H(A) \geqslant d - \limsup_{n\to\infty} \sum_{j=1}^{n} \frac{|\log \Delta_j|}{|\log d_n|}.$$

3. *The escaping parameters*

In contrast to the exponential and tangent families, there are no known examples of $\wp_\Lambda$-Weierstrass functions with critical values approaching infinity. In this section, we review results from [8] on how one can find elliptic functions with critical values eventually mapped onto poles (Lemma 3·1) and maps with critical values escaping to infinity (Lemma 3·2).

We consider a one-parameter family of functions

$$g_\beta(z) = \beta\wp_\Lambda(z),$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

where $\beta \in \mathbb{C} \setminus \{0\}$, $\Lambda = [\lambda_1, e^{2\pi i/3}\lambda_1]$ is a triangular lattice such that all critical values of $\wp_\Lambda$ are poles (see e.g. Example 2·2). The functions under consideration g_β are periodic and their critical points are the same as for the Weierstrass function $\wp_\Lambda$. It was shown in [**8**] that the critical orbits of g_β behave symmetrically, i.e.

$$g_\beta^n(c_2) = \gamma^2 g_\beta^n(c_1), \quad g_\beta^n(c_3) = \gamma g_\beta^n(c_1) \tag{3·1}$$

for all $n \in \mathbb{N}$, where $\gamma = e^{2\pi i/3}$. Therefore it is enough to consider only one critical orbit, so we consider the trajectory of the critical value $g_\beta(c_1)$. Denote $B_\rho(\infty) := \{z \in \overline{\mathbb{C}}: |z| > \rho\}$, $\rho > 0$. In order to prove the next lemma, we consider auxiliary functions $h_n(\beta) = g_\beta^n(c_1)$, $n \in \mathbb{N}$. It will become apparent that these functions are defined outside a countable set of parameters.

LEMMA 3·1. *Let Λ be a triangular lattice such that all critical values of $\wp_\Lambda$ are poles. For every $r > 0$ and each $n \geqslant 2$, there is $\beta \in B(1, r)$, such that $g_\beta^n(c_1) = \infty$.*

Proof. Consider the function h_1 defined above, i.e. $h_1: B(1, r) \to \mathbb{C}$, $h_1(\beta) = g_\beta(c_1)$, where $0 < r < 1/2$. By assumption, $h_1(1) = g_1(c_1) = \wp_\Lambda(c_1)$ is a pole of $\wp_\Lambda$. Now, we define $h_2: B(1, r) \to \overline{\mathbb{C}}$ by the formula $h_2(\beta) = g_\beta^2(c_1)$ and denote by $\mathcal{P}(h_2)$ the set of its poles. Since $h_2(1) = g_1^2(c_1) = \wp_\Lambda^2(c_1) = \infty$, then $1 \in \mathcal{P}(h_2)$. Thus, the theorem is true for $n = 2$. We can take r so small that 1 is a unique pole of h_2 in $B(1, r)$. Actually, let $\beta \in B(1, r) \setminus \{1\}$ be a pole of h_2. Thus, $h_2(\beta) = g_\beta^2(c_1) = \beta\wp_\Lambda(\beta\wp_\Lambda(c_1)) = \infty$, so $\wp_\Lambda(\beta\wp_\Lambda(c_1)) = \infty$, which implies $\beta\wp_\Lambda(c_1) \in \Lambda$. However $\wp_\Lambda(c_1) \in \Lambda$, so taking r small enough we have $\beta\wp_\Lambda(c_1) \notin \Lambda$ for $\beta \in B(1, r) \setminus \{1\}$. Then, h_2 is a non-constant meromorphic function. Since 1 is a pole of the function h_2, then there exists $R_2 \geqslant 2^2$ such that $B_{R_2}(\infty) \subset h_2(B(1, r))$. The set $B_{R_2}(\infty)$ contains infinitely many lattice points $b_{l,m}^{(2)}$ of Λ and each of them (being a pole of $\wp_\Lambda$) is the image of a parameter $\beta_{l,m}^{(2)} \in B(1, r) \setminus \{1\}$ under h_2. Choose one of $\beta_{l,m}^{(2)}$ and denote it, for simplicity, by β_2. We denote the corresponding pole by b_2. We have constructed the map g_{β_2}, such that the orbit of the critical point c_1 is the following

$$c_1 \longmapsto g_{\beta_2}(c_1) \longmapsto g_{\beta_2}^2(c_1) = b_2 \longmapsto g_{\beta_2}^3(c_1) = \infty,$$

where $g_{\beta_2}(c_1)$ is close to (but not equal to) the critical value $\wp_\Lambda(c_1)$ and $g_{\beta_2}^2(c_1) \in B_{R_2}(\infty)$. Let $r_1 := r$ and take $0 < r_2 < r_1/2$ so small that $\overline{B(\beta_2, r_2)} \subset B(1, r) \setminus \mathcal{P}(h_2)$ and $h_3(B(\beta_2, r_2)) \subset B_{R_2}(\infty)$, where $h_3(\beta) = g_\beta^3(c_1)$. Restricting h_3 to $B(\beta_2, r_2)$, we take $R_3 \geqslant 2R_2 \geqslant 2^3$ such that $B_{R_3}(\infty) \subset h_3(B(\beta_2, r_2))$. Each lattice point $b_{l,m}^{(3)} \in B_{R_3}(\infty)$ is the image of a parameter $\beta_{l,m}^{(3)} \in B(\beta_2, r_2) \setminus \{\beta_2\}$. Note that this proves the existence of a parameter β_3 such that

$$c_1 \longmapsto g_{\beta_3}(c_1) \approx \wp_\Lambda(c_1) \longmapsto g_{\beta_3}^2(c_1) \approx b_2 \longmapsto g_{\beta_3}^2(c_1) = b_3 \longmapsto g_{\beta_3}^4(c_1) = \infty,$$

where none of the $\approx$ are equality and $b_i \in \Lambda \bigcap B_{R_i}(\infty)$ with $R_i \geqslant 2^i$, $i = 2, 3$. Now, by induction we define a map with the property that the critical point is a prepole of order $n \geqslant 4$. Fix $n \geqslant 4$ and suppose for all $k < n$ we have constructed the maps

$$h_k: B(1, r) \setminus \bigcup_{1<i<k} \mathcal{P}(h_i) \longrightarrow \overline{\mathbb{C}}$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

by the formulas $h_k(\beta) = g_\beta^k(c_1)$, where $\mathcal{P}(h_i)$ is the set of poles of h_i. We define a map

$$h_n\colon B(1,r) \setminus \bigcup_{1<k<n} \mathcal{P}(h_k) \longrightarrow \overline{\mathbb{C}}$$

such that $h_n(\beta) = g_\beta^n(c_1)$. The set $\bigcup_{1<k<n} \mathcal{P}(h_k)$ is a set of essential singularities of h_n. In its complement the map h_n is meromorphic, denote by $\mathcal{P}(h_n)$ its set of poles. Set a pole $\beta_{n-1} \in \mathcal{P}(h_n)$. The equality $h_n(\beta_{n-1}) = g_{\beta_{n-1}}^n(c_1) = \infty$ implies that there is a small enough constant $0 < r_{n-1} < r_{n-2}/2$ such that $\overline{B(\beta_{n-1}, r_{n-1})} \subset B(\beta_{n-2}, r_{n-2}) \setminus \bigcup_{1<k<n} \mathcal{P}(h_k)$ and $h_n(B(\beta_{n-1}, r_{n-1})) \subset B_{R_{n-1}}(\infty)$. Now, we can take $R_n \geqslant 2R_{n-1} \geqslant 2^n$ such that $B_{R_n}(\infty) \subset h_n(B(\beta_{n-1}, r_{n-1}))$. Next, we choose one of the lattice points of Λ from $B_{R_n}(\infty)$ and denote it by b_n. We know that b_n is the image of a parameter $\beta_n \in B(\beta_{n-1}, r_{n-1}) \setminus \{\beta_{n-1}\}$, i.e. $b_n = h_n(\beta_n) = g_{\beta_n}^n(c_1)$. The orbit of the critical point c_1 for the map g_{β_n} is the following

$$c_1 \longmapsto g_{\beta_n}(c_1) \approx \wp_\Lambda(c_1) \longmapsto g_{\beta_n}^2(c_1) \approx b_2 \longmapsto \cdots \longmapsto g_{\beta_n}^n(c_1) = b_n \longmapsto g_{\beta_n}^{n+1}(c_1) = \infty,$$

where $g_{\beta_n}^i(c_1) \in B_{R_i}(\infty)$, $i = 1, \ldots, n$. This completes the proof.

LEMMA 3·2. *Let Λ be a triangular lattice such that all critical values of $\wp_\Lambda$ are poles. Then, for every $r > 0$ there is a parameter $\beta \in B(1, r)$ such that* $\lim_{n\to\infty} g_\beta^n(c_i) = \infty$, $i = 1, 2, 3$.

Proof. We show that $\lim_{n\to\infty} g_\beta^n(c_1) = \infty$. The 'symmetry' of the critical orbits given in (3·1) implies the lemma is true for c_2 and c_3. By Lemma 3·1, there is a sequence of parameters $\{\beta_n\}_{n\geqslant 2}$ such that

$$\left|g_{\beta_n}^n(c_1)\right| > R_n \text{ and } g_{\beta_n}^{n+1}(c_1) = \infty,$$

where $R_n \geqslant 2^n$ and a decreasing sequence of balls $B(\beta_n, r_n) \subset B(1, r_1) \setminus \bigcup_{1<k<n} \mathcal{P}(h_k)$ such that $r_n < 2^{-n}$. Since $r_n \to 0$, then there is the parameter $\beta = \bigcap_{n\geqslant 2} \overline{B(\beta_n, r_n)}$. By the construction from the proof of Lemma 3·1, β is an accumulation point of the set $\bigcup_{n>1} \mathcal{P}(h_n)$. The iterates of the critical point under g_β satisfy the conditions $\left|g_\beta^n(c_1)\right| > R_n \geqslant 2^n$ for all $n \geqslant 2$. Hence, $\lim_{n\to\infty} R_n = \infty$ which implies $\lim_{n\to\infty} g_\beta^n(c_1) = \infty$.

4. *Escaping parameters with a prescribed growth rate of critical orbits*

In this section, we construct a collection of subsets of $\mathcal{E}$ with a prescribed growth rate of the critical orbits of g_β. We fix a function $\wp_\Lambda$ such that

$$\Lambda = [\lambda_1, e^{2\pi i/3}\lambda_1]$$

is a triangular lattice and all critical values of $\wp_\Lambda$ are poles. We consider the one-parameter family of functions

$$g_\beta(z) = \beta\wp_\Lambda(z), \;\; \beta \in B(1, r) \text{ for } 0 < r < \frac{1}{4} - \frac{1}{2\alpha + 4} \approx 0.04, \tag{4·1}$$

where $\alpha = \sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2$. The functions g_β are periodic and their critical points are the same as the critical points of the Weierstrass function $\wp_\Lambda$. It follows from (3·1) that the critical orbits of g_β behave symmetrically, i.e.

$$g_\beta^n(c_2) = \gamma^2 g_\beta^n(c_1), \;\; g_\beta^n(c_3) = \gamma g_\beta^n(c_1)$$

for all $n \in \mathbb{N}$, where $\gamma = e^{2\pi i/3}$. Since $\wp_\Lambda$ is periodic, there exists a constant

$$0 < \varepsilon_0 < \min\{1, |\lambda_1|/3\}$$

and holomorphic functions G, H such that for each pole $b_{l,m} \in \Lambda$

$$\wp_\Lambda(z) = \frac{a_{-2}}{(z - b_{l,m})^2} + \frac{a_{-1}}{z - b_{l,m}} + \sum_{k=0}^{\infty} a_k (z - b_{l,m})^k =: \frac{G(z)}{(z - b_{l,m})^2},$$

$$\wp'_\Lambda(z) = \frac{b_{-3}}{(z - b_{l,m})^3} + \frac{b_{-2}}{(z - b_{l,m})^2} + \frac{b_{-1}}{z - b_{l,m}} + \sum_{k=0}^{\infty} b_k (z - b_{l,m})^k =: \frac{H(z)}{(z - b_{l,m})^3}$$

for all $z \in B(b_{l,m}, \varepsilon_0)$, where $G(b_{l,m}) = a_{-2} \neq 0$, $H(b_{l,m}) = b_{-3} \neq 0$. Shrinking ε_0, if necessary, we may assume that $G(z) \neq 0$ and $H(z) \neq 0$ for $z \in B(b_{l,m}, \varepsilon_0)$. The periodicity of $\wp_\Lambda$ implies that there exist universal constants $K_1, K_2 > 0$ such that

$$K_1^{-1} \leqslant |G(z)| \leqslant K_1, \quad K_2^{-1} \leqslant |H(z)| \leqslant K_2$$

on all balls $B(b_{l,m}, \varepsilon_0)$. Hence,

$$\frac{K_1^{-1}}{|z - b_{l,m}|^2} \leqslant |\wp_\Lambda(z)| = \left| \frac{G(z)}{(z - b_{l,m})^2} \right| \leqslant \frac{K_1}{|z - b_{l,m}|^2}$$

and

$$\frac{K_2^{-1}}{|z - b_{l,m}|^3} \leqslant |\wp'_\Lambda(z)| = \left| \frac{H(z)}{(z - b_{l,m})^3} \right| \leqslant \frac{K_2}{|z - b_{l,m}|^3}$$

for all $l, m \in \mathbb{Z}$ and $z \in B(b_{l,m}, \varepsilon_0)$. For every $\beta \in B(1, r)$, where r is defined in (4·1) and for all $z \in B(b_{l,m}, \varepsilon_0)$, $l, m \in \mathbb{Z}$, we have

$$\frac{C_1^{-1}}{|z - b_{l,m}|^2} \leqslant \left| g_\beta(z) \right| = |\beta \wp_\Lambda(z)| \leqslant \frac{C_1}{|z - b_{l,m}|^2} \tag{4·2}$$

and

$$\frac{C_2^{-1}}{|z - b_{l,m}|^3} \leqslant \left| g'_\beta(z) \right| = |\beta \wp'_\Lambda(z)| \leqslant \frac{C_2}{|z - b_{l,m}|^3} \tag{4·3}$$

where $C_1 = 2K_1$, $C_2 = 2K_2$. Since $0 < r < 1/4 - 1/(2\alpha + 4)$, then $|\mathrm{Arg}\beta| \leqslant \arcsin(1/4 - 1/(2\alpha + 4)) \approx 0.04$ for $\beta \in B(1, r)$. Hence, shrinking ε_0 if necessary, we can choose constants M_1, M_2, $0 < M_2 - M_1 < \pi/4$ such that

$$M_1 \leqslant \arg(\beta G(z)) \leqslant M_2 \tag{4·4}$$

for all $\beta \in B(1, r)$ and $z \in B(b_{l,m}, \varepsilon_0)$, $l, m \in \mathbb{Z}$. We recall from Section 3 that

$$h_1 : B(1, r) \to \mathbb{C}, \ \ h_1(\beta) = g_\beta(c_1),$$

where c_1 is the critical point of $\wp_\Lambda$. We choose $\varepsilon > 0$ such that the following conditions are simultaneously satisfied

$$\begin{aligned} &\varepsilon < \min\{\varepsilon_0, |\wp_\Lambda(c_1)|/3\}, \\ &B(\wp_\Lambda(c_1), \varepsilon) \subset h_1(B(1, r)), \\ &\wp_\Lambda \text{ is one-to-one in each of the segments defined in (4·6).} \end{aligned} \tag{4·5}$$

Let

$$U(z_0, \varepsilon) := \left\{ z \in \mathbb{C} : \ -\frac{3\pi}{8} \leqslant \mathrm{Arg}(z - z_0) \leqslant \frac{3\pi}{8}, \ |z - z_0| \leqslant \varepsilon \right\}, \tag{4·6}$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

where $z_0 \in \Lambda$ and ε is defined above. Next, we take $R_1 > 0$ such that

$$U(\wp_\Lambda(c_1), \varepsilon) \subset P(0, R_1, 2R_1) := \{z \in \mathbb{C} : R_1 < |z| < 2R_1\}.$$

Using (4·2) and (4·4), we get

$$\left\{z \in \overline{\mathbb{C}}\colon |z| \geqslant \frac{C_1}{\varepsilon^2},\ -\frac{3\pi}{4} + M_2 \leqslant \arg z \leqslant \frac{3\pi}{4} + M_1\right\} \subset g_\beta(U(b_{l,m}, \varepsilon))$$
$$\subset \left\{z \in \overline{\mathbb{C}}\colon |z| \geqslant \frac{C_1^{-1}}{\varepsilon^2},\ -\frac{3\pi}{4} + M_1 \leqslant \arg z \leqslant \frac{3\pi}{4} + M_2\right\}$$

for all $l, m \in \mathbb{Z}$, $\beta \in B(1, r)$. Since $0 < M_2 - M_1 < \pi/4$, there exists $\phi \in \mathbb{R}$ such that

$$\left\{z \in \overline{\mathbb{C}}\colon |z| \geqslant \frac{C_1}{\varepsilon^2},\ \phi - \frac{\pi}{8} \leqslant \arg z \leqslant \phi + \frac{9\pi}{8}\right\} \subset g_\beta(U(b_{l,m}, \varepsilon)). \tag{4·7}$$

We choose $\tilde{R}_2$ such that

$$\tilde{R}_2 > \frac{C_1}{(1-\alpha)\varepsilon^2}, \tag{4·8}$$

where $\alpha = \sin(\pi/8)$. Let $a_1 = \tilde{R}_2/R_1 > C_1/((1-\alpha)\varepsilon^2 R_1)$. Now, we define a constant

$$a_0 = \max\left\{2, a_1, \frac{1}{R_1}, \frac{3C_1^{3/2}}{C_2 R_1}, \frac{6^4 C_1^6}{C_2^4 R_1^5}, \left(\frac{4\varepsilon(1+r)C_1^{3/2}}{C_2 R_1^{5/2}}\right)^{2/3}, \frac{\sqrt{C_1}}{\sqrt[3]{C_2}\sqrt{R_1}}\right\}. \tag{4·9}$$

Fix

$$a > a_0$$

and consider a sequence of radii

$$R_n := a^{n-1} R_1, \quad n \geqslant 2.$$

Let

$$P(0, R_n, 2R_n) := \{z \in \mathbb{C}\colon R_n < |z| < 2R_n\}, \quad n \geqslant 2,$$

and

$$P^+(0, R_n, 2R_n) := \{z \in \mathbb{C}\colon R_n < |z| < 2R_n,\ \phi < \arg z < \phi + \pi\}, \quad n \geqslant 2, \tag{4·10}$$

where ϕ is as in (4·7). The condition $a > a_0 \geqslant 2$ guarantees that the annuli $P(0, R_n, 2R_n)$ are pairwise disjoint. It follows from (4·7) that

$$\{z \in \mathbb{C} : |z| > R_2 \geqslant \tilde{R}_2,\ \phi \leqslant \arg z \leqslant \phi + \pi\} \subset g_\beta(U(b_{l,m}, \varepsilon)) \tag{4·11}$$

for all poles $b_{l,m}$ and $\beta \in B(1, r)$. Recall that in the previous section we defined the auxiliary functions $h_n(\beta) = g_\beta^n(c_1)$, $n \in \mathbb{N}$.

Definition 4·1. We define the following family of sets

$\mathcal{A}_0(a) = \{A_0 = B(1, r)\}$,

$\mathcal{A}_1(a) = \{A_1 = h_1^{-1}(U(\wp_\Lambda(c_1), \varepsilon)) \subset A_0\}$,

$\mathcal{A}_2(a) = \{A_2 \subset A_1 \mid \exists b_{l,m}^{(2)} \in \Lambda\colon U(b_{l,m}^{(2)}, \varepsilon) \subset P^+(0, R_2, 2R_2),\ A_2 = h_2^{-1}(U(b_{l,m}^{(2)}, \varepsilon))\}$,

. . .

$\mathcal{A}_n(a) = \{A_n \subset A_{n-1} \mid \exists b_{l,m}^{(n)} \in \Lambda\colon U(b_{l,m}^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n),\ A_n = h_n^{-1}(U(b_{l,m}^{(n)}, \varepsilon))\}$,

. . .

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

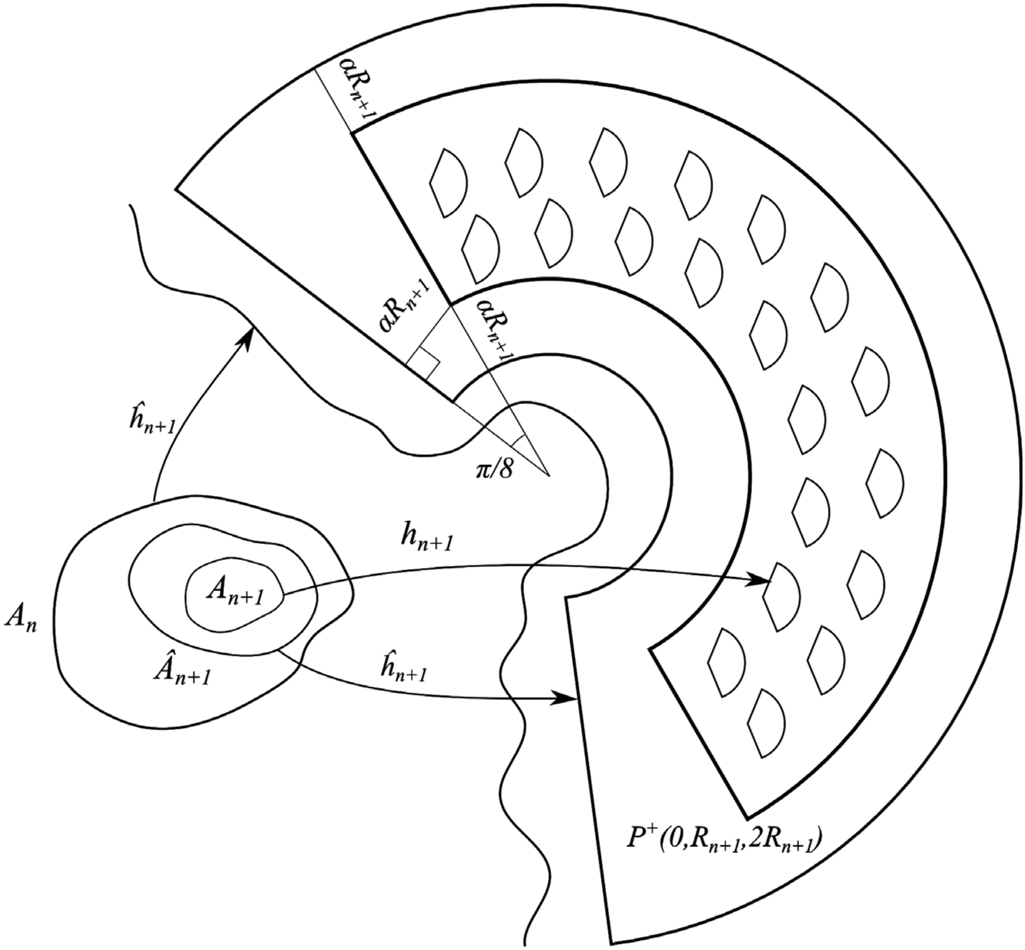

Fig. 1.

where $h_n^{-1}(U(b_{l,m}^{(n)}, \varepsilon))$ denotes a component of the preimage of $U(b_{l,m}^{(n)}, \varepsilon)$. Let

$$\mathcal{U}_n(a) = \bigcup_{A_n \in \mathcal{A}_n(a)} A_n, \quad A(a) = \bigcap_{n=1}^{\infty} \mathcal{U}_n(a).$$

The sets defined above are illustrated in Figure 1.

PROPOSITION 4·2. *For each $n \in \mathbb{N}$, the set $\mathcal{A}_n(a)$ defined above is non-empty.*

Proof. In the previous section, we showed that the function h_2 has a pole at $\beta = 1 = h_1^{-1}(\wp_\Lambda(c_1)) \in \partial A_1$. Thus, $\mathcal{A}_1(a) \neq \emptyset$. Since $h_1(A_1) = U(\wp_\Lambda(c_1), \varepsilon)$, it follows from (4·11) that

$$h_2(A_1) = \{g_\beta(h_1(\beta)) | \beta \in A_1\} \supset P^+(0, R_2, 2R_2).$$

Take a pole $b_{l,m}^{(2)} \in \Lambda \cap P^+(0, R_2, 2R_2)$ with $U(b_{l,m}^{(2)}, \varepsilon) \subset P^+(0, R_2, 2R_2)$. Since $h_2(A_1) \supset P^+(0, R_2, 2R_2)$, there exists $\beta_{l,m}^{(2)} \in A_1$ such that $h_2(\beta_{l,m}^{(2)}) = b_{l,m}^{(2)}$. Thus, the set $\mathcal{A}_2(a)$ is non-empty. Now, we fix $n \geqslant 3$ and suppose that $\mathcal{A}_{n-1}(a) \neq \emptyset$. We will show that $\mathcal{A}_n(a) \neq \emptyset$. Since $h_{n-1}(A_{n-1}) = U(b_{l,m}^{(n-1)}, \varepsilon)$ for some $b_{l,m}^{(n-1)} \in \Lambda \cap P^+(0, R_{n-1}, 2R_{n-1})$, it follows from

(4·11) that

$$h_n(A_{n-1}) = \{g_\beta(h_{n-1}(\beta))|\beta \in A_{n-1}\} \supset P^+(0, R_n, 2R_n),$$

as $R_n = a^{n-2}R_2$ and $a > a_0 \geqslant 2$ in view of (4·9). Choosing $\beta_{l,m}^{(n)} \in A_{n-1}$ such that $h_n(\beta_{l,m}^{(n)}) = b_{l,m}^{(n)} \in \Lambda \cap P^+(0, R_n, 2R_n)$ and $U(b_{l,m}^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n)$, we get $\mathcal{A}_n(a) \neq \emptyset$. By induction, the lemma is true for all $n \in \mathbb{N}$.

THEOREM 4·3. *Let g_β be the family of maps defined in (4·1) and let a_0 be the constant given in* (4·9). *Then, for every $a > a_0$ there is a Cantor subset $A(a)$ of $\mathcal{E}$, and for this subset*

$$\dim_H(A(a)) \geqslant \frac{4}{3} - \frac{6\log 2}{\log a}.$$

COROLLARY 4·4. *For $a \nearrow +\infty$ we have* $\dim_H(A(a)) \geqslant 4/3 - 6\log 2/\log a \nearrow 4/3$ *and* $\dim_H(\mathcal{E}) \geqslant 4/3$.

5. *The proofs*

In this section, we prove Theorem 4·3. We fix $a > a_0$ and consider the sets $\mathcal{A}_n(a)$, $n \geqslant 1$, given in Definition 4·1. We drop the parameter a and keep notation from the last section.

The first two lemmas include the estimates of the derivatives h_n', $n \geqslant 2$.

LEMMA 5·1. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 2$. Then, for every $\beta \in A_n$*

$$h_n'(\beta) = \frac{1}{\beta}\prod_{k=1}^{n-1} g_\beta'(g_\beta^k(c_1))\left[g_\beta(c_1) + \sum_{k=2}^{n}\frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right].$$

Proof. Let $n = 2$. Then:

$$h_1(\beta) = g_\beta(c_1) = \beta\wp_\Lambda(c_1);$$
$$h_2(\beta) = g_\beta^2(c_1) = \beta\wp_\Lambda(\beta\wp_\Lambda(c_1));$$
$$h_2'(\beta) = \wp_\Lambda(\beta\wp_\Lambda(c_1)) + \beta\wp_\Lambda'(\beta\wp_\Lambda(c_1))\wp_\Lambda(c_1) = \frac{g_\beta^2(c_1)}{\beta} + \frac{g_\beta'(\beta\wp_\Lambda(c_1))g_\beta(c_1)}{\beta}$$
$$= \frac{1}{\beta}g_\beta'(g_\beta(c_1))\left[g_\beta(c_1) + \frac{g_\beta^2(c_1)}{g_\beta'(g_\beta(c_1))}\right].$$

Suppose that the lemma is true for some $n \geqslant 2$. We show that it is true for $n+1$.

$$h_{n+1}(\beta) = \beta\wp_\Lambda(h_n(\beta)),$$
$$h_{n+1}'(\beta) = \wp_\Lambda(h_n(\beta)) + \beta\wp_\Lambda'(h_n(\beta)) \cdot h_n'(\beta)$$
$$= \frac{g_\beta^{n+1}(c_1)}{\beta} + g_\beta'(g_\beta^n(c_1)) \cdot \frac{1}{\beta} \cdot \prod_{k=1}^{n-1} g_\beta'(g_\beta^k(c_1)) \cdot \left[g_\beta(c_1) + \sum_{k=2}^{n}\frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right]$$
$$= \frac{g_\beta^{n+1}(c_1)}{\beta} + \frac{1}{\beta} \cdot \prod_{k=1}^{n} g_\beta'(g_\beta^k(c_1)) \cdot \left[g_\beta(c_1) + \sum_{k=2}^{n}\frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right]$$
$$= \frac{1}{\beta} \cdot \prod_{k=1}^{n} g_\beta'(g_\beta^k(c_1)) \cdot \left[g_\beta(c_1) + \sum_{k=2}^{n}\frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))} + \frac{g_\beta^{n+1}(c_1)}{\prod_{i=1}^{n} g_\beta'(g_\beta^i(c_1))}\right]$$
$$= \frac{1}{\beta} \cdot \prod_{k=1}^{n} g_\beta'(g_\beta^k(c_1)) \cdot \left[g_\beta(c_1) + \sum_{k=2}^{n+1}\frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right].$$

As can be seen from the previous section (see (4·2), (4·3)), there are universal constants $C_1,\ C_2 > 0$ such that

$$\frac{C_1^{-1}}{|z-b_{l,m}|^2} \leqslant \left|g_\beta(z)\right| \leqslant \frac{C_1}{|z-b_{l,m}|^2}, \quad \frac{C_2^{-1}}{|z-b_{l,m}|^3} \leqslant \left|g'_\beta(z)\right| \leqslant \frac{C_2}{|z-b_{l,m}|^3}$$

for all $l, m \in \mathbb{Z}$, every $z \in B(b_{l,m}, \varepsilon)$ and all $\beta \in B(1, r)$. To simplify the formulas in the following part of the paper, we write

$$|g_\beta(z)| \asymp \frac{C_1}{|z-b_{l,m}|^2}, \quad |g'_\beta(z)| \asymp \frac{C_2}{|z-b_{l,m}|^3}. \tag{5·1}$$

Note that if $\beta \in \mathcal{U}_n,\ n \geqslant 2$ and $z = g_\beta^j(c_1)$ with $j \in \{1, 2, \ldots, n-1\}$ we have $g_\beta(z) = g_\beta^{j+1}(c_1) = h_{j+1}(\beta) \in U(b_{l,m}^{(j+1)}, \varepsilon) \subset P^+(0, R_{j+1}, 2R_{j+1})$ and moreover, using (5·1),

$$R_{j+1} \leqslant |g_\beta(z)| \asymp \frac{C_1}{|z-b_{l,m}|^2} \leqslant 2R_{j+1} \tag{5·2}$$

for some $b_{l,m} \in \Lambda \bigcap P(0, R_j, 2R_j)$. The inequality (5·2) implies that

$$\frac{C_1}{2R_{j+1}} \leqslant |z-b_{l,m}|^2 \leqslant \frac{C_1}{R_{j+1}},$$

which is equivalent to

$$\left(\frac{C_1}{2R_{j+1}}\right)^{3/2} \leqslant \left|z-b_{l,m}\right|^3 \leqslant \left(\frac{C_1}{R_{j+1}}\right)^{3/2}.$$

Then,

$$\frac{C_2}{\left(\frac{C_1}{R_{j+1}}\right)^{3/2}} \leqslant |g'_\beta(z)| \asymp \frac{C_2}{\left|z-b_{l,m}\right|^3} \leqslant \frac{C_2}{\left(\frac{C_1}{2R_{j+1}}\right)^{3/2}}$$

or, equivalently,

$$\frac{C_2 R_{j+1}^{3/2}}{C_1^{3/2}} \leqslant |g'_\beta(z)| \leqslant \frac{2^{3/2} C_2 R_{j+1}^{3/2}}{C_1^{3/2}} \tag{5·3}$$

for $\beta \in \mathcal{U}_n,\ n \geqslant 2$ and $z = g_\beta^j(c_1)$ with $j \in \{1, 2, \ldots n-1\}$.

LEMMA 5·2. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 2$. Then, for every $\beta \in A_n$*

$$\frac{1}{2(1+r)}\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}} \leqslant \left|h'_n(\beta)\right| \leqslant \frac{5}{2(1-r)}\left(\frac{2^{3/2}C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}.$$

Proof. In Lemma 5·1, we proved that

$$h'_n(\beta) = \frac{1}{\beta}\prod_{k=1}^{n-1} g'_\beta(g_\beta^k(c_1)) \left[g_\beta(c_1) + \sum_{k=2}^{n} \frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g'_\beta(g_\beta^i(c_1))}\right]$$

for all $n \geqslant 2$ and every $\beta \in A_n$. First, we estimate the product $\prod_{k=1}^{n-1} g'_\beta(g_\beta^k(c_1))$. Observe that

$$g_\beta(g_\beta^k(c_1)) = g_\beta^{k+1}(c_1) = h_{k+1}(\beta),\ k = 1, 2, \ldots, n-1.$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

The functions $h_2, \ldots, h_n$ are well-defined for $\beta \in A_n$, because $A_n \subset A_k$, $k = 2, \ldots, n$. Since $h_{k+1}(\beta) \in P(0, R_{k+1}, 2R_{k+1})$, then using (5·3), we get

$$\left|\prod_{k=1}^{n-1} g_\beta'(g_\beta^k(c_1))\right| \leqslant \frac{2^{3/2}C_2R_2^{3/2}}{C_1^{3/2}} \cdot \ldots \cdot \frac{2^{3/2}C_2R_n^{3/2}}{C_1^{3/2}} = \left(\frac{2^{3/2}C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3(n-1)}{2}}.$$

Analogously, we get the estimate from below

$$\left|\prod_{k=1}^{n-1} g_\beta'(g_\beta^k(c_1))\right| \geqslant \left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3(n-1)}{2}}.$$

Finally,

$$\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3(n-1)}{2}} \leqslant \left|\prod_{k=1}^{n-1} g_\beta'(g_\beta^k(c_1))\right| \leqslant \left(\frac{2^{3/2}C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3(n-1)}{2}}. \tag{5·4}$$

Now, using (5·4), we estimate the sum $\sum_{k=2}^n \frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}$.

$$\left|\sum_{k=2}^{n} \frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right| \leqslant \sum_{k=2}^{n} \frac{2R_k}{\left(\frac{C_2}{C_1^{3/2}}\right)^{k-1} a^{\frac{3k(k-1)}{4}} R_1^{\frac{3(k-1)}{2}}}$$

$$= \sum_{k=2}^{n} \frac{2}{\left(\frac{C_2}{C_1^{3/2}}\right)^{k-1} a^{\frac{(k-1)(3k-4)}{4}} R_1^{\frac{3k-5}{2}}} = \frac{2C_1^{3/2}}{C_2\sqrt[4]{aR_1}} \sum_{k=2}^{n} \left(\frac{C_1^{3/2}}{C_2}\right)^{k-2} \frac{1}{a^{\frac{3k^2-7k+3}{4}} R_1^{\frac{6k-11}{4}}}.$$

Since $a > a_0 \geqslant 2$ and $3k^2 - 7k + 3 \geqslant 6k - 11$ for $k = 2, 3, \ldots$, then

$$\sum_{k=2}^{n} \left(\frac{C_1^{3/2}}{C_2}\right)^{k-2} \frac{1}{a^{\frac{3k^2-7k+3}{4}} R_1^{\frac{6k-11}{4}}} \leqslant \sum_{k=2}^{n} \left(\frac{C_1^{3/2}}{C_2}\right)^{k-2} \frac{1}{(aR_1)^{\frac{6k-11}{4}}}.$$

Using the inequality $(6k - 11)/4 \geqslant k - 2$ for $k \geqslant 3/2$ and the fact that $a > a_0 \geqslant \max\{1/R_1, 3C_1^{3/2}/(C_2R_1)\}$, we get

$$\sum_{k=2}^{n} \left(\frac{C_1^{3/2}}{C_2}\right)^{k-2} \frac{1}{(aR_1)^{\frac{6k-11}{4}}} \leqslant \sum_{k=2}^{n} \left(\frac{C_1^{3/2}}{C_2aR_1}\right)^{k-2} \leqslant \sum_{k=2}^{\infty} \left(\frac{C_1^{3/2}}{C_2aR_1}\right)^{k-2} = \frac{1}{1 - \frac{C_1^{3/2}}{C_2aR_1}} < \frac{3}{2}.$$

Hence,

$$\left|\sum_{k=2}^{n} \frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right| \leqslant \frac{3C_1^{3/2}}{C_2\sqrt[4]{aR_1}} \leqslant \frac{R_1}{2}, \tag{5·5}$$

because $a > a_0 \geqslant 6^4C_1^6/(C_2^4R_1^5)$. Using (5·5), we get

$$\frac{R_1}{2} = R_1 - \frac{R_1}{2} \leqslant \left|g_\beta(c_1) + \sum_{k=2}^{n} \frac{g_\beta^k(c_1)}{\prod_{i=1}^{k-1} g_\beta'(g_\beta^i(c_1))}\right| \leqslant 2R_1 + \frac{R_1}{2} = \frac{5R_1}{2}. \tag{5·6}$$

Plugging (5·4), (5·6) into the formula for h_n' from Lemma 5·1, we obtain

$$|h_n'(\beta)| \leqslant \frac{5}{2(1-r)} \left(\frac{2^{3/2}C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

and

$$|h_n'(\beta)| \geqslant \frac{1}{2(1+r)} \left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}.$$

Both estimates prove the lemma.

In Proposition 4·2, we showed that each set $\mathcal{A}_n$ provided in Definition 4·1 is non-empty and each of its elements, which are sets A_n, contains boundary parameters β_n such that $h_n(\beta_n) \in \Lambda \cap P(0, R_n, 2R_n)$. Later in this section, we estimate the diameters of A_n and the ratios $\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)/\mathrm{vol}(A_n)$ and in order to do that we have to prove that the functions h_n are conformal on $A_n \in \mathcal{A}_n$. Note that the maps $h_n, n \geqslant 2$, are holomorphic outside a countable set of points and have poles at $\beta_{n-1} \in \partial A_{n-1}$.

LEMMA 5·3. *For each $A_n \in \mathcal{A}_n$, $n \geqslant 1$, the map h_n is conformal on A_n.*

Proof. The map h_1 is one-to-one and holomorphic on A_1. By induction, we show that the maps $h_n, n \geqslant 2$ are conformal. Suppose that $h_n, n \geqslant 1$ is conformal on A_n, we prove that h_{n+1} is conformal on $A_{n+1} \subset A_n$. If $n = 1$ then we take the segment

$$U(b_{l,m}^{(1)}, \varepsilon) \subset P(0, R_1, 2R_1)$$

with $b_{l,m}^{(1)} = \wp_\Lambda(c_1)$ and if $n \geqslant 2$ we consider a segment

$$U(b_{l,m}^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n).$$

We know that $A_n = h_n^{-1}(U(b_{l,m}^{(n)}, \varepsilon)), n \geqslant 1$. Let $b_{l,m}^{(n)} = b_n$, $\beta_n = h_n^{-1}(b_n) \in \partial A_n$ and $b_{l,m}^{(n+1)} = b_{n+1}$. If $U(b_{n+1}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1})$, then $h_{n+1}^{-1}(U(b_{n+1}, \varepsilon)) = A_{n+1} \subset A_n$. We define a map $\hat{h}_{n+1}(\beta) = \beta_n \wp_\Lambda(h_n(\beta)), \beta \in A_n$. It follows from (4·7) that

$$\hat{h}_{n+1}(A_n) \supset \left\{z \in \overline{\mathbb{C}}\colon |z| \geqslant \frac{C_1}{\varepsilon^2},\ \phi - \frac{\pi}{8} \leqslant \arg z \leqslant \phi + \frac{9\pi}{8}\right\}.$$

We show that $\hat{h}_{n+1}$ is one-to-one in A_n. Take $\beta', \beta'' \in A_n$ such that $\hat{h}_{n+1}(\beta') = \hat{h}_{n+1}(\beta'')$. By definition of the map $\hat{h}_{n+1}$, we have $\wp_\Lambda(h_n(\beta')) = \wp_\Lambda(h_n(\beta''))$, where $h_n(\beta'), h_n(\beta'') \in h_n(A_n) = U(b_n, \varepsilon)$. Since $\wp_\Lambda$ is one-to-one in $U(b_n, \varepsilon)$, then $h_n(\beta') = h_n(\beta'')$ and this implies that $\beta' = \beta''$. This follows from the injectivity of the map h_n. There is a set $\hat{A}_{n+1} \subset A_n$ such that

$$\hat{h}_{n+1}(\hat{A}_{n+1}) = \left\{z \in \mathbb{C}\colon (1-\alpha)R_{n+1} < |z| < (2+\alpha)R_{n+1},\ \phi - \frac{\pi}{8} < \arg z < \phi + \frac{9\pi}{8}\right\} \tag{5·7}$$

for $\alpha = \sin(\pi/8)$ and ϕ as in (4·7). Now, we show that $A_{n+1} \subset \hat{A}_{n+1}$. Note that $\hat{h}_{n+1}(\beta) = (\beta_n/\beta)h_{n+1}(\beta)$. Since $h_{n+1}(A_{n+1}) = U(b_{n+1}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1})$ and

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

$0 < r < 1/4 - 1/(2\alpha+4)$, then for $\beta \in A_{n+1}$ we have:

$$|\hat{h}_{n+1}(\beta)| > \frac{1-r}{1+r}R_{n+1} > \frac{3\alpha+8}{5\alpha+8}R_{n+1} \approx 0.92R_{n+1} > (1-\alpha)R_{n+1} \approx 0.62R_{n+1};$$

$$|\hat{h}_{n+1}(\beta)| < \frac{1+r}{1-r}2R_{n+1} < \frac{2(5\alpha+8)}{3\alpha+8}R_{n+1} \approx 2.17R_{n+1} < (2+\alpha)R_{n+1} \approx 2.38R_{n+1};$$

$$\arg\hat{h}_{n+1}(\beta) < \phi + \pi + 2\max_{\beta\in B(1,r)} \operatorname{Arg}\beta < \phi + \pi + 2\arcsin\left(\frac{1}{4} - \frac{1}{2\alpha+4}\right)$$
$$\approx \phi + \pi + 0.08 < \phi + \frac{9\pi}{8};$$

$$\arg\hat{h}_{n+1}(\beta) > \phi - 2\max_{\beta\in B(1,r)} \operatorname{Arg}\beta > \phi - 2\arcsin\left(\frac{1}{4} - \frac{1}{2\alpha+4}\right) \approx \phi - 0.08 > \phi - \frac{\pi}{8}.$$

Thus, $\hat{h}_{n+1}(A_{n+1}) \subset \hat{h}_{n+1}(\hat{A}_{n+1})$. Since the map $\hat{h}_{n+1}$ is one-to-one in A_n, then $A_{n+1} \subset \hat{A}_{n+1}$. It follows from (4·10) and (5·7) that

$$U(b_{n+1},\varepsilon) = h_{n+1}(A_{n+1}) \subset P^{+}(0, R_{n+1}, 2R_{n+1})$$
$$\subset \left\{z \in \mathbb{C}\colon (1-\alpha)R_{n+1} < |z| < (2+\alpha)R_{n+1},\ \phi - \frac{\pi}{8} < \arg z < \phi + \frac{9\pi}{8}\right\} = \hat{h}_{n+1}(\hat{A}_{n+1}).$$

Since $0 < r < 1/4 - 1/(2\alpha+4)$ then, taking $\zeta = h_n(\beta)$ for $\beta \in \partial\hat{A}_{n+1}$, we have

$$2r|\wp_\Lambda(\zeta)| < \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)|\wp_\Lambda(\zeta)| = \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|\frac{\hat{h}_{n+1}(\beta)}{\beta_n}\right|$$
$$\leqslant \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\frac{(2+\alpha)R_{n+1}}{|\beta_n|} = \frac{\alpha R_{n+1}}{2|\beta_n|} < \alpha R_{n+1},$$

as $|\beta_n| \geqslant 1 - r > 1/2$. Hence (see Figure 1),

$$\operatorname{dist}(\hat{h}_{n+1}(\beta), h_{n+1}(A_{n+1})) \geqslant \alpha R_{n+1} > 2r|\wp_\Lambda(\zeta)|.$$

We define auxiliary maps $H_{n+1}(\beta) = h_{n+1}(\beta) - w$, $\hat{H}_{n+1}(\beta) = \hat{h}_{n+1}(\beta) - w$ with $w \in h_{n+1}(A_{n+1})$. Thus, for $\beta \in \partial\hat{A}_{n+1}$ we have

$$|\hat{H}_{n+1}(\beta)| = |\hat{h}_{n+1}(\beta) - w| \geqslant \operatorname{dist}(\hat{h}_{n+1}(\beta), h_{n+1}(A_{n+1})) > 2r|\wp_\Lambda(\zeta)|$$

and

$$|H_{n+1}(\beta) - \hat{H}_{n+1}(\beta)| = |h_{n+1}(\beta) - \hat{h}_{n+1}(\beta)| = |\beta\wp_\Lambda(\zeta) - \beta_n\wp_\Lambda(\zeta)|$$
$$= |\beta - \beta_n||\wp_\Lambda(\zeta)| < 2r|\wp_\Lambda(\zeta)|.$$

Hence, $|\hat{H}_{n+1}(\beta)| > |H_{n+1}(\beta) - \hat{H}_{n+1}(\beta)|$ in the set $\partial\hat{A}_{n+1}$. Since the map h_{n+1} is holomorphic on $\operatorname{int}A_n$, then the maps H_{n+1}, $\hat{H}_{n+1}$ are holomorphic on $\overline{\hat{A}_{n+1}}$. Thus, the assumptions of the Rouché theorem are satisfied. It implies that $\hat{H}_{n+1}$ and $H_{n+1} = \hat{H}_{n+1} + H_{n+1} - \hat{H}_{n+1}$ have the same number of zeros in $\hat{A}_{n+1}$, or, equivalently, the equations $\hat{h}_{n+1}(\beta) = w$ and $h_{n+1}(\beta) = w$ have the same number of roots in $\hat{A}_{n+1}$. Since the map $\hat{h}_{n+1}$ is one-to-one in $\hat{A}_{n+1}$, then the former equation has a unique root for a given w. Thus, so does the latter. Since $A_{n+1} \subset \hat{A}_{n+1}$, then h_{n+1} is one-to-one in A_{n+1}. The map h_{n+1} is holomorphic on $\operatorname{int}A_n$, so is conformal on A_{n+1}.

Remark 5·4. In Lemma 5·3, we showed in fact that there is a unique set

$$A_1 = h_1^{-1}(U(\wp_\Lambda(c_1),\varepsilon))$$

and the segments $U(b_n, \varepsilon) \subset P^+(0, R_n, 2R_n)$, $n \geqslant 2$, are in one-to-one correspondence with the sets $A_n \in \mathcal{A}_n$. Hence, each $\mathcal{A}_n$, $n \geqslant 1$, is a finite collection of the sets A_n.

LEMMA 5·5. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 2$. Then*

$$L(h_n, A_n) \leqslant \frac{5(1+r)}{1-r} \cdot 2^{\frac{3(n-1)}{2}}.$$

Proof. Using the definition of distortion and Lemma 5·2, we get

$$L(h_n, A_n) = \frac{\sup_{\beta \in A_n} |h_n'(\beta)|}{\inf_{\beta \in A_n} |h_n'(\beta)|} \leqslant \frac{\frac{5}{2(1-r)} \left(\frac{2^{3/2} C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}}{\frac{1}{2(1+r)} \left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}} = \frac{5(1+r)}{1-r} \cdot 2^{\frac{3(n-1)}{2}}.$$

LEMMA 5·6. *For each $A_n \in \mathcal{A}_n$, $n \geqslant 2$,*

$$\operatorname{diam}(A_n) \leqslant \frac{4\varepsilon(1+r)}{\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}},$$

where ε is as in (4·5).

Proof. From Definition 4·1 we know that each set of the form $h_n(A_n)$ is a segment of radius ε, so $\operatorname{diam}(h_n(A_n)) \leqslant 2\varepsilon$. Using Lemma 5·2, we get

$$\operatorname{diam}(A_n) \leqslant \frac{\operatorname{diam}(h_n(A_n))}{\inf_{\beta \in A_n} |h_n'(\beta)|} \leqslant \frac{2\varepsilon}{\frac{1}{2(1+r)} \left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}}$$
$$= \frac{4\varepsilon(1+r)}{\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}}.$$

Remark 5·7. Observe that $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$, since $a > a_0 \geqslant 2$. This proves that the set A from Definition 4·1 is a Cantor set of parameters.

By Lemma 5·6, the numbers d_n given in Proposition 2·10 are equal to

$$d_n = \frac{4\varepsilon(1+r)}{\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}}, \quad n \geqslant 2 \tag{5·8}$$

and $d_1 = \operatorname{diam}(A_1) \leqslant 2r < 1$ by (4·1). A straightforward calculation shows that the condition $d_2 < 1$ is equivalent to $a > (4\varepsilon(1+r)C_1^{3/2}/(C_2 R_1^{5/2}))^{2/3}$. Using (5·8), we get $d_{n+1}/d_n = C_1^{3/2}/(C_2 a^{3n/2} R_1^{3/2})$ and

$$\frac{d_3}{d_2} = \frac{C_1^{3/2}}{C_2 a^3 R_1^{3/2}} < 1 \iff a > \frac{\sqrt{C_1}}{\sqrt[3]{C_2}\sqrt{R_1}}.$$

Since $a > a_0 \geqslant \max\left\{1, \left(\frac{4\varepsilon(1+r)C_1^{3/2}}{C_2 R_1^{5/2}}\right)^{2/3}, \frac{\sqrt{C_1}}{\sqrt[3]{C_2}\sqrt{R_1}}\right\}$ and $d_{n+1}/d_n < d_3/d_2$ for $n \geqslant 3$, we get $d_n < 1$, $n = 2, 3, \ldots$ as required in Proposition 2·10.

Next, we estimate the density of the sets $\mathcal{U}_{n+1} \cap A_n$ in the set $A_n \in \mathcal{A}_n$ from below for all $n \geqslant 1$.

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

LEMMA 5·8. *There exists $M > 0$ such that*

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geqslant \frac{M}{2^{9n} R_{n+1}},$$

for each $A_n \in \mathcal{A}_n$, $n \geqslant 2$. Moreover,

$$\frac{\mathrm{vol}(\mathcal{U}_2 \cap A_1)}{\mathrm{vol}(A_1)} \geqslant \frac{M'}{R_2},$$

for some $M' > 0$.

Proof. First, we estimate the number N_n of parallelograms of the lattice Λ in the half-annulus $P^+(0, R_n, 2R_n)$ for $n \geqslant 2$. We have

$$N_n \asymp \frac{4\pi R_n^2 - \pi R_n^2}{2a^2(\Lambda)} = \frac{3\pi R_n^2}{2a^2(\Lambda)}, \tag{5·9}$$

where $a^2(\Lambda)$ is the measure of each parallelogram of Λ. Recall that in Definition 4·1 we considered the segments

$$U(b_{l,m}, \varepsilon) = \left\{ z \in \mathbb{C}\colon -\frac{3\pi}{8} \leqslant \mathrm{Arg}(z - b_{l,m}) \leqslant \frac{3\pi}{8},\ |z - b_{l,m}| \leqslant \varepsilon \right\},$$

where $b_{l,m} \in \Lambda$ and $\varepsilon > 0$ as in (4·5). Hence, $\mathrm{vol}(U(b_{l,m}, \varepsilon)) = 3\pi\varepsilon^2/8$.

Fix $n \geqslant 2$ and $A_n \in \mathcal{A}_n$. There exist $l, m \in \mathbb{Z}$ such that $A_n = h_n^{-1}(U(b_{l,m}^{(n)}, \varepsilon))$, where $U(b_{l,m}^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n)$. Moreover, for each $A_k \in \mathcal{A}_{n+1}$ there are $l' = l'(k), m' = m'(k) \in \mathbb{Z}$ such that $A_k = h_{n+1}^{-1}(U(b_{l',m'}^{(n+1)}, \varepsilon))$, where $U(b_{l',m'}^{(n+1)}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1})$. To simplify the formulas, we denote $b_{l,m}^{(n)}$ by b_n. There are finitely many sets $A_k \in \mathcal{A}_{n+1}$ contained in A_n. We denote by b_k the pole corresponding to A_k. Let $\beta_n := h_n^{-1}(b_n) \in A_n$, $\beta_k := h_{n+1}^{-1}(b_k) \in A_k$. Lemma 5·3 implies that h_n are conformal on A_n. Using (2·4), we get

$$L(h_n, A_n) = L(h_n^{-1}, h_n(A_n)).$$

Hence,

$$\begin{aligned}
\mathrm{vol}(A_n) &= \iint_{U(b_n,\varepsilon)} \left|(h_n^{-1})'(z)\right|^2 dz \leqslant \iint_{U(b_n,\varepsilon)} \left(\sup_{z \in U(b_n,\varepsilon)} |(h_n^{-1})'(z)| \right)^2 dz \\
&= \mathrm{vol}(U(b_n,\varepsilon)) \left(L(h_n^{-1}, U(b_n,\varepsilon)) \inf_{z \in U(b_n,\varepsilon)} |(h_n^{-1})'(z)| \right)^2 \\
&\leqslant \frac{3\pi\varepsilon^2}{8} \left(L(h_n, A_n) |(h_n^{-1})'(b_n)| \right)^2 = \frac{3\pi\varepsilon^2}{8} \left(\frac{L(h_n, A_n)}{|h_n'(\beta_n)|} \right)^2.
\end{aligned} \tag{5·10}$$

Set $P_{n+1} := P^+(0, R_{n+1}, 2R_{n+1})$.

$$\begin{aligned}
&\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n) \\
&= \sum_{A_k \subset A_n} \mathrm{vol}(A_k) = \sum_{b_k \in P_{n+1}} \mathrm{vol}(h_{n+1}^{-1}(U(b_k, \varepsilon))) \\
&= \sum_{b_k \in P_{n+1}} \iint_{U(b_k,\varepsilon)} \left|(h_{n+1}^{-1})'(z)\right|^2 dz \geqslant \sum_{b_k \in P_{n+1}} \iint_{U(b_k,\varepsilon)} \left(\inf_{z \in U(b_k,\varepsilon)} |(h_{n+1}^{-1})'(z)|\right)^2 dz \\
&= \frac{3\pi\varepsilon^2}{8} \sum_{b_k \in P_{n+1}} \left(\frac{\sup_{z \in U(b_k,\varepsilon)} |(h_{n+1}^{-1})'(z)|}{L(h_{n+1}^{-1}, U(b_k, \varepsilon))}\right)^2 \geqslant \frac{3\pi\varepsilon^2}{8} \sum_{b_k \in P_{n+1}} \left(\frac{|(h_{n+1}^{-1})'(b_k)|}{L(h_{n+1}^{-1}, U(b_k, \varepsilon))}\right)^2 \\
&= \frac{3\pi\varepsilon^2}{8} \sum_{\beta_k \in A_k \subset A_n} \left(L(h_{n+1}, A_k)|h_{n+1}'(\beta_k)|\right)^{-2}. \qquad (5{\cdot}11)
\end{aligned}$$

Now, using (5·10) and (5·11), we estimate the density of the sets $\mathcal{U}_{n+1} \cap A_n$ in A_n.

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geqslant \frac{|h_n'(\beta_n)|^2}{(L(h_n, A_n))^2} \sum_{\beta_k \in A_k \subset A_n} \left(L(h_{n+1}, A_k)|h_{n+1}'(\beta_k)|\right)^{-2}. \qquad (5{\cdot}12)$$

Lemma 5·1 and the inequalities (5·6) give

$$|h_n'(\beta_n)| \geqslant \frac{R_1}{2(1+r)} \left|\prod_{j=1}^{n-1} g_{\beta_n}'(g_{\beta_n}^j(c_1))\right| \qquad (5{\cdot}13)$$

and

$$|h_{n+1}'(\beta_k)| \leqslant \frac{5R_1}{2(1-r)} \left|\prod_{j=1}^{n} g_{\beta_k}'(g_{\beta_k}^j(c_1))\right|. \qquad (5{\cdot}14)$$

It follows from Lemma 5·5 that

$$(L(h_n, A_n))^2 \leqslant \left(\frac{1+r}{1-r}\right)^2 5^2 2^{3(n-1)} \qquad (5{\cdot}15)$$

and

$$(L(h_{n+1}, A_k))^2 \leqslant \left(\frac{1+r}{1-r}\right)^2 5^2 2^{3n}. \qquad (5{\cdot}16)$$

Plugging (5·13)–(5·16) into (5·12), we have

$$\begin{aligned}
\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geqslant &\frac{\left(\frac{R_1}{2(1+r)}\right)^2 \left|\prod_{j=1}^{n-1} g_{\beta_n}'(g_{\beta_n}^j(c_1))\right|^2}{\left(\frac{1+r}{1-r}\right)^2 5^2 2^{3(n-1)}} \\
&\times \sum_{\beta_k \in A_k \subset A_n} \frac{1}{\left(\frac{1+r}{1-r}\right)^2 5^2 2^{3n} \left(\frac{5R_1}{2(1-r)}\right)^2 \left|\prod_{j=1}^{n} g_{\beta_k}'(g_{\beta_k}^j(c_1))\right|^2}
\end{aligned}$$

https://doi.org/10.1017/S0305004112000448 Published online by Cambridge University Press

$$= \left(\frac{1-r}{1+r}\right)^6 \frac{1}{5^6 2^{3(2n-1)}} \left|\prod_{j=1}^{n-1} g'_{\beta_n}(g^j_{\beta_n}(c_1))\right|^2 \sum_{\beta_k \in A_k \subset A_n} \frac{1}{\left|\prod_{j=1}^{n} g'_{\beta_k}(g^j_{\beta_k}(c_1))\right|^2}$$

$$= \left(\frac{1-r}{1+r}\right)^6 \frac{1}{5^6 2^{3(2n-1)}} \sum_{\beta_k \in A_k \subset A_n} \left(\prod_{j=1}^{n-1} \frac{\left|g'_{\beta_n}(g^j_{\beta_n}(c_1))\right|}{\left|g'_{\beta_k}(g^j_{\beta_k}(c_1))\right|}\right)^2 \cdot \frac{1}{\left|g'_{\beta_k}(g^n_{\beta_k}(c_1))\right|^2}. \quad (5{\cdot}17)$$

For each $j = 1, 2, \ldots, n-1$

$$g_{\beta_n}(g^j_{\beta_n}(c_1)) = g^{j+1}_{\beta_n}(c_1) = h_{j+1}(\beta_n) \in P^+(0, R_{j+1}, 2R_{j+1})$$

and

$$g_{\beta_k}(g^j_{\beta_k}(c_1)) = g^{j+1}_{\beta_k}(c_1) = h_{j+1}(\beta_k) \in P^+(0, R_{j+1}, 2R_{j+1}),$$

since $\beta_n \in A_n \subset A_{j+1}$ and $\beta_k \in A_k \subset A_n \subset A_{j+1}$. Thus, by (5·3), for $j = 1, 2, \ldots, n-1$ we have

$$|g'_{\beta_n}(g^j_{\beta_n}(c_1))| \geqslant \frac{C_2 R^{3/2}_{j+1}}{C_1^{3/2}} \quad \text{and} \quad |g'_{\beta_k}(g^j_{\beta_k}(c_1))| \leqslant \frac{2^{3/2} C_2 R^{3/2}_{j+1}}{C_1^{3/2}}.$$

This implies that

$$\frac{\left|g'_{\beta_n}(g^j_{\beta_n}(c_1))\right|}{\left|g'_{\beta_k}(g^j_{\beta_k}(c_1))\right|} \geqslant \frac{1}{2^{3/2}}, \quad j = 1, 2, \ldots, n-1. \quad (5{\cdot}18)$$

Analogously,

$$g_{\beta_k}(g^n_{\beta_k}(c_1)) = g^{n+1}_{\beta_k}(c_1) = h_{n+1}(\beta_k) \in P^+(0, R_{n+1}, 2R_{n+1})$$

as $\beta_k \in A_k \in \mathcal{A}_{n+1}$. By applying this to (5·3), we get

$$|g'_{\beta_k}(g^n_{\beta_k}(c_1))| \leqslant \frac{2^{3/2} C_2 R^{3/2}_{n+1}}{C_1^{3/2}}. \quad (5{\cdot}19)$$

Putting (5·18), (5·19) into (5·17), by Remark 5·4 and (5·9), we obtain

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol} A_n} \geqslant \left(\frac{1-r}{1+r}\right)^6 \frac{1}{5^6 2^{3(2n-1)}} \left(\frac{1}{2^{3/2}}\right)^{2(n-1)} \frac{C_1^3}{2^3 C_2^2 R^3_{n+1}} \sum_{\beta_k \in A_k \subset A_n} 1$$

$$\asymp \left(\frac{1-r}{1+r}\right)^6 \frac{2^3}{5^6 2^{9n}} \frac{C_1^3}{C_2^2 R^3_{n+1}} N_{n+1} \asymp \left(\frac{1-r}{1+r}\right)^6 \frac{2^3}{5^6 2^{9n}} \frac{C_1^3}{C_2^2 R^3_{n+1}} R^2_{n+1}$$

$$= \frac{M}{2^{9n} R_{n+1}},$$

where $M = 2^3(1-r)^6 C_1^3/(5^6(1+r)^6 C_2^2)$.

Similarly, we consider the case $n = 1$. By Definition 4·1, the set $\mathcal{A}_1$ has only one element, i.e. A_1 and its Lebesgue measure $\mathrm{vol}(A_1) \leqslant \pi r^2$. The set A_1 contains finitely many subsets $A_k \in \mathcal{A}_2$. As for $n \geqslant 2$, we denote by b_k the pole corresponding to A_k. Arguing as in (5·11), we get

$$\mathrm{vol}(\mathcal{U}_2 \cap A_1) \geqslant \frac{3\pi\varepsilon^2}{8} \sum_{\beta_k \in A_k \subset A_1} \left(L(h_2, A_k)|h'_2(\beta_k)|\right)^{-2}.$$

Setting $n = 1$ in bounds (5·14), (5·16) we have

$$|h_2'(\beta_k)| \leqslant \frac{5R_1}{2(1-r)}|g_{\beta_k}'(g_{\beta_k}(c_1))| \quad \text{and} \quad (L(h_2, A_k))^2 \leqslant \left(\frac{1+r}{1-r}\right)^2 5^2 2^3,$$

which implies that

$$\mathrm{vol}(\mathcal{U}_2 \cap A_1) \geqslant \frac{3\pi\varepsilon^2}{8} \sum_{\beta_k \in A_k \subset A_1} \frac{1}{\left(\frac{5R_1}{2(1-r)}\right)^2 |g_{\beta_k}'(g_{\beta_k}(c_1))|^2 \left(\frac{1+r}{1-r}\right)^2 5^2 2^3}.$$

Analogously as in (5·19), we obtain

$$|g_{\beta_k}'(g_{\beta_k}(c_1))| \leqslant \frac{2^{3/2} C_2 R_2^{3/2}}{C_1^{3/2}}$$

and, using Remark 5·4 and (5·9), we conclude that

$$\frac{\mathrm{vol}(\mathcal{U}_2 \cap A_1)}{\mathrm{vol}(A_1)} \geqslant \frac{3\pi\varepsilon^2}{8\pi r^2} \sum_{\beta_k \in A_k \subset A_1} \frac{1}{\left(\frac{5R_1}{2(1-r)}\right)^2 \left(\frac{2^{3/2} C_2 R_2^{3/2}}{C_1^{3/2}}\right)^2 \left(\frac{1+r}{1-r}\right)^2 5^2 2^3}$$

$$= \frac{3\varepsilon^2 (1-r)^4 C_1^3}{2^7 5^4 r^2 (1+r)^2 C_2^2 R_1^2} \frac{\sum_{\beta_k \in A_k \subset A_1} 1}{R_2^3} \asymp M' \frac{N_2}{R_2^3} \asymp M' \frac{R_2^2}{R_2^3} = \frac{M'}{R_2},$$

where $M' = 3\varepsilon^2(1-r)^4 C_1^3/(2^7 5^4 r^2 (1+r)^2 C_2^2 R_1^2)$.

By Lemma 5·8, the numbers Δ_n from Proposition 2·10 are equal to

$$\Delta_1 = \frac{M'}{R_2}, \quad \Delta_n = \frac{M}{2^{9n} R_{n+1}}, \quad n \geqslant 2.$$

Assembling the preceding lemmas, we may now prove Theorem 4·3.

Proof of Theorem 4·3. Lemma 5·8 implies that

$$\sum_{j=1}^{n} |\log \Delta_j| = |\log \Delta_1| + \sum_{j=2}^{n} |\log \Delta_j| = \left|\log \frac{M'}{R_2}\right| + \sum_{j=2}^{n} \left|\log \frac{M}{2^{9j} R_{j+1}}\right|$$

$$= \log(aR_1) - \log M' + \sum_{j=2}^{n} \log(2^{9j} a^j R_1) - (n-1)\log M$$

$$= \log M - \log M' + n \log R_1 - n \log M + 9 \log 2 \sum_{j=2}^{n} j + \log a \sum_{j=1}^{n} j$$

$$= \log \frac{M}{M'} + n \log \frac{R_1}{M} + \frac{9(n+2)(n-1)}{2} \log 2 + \frac{n(n+1)}{2} \log a. \tag{5·20}$$

In view of Lemma 5·6, we have

$$|\log d_n| = \left|\log \frac{4\varepsilon(1+r)}{\left(\frac{C_2}{C_1^{3/2}}\right)^{n-1} a^{\frac{3n(n-1)}{4}} R_1^{\frac{3n-1}{2}}}\right|$$

$$= (n-1)\log \frac{C_2}{C_1^{3/2}} + \frac{3n(n-1)}{4} \log a + \frac{3n-1}{2} \log R_1 - \log 4\varepsilon(1+r). \tag{5·21}$$

The final estimate follows from (5·20) and (5·21).

$$\dim_H(A(a)) \geqslant 2 - \limsup_{n\to\infty} \frac{\log \frac{M}{M'} + n\log\frac{R_1}{M} + \frac{9(n+2)(n-1)}{2}\log 2 + \frac{n(n+1)}{2}\log a}{(n-1)\log\frac{C_2}{C_1^{3/2}} + \frac{3n(n-1)}{4}\log a + \frac{3n-1}{2}\log R_1 - \log 4\varepsilon(1+r)}$$

$$= 2 - \frac{\frac{1}{2}\log a + \frac{9}{2}\log 2}{\frac{3}{4}\log a} = \frac{4}{3} - \frac{6\log 2}{\log a}.$$

Thus, the theorem stated in Section 1 follows from Theorem 4·3.

Question. Is the Hausdorff dimension of the escaping set $\mathcal{E}$ equal to $4/3$?

Acknowledgements. We are grateful to the referee for helpful comments.

REFERENCES

[**1**] W. BERGWEILER, B. KARPIŃSKA and G. M. STALLARD. The growth rate of an entire function and the Hausdorff dimension of its Julia set. *J. Lond. Math. Soc.* **80** (2009), 680–698.
[**2**] W. BERGWEILER and J. KOTUS. On the Hausdorff dimension of the escaping set of certain meromorphic functions, to appear in *Trans. Amer. Math. Soc.*, arxiv: 0901.3014.
[**3**] W. BERGWEILER, P. J. RIPPON and G. M. STALLARD. Dynamics of meromorphic functions with direct or logarithmic singularities. *Proc. Lond. Math. Soc.* **97** (2008), 368–400.
[**4**] P. DOMÍNGUEZ. Dynamics of transcendental meromorphic functions. *Ann. Acad. Sci. Fenn. Math.* **23** (1998), 225–250.
[**5**] J. HAWKINS and L. KOSS. Connectivity of Julia sets of Weierstrass elliptic functions. *Topology Appl.* **152** (2002), 107–137.
[**6**] J. HAWKINS and L. KOSS. Ergodic properties and Julia sets of Weierstrass elliptic functions. *Monatsh. Math.* **137** (2002), 273–300.
[**7**] J. HAWKINS and L. KOSS. Parametrized dynamics of the Weierstrass elliptic functions. *Conform. Geom. Dyn.* **8** (2004), 1–35.
[**8**] J. HAWKINS, L. KOSS and J. KOTUS. Elliptic functions with critical orbits approaching infinity. *J. Difference Equ. Appl.* **16** (2010), 613–630.
[**9**] J. M. HEMKE. Recurrence of entire transcendental functions with simple post-singular sets. *Fund. Math.* **187** (2005), 255–289.
[**10**] J. KOTUS. On the Hausdorff dimension of Julia sets of meromorphic functions II. *Bull. Soc. Math. France* **123** (1995), 33–46.
[**11**] J. KOTUS and M. URBAŃSKI. Existence of invariant measures for transcendental subexpanding functions. *Math. Zeit.* **243** (2003), 25–36.
[**12**] J. KOTUS and M. URBAŃSKI. Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. *Bull. Lond. Math. Soc.* **35** (2003), 269–275.
[**13**] J. KOTUS and M. URBAŃSKI. Geometry and ergodic theory of non-recurrent elliptic functions. *J. Anal. Math.* **93** (2004), 35–102.
[**14**] J. KOTUS and M. URBAŃSKI. The class of pseudo non-recurrent elliptic functions; geometry and dynamics, preprint available at *www.math.unt.edu/~urbanski*
[**15**] M. YU. LYUBICH. Measurable dynamics of the exponential. *Sib. Math. J.* **28** (1988), 780–793.
[**16**] C. MCMULLEN. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.* **300** (1987), 329–342.
[**17**] P. J. RIPPON and G. M. STALLARD. Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.* **127** (1999), 3251–3258.
[**18**] P. J. RIPPON and G. M. STALLARD. Escaping points of meromorphic functions with a finite number of poles. *J. Anal. Math.* **96** (2005), 225–245.
[**19**] P. J. RIPPON and G. M. STALLARD. Escaping points of entire functions of small growth. *Math. Zeit.* **261** (2009), 557–570.
[**20**] M. URBAŃSKI and A. ZDUNIK. Geometry and ergodic theory of non-hyperbolic exponential maps. *Trans. Amer. Math. Soc.* **359** (2007), 3973–3997.