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THE DENSE REGION IN SCATTERING DIAGRAMS

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Abstract

We use deformations and mutations of scattering diagrams to show that a scattering diagram with initial functions $f_1 = (1 + tx)^{\mu}$ and $f_2 = (1 + ty)^{\nu}$ has a dense region. This answers a question asked by Gross and Pandharipande ['Quivers, curves, and the tropical vertex', *Port. Math.* **67**(2) (2010), 211–259] which had been proved only for the case $\mu = \nu$.

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1. Introduction

Scattering diagrams (introduced in [13]) are a method to combinatorially encode families of automorphisms of an algebraic torus (or, more generally, elements of the Lie group associated to a pronilpotent Lie algebra). They are related to various subjects such as curve counting [1, 2, 4, 6, 10, 14], quiver representations [9, 16], stability conditions [3, 5], cluster algebras [8, 14] and mirror symmetry [7, 11–13].

A *scattering diagram* $\mathfrak{D}$ (in dimension two) is a collection of rays $\mathfrak{d} \subset \mathbb{R}^2$ with attached functions $f_{\mathfrak{d}} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}][\![t]\!]$. It is completely described by the coefficients $c_{a,b}$ of its functions. We use the factorised representation

$$f_{\mathfrak{d}} = \prod_{k>0} (1 + tx^a y^b)^{c_{ka,kb}}.$$

Each ray or line $\mathfrak{d}$ induces an automorphism $\theta_{\mathfrak{d}} \in \mathrm{Aut}_{\mathbb{C}[\![t]\!]}\mathbb{C}[x^{\pm 1}, y^{\pm 1}][\![t]\!]$. Starting with an initial diagram $\mathfrak{D}_0$, there is a scattering algorithm that iteratively produces scattering diagrams $\mathfrak{D}_k$ such that the composition of the automorphisms $\theta_{\mathfrak{d}}$ is trivial modulo t^k. Taking the formal limit, one obtains a *consistent* scattering diagram $\mathfrak{D}_\infty$. (See Section 2.1 for more details.)

The aim of this paper is to show that most consistent scattering diagrams have a *dense region* in which, for every slope, there exists a ray with nontrivial function

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$f_{\mathfrak{d}} \neq 1$. To achieve this, we make use of three different techniques using certain properties of scattering diagrams.

First (Section 2.2 and [8, 10]), we can assume that no more than two rays or lines intersect in the same point. Otherwise, one can *deform* the diagram by slightly moving its rays. If a ray $\mathfrak{d}$ has a reducible function $f_{\mathfrak{d}} = f_1 \cdots f_r$, one can also split it into several rays with irreducible functions $f_1, \ldots, f_r$ and then deform the diagram to obtain nonintersecting parallel rays $\mathfrak{d}_1, \ldots, \mathfrak{d}_r$.

Second (Section 2.3 and [8]), using a *change of lattice* we can assume that the two rays or lines intersect transversally, so that the attached functions are $f_1 = (1+tx)^\mu$ and $f_2 = (1+ty)^\nu$. The corresponding consistent diagram is called a *standard scattering diagram* $\mathfrak{D}^{\mu,\nu}$. By the above, it suffices to study standard scattering diagrams.

Third (Section 2.4 and [8, 9]), the rays in a standard scattering diagram $\mathfrak{D}^{\mu,\nu}$ obey certain symmetries related to the notion of *mutations*. Precisely, the corresponding coefficients satisfy $c^{\mu,\nu}_{a,b} = c^{\mu,\nu}_{\mu b-a,b} = c^{\mu,\nu}_{a,\nu a-b}$.

We use the above techniques to prove the theorem stated below.

DEFINITION 1.1. Let $\mathfrak{D}^{\mu,\nu}$ be a standard scattering diagram, that is, the consistent diagram obtained from the initial diagram consisting of two lines with functions $f_1 = (1+tx)^\mu$ and $f_2 = (1+ty)^\nu$. The function attached to the ray in $\mathfrak{D}^{\mu,\nu}$ with direction (a,b), $\gcd(a,b) = 1$, can be factorised as

$$f^{\mu,\nu}_{(a,b)} = \prod_{k>0} (1 + tx^a y^b)^{c^{\mu,\nu}_{ka,kb}},$$

defining positive integers (by Proposition 2.9) $c^{\mu,\nu}_{ka,kb} \in \mathbb{Z}_{>0}$.

DEFINITION 1.2. Define (μ,ν)-mutations $\mathbf{T}^{\mu,\nu}_1(a,b) = (\mu b - a, b)$, $\mathbf{T}^{\mu,\nu}_2(a,b) = (a, \nu a - b)$.

DEFINITION 1.3. We say that $(a,b) \in \mathbb{Z}^2_{>0}$ is in the dense region $\Phi^{\mu,\nu}$ if

$$\frac{\mu\nu - \sqrt{\mu\nu(\mu\nu-4)}}{2\mu} < \frac{b}{a} < \frac{\mu\nu + \sqrt{\mu\nu(\mu\nu-4)}}{2\mu}.$$

THEOREM 1.4.

(a) *If $(a,b) \in \mathbb{Z}^2_{>0}$ is in the dense region $\Phi^{\mu,\nu}$, then $c^{\mu,\nu}_{a,b} \neq 0$.*

(b) *Otherwise, $c^{\mu,\nu}_{a,b} \neq 0$ if and only if (a,b) is obtained from $(1,0)$ or $(0,1)$ via a sequence of (μ,ν)-mutations. In particular, (a,b) must be primitive in this case.*

Theorem 1.4 will be proved in Section 3. The idea is as follows. It is enough to show density inside a fundamental domain $\phi^{\mu,\nu}_0$ for the mutation actions (Section 3.1). One can show that $\phi^{\mu+1,\nu}_0$ (respectively, $\phi^{\mu,\nu+1}_0$) is contained in $\Phi^{\mu,\nu}$ if $\mu\nu > 4$ and $\mu > 1$ (respectively, $\nu > 1$). Then, by induction and symmetry $\mu \leftrightarrow \nu$, it is enough to show density for $\Phi^{2,3}$ and $\Phi^{1,5}$ (Section 3.2). We show this explicitly by deforming $\mathfrak{D}^{2,3}$ to $\mathfrak{D}^{2,2}$ plus $\mathfrak{D}^{2,1}$ and deforming $\mathfrak{D}^{1,5}$ to $\mathfrak{D}^{1,3}$ plus $\mathfrak{D}^{1,2}$ (Section 3.3). Part (b) follows from the mutation actions (Section 3.4).

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REMARK 1.5. Theorem 1.4 answers [9, Question 4]. It was proved in the case $\mu = \nu$ in [9, Section 4.7], using an existence statement for quiver representations from [15]. Reineke stated that there should be a similar argument in the case $\mu \neq \nu$ using bipartite quivers [18], but this has not been worked out in detail. Our proof is purely combinatorial.

2. Preliminaries

2.1. Scattering diagrams. We provide a definition for scattering diagrams, based on [10]. (See [8, 10] for more general definitions.)

Let $M \cong \mathbb{Z}^2$ be a lattice with basis $e_1 = (1,0), e_2 = (0,1)$, and let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. For $m \in M$, let $z^m \in \mathbb{C}[M]$ denote the corresponding element in the group ring. If $x = z^{e_1}, y = z^{e_2}$, then $\mathbb{C}[M] = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is the ring of Laurent polynomials in x and y.

Let R be an Artin local $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}_R$, and let

$$\mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}R = \varprojlim \mathbb{C}[M] \otimes_{\mathbb{C}} R/\mathfrak{m}_R^k.$$

We take $M = N = \mathbb{Z}^2$ and $R = \mathbb{C}[\![t]\!]$, so that $\mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}][\![t]\!]$ and $\mathfrak{m}_R = (t)$.

DEFINITION 2.1. A *ray* or *line* is a pair $\mathfrak{d} = (\underline{\mathfrak{d}}, f_{\mathfrak{d}})$, where $\underline{\mathfrak{d}} = b_{\mathfrak{d}} + \mathbb{R}_{\geq 0} m_{\mathfrak{d}}$ if it is a ray or $\underline{\mathfrak{d}} = b_{\mathfrak{d}} + \mathbb{R} m_{\mathfrak{d}}$ if it is a line, and $f_{\mathfrak{d}} \in \mathbb{C}[z^{m_{\mathfrak{d}}}]\widehat{\otimes}_{\mathbb{C}}R \subseteq \mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}R$ is a function such that

$$f_{\mathfrak{d}} \equiv 1 \pmod{z^{m_{\mathfrak{d}}}\mathfrak{m}_R}.$$

A *scattering diagram* $\mathfrak{D}$ is a collection of rays and lines such that, for every $k > 0$, there are finitely many rays and lines $(\underline{\mathfrak{d}}, f_{\mathfrak{d}})$ with $f_{\mathfrak{d}} \not\equiv 1 \pmod{\mathfrak{m}_R^k}$.

DEFINITION 2.2. For a ray $\mathfrak{d}$ and a curve γ in $M_{\mathbb{R}}$ intersecting $\mathfrak{d}$ transversally at p, let $n_{\mathfrak{d}} \in N$ annihilate $m_{\mathfrak{d}}$ and evaluate positively on $\gamma'(p)$. Define $\theta_{\mathfrak{d}} = \theta_{\gamma,p,\mathfrak{d}} \in \mathrm{Aut}_{\mathbb{C}[\![t]\!]}(\mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}R)$ by

$$\theta_{\mathfrak{d}} : z^m \mapsto z^m f_{\mathfrak{d}}^{\langle m, n_{\mathfrak{d}} \rangle}.$$

DEFINITION 2.3. A *singularity* of a scattering diagram $\mathfrak{D}$ is either a base point of a ray or an intersection between two rays or lines that consists of a single point.

Let $\gamma : [0,1] \to M_{\mathbb{R}}$ be a smooth curve that does not pass through any singularities and whose endpoints are not in any ray or line in the diagram. If all intersections of γ with rays or lines are transverse, then we define the *γ-ordered product* $\theta_{\gamma,\mathfrak{D}} \in \mathrm{Aut}_R(\mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}R)$ in the following way. For each k, as there are finitely many rays or lines with functions $f_{\mathfrak{d}} \not\equiv 1 \pmod{\mathfrak{m}_R^k}$, let $0 < p_1 \leq p_2 \leq \cdots \leq p_s < 1$ be such that, at each p_i, $\gamma(p_i) \in \mathfrak{d}_i$ for some ray or line $(\mathfrak{d}_i, f_{\mathfrak{d}_i})$, and when $p_i = p_j$ for $i \neq j$, $\mathfrak{d}_i \neq \mathfrak{d}_j$ are different rays of the diagram. Then let $\theta_i = \theta_{\gamma,p_i,\mathfrak{d}_i}$ and

$$\theta^k_{\gamma,\mathfrak{D}} = \theta_s \circ \cdots \circ \theta_2 \circ \theta_1.$$

Then we define $\theta_{\gamma,\mathfrak{D}}$ as the formal limit $\theta_{\gamma,\mathfrak{D}} = \lim_{k\to\infty}(\theta^k_{\gamma,\mathfrak{D}})$.

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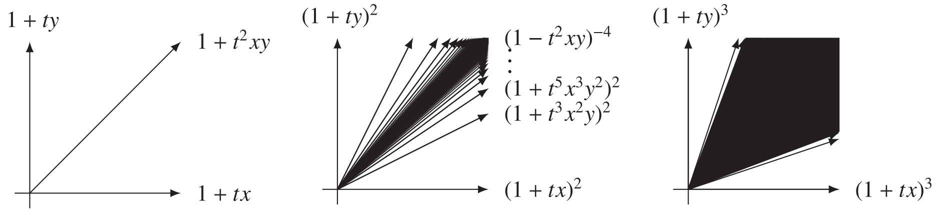

FIGURE 1. The standard scattering diagrams $\mathfrak{D}^{1,1}$, $\mathfrak{D}^{2,2}$ and $\mathfrak{D}^{3,3}$.

We say that a diagram $\mathfrak{D}$ is *consistent* if $\theta_{\gamma,\mathfrak{D}}$ is the identity map for every closed curve γ (for which $\theta_{\gamma,\mathfrak{D}}$ is defined). Two diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ are *equivalent* if $\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathfrak{D}'}$ for every curve γ.

PROPOSITION 2.4 [13], [10, Theorem 1.4]. *For a scattering diagram $\mathfrak{D}$, there exists a consistent scattering diagram $\mathfrak{D}_\infty \supseteq \mathfrak{D}$ such that $\mathfrak{D}_\infty \setminus \mathfrak{D}$ consists only of rays.*

REMARK 2.5. The consistent diagram $\mathfrak{D}_\infty$ obtained from $\mathfrak{D}$ is unique (up to equivalence) if we require that it has no two rays $\mathfrak{d}, \mathfrak{d}'$ with the same support $\underline{\mathfrak{d}} = \underline{\mathfrak{d}}'$.

DEFINITION 2.6. If a consistent diagram $\mathfrak{D} = \mathfrak{D}_\infty$ has only one singularity, then (by Remark 2.5, up to equivalence) there is at most one ray in each direction $m \in \mathbb{Z}^2$. We write the function of this ray as $f_m^{\mathfrak{D}}$. If $f_m^{\mathfrak{D}} = 1$, then we can omit the ray.

DEFINITION 2.7. The *standard scattering diagram* $\mathfrak{D}^{\mu,\nu} = \mathfrak{D}_\infty^{\mu,\nu}$ is the diagram obtained by performing scattering on the initial diagram

$$\mathfrak{D}_0^{\mu,\nu} = \{(\mathbb{R}(1,0),(1+tx)^\mu),(\mathbb{R}(0,1),(1+ty)^\nu)\}.$$

The scattering only produces rays in the first quadrant, that is, with $m_{\mathfrak{d}} = (a,b) \in \mathbb{Z}^2_{>0}$. Consider an equivalent diagram to a standard scattering diagram such that there is a unique ray in each direction (see Remark 2.5). We can express the function $f_{\mathfrak{d}}$ of the ray $\mathfrak{d}$ in direction $(a,b) \in \mathbb{Z}^2_{>0}$ as

$$f_{(a,b)}^{\mu,\nu} := f_{(a,b)}^{\mathfrak{D}^{\mu,\nu}} = \prod_{k=1}^{\infty}(1+t^{ka+kb}x^{ka}y^{kb})^{c_{ka,kb}^{\mu,\nu}}.$$

DEFINITION 2.8. The *coefficients* for $\mathfrak{D}^{\mu,\nu}$ are these $c_{a,b}^{\mu,\nu}$.

PROPOSITION 2.9 [8, Proposition C.13]. *The coefficients of a standard scattering diagram are positive integers: $c_{a,b}^{\mu,\nu} \in \mathbb{Z}_{>0}$.*

REMARK 2.10. Note that $c_{a,b}^{\mu,\nu} = c_{b,a}^{\nu,\mu}$ by symmetry: as $\mathfrak{D}_\infty^{\mu,\nu}$ is consistent, its reflection along the diagonal $\mathbb{R}(1,1)$ is as well, which gives a consistent diagram containing $\mathfrak{D}_0^{\nu,\mu}$.

EXAMPLE 2.11. Figure 1 shows the standard scattering diagrams $\mathfrak{D}^{1,1}$, $\mathfrak{D}^{2,2}$ and $\mathfrak{D}^{3,3}$.

The diagram $\mathfrak{D}^{1,1}$ has, apart from the initial lines, only one ray in direction $(1,1)$ with function $f_{(1,1)}^{1,1} = 1+t^2xy$. Hence, the only nontrivial coefficient is $c_{1,1}^{1,1} = 1$.

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The diagram $\mathfrak{D}^{2,2}$ has only rays in directions $(1,1)$, $(n,n+1)$ and $(n+1,n)$ for $n \in \mathbb{N}$, with

$$f^{2,2}_{(1,1)} = (1-xy)^{-4}, \quad f_{(n,n+1)} = (1+x^n y^{n+1})^2, \quad f_{(n+1,n)} = (1+x^{n+1}y^n)^2.$$

Hence, the nonzero coefficients $c^{2,2}_{a,b}$ are

$$c^{2,2}_{n,n} = \begin{cases} 4 & n = 2^k, \\ 0 & \text{otherwise}, \end{cases} \qquad c^{2,2}_{n,n+1} = 2, \quad c^{2,2}_{n+1,n} = 2.$$

In particular, the rays are discrete.

For $\mathfrak{D}^{3,3}$ there is a *dense region* in which each ray appears with nontrivial function. This is the statement of Theorem 1.4. The functions $f^{3,3}_{(a,b)}$ and coefficients $c^{3,3}_{a,b}$ are very complicated and unknown in general. Only for the slope 1 coefficients $c^{\mu,\nu}_{k,k}$ is there a known formula, which was proved for $\mu = \nu$ in [17, Theorem 6.4] and for $\mu \neq \nu$ in [18, Corollary 11.2]. (See also [10, Example 1.6] and [9, Section 1.4].)

2.2. Deformations. Given a consistent scattering diagram $\mathfrak{D}$, we can form the asymptotic diagram $\mathfrak{D}_{\mathrm{as}}$ by replacing every ray $(b_{\mathfrak{d}} + \mathbb{R}_{\geq 0} m_{\mathfrak{d}}, f_{\mathfrak{d}})$ with $(\mathbb{R}_{\geq 0} m_{\mathfrak{d}}, f_{\mathfrak{d}})$, and similarly for lines. By considering sufficiently large curves in $\mathfrak{D}$ around the origin containing all singularities, we see that $\mathfrak{D}_{\mathrm{as}}$ is also consistent. We can use this to consider *deformations* as follows. (For more details see [10, Section 1.4] and [8, Proposition C.13, Step III].)

Definition 2.12. The *full deformation* of $\mathfrak{D}^{\mu,\nu}$ consists of general lines $\mathfrak{d}_{1,1}, \ldots, \mathfrak{d}_{1,\mu}$, $\mathfrak{d}_{2,1}, \ldots, \mathfrak{d}_{2,\nu}$ with functions

$$f_{\mathfrak{d}_{1,i}} = 1 + tx, \quad f_{\mathfrak{d}_{2,i}} = 1 + ty.$$

Here the lines being general means that all rays in the consistent diagram intersect in points, not in rays. We will also consider *partial deformations* by pulling out only one factor.

Proposition 2.13 [10, Section 1.4]. *Let $\mathfrak{D}'$ be a partial or full deformation of $\mathfrak{D}$. Then $(\mathfrak{D}'_\infty)_{\mathrm{as}} = \mathfrak{D}_\infty$.*

Example 2.14. Figure 2 shows a full deformation of $\mathfrak{D}^{3,1}$ and a partial deformation by pulling out one factor of $(1+x)^3$. This gives $c^{3,1}_{1,1} = 3$, $c^{3,1}_{2,1} = 3$, $c^{3,1}_{3,1} = 1$, $c^{3,1}_{3,2} = 3$.

Lemma 2.15. *If $\mu \leq \mu'$ and $\nu \leq \nu'$, then $c^{\mu,\nu}_{a,b} \leq c^{\mu',\nu'}_{a,b}$.*

Proof. Deform $\mathfrak{D}^{\mu',\nu'}$ in such a way that we have a horizontal line with function $(1+tx)^{\mu}$ and a vertical line with function $(1+ty)^{\nu}$. From this, we get a ray contributing $c^{\mu,\nu}_{a,b}$. As all coefficients are positive by Proposition 2.9, we see that $c^{\mu,\nu}_{a,b} \leq c^{\mu',\nu'}_{a,b}$. □

2.3. The change of lattice trick. There is a useful way to reduce to only needing to consider standard diagrams (found in [8, Proof of Proposition C.13, Step IV]).

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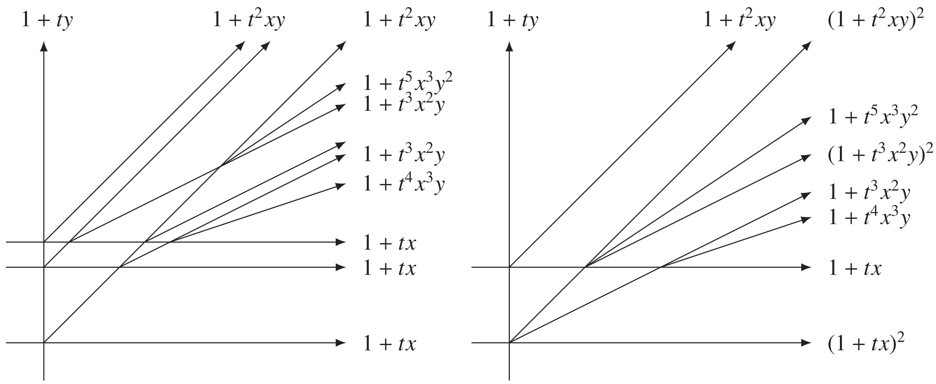

FIGURE 2. A full (left) and partial (right) deformation of $\mathfrak{D}^{3,1}$.

PROPOSITION 2.16. *Let $\mathfrak{D}$ be the consistent diagram obtained from the scattering diagram consisting of two lines $\mathfrak{d}_1$ and $\mathfrak{d}_2$ with functions $f_1 = (1+tz^{m_1})^{d_1}$ and $f_2 = (1+tz^{m_2})^{d_2}$. Let $M' \subseteq M$ be the sublattice generated by m_1 and m_2 and let $N' \supseteq N$ be the dual lattice. If $m \in M \setminus M'$, then $f_m = 1$. Otherwise, write $m = am_1 + bm_2$. Then*

$$f_m^{\mathfrak{D}} = (f_{(a,b)}^{d_1 e(m_2^*), d_2 e(m_1^*)})^{1/e(n)},$$

where $n \in N'$ is orthogonal to $m \in M'$ and primitive and, for any $n' \in N'$,

$$e(n') := \min\{k \in \mathbb{N} \mid kn' \in N\}.$$

In particular, the scattering of any scattering diagram consisting of two lines can be computed from a standard scattering diagram.

PROOF. Any ray $\mathfrak{d}$ in $\mathfrak{D}$ has direction vector $m_{\mathfrak{d}}$ contained in $M' \subseteq M$. Hence, we can consider $\mathfrak{d}$, $\mathfrak{D}$ and $\mathfrak{D}_\infty$ in the lattice M or in the lattice M'. In the latter case, we write $\mathfrak{d}'$, $\mathfrak{D}'$ and $\mathfrak{D}'_\infty$. By definition, the automorphism $\theta_{\mathfrak{d}} \in \mathrm{Aut}_{\mathbb{C}[\![t]\!]}(\mathbb{C}[M]\widehat{\otimes}_{\mathbb{C}}\mathbb{C}[\![t]\!])$ defined by a ray $\mathfrak{d} \in \mathfrak{D}_\infty$ is given by

$$\theta_{\mathfrak{d}} : z^m \mapsto z^m f_{\mathfrak{d}}^{\langle m, n_{\mathfrak{d}} \rangle}.$$

Let $e(n')$ be defined as above. Then we have $n_{\mathfrak{d}} = e(n_{\mathfrak{d}'})n_{\mathfrak{d}'}$ and the corresponding automorphism $\theta_{\mathfrak{d}'} \in \mathrm{Aut}_{\mathbb{C}[\![t]\!]}(\mathbb{C}[M']\widehat{\otimes}_{\mathbb{C}}\mathbb{C}[\![t]\!])$ defined by $\mathfrak{d}' \in \mathfrak{D}'_\infty$ is given by

$$\theta_{\mathfrak{d}'} : z^{m'} \mapsto z^{m'} f_{\mathfrak{d}}^{\langle m', n_{\mathfrak{d}} \rangle} = z^{m'} f_{\mathfrak{d}}^{e(n_{\mathfrak{d}'})\langle m', n_{\mathfrak{d}'} \rangle} = z^{m'} f_{\mathfrak{d}'}^{\langle m', n_{\mathfrak{d}'} \rangle}.$$

This shows that $f_{\mathfrak{d}'} = f_{\mathfrak{d}}^{e(n_{\mathfrak{d}'})}$. In particular, the initial functions f_1 and f_2 considered in the lattice M' are $f_1' = (1+tx)^{d_1 e(m_2^*)}$ and $f_2' = (1+ty)^{d_2 e(m_1^*)}$, where $x = z^{m_1}$ and $y = z^{m_2}$. These are the initial functions of the standard scattering diagram $\mathfrak{D}^{d_1 e(m_2^*), d_2 e(m_1^*)}$.

We know that scattering gives a consistent diagram $\mathfrak{D}_\infty^{d_1 e(m_2^*), d_2 e(m_1^*)}$. We get a consistent diagram containing $\mathfrak{D}$ by replacing any ray $\mathfrak{d}' \in \mathfrak{D}_\infty^{e(n_1), e(n_2)}$ by $\mathfrak{d}$ with function $f_{\mathfrak{d}} = f_{\mathfrak{d}'}^{1/e(n_{\mathfrak{d}'})}$. By uniqueness of consistent diagrams up to equivalence (Remark 2.5), this completes the proof. □

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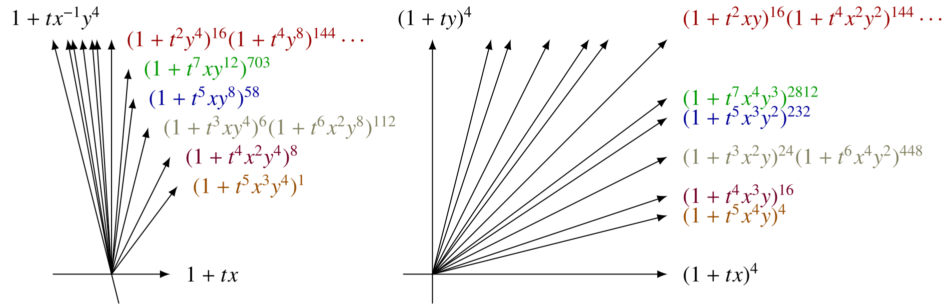

FIGURE 3. The diagrams $\mathfrak{D}^4_{\mathrm{det}}$ and $\mathfrak{D}^{4,4}$ to order seven.

EXAMPLE 2.17. Let $\mathfrak{D}^k_{\mathrm{det}}$ be the scattering diagram consisting of two lines with attached functions $f_1 = 1 + tx$ and $f_2 = 1 + tx^{-1}y^k$. Let M' be the sublattice generated by $m_1 = (1, 0)$ and $m_2 = (-1, k)$ and consider $m = am_1 + bm_2 \in M'$ primitive with dual $n \in N'$. Then

$$f_m^{\mathfrak{D}^k_{\mathrm{det}}} = (f^{k,k}_{(a,b)})^{\gcd(k,m_{(1)})/k},$$

where $m_{(1)}$ is the first component of $m \in M$. This is because we have $e(m_1^*) = e(m_2^*) = k$ and $e(n) = \gcd(k, m_{(1)})/k$. Figure 3 shows $\mathfrak{D}^k_{\mathrm{det}}$ and $\mathfrak{D}^{k,k}$ for $k = 4$ to t-order seven.

2.4. Mutations

DEFINITION 2.18. For $\mu, \nu \in \mathbb{Z}_{>0}$ define two *mutation* actions on $\mathbb{Z}^2$ by

$$\mathbf{T}_1^{\mu,\nu} : (a, b) \mapsto \begin{cases} (\mu b - a, b) & b > 0, \\ (a, b) & b \leq 0, \end{cases} \qquad \mathbf{T}_2^{\mu,\nu} : (a, b) \mapsto \begin{cases} (a, \nu a - b) & a > 0, \\ (a, b) & a \leq 0. \end{cases}$$

Here $\mathbb{Z}^2$ is the space of direction vectors (a, b) of rays in a scattering diagram, and (a, b) will usually be assumed to be primitive, that is, $\gcd(a, b) = 1$, and such that (a, b), $\mathbf{T}_1^{\mu,\nu}(a, b)$ and $\mathbf{T}_2^{\mu,\nu}(a, b)$ are all contained in the first quadrant $\mathbb{Z}^2_{\geq 0}$.

PROPOSITION 2.19 [9, Theorem 7] and [8, Theorem 1.24]. *We have*

$$f^{\mu,\nu}_{(a,b)} = f^{\mu,\nu}_{\mathbf{T}_1^{\mu,\nu}(a,b)} = f^{\mu,\nu}_{\mathbf{T}_2^{\mu,\nu}(a,b)}.$$

3. Proof of Theorem 1.4

Consider a standard scattering diagram $\mathfrak{D}^{\mu,\nu}$ (Definition 2.7). Mutations (Section 2.4) act on the directions $\mathbb{Z}^2$ (or slopes $\mathbb{Q}$). They have some fixed points and naturally divide the scattering diagram into certain *regions*. We will show the following. All rays produced from scattering have slope $1/\mu < \rho < \nu$ (Proposition 3.13). For $\mu\nu > 4$, there is a *dense region* $\Phi^{\mu,\nu}$ between slopes $\rho_-^{\mu,\nu}$ and $\rho_+^{\mu,\nu}$ (Definition 3.1) in which

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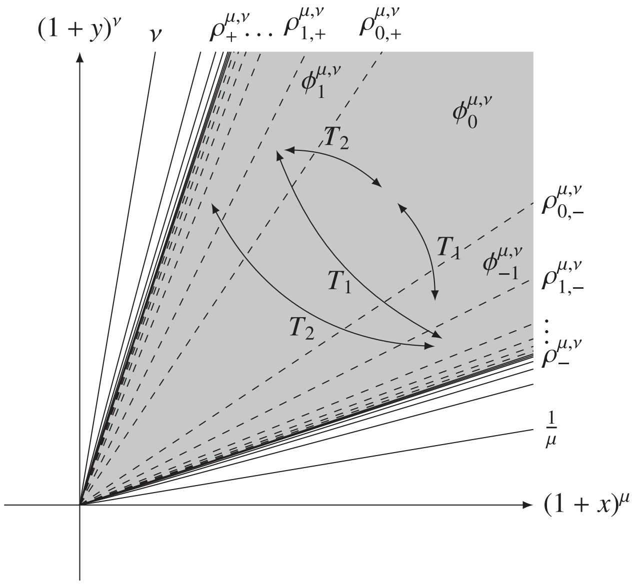

FIGURE 4. The regions of the scattering diagram.

every ray occurs with a nontrivial function (Theorem 3.3). It is made up of an infinite number of fundamental domains $\phi_k^{\mu,\nu}$ for the mutation action. The central domain $\phi_0^{\mu,\nu}$ lies between the slopes $\rho_{0,\pm}^{\mu,\nu}$ (Lemma 3.5). Outside the dense region there is a discrete number of rays and each of them appears with coefficients μ or ν, because they come from mutation of the initial rays (Proposition 3.14). The situation is summarised in Figure 4.

DEFINITION 3.1. In a standard scattering diagram $\mathfrak{D}^{\mu,\nu}$ where $\mu\nu > 4$, the *dense region* $\Phi^{\mu,\nu}$ is the cone spanned by the rays from the origin with slope

$$\rho_\pm^{\mu,\nu} = \frac{\mu\nu \pm \sqrt{\mu\nu(\mu\nu-4)}}{2\mu} = \frac{2\nu}{\mu\nu \mp \sqrt{\mu\nu(\mu\nu-4)}}.$$

A ray in direction $(a,b) \in \mathbb{Z}_{>0}^2$ is in the dense region if $\rho_-^{\mu,\nu} < b/a < \rho_+^{\mu,\nu}$.

In this section, we will prove the theorem stated below, by induction.

DEFINITION 3.2. A cone $\phi^{\mu,\nu} \subset \mathbb{R}_{>0}^2$ is *full* if $c_{a,b}^{\mu,\nu} \neq 0$ for every $(a,b) \in \mathbb{Z}_{>0}^2$ (not necessarily primitive) such that the ray in direction (a,b) lies in $\phi^{\mu,\nu}$.

THEOREM 3.3 (Theorem 1.4(a)). *$\Phi^{\mu,\nu}$ is full (and, in particular, dense with rays) when $\mu\nu > 4$.*

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3.1. Fundamental domains

DEFINITION 3.4. The *fundamental region* $\phi_0^{\mu,\nu}$ is the cone between the directions

$$\rho_{0,+}^{\mu,\nu} = \frac{\nu}{2}, \quad \rho_{0,-}^{\mu,\nu} = \frac{2}{\mu}.$$

A ray in direction $(a, b) \in \mathbb{Z}_{>0}^2$ is in $\phi_0^{\mu,\nu}$ if $2/\mu \le b/a \le \nu/2$. For $k > 0$ and $k < 0$, define recursively

$$\rho_{k+1,+}^{\mu,\nu} = \mathbf{T}_1^{\mu,\nu}(\rho_{k,-}^{\mu,\nu}), \quad \rho_{k-1,-}^{\mu,\nu} = \mathbf{T}_2^{\mu,\nu}(\rho_{k,+}^{\mu,\nu}).$$

LEMMA 3.5. *The actions of* $\mathbf{T}_1^{\mu,\nu}$ *and* $\mathbf{T}_2^{\mu,\nu}$ *on the slope* $\rho = b/a$ *are order reversing (or strictly decreasing) for* $1/\mu < \rho < \nu$ *and have fixed points* $\rho_{0,-}^{\mu,\nu} = 2/\mu$ *and* $\rho_{0,+}^{\mu,\nu} = \nu/2$, *respectively.*

PROOF. This is clear from the definition (see Definition 2.18). The actions on the slope are given by

$$\mathbf{T}_1(\rho) = \frac{1}{\mu - 1/\rho}, \quad \mathbf{T}_2(\rho) = \nu - \rho.$$ □

LEMMA 3.6. *We have* $\mathbf{T}_1^{\mu,\nu}(\rho_+^{\mu,\nu}) = \rho_-^{\mu,\nu}$ *and* $\mathbf{T}_2^{\mu,\nu}(\rho_-^{\mu,\nu}) = \rho_+^{\mu,\nu}$.

PROOF. The second statement is

$$\mathbf{T}_2^{\mu,\nu}(\rho_-^{\mu,\nu}) = \nu - \frac{\mu\nu - \sqrt{\mu\nu(\mu\nu-4)}}{2\mu} = \frac{\mu\nu + \sqrt{\mu\nu(\mu\nu-4)}}{2\mu} = \rho_+^{\mu,\nu}.$$

For the first statement, we show that the reciprocals are equal,

$$\frac{1}{\mathbf{T}_1^{\mu,\nu}(\rho_+^{\mu,\nu})} = \mu - \frac{\mu\nu - \sqrt{\mu\nu(\mu\nu-4)}}{2\nu} = \frac{\mu\nu + \sqrt{\mu\nu(\mu\nu-4)}}{2\nu} = \frac{1}{\rho_-^{\mu,\nu}}.$$ □

LEMMA 3.7. *We have* $\lim_{k\to\infty} \rho_{k,\pm}^{\mu,\nu} = \rho_\pm^{\mu,\nu}$.

PROOF. The compositions $\mathbf{T}_1^{\mu,\nu}\mathbf{T}_2^{\mu,\nu}$ and $\mathbf{T}_2^{\mu,\nu}\mathbf{T}_1^{\mu,\nu}$ are strictly increasing continuous functions for $1/\mu < \rho < \nu$. Hence, they have unique fixed points, which, by Lemma 3.6, are given by $\rho_+^{\mu,\nu}$ and $\rho_-^{\mu,\nu}$. The limit of the recursively defined sequences $\rho_{k,-}^{\mu,\nu}$ and $\rho_{k,+}^{\mu,\nu}$ have to be fixed points of $\mathbf{T}_1^{\mu,\nu}\mathbf{T}_2^{\mu,\nu}$ and $\mathbf{T}_2^{\mu,\nu}\mathbf{T}_1^{\mu,\nu}$, respectively. Hence, they are given by $\rho_-^{\mu,\nu}$ and $\rho_+^{\mu,\nu}$. □

LEMMA 3.8. *If* $\phi_0^{\mu,\nu}$ *is full, then so is* $\Phi^{\mu,\nu}$.

PROOF. By Lemmas 3.5, 3.6 and 3.7, $\Phi^{\mu,\nu}$ is the union of images of $\phi_0^{\mu,\nu}$ under repeated application of $\mathbf{T}_1^{\mu,\nu}$ and $\mathbf{T}_2^{\mu,\nu}$. This proves the claim. □

3.2. Induction step

LEMMA 3.9. *If* $\mu\nu > 4$ *and* $\mu > 1$, *then* $\phi_0^{\mu+1,\nu}$ *is contained in* $\Phi^{\mu,\nu}$. *Similarly, if* $\mu\nu > 4$ *and* $\nu > 1$, *then* $\phi_0^{\mu,\nu+1}$ *is contained in* $\Phi^{\mu,\nu}$.

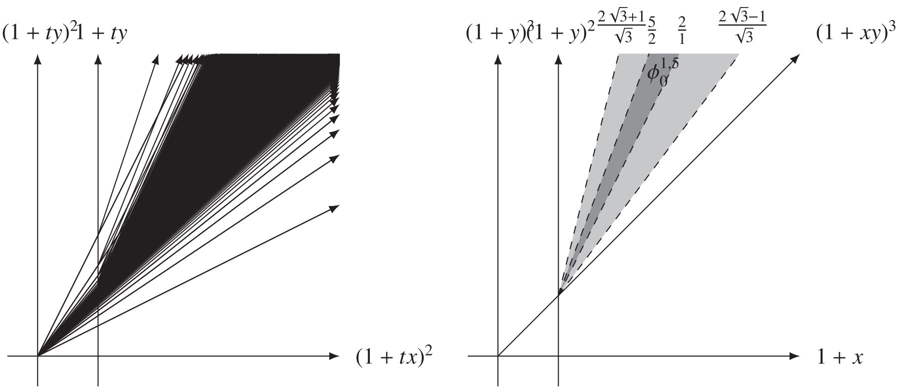

FIGURE 5. Partial deformations showing that $\phi_0^{2,3}$ (left) and $\phi_0^{1,5}$ (right) are full.

PROOF. By symmetry, we only have to show the second statement, which is equivalent to $\rho_{0,+}^{\mu,\nu+1} < \rho_+^{\mu,\nu}$ and $1/\rho_{0,-}^{\mu,\nu+1} < 1/\rho_-^{\mu,\nu}$. The second inequality is

$$\frac{\mu}{2} < \frac{\mu + \sqrt{\mu^2 - 4\frac{\mu}{\nu}}}{2} \iff 0 < \mu^2 - 4\frac{\mu}{\nu} \iff \mu\nu > 4.$$

The first inequality is

$$\frac{\nu+1}{2} < \frac{\nu + \sqrt{\nu^2 - 4\frac{\nu}{\mu}}}{2} \iff 1 < \nu^2 - 4\frac{\nu}{\mu} \iff \mu > \frac{4\nu}{\nu^2 - 1}.$$

For $\nu \geq 5$, the right-hand side is smaller than one, and hence any $\mu \in \mathbb{Z}_{>0}$ satisfies the inequality. For $\nu = 4, 3, 2$, the inequality is satisfied by all $\mu \geq 2, 2, 3$, respectively. Hence, all $\mu, \nu \in \mathbb{Z}_{>0}$ satisfying $\mu\nu > 4$ and $\nu > 1$ also satisfy this inequality. This completes the proof. □

PROPOSITION 3.10. *If $\Phi^{2,3}$ and $\Phi^{1,5}$ are full, then $\Phi^{\mu,\nu}$ is full for all $\mu\nu > 4$.*

PROOF. Start with a pair (μ, ν) with $\mu\nu > 4$. With $\mu\nu > 4$ do the following. If $\mu \geq \nu$, replace (μ, ν) by $(\mu - 1, \nu)$. Otherwise, replace (μ, ν) by $(\mu, \nu - 1)$. Doing this repeatedly we eventually arrive at $(1, 5)$ or $(5, 1)$ or $(2, 3)$ or $(3, 2)$. Note that, if $\mu\nu > 4$ and $\mu \geq \nu$, then $\mu > 1$. Similarly, if $\mu\nu > 4$ and $\nu > \mu$, then $\nu > 1$. By Lemmas 3.8 and 3.9, we conclude that $\Phi^{\mu,\nu}$ is full if $\Phi^{1,5}$, $\Phi^{5,1}$, $\Phi^{2,3}$ and $\Phi^{3,2}$ are full. By Remark 2.10, $\Phi^{5,1}$ is full if and only if $\Phi^{1,5}$ is full and $\Phi^{3,2}$ is full if and only if $\Phi^{2,3}$ is full. Hence, the statement follows. □

3.3. The base cases

LEMMA 3.11. *$\Phi^{2,3}$ is full.*

PROOF. Consider a partial deformation of $\mathfrak{D}^{2,3}$ into $\mathfrak{D}^{2,2}$ and $\mathfrak{D}^{2,1}$ by pulling out a vertical line (see Figure 5). In $\mathfrak{D}^{2,2}$, for every $n \in \mathbb{N}$, there is a ray in direction $(n, n+1)$

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with function $(1+t^{2n+1}x^ny^{n+1})^2$ (see Example 2.11). This intersects the pulled out vertical line, which has function $1+ty$. By the change of lattice trick (Proposition 2.16), the diagram at the intersection corresponds to a standard diagram $\mathfrak{D}^{2n,n}$. By a full deformation, one easily sees that $\mathfrak{D}^{2n,n}$ has rays in directions $(1,1),(1,2),\ldots,(1,n)$, which correspond to rays in $\mathfrak{D}^{2,3}$ with directions $(n,n+2),(n,n+3),\ldots,(n,2n+1)$. Hence, for each direction (a,b) with $a \le b \le 2a+1$, there exists a nontrivial ray in $\mathfrak{D}^{3,2}$. These include all directions with $1 \le b/a \le 2$. But this contains the fundamental region $\phi_0^{3,2}$, which lies between $\rho_{0,-}^{3,2}=1$ and $\rho_{0,+}^{3,2}=\frac{3}{2}$. Therefore, $\phi_0^{3,2}$ is full, and so, by Lemma 3.8, $\Phi^{3,2}$ is full. □

LEMMA 3.12. *$\Phi^{1,5}$ is full.*

PROOF. Consider a partial deformation of $\mathfrak{D}^{1,5}$ to $\mathfrak{D}^{1,3}$ and $\mathfrak{D}^{1,2}$ (see Figure 5). As $c_{1,1}^{1,3}=3$ (see Example 2.14), we get a subdiagram corresponding to $\mathfrak{D}^{3,2}$, with rays (a,b) in $\mathfrak{D}^{3,2}$ corresponding to rays $(a,a+b)$ in $\mathfrak{D}^{1,5}$. This maps the slope $\rho \mapsto \rho+1$, so sends the dense region $\Phi^{3,2}$ between

$$\rho_{\pm}^{3,2} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{1}{\sqrt{3}}$$

to the region between $2 \pm 1/\sqrt{3}$ in $\mathfrak{D}^{1,5}$. This contains the fundamental region $\phi_0^{1,5}$ which lies between $\rho_{0,-}^{1,5}=2$ and $\rho_{0,+}^{1,5}=\frac{5}{2}$. So $\phi_0^{1,5}$ is full, and hence $\Phi^{1,5}$ is full by Lemma 3.8. □

Now Theorem 3.3 follows from Lemmas 3.11, 3.12 and Proposition 3.10.

3.4. Outside the dense region

PROPOSITION 3.13. *In a standard scattering diagram $\mathfrak{D}^{\mu,\nu}$, every ray with direction $(a,b)\in\mathbb{Z}_{>0}^2$ satisfies*

$$\frac{1}{\mu} \le \frac{b}{a} \le \nu.$$

PROOF. We show that $b/a \le \nu$. Then $1/\mu \le b/a$ follows by symmetry under exchanging (μ,a) and (ν,b). For $\nu=1$, a full deformation of $\mathfrak{D}^{\mu,1}$ shows that only rays with slope $b/a \le 1$ appear. Hence, we can assume that $\nu>1$. We proceed by induction on $a+b$. The statement is clear for $a+b=1$. For $a+b=2$, we have $a=b=1$, so $b/a=1$, and the statement is also true. For $a+b>2$, consider the partial deformation of $\mathfrak{D}^{\mu,\nu}$ into $\mathfrak{D}^{\mu,\nu-1}$ and $\mathfrak{D}^{\mu,1}$ by pulling out a vertical line. Consider the ray in $\mathfrak{D}^{\mu,\nu-1}$ with direction $(a_0,b_0)\in\mathbb{Z}_{>0}^2$. Its attached function is $(1+t^{a_0+b_0}x^{a_0}y^{b_0})^{c_{a_0,b_0}^{\mu,\nu-1}}$. It intersects the pulled-out vertical line, which has function $1+ty$. By the change of lattice trick (Example 2.17), the diagram at the intersection point is equivalent to $\mathfrak{D}^{a_0c_{a_0,b_0}^{\mu,\nu-1},a_0}$. It produces rays with directions $(a,b)=\alpha(a_0,b_0)+\beta(0,1)$ for some $\alpha,\beta\in\mathbb{Z}_{>0}$. We have

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$\alpha \le a$ and $\beta < b$, so $\alpha + \beta < a + b$ and $a_0 + b_0 < a + b$. By the induction hypothesis, we have $\beta/\alpha \le a_0$ and $b_0/a_0 \le \nu - 1$. Then

$$\frac{b}{a} = \frac{\alpha b_0 + \beta}{\alpha a_0} = \frac{b_0}{a_0} + \frac{\beta}{\alpha}\frac{1}{a_0} \le \nu.$$

This completes the proof. □

PROPOSITION 3.14 (Theorem 1.4(b)). *Outside $\Phi^{\mu,\nu}$, the only rays that occur are those given by mutations of the initial rays. In particular, rays cannot be dense.*

PROOF. Let $\alpha_0 = 0, \alpha_1 = 1/\mu, \alpha_{n+1} = \mathbf{T}_2(\beta_n)$ and $\beta_0 = \infty, \beta_1 = \nu, \beta_{n+1} = \mathbf{T}_1(\alpha_n)$.

We know that there are no rays with slope $\alpha_0 < \rho < \alpha_1$ or $\beta_0 > \rho > \beta_1$, and, under mutations, if there are no rays with slope $\alpha_{n-1} < \rho < \alpha_n$ or $\beta_{n-1} > \rho > \beta_n$, then there are none with slope $\alpha_n < \rho < \alpha_{n+1}$ or $\beta_n > \rho > \beta_{n+1}$.

Note also that $\alpha_0 < \rho_-^{\mu,\nu}$ and $\beta_0 > \rho_+^{\mu,\nu}$ and that $\mathbf{T}_1, \mathbf{T}_2 : \rho_\pm^{\mu,\nu} \mapsto \rho_\mp^{\mu,\nu}$ are order reversing, so $\alpha_n < \rho_-^{\mu,\nu}$ and $\beta_n > \rho_+^{\mu,\nu}$. So we get bounded monotone sequences α_n, β_n and they converge to α, β, respectively. As $\mathbf{T}_2\mathbf{T}_1$ is continuous and maps α_n to α_{n+2} and β_n to β_{n+2}, respectively, α, β must be fixed points of $\mathbf{T}_2\mathbf{T}_1$. But the fixed points of

$$\mathbf{T}_2\mathbf{T}_1 : \rho \mapsto \nu - \frac{1}{\mu - 1/\rho}$$

are exactly $\rho_\pm^{\mu,\nu}$. So $\alpha = \rho_-^{\mu,\nu}$ and $\beta = \rho_+^{\mu,\nu}$, and we get the claim. □

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References

[1] H. Argüz and M. Gross, 'The higher-dimensional tropical vertex', *Geom. Topol.* **26** (2022), 2135–2235.

[2] P. Bousseau, 'The quantum tropical vertex', *Geom. Topol.* **24**(3) (2020), 1297–1379.

[3] P. Bousseau, 'Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$', *J. Algebraic Geom.* **31** (2022), 593–686.

[4] P. Bousseau, A. Brini and M. van Garrel, 'Stable maps to Looijenga pairs', *Geom. Topol.* **28** (2024), 393–496.

[5] T. Bridgeland, 'Scattering diagrams, Hall algebras and stability conditions', *Algebr. Geom.* **4**(5) (2017), 523–561.

[6] T. Gräfnitz, 'Tropical correspondence for smooth del Pezzo log Calabi–Yau pairs', *J. Algebraic Geom.* **31**(4) (2022), 687–749.

[7] M. Gross, P. Hacking and S. Keel, 'Mirror symmetry for log Calabi–Yau surfaces I', *Publ. Math. Inst. Hautes Études Sci.* **122** (2015), 65–168.

[8] M. Gross, P. Hacking, S. Keel and M. Kontsevich, 'Canonical bases for cluster algebras', *J. Amer. Math. Soc.* **31**(2) (2018), 497–608.

https://doi.org/10.1017/S000497272400128X Published online by Cambridge University Press

[9] M. Gross and R. Pandharipande, 'Quivers, curves, and the tropical vertex', *Port. Math.* **67**(2) (2010), 211–259.
[10] M. Gross, R. Pandharipande and B. Siebert, 'The tropical vertex', *Duke Math. J.* **153**(2) (2010), 297–362.
[11] M. Gross and B. Siebert, 'From real affine geometry to complex geometry', *Ann. of Math. (2)* **174**(3) (2011), 1301–1428.
[12] M. Gross and B. Siebert, 'The canonical wall structure and intrinsic mirror symmetry', *Invent. Math.* **229** (2022), 1101–1202.
[13] M. Kontsevich and Y. Soibelman, 'Affine structures and non-Archimedean analytic spaces', in: *The Unity of Mathematics*, Progress in Mathematics, 244 (eds. P. Etingof, V. Retakh and I. M. Singer) (Birkhäuser, Boston, MA, 2006), 321–385.
[14] M. Kontsevich and Y. Soibelman, 'Stability structures, motivic Donaldson–Thomas invariants and cluster transformations', Preprint, 2008, arXiv:0811.2435.
[15] M. Reineke, 'The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli', *Invent. Math.* **152** (2003), 349–368.
[16] M. Reineke, 'Poisson automorphisms and quiver moduli', *J. Inst. Math. Jussieu* **9**(3) (2010), 653–667.
[17] M. Reineke, 'Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants', *Compos. Math.* **147**(3) (2011), 943–964.
[18] M. Reineke and T. Weist, 'Refined GW/Kronecker correspondence', *Math. Ann.* **355** (2013), 17–56.

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