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The fragmentation equation with size diffusion: Well posedness and long-term behaviour

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The dynamics of the fragmentation equation with size diffusion is investigated when the size ranges in $(0,\infty)$. The associated linear operator involves three terms and can be seen as a nonlocal perturbation of a Schrödinger operator. A Miyadera perturbation argument is used to prove that it is the generator of a positive, analytic semigroup on a weighted L_1-space. Moreover, if the overall fragmentation rate does not vanish at infinity, then there is a unique stationary solution with given mass. Assuming further that the overall fragmentation rate diverges to infinity for large sizes implies the immediate compactness of the semigroup and that it eventually stabilizes at an exponential rate to a one-dimensional projection carrying the information of the mass of the initial value.

Keywords: Fragmentation, size diffusion, well posedness, convergence, semigroup, perturbation

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1 Introduction

The well posedness of the fragmentation equation with size diffusion

$$\partial_t\phi(t,x) = D\partial_x^2\phi(t,x) - a(x)\phi(t,x) + \int_x^\infty a(y)b(x,y)\phi(t,y)\,\mathrm{d}y\,, \qquad (t,x)\in(0,\infty)^2\,, \tag{1.1a}$$

$$\phi(t,0)=0\,, \qquad t>0\,, \tag{1.1b}$$

$$\phi(0,x)=f(x)\,, \qquad x\in(0,\infty)\,, \tag{1.1c}$$

along with the long-term behaviour of its solutions is investigated by a semigroup approach. In (1.1), $\phi=\phi(t,x)\geqslant 0$ denotes the size distribution function of particles of size $x\in(0,\infty)$ at time $t>0$, while $a(x)\geqslant 0$ is the overall fragmentation rate of particles of size x, and $b(x,y)$ is the daughter distribution function that describes the distribution of fragments resulting from the breakup of a particle of size y. Besides undergoing fragmentation events, particles are also assumed to modify their size by diffusion at a constant diffusion rate $D>0$. Finally, nucleation is not taken into account in this model that leads to the homogeneous boundary condition (1.1b) at $x=0$.

An interplay between diffusion and fragmentation as depicted by (1.1) is met in the growth of ice crystals, see [16, 27]. Indeed, on the one hand, ice crystals grow or shrink in a way which looks like diffusion and break apart due to internal stresses, the latter process being referred to as *polygonization* or *rotation recrystallization* in ice physics. The fragmentation equation with size diffusion (1.1) is also derived in [18] to describe the growth of microtubules. In the absence of diffusion (i.e. $D=0$), equation (1.1) is the spontaneous fragmentation equation which has a long and rich history and has been extensively studied in the mathematical and physical literature since the pioneering works [17, 28, 32], see [5, 7–10], and the references therein.

An important role is played in the dynamics by the total mass of the particles' distribution

$$M_1(\phi(t)) := \int_0^\infty x\phi(t,x)\,\mathrm{d}x\,, \qquad t \geqslant 0\,,$$

which is expected to be conserved throughout time evolution when there is no loss of matter during fragmentation events; that is, when b satisfies

$$\int_0^y xb(x,y)\,\mathrm{d}x = y\,, \qquad y \in (0,\infty)\,.$$

Thus, $X_1 := L_1((0,\infty), x\mathrm{d}x)$ is a natural functional framework for the study of the fragmentation operator. We further observe that the homogeneous Dirichlet boundary condition (1.1b) corresponds actually to a no-flux boundary condition for the Laplace operator in X_1, so that this space turns out to be also well suited for diffusion. However, the analysis already performed on the fragmentation equation without diffusion reveals that a complete scale of weighted L_1-spaces is needed besides X_1. In this regard, we introduce the spaces

$$X_m := L_1((0,\infty), x^m\mathrm{d}x) \quad \text{and} \quad X_{1,m} := X_1 \cap X_m$$

for $m \in \mathbb{R}$. We denote the positive cone of $X_{1,m}$ by $X_{1,m}^+$. For $f \in X_m$ and $m \in \mathbb{R}$, we also define the moment $M_m(f)$ of order m of f by

$$M_m(f) := \int_0^\infty x^m f(x)\,\mathrm{d}x\,,$$

so that $\|f\|_{X_m} = M_m(|f|)$. For definiteness, we equip $X_{1,m}$ with the norm

$$\|\cdot\|_{X_{1,m}} := \|\cdot\|_{X_1} + \|\cdot\|_{X_m}$$

and note that $X_1 \doteq X_{1,1}$. Finally, we set $BC([0,\infty)) := C([0,\infty)) \cap L_\infty(0,\infty)$ and recall that $C_c^\infty((0,\infty))$ stands for the space of C^∞-smooth functions on $(0,\infty)$ with compact support in $(0,\infty)$.

Our strategy to study the well posedness and the long-term behaviour of (1.1) is to write it as an abstract Cauchy problem in $X_{1,m}$ for $m \geqslant 1$ and show that the corresponding operator generates a semigroup with properties depending on m, a, and b. To this end, we assume throughout the paper that

$$a \in L_{\infty,loc}([0,\infty))\,, \qquad a \geqslant 0 \text{ a.e. in } (0,\infty)\,, \tag{1.2}$$

and that the daughter distribution function b is a nonnegative measurable function on $(0,\infty)^2$ satisfying

$$\int_0^y xb(x,y)\,\mathrm{d}x = y\,, \qquad y \in (0,\infty)\,. \tag{1.3}$$

Moreover, the diffusion rate D is normalized to $D=1$.

For $m \geqslant 1$ we then define the (Schrödinger) operator $A_{a,m}$ on $X_{1,m}$ by

$$\begin{aligned} \mathrm{dom}(A_{a,m}) &:= \{f \in X_{1,m} \ : \ f'' \in X_{1,m}\,,\ af \in X_{1,m}\,,\ f(0)=0\}\,, \\ A_{a,m}f &:= f'' - af\,, \qquad f \in \mathrm{dom}(A_{a,m})\,, \end{aligned} \tag{1.4}$$

as well as the nonlocal operator B_m on $X_{1,m}$ by

$$\begin{aligned} \mathrm{dom}(B_m) &:= \{f \in X_{1,m} \ : \ af \in X_{1,m}\} \subset \mathrm{dom}(A_{a,m})\,, \\ B_m f(x) &:= \int_x^\infty a(y)b(x,y)f(y)\,\mathrm{d}y\,, \quad x \in (0,\infty)\,, \qquad f \in \mathrm{dom}(B_m)\,. \end{aligned} \tag{1.5}$$

Owing to (1.3) the operator B_m turns out to be well defined, see Lemma 5.1. Setting

$$\mathbb{A}_m := A_{a,m} + B_m \ \text{ with } \ \mathrm{dom}(\mathbb{A}_m) := \mathrm{dom}(A_{a,m})\,, \tag{1.6}$$

equation (1.1) can be equivalently formulated as the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi = \mathbb{A}_m\phi\,, \quad t>0\,, \qquad \phi(0)=f\,, \tag{1.7}$$

in $X_{1,m}$, and we shall investigate generation properties of the operator $\mathbb{A}_m$.

For concise statements we introduce the following notation: Given $\kappa \geqslant 1$ and $\omega \in \mathbb{R}$, we write $A \in \mathcal{G}(X_{1,m}, \kappa, \omega)$ if the (unbounded) linear operator A on $X_{1,m}$ is the generator of a strongly continuous semigroup $(e^{tA})_{t\geqslant 0}$ on $X_{1,m}$ and

$$\|e^{tA}\|_{\mathcal{L}(X_{1,m})} \leqslant \kappa e^{\omega t}\,, \qquad t \geqslant 0\,.$$

We set

$$\mathcal{G}(X_{1,m}) := \bigcup_{\kappa\geqslant 1\,,\,\omega\in\mathbb{R}} \mathcal{G}(X_{1,m}, \kappa, \omega)\,.$$

Moreover, we write $A \in \mathcal{G}_+(X_{1,m})$ if the semigroup $(e^{tA})_{t\geqslant 0}$ is positive on the Banach lattice $X_{1,m}$. We denote the domain of the (unbounded) operator A in $X_{1,m}$ by $\mathrm{dom}(A)$ and set

$$D(A) := \big(\mathrm{dom}(A), \|\cdot\|_A\big)\,,$$

where $\|f\|_A := \|f\|_{X_{1,m}} + \|Af\|_{X_{1,m}}$ for $f \in \mathrm{dom}(A)$ is the graph norm. Finally, we write $A \in \mathcal{H}(X_{1,m})$ if $A \in \mathcal{G}(X_{1,m})$ and the semigroup $(e^{tA})_{t\geqslant 0}$ is analytic.

With this notation we may formulate the generation result in $X_{1,m}$ for $m \geqslant 1$:

Theorem 1.1 *Assume that a and b satisfy* (1.2) *and* (1.3).

(a) *There is an extension* $\tilde{\mathbb{A}}_1 \in \mathcal{G}_+(X_1, 1, 0)$ *of* $\mathbb{A}_1$.

(b) *Assume further that there is* $\delta_2 \in (0,1)$ *such that*

$$(1-\delta_2)y^2 \geqslant \int_0^y x^2 b(x,y)\,\mathrm{d}x\,, \qquad y \in (0,\infty)\,. \tag{1.8}$$

If $m>1$, *then* $\mathbb{A}_m \in \mathcal{G}_+(X_{1,m}) \cap \mathcal{H}(X_{1,m})$. *In addition, for all* $f \in X_{1,m}$,

$$M_1\left(e^{t\mathbb{A}_m}f\right) = M_1(f)\,, \qquad t \geqslant 0\,. \tag{1.9}$$

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(c) *Assume that a satisfies*

$$\lim_{x\to\infty} a(x) = \infty \tag{1.10}$$

and that b satisfies (1.8). *Then,* $(e^{t\mathbb{A}_m})_{t\geqslant 0}$ *is immediately compact on* $X_{1,m}$ *for* $m > 1$.

Assumption (1.8) is commonly encountered in the investigation of the fragmentation equation and somehow excludes the concentration of b along the diagonal. Recall that the exponent 2 in (1.8) plays no particular role and may be replaced by any exponent $p > 1$, since (1.8) is equivalent to the existence of $p > 1$ and $\delta_p \in (0, 1)$ such that

$$(1-\delta_p)y^p \geqslant \int_0^y x^p b(x,y)\,\mathrm{d}x\,, \qquad y\in(0,\infty)\,,$$

see [8, Theorem 5.1.47 (c)].

At this stage, the extension $\tilde{\mathbb{A}}_1$ of $\mathbb{A}_1$ is not completely identified. In particular, we do not know whether or not $\tilde{\mathbb{A}}_1$ coincides with $\mathbb{A}_1$. Still, it is a question worth of investigation and we refer to [8] for a thorough discussion of this issue for the fragmentation equation without diffusion. Anyway, a positive answer is straightforward when $a \in L_\infty(0,\infty)$ and reported in the next result.

Proposition 1.2 *Assume that* $a \in L_\infty(0,\infty)$ *is nonnegative and that b satisfies* (1.3).

(a) *For* $m \geqslant 1$, $\mathbb{A}_m \in \mathcal{G}_+(X_{1,m}) \cap \mathcal{H}(X_{1,m})$ *and* (1.9) *is satisfied. In addition,* $\mathbb{A}_1 \in \mathcal{G}_+(X_1, 1, 0)$.

(b) *The semigroup* $(e^{t\mathbb{A}_m})_{t\geqslant 0}$ *is not compact on* $X_{1,m}$.

We immediately obtain the well posedness of the Cauchy problem (1.7) in $X_{1,m}$ and, equivalently, of (1.1) in a classical sense. Since we shall see that $D(\mathbb{A}_m) \doteq D(A_{a,m})$, we can formulate the result as follows:

Corollary 1.3 *Assume that a and b satisfy* (1.2) *and* (1.3). *Assume further that, either* $m = 1$ *and* $a \in L_\infty(0,\infty)$, *or* $m > 1$ *and b satisfies* (1.8). *Then, for any* $f \in X_{1,m}$, *there is a unique classical solution*

$$\phi \in C([0,\infty), X_{1,m}) \cap C^1((0,\infty), X_{1,m}) \cap C((0,\infty), D(A_{a,m}))$$

to (1.1) *which is given by* $\phi(t) = e^{t\mathbb{A}_m} f$ *for* $t \geqslant 0$ *and satisfies*

$$M_1(\phi(t)) = M_1(f)\,, \qquad t \geqslant 0\,. \tag{1.11}$$

Moreover, if $f \geqslant 0$ *a.e. in* $(0,\infty)$, *then* $\phi(t) \geqslant 0$ *for all* $t > 0$.

Remark 1.4 Given $f \in X_{1,m}^+$, the corresponding solution ϕ to (1.1) provided by Corollary 1.3 satisfies the mass conservation (1.11), a feature which is in particular due to the assumed boundedness (1.2) of a on (0,1). Indeed, even when b satisfies (1.3), infringement of mass conservation is known to occur when the overall fragmentation a is unbounded for small sizes. In that case, the total mass is a decreasing function of time, a phenomenon usually referred to as shattering which is closely related to the honesty property of the associated semigroup, see [4, 8, 11, 17, 21, 22, 28]. In fact, shattering takes place as soon as $x \mapsto 1/(xa(x))$ fails to be integrable at $x = 0$.

Having settled the well posedness of (1.1), we next turn to qualitative properties of its dynamics. As a guideline, it was pointed out in [16] that the interplay between diffusion and fragmentation results in the stabilization of solutions to (1.1) to a stationary solution. This is in sharp contrast to the fragmentation equation without diffusion, since fragmentation is an irreversible process driving the particle distribution to zero. When diffusion is turned on, a closed-form stationary solution to (1.1) can be computed for the particular choice $a(x)=x$ and $b(x,y)=2y^{-1}\mathbf{1}_{(0,y)}(x)$, see [16]. The existence of stationary solutions is also established in [25] for an overall fragmentation rate a obeying a power law ($a(x)=x^\gamma$, $\gamma \geqslant 0$) and for a specific class of daughter distribution functions b. Here we extend this existence result to a broader class of fragmentation coefficients a and b, see Proposition 1.6. In addition, when a diverges to infinity as $x\to\infty$, we provide the exponential decay of the solution to (1.1) to the steady state with the total mass of the initial value.

Theorem 1.5 *Assume that a and b satisfy* (1.2), (1.3), (1.8), *and* (1.10), *and that* $a>0$ *and* $b>0$. *There is a unique nonnegative*

$$\psi_1 \in \bigcap_{r\geqslant 1} \mathrm{dom}(\mathbb{A}_r)$$

such that $M_1(\psi_1)=1$ *and* $\ker(\mathbb{A}_m)=\mathbb{R}\psi_1 := \{r\psi_1 \ :\ r\in\mathbb{R}\}$ *for every* $m>1$. *In particular, for* $m>1$, *the spectral bound* $s(\mathbb{A}_m)=0$ *is a dominant eigenvalue of* $\mathbb{A}_m$ *and there are* $N_m\geqslant 1$ *and* $\nu_m>0$ *such that, for all* $f\in X_{1,m}$,

$$\|e^{t\mathbb{A}_m}f - M_1(f)\psi_1\|_{X_{1,m}} \leqslant N_m e^{-\nu_m t}\|f\|_{X_{1,m}}\,, \quad t\geqslant 0\,.$$

It is worth pointing out that the stationary solution ψ_1 decays faster than algebraically at infinity, a property which is perfectly consistent with the exponentially decaying tail experimentally observed in [27]. Also, combining Theorem 1.5 with Lemma 2.1 below implies that $\psi_1\in X_r$ for all $r>-1$.

Theorem 1.5 provides a complete description of the long-term behaviour of solutions to (1.1) when a diverges to infinity as $x\to\infty$. However, the unboundedness of a at infinity is not a necessary condition for the existence of stationary solutions. In fact, when

$$a(x)=1\,, \qquad b(x,y)=\frac{2}{y}\mathbf{1}_{(0,y)}(x)\,, \qquad 0<x<y\,,$$

we notice that equation (1.1) has an explicit stationary solution $\psi_1(x)=xe^{-x}$, $x>0$.[1] This particular example is not peculiar, and we actually obtain the existence of stationary solutions to (1.1) as soon as there is a positive lower bound for a as $x\to\infty$.

Proposition 1.6 *Assume that a and b satisfy* (1.2), (1.3), *and* (1.8) *and that* $a>0$ *and* $b>0$. *Assume further that*

$$\alpha := \frac{1}{2}\liminf_{x\to\infty} a(x) \in (0,\infty)\,. \tag{1.12}$$

[1] We actually compute explicit stationary solutions to (1.11) when $a(x)=x^\gamma$ and $b(x,y)=(\nu+2)x^\nu y^{-\nu-1}$ for $\gamma\geqslant 0$, $\nu\in(-2,0]$ in [26].

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There is a unique nonnegative

$$\psi_1 \in \bigcap_{r \geqslant 1} \mathrm{dom}(\mathbb{A}_r)$$

such that $M_1(\psi_1) = 1$ *and* $\ker(\mathbb{A}_m) = \mathbb{R}\psi_1$ *for every* $m > 1$.

Let us mention here that there is no loss of generality in assuming the finiteness of $\liminf_{x\to\infty} a(x)$ in (1.12). Indeed, if $\liminf_{x\to\infty} a(x) = \infty$, then a satisfies (1.10), a situation which is dealt with in Theorem 1.5.

When a only satisfies (1.12), the associated semigroup $\left(e^{t\mathbb{A}_m}\right)_{t\geqslant 0}$ need not be compact, see Proposition 1.2. We thus take a different route to prove Proposition 1.6 by an approximation procedure. This approach does not allow us to retrieve information on the long-term behaviour and it is likely that, either a more precise study of the operator $\mathbb{A}_1$ or a different approach (such as the one developed in [29]) is required to fully identify the long-term behaviour when a only satisfies (1.12).

Let us end this introduction with a brief outline of the paper. Auxiliary results are gathered in the next section, which includes integrability properties of elements of $\mathrm{dom}(A_{0,1})$ on the one hand and a weighted version of Kato's inequality on the other hand. In Section 3, we recall some properties of the heat semigroup in the weighted L_1-space $X_{1,m}$ with $m \geqslant 1$, relying on the explicit representation formula which is available in that case. Section 4 is devoted to the Schrödinger operator $A_{a,m}$ and the associated absorption semigroup and is mostly a consequence of the thorough study performed in [3]. We then use a perturbation argument in Section 5 to study the full fragmentation-diffusion operator $\mathbb{A}_m = A_{a,m} + B_m$. On the one hand, for $m = 1$, the existence of an extension $\tilde{\mathbb{A}}_1 \in \mathcal{G}_+(X_1, 1, 0)$ of $\mathbb{A}_1$ is a consequence of [31]. On the other hand, if $m > 1$, then we can use a Miyadera perturbation technique to prove that $\mathbb{A}_m \in \mathcal{H}(X_{1,m})$. We recall that this approach has already proved successful for the fragmentation equation without size diffusion, see [6]. The remainder of the paper is then devoted to the long-term dynamics. As a preliminary step, we establish in Section 6 the immediate compactness of the semigroup in $X_{1,m}$ for $m > 1$ when a diverges to infinity as $x \to \infty$. We then construct in Section 7 a bounded convex subset of $X_{1,m}$ which is invariant with respect to the semigroup. This feature, along with the immediate compactness of the semigroup, implies the existence of at least one stationary solution for any given mass. After showing that this stationary solution is unique, we perform a detailed study of the spectrum of $\mathbb{A}_m$ and end up with the announced convergence at a (yet non-explicit) exponential rate. Building upon the analysis performed in Section 7, we turn to the proof of Proposition 1.6 in Section 8 which relies on an approximation procedure. Specifically, introducing $a_n(x) := a(x) + x/n$ for $x > 0$ and $n \geqslant 1$, we deduce from Theorem 1.5 that there is a unique nonnegative stationary solution $\psi_{1,n}$ to (1.1) with a_n instead of a. We then show that cluster points as $n \to \infty$ of this sequence are stationary solutions to (1.1).

2 Auxiliary results

According to the definition of $\mathrm{dom}(A_{0,1})$, an important role is played in the forthcoming analysis by functions $f \in X_1$ such that $f'' \in X_1$. We collect useful properties of this class of functions in the next lemma and show, in particular, that the boundary condition (1.1c) is well defined for such functions.

Lemma 2.1 *Consider $f \in X_1$ such that $f'' \in X_1$. Then $f \in BC([0,\infty)) \cap C^1((0,\infty))$, $f' \in L_1(0,\infty)$, and, for $x > 0$,*

$$|f(x)| \leqslant \|f''\|_{X_1}, \qquad x|f'(x)| \leqslant \|f''\|_{X_1}, \qquad \|f'\|_{L_1(0,\infty)} \leqslant \|f''\|_{X_1}. \tag{2.1a}$$

Moreover,

$$\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} xf'(x) = 0. \tag{2.1b}$$

In fact, f and f' are given by

$$f(x) = -\int_x^\infty f'(y)\,\mathrm{d}y, \quad f'(x) = -\int_x^\infty f''(y)\,\mathrm{d}y, \qquad x \in (0,\infty). \tag{2.1c}$$

Also, $f \in X_m$ for any $m \in (-1,1)$ and, for all $\varepsilon > 0$,

$$\|f\|_{X_m} \leqslant \frac{\varepsilon^{m+1}}{m+1}\|f''\|_{X_1} + \varepsilon^{m-1}\|f\|_{X_1}. \tag{2.2}$$

Equivalently,

$$\|f\|_{X_m} \leqslant \frac{2(1-m)^{(m-1)/2}}{m+1}\|f''\|_{X_1}^{(1-m)/2}\|f\|_{X_1}^{(m+1)/2}, \qquad m \in (-1,1). \tag{2.3}$$

Proof. Introducing

$$F(x) := \int_x^\infty (y-x)f''(y)\,\mathrm{d}y, \qquad x \in (0,\infty),$$

it follows from the integrability of f'' that

$$-\|f''\|_{X_1} \leqslant -\int_x^\infty (y-x)|f''(y)|\mathrm{d}y \leqslant F(x) \leqslant \int_x^\infty (y-x)|f''(y)|\mathrm{d}y \leqslant \|f''\|_{X_1},$$

so that $F(x)$ is well defined for $x \geqslant 0$. Moreover,

$$F \in BC([0,\infty)) \cap C^1((0,\infty)) \cap W^2_{1,loc}(0,\infty)$$

satisfies

$$F'(x) = -\int_x^\infty f''(y)\,\mathrm{d}y, \qquad F''(x) = f''(x), \qquad x \in (0,\infty), \tag{2.4}$$

and

$$\lim_{x\to\infty} xF'(x) = \lim_{x\to\infty} F(x) = 0. \tag{2.5}$$

In particular, we infer from (2.4) that there is $(\alpha,\beta) \in \mathbb{R}^2$ such that $(f-F)(x) = \alpha + \beta x$ for $x > 0$. Moreover, since $f \in X_1$, it follows from (2.5) that

$$\lim_{x\to\infty}\int_x^{x+1} |\alpha+\beta y|\,\mathrm{d}y \leqslant \lim_{x\to\infty}\int_x^{x+1} (|f(y)|+|F(y)|)\,\mathrm{d}y = 0,$$

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which readily gives $\alpha=\beta=0$ and $F=f$, thereby establishing (2.1), except for the limiting behaviour of f at infinity. To this end, we observe that, since $f\in L_\infty(0,\infty)$ and $f'\in L_1(0,\infty)$, a formula for $f(x)^2$ reads

$$f(x)^2=-2\int_x^\infty f(y)f'(y)\,\mathrm{d}y\,,\qquad x\geqslant 0\,.$$

We then deduce from (2.1a) that

$$f(x)^2\leqslant 2\int_x^\infty y|f'(y)||f(y)|\,\frac{\mathrm{d}y}{y}\leqslant\frac{2}{x^2}\|f''\|_{X_1}\int_x^\infty y|f(y)|\,\mathrm{d}y\,,$$

which implies that $xf(x)\to 0$ as $x\to\infty$ due to $f\in X_1$.

Next, let $\varepsilon>\delta>0$. It follows from (2.1c) that

$$\begin{aligned}\int_\delta^\infty x^m|f(x)|\,\mathrm{d}x&\leqslant\int_\delta^\varepsilon x^m|f(x)|\,\mathrm{d}x+\varepsilon^{m-1}\int_\varepsilon^\infty x\,|f(x)|\,\mathrm{d}x\\&\leqslant\int_\delta^\varepsilon x^m\int_x^\infty|f'(y)|\,\mathrm{d}y\mathrm{d}x+\varepsilon^{m-1}\|f\|_{X_1}\\&\leqslant\int_\delta^\varepsilon x^m\int_x^\varepsilon|f'(y)|\,\mathrm{d}y\mathrm{d}x+\frac{\varepsilon^{m+1}}{m+1}\int_\varepsilon^\infty|f'(y)|\,\mathrm{d}y+\varepsilon^{m-1}\|f\|_{X_1}\,.\end{aligned}$$

By Fubini's theorem,

$$\int_\delta^\varepsilon x^m\int_x^\varepsilon|f'(y)|\,\mathrm{d}y\mathrm{d}x=\frac{1}{m+1}\int_\delta^\varepsilon\left(y^{m+1}-\delta^{m+1}\right)|f'(y)|\,\mathrm{d}y\leqslant\frac{\varepsilon^{m+1}}{m+1}\int_0^\varepsilon|f'(y)|\,\mathrm{d}y\,.$$

Combining the above inequalities with (2.1a) gives

$$\int_\delta^\infty x^m|f(x)|\,\mathrm{d}x\leqslant\frac{\varepsilon^{m+1}}{m+1}\int_0^\infty|f'(y)|\,\mathrm{d}y+\varepsilon^{m-1}\|f\|_{X_1}\leqslant\frac{\varepsilon^{m+1}}{m+1}\|f''\|_{X_1}+\varepsilon^{m-1}\|f\|_{X_1}\,.$$

Letting $\delta\to 0$ completes the proof of (2.2). We finally optimize (2.2) with respect to $\varepsilon\in(0,\infty)$ to derive (2.3). □

We next state for the sake of completeness the density in $X_{1,m}$ of smooth functions with compact support in $(0,\infty)$, which is actually a straightforward consequence of the density of $C_c^\infty((0,\infty))$ in $L_1(0,\infty)$.

Lemma 2.2 *The space $C_c^\infty((0,\infty))$ is dense in $X_{1,m}$ for $m\geqslant 1$.*

We finally recall a variant of the celebrated inequality of Kato [23, Lemma A].

Lemma 2.3 *Let ℓ be a nonnegative function in $W^1_{\infty,loc}([0,\infty))$ with $\ell(0)=0$ and consider $f\in W^2_{1,loc}(0,\infty)\cap X_1$ such that $f'\ell'\in L_1(0,\infty)$ and $f''\in L_1((0,\infty),\ell(x)\mathrm{d}x)$. Then*

$$-\int_0^\infty\ell(x)\,\mathrm{sign}(f(x))f''(x)\,\mathrm{d}x\geqslant\int_0^\infty\ell'(x)\,|f|'(x)\,\mathrm{d}x\,.\qquad(2.6)$$

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Proof. For $\varepsilon \in (0,1)$, we define $\beta_\varepsilon \in W^2_{\infty,loc}(\mathbb{R})$ by $\beta_\varepsilon(0) = 0$ and

$$\beta_\varepsilon'(r) := \mathrm{sign}(r)\,, \quad |r| > \varepsilon\,, \quad \text{and} \quad \beta_\varepsilon'(r) := \frac{r}{\varepsilon}\,, \quad r \in [-\varepsilon, \varepsilon]\,.$$

Since $f \in X_1$, we note that

$$|\beta_\varepsilon'(f)| \leqslant \frac{|f|}{\varepsilon} \in X_1\,,$$

while the integrability assumptions on f and ℓ imply that

$$\begin{aligned}|(\ell f')(x)| &\leqslant |(\ell f')(1)| + \int_1^x |(\ell' f' + \ell f'')(y)|\,\mathrm{d}y \\ &\leqslant |(\ell f')(1)| + \|\ell' f'\|_{L_1(0,\infty)} + \|\ell f''\|_{L_1(0,\infty)}\,, \qquad x \geqslant 1\,.\end{aligned}$$

Next, integration by parts gives

$$-\int_0^\infty \ell(x)\, \beta_\varepsilon'(f(x)) f''(x)\,\mathrm{d}x = \int_0^\infty \left[\ell(x)\, \beta_\varepsilon''(f(x))|f'(x)|^2 + \ell'(x)\beta_\varepsilon'(f(x))f'(x)\right]\,\mathrm{d}x\,,$$

observing that the boundary terms vanish due to $\ell(0)=0$, $\ell f' \in L_\infty(1,\infty)$, and $\beta_\varepsilon'(f) \in L_1(1,\infty)$. We then deduce from the nonnegativity of β_ε'' on $\mathbb{R}$ that

$$-\int_0^\infty \ell(x)\, \beta_\varepsilon'(f(x)) f''(x)\,\mathrm{d}x \geqslant \int_0^\infty \ell'(x)\beta_\varepsilon'(f(x))f'(x)\,\mathrm{d}x\,.$$

Since $|\beta_\varepsilon(r) - |r|| \leqslant \varepsilon$ for $r \in \mathbb{R}$ and since β_ε' converges pointwise to the sign function in $\mathbb{R}$ as $\varepsilon \to 0$ with $|\beta_\varepsilon'| \leqslant 1$, we may pass to the limit as $\varepsilon \to 0$ in the previous inequality with the help of Lebesgue's dominated convergence theorem and the integrability properties of f' and f'' and find

$$-\int_0^\infty \ell(x)\, \mathrm{sign}(f(x)) f''(x)\,\mathrm{d}x \geqslant \int_0^\infty \ell'(x)\mathrm{sign}(f(x))f'(x)\,\mathrm{d}x\,.$$

We finally use the classical property $|f|' = \mathrm{sign}(f) f'$ a.e. in $(0,\infty)$, see [24, Chapter II, Theorem A.2], to complete the proof. □

3 The heat semigroup

It is well known, see [13, Section 3.4] for instance, that the solution to the heat equation with homogeneous Dirichlet boundary conditions at $x=0$,

$$\begin{aligned}\partial_t w - \partial_x^2 w &= 0\,, \qquad (t,x) \in (0,\infty)\times(0,\infty)\,,\\ w(t,0) &= 0\,, \qquad t \in (0,\infty)\,,\\ w(0,x) &= f(x)\,, \qquad x \in (0,\infty)\,,\end{aligned}$$

is given by the representation formula

$$w(t,x) = W(t)f(x) := \int_0^\infty [k(t,x-y) - k(t,x+y)]f(y)\,\mathrm{d}y\,, \quad (t,x) \in (0,\infty)^2\,, \tag{3.1}$$

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where

$$k(t,x) := \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}\,, \qquad (t,x) \in (0,\infty) \times \mathbb{R}\,. \tag{3.2}$$

Moreover, $(W(t))_{t \geqslant 0}$ (with $W(0) := \mathrm{id}_{L_1(0,\infty)}$) is a positive analytic semigroup of contractions on $L_1(0,\infty)$ with generator G given by

$$\begin{aligned} \mathrm{dom}(G) &:= \{f \in L_1(0,\infty)\ :\ f'' \in L_1(0,\infty)\ \text{ and } f(0) = 0\}\,,\\ Gf &:= f''\,, \qquad f \in \mathrm{dom}(G)\,. \end{aligned}$$

That is, $G \in \mathcal{G}_+(L_1(0,\infty),1,0) \cap \mathcal{H}(L_1(0,\infty))$ and $e^{tG} = W(t)$ for $t \geqslant 0$. We now consider this semigroup in the weighted space $X_{1,m}$. More precisely, for $m \geqslant 1$, we recall the definition (1.4) (with $a = 0$) of the unbounded operator $A_{0,m}$ on $X_{1,m}$, given by

$$\begin{aligned} \mathrm{dom}(A_{0,m}) &= \{f \in X_{1,m}\ :\ f'' \in X_{1,m}\ \text{ and } f(0) = 0\}\,,\\ A_{0,m} f &= f''\,, \qquad f \in \mathrm{dom}(A_{0,m})\,, \end{aligned} \tag{3.3}$$

and show that it is the generator of the heat semigroup in $X_{1,m}$.

Proposition 3.1 *Let $m \geqslant 1$. There is $\omega_m \geqslant 0$ such $A_{0,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m) \cap \mathcal{H}(X_{1,m})$ with $(0,\infty) \subset \rho(A_{0,m})$. The semigroup $\left(e^{tA_{0,m}}\right)_{t\geqslant 0}$ is given by*

$$e^{tA_{0,m}} f(x) = \int_0^\infty [k(t,x-y) - k(t,x+y)] f(y)\,\mathrm{d}y\,, \qquad (t,x) \in (0,\infty)^2\,, \tag{3.4}$$

for $f \in X_{1,m}$, where k is defined in (3.2). Moreover, $e^{tA_{0,m}} = e^{tA_{0,1}}|_{X_{1,m}}$ for $t \geqslant 0$ and

$$\left(\lambda - A_{0,m}\right)^{-1} = \left(\lambda - A_{0,1}\right)^{-1}|_{X_{1,m}}\,, \qquad \lambda > 0\,.$$

Two steps are needed to show Proposition 3.1. We first establish Proposition 3.1 for $m = 1$ and $m \geqslant 3$, see Lemma 3.2 below. An interpolation argument then completes the proof for $m \in (1,3)$.

Lemma 3.2 *Let $m \in \{1\} \cup (3,\infty)$. Then $A_{0,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m) \cap \mathcal{H}(X_{1,m})$ with*

$$\omega_1 := 0 \ \text{ and } \ \omega_m := 4^{1/(m-1)} m(m-3)^{(m-3)/(m-1)}\,, \qquad m \geqslant 3\,.$$

Moreover, $e^{tA_{0,m}} = e^{tA_{0,1}}|_{X_{1,m}}$ for $t \geqslant 0$, $(0,\infty) \subset \rho(A_{0,m})$, and

$$(\lambda - A_{0,m})^{-1} = (\lambda - A_{0,1})^{-1}|_{X_{1,m}}\,, \qquad \lambda > 0\,.$$

Proof. We first note that $A_{0,m}$ is a closed operator on $X_{1,m}$ and that its domain is dense in $X_{1,m}$ due to Lemmas 2.1 and 2.2. We divide the remainder of the proof into several steps.

Step 1. We first show the dissipativity of $A_{0,m} - \omega_m$ on $X_{1,m}$. To this end, let $\lambda > 0$ and $f \in \mathrm{dom}(A_{0,m})$. Using the inequality

$$|r - s| \geqslant \mathrm{sign}(r)(r-s)\,, \qquad (r,s) \in \mathbb{R}^2\,,$$

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along with the nonnegativity of λ and ω_m, gives

$$\|\lambda f-(A_{0,m}-\omega_m)f\|_{X_m}=\|(\lambda+\omega_m)f-A_{0,m}f\|_{X_m} \geqslant \int_0^\infty x^m\ \mathrm{sign}(f(x))[(\lambda+\omega_m)f(x)-f''(x)]\ \mathrm{d}x\,.$$

By Kato's inequality (2.6) (with $\ell(x)=x^m$), Lemma 2.1, and the boundary condition $f(0)=0$, we further obtain

$$\|\lambda f-(A_{0,m}-\omega_m)f\|_{X_m}\geqslant(\lambda+\omega_m)\|f\|_{X_m}+m\int_0^\infty x^{m-1}\ |f|'(x)\ \mathrm{d}x = (\lambda+\omega_m)\|f\|_{X_m}-m(m-1)\|f\|_{X_{m-2}}\,.$$

In particular, when $m=1$,

$$\|\lambda f-A_{0,1}f\|_{X_1}\geqslant\lambda\|f\|_{X_1}\,, \tag{3.5}$$

so that $A_{0,1}$ is dissipative on X_1. We next handle the case $m\geqslant 3$. Then $m-2\in[1,m)$ and we infer from Young's inequality that, for $\varepsilon>0$,

$$m(m-1)\|f\|_{X_{m-2}}\leqslant m(m-3)\varepsilon\|f\|_{X_m}+2m\varepsilon^{(3-m)/2}\|f\|_{X_1}\,.$$

Hence,

$$(\lambda+\omega_m)\|f\|_{X_m}\leqslant m(m-3)\varepsilon\|f\|_{X_m}+2m\varepsilon^{(3-m)/2}\|f\|_{X_1}+\|\lambda f-(A_{0,m}-\omega_m)f\|_{X_m}\,.$$

Combining (3.5) with this inequality gives

$$(\lambda+\omega_m)\|f\|_{X_{1,m}}\leqslant\|\lambda f-(A_{0,m}-\omega_m)f\|_{X_{1,m}} +m(m-3)\varepsilon\|f\|_{X_m}+2m\varepsilon^{(3-m)/2}\|f\|_{X_1}\,.$$

We now choose $\varepsilon=\varepsilon_m:=(2/(m-3))^{2/(m-1)}$. Since

$$\omega_m=m(m-3)\varepsilon_m=m\varepsilon_m^{(3-m)/2}\,,$$

we readily conclude

$$\lambda\|f\|_{X_{1,m}}\leqslant\|\lambda f-(A_{0,m}-\omega_m)f\|_{X_{1,m}}\,,$$

so that $A_{0,m}-\omega_m I$ is a dissipative operator on $X_{1,m}$.

Step 2. We next show that $\mathrm{rg}(\lambda-A_{0,m})=X_{1,m}$ for $\lambda>0$. Consider $g\in X_{1,m}$. According to Lemma 2.2, there is a sequence $(g_n)_{n\geqslant 1}$ in $C_c^\infty((0,\infty))$ such that

$$\lim_{n\to\infty}\|g_n-g\|_{X_{1,m}}=0\,. \tag{3.6}$$

Since $g_n\in L_1(0,\infty)$ and $(0,\infty)\subset\rho(G)$, there is a unique $f_n\in\mathrm{dom}(G)$ such that $\lambda f_n-Gf_n=g_n$; that is,

$$\lambda f_n-f_n''=g_n\quad\text{in}\quad(0,\infty)\,,\qquad f_n(0)=0\,. \tag{3.7}$$

Now, let $R>1$. We multiply (3.7) by $(x\wedge R)^m\ \mathrm{sign}(f_n(x))$ and integrate over $(0,\infty)$. Using Kato's inequality (2.6) (with $\ell(x)=(x\wedge R)^m$), we obtain

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$$\begin{aligned}\|g_n\|_{X_m} &\geqslant \int_0^\infty (x\wedge R)^m \operatorname{sign}(f_n(x))[\lambda f_n(x) - f_n''(x)]\,\mathrm{d}x \\ &\geqslant \lambda \int_0^\infty (x\wedge R)^m\ |f_n(x)|\,\mathrm{d}x + m\int_0^R x^{m-1}|f_n|'(x)\,\mathrm{d}x \\ &= \lambda \int_0^\infty (x\wedge R)^m\ |f_n(x)|\,\mathrm{d}x - m(m-1)\int_0^R x^{m-2}\ |f_n(x)|\,\mathrm{d}x\,.\end{aligned}$$

In particular, for $m=1$ we get

$$\|g_n\|_{X_1} \geqslant \lambda \int_0^\infty (x\wedge R)\ |f_n(x)|\,\mathrm{d}x\,,$$

so that, letting $R\to\infty$ and using Fatou's lemma,

$$\lambda\|f_n\|_{X_1} \leqslant \|g_n\|_{X_1}\,. \tag{3.8}$$

The same argument entails that

$$\lambda\|f_n - f_l\|_{X_1} \leqslant \|g_n - g_l\|_{X_1}\,, \qquad n,l\geqslant 1\,. \tag{3.9}$$

If $m\geqslant 3$, then $m-2\in[1,m)$, and we use Young's inequality to deduce that, for $\varepsilon>0$,

$$\begin{aligned}\lambda \int_0^\infty (x\wedge R)^m\ |f_n(x)|\,\mathrm{d}x &\leqslant \|g_n\|_{X_m} + m(m-3)\varepsilon \int_0^R x^m\ |f_n(x)|\,\mathrm{d}x \\ &\qquad + 2m\varepsilon^{(3-m)/2}\int_0^R x\ |f_n(x)|\,\mathrm{d}x \\ &\leqslant \|g_n\|_{X_m} + m(m-3)\varepsilon\int_0^\infty (x\wedge R)^m\ |f_n(x)|\,\mathrm{d}x \\ &\qquad + 2m\varepsilon^{(3-m)/2}\|f_n\|_{X_1}\,.\end{aligned}$$

Choosing $\varepsilon = \lambda/(2m(m-3))$, we combine (3.8) and the above inequality to conclude that

$$\frac{\lambda}{2}\int_0^\infty (x\wedge R)^m\ |f_n(x)|\,\mathrm{d}x \leqslant \|g_n\|_{X_m} + \frac{2m}{\lambda}\left(\frac{2m(m-3)}{\lambda}\right)^{(m-3)/2}\|g_n\|_{X_1}\,.$$

We now let $R\to\infty$ in this inequality and deduce from Fatou's lemma that $f_n\in X_m$ with

$$\lambda\|f_n\|_{X_m} \leqslant 2\|g_n\|_{X_m} + \frac{4m}{\lambda}\left(\frac{2m(m-3)}{\lambda}\right)^{(m-3)/2}\|g_n\|_{X_1}\,. \tag{3.10}$$

The same argument entails that

$$\lambda\|f_n - f_l\|_{X_m} \leqslant 2\|g_n - g_l\|_{X_m} + \frac{4m}{\lambda}\left(\frac{2m(m-3)}{\lambda}\right)^{(m-3)/2}\|g_n - g_l\|_{X_1} \tag{3.11}$$

for $n\geqslant 1$ and $l\geqslant 1$. Therefore, for $m\in\{1\}\cup[3,\infty)$, the estimates (3.9) and (3.11) along with (3.6) guarantee that $(f_n)_{n\geqslant 1}$ is a Cauchy sequence in $X_{1,m}$ and that there is $f\in X_{1,m}$ such that

$$\lim_{n\to\infty}\|f_n - f\|_{X_{1,m}} = 0\,. \tag{3.12}$$

Since $f_n'' = \lambda f_n - g_n$ for all $n \geqslant 1$ by (3.7), it readily follows from (3.6) and (3.12) that $(f_n'')_{n\geqslant 1}$ converges to $\lambda f - g$ in $X_{1,m}$ and to f'' in the sense of distributions. Therefore, $f'' \in X_{1,m}$ with $f'' = \lambda f - g$ and $\|f_n'' - f''\|_{X_{1,m}} \to 0$ as $n \to \infty$. Finally, by (2.1c),

$$\begin{aligned}|f(0)| = |f(0) - f_n(0)| &\leqslant \int_0^\infty |(f' - f_n')(x)|\, \mathrm{d}x \\ &\leqslant \int_0^\infty \int_x^\infty |(f'' - f_n'')(y)|\, \mathrm{d}y\mathrm{d}x = \int_0^\infty y\, |(f'' - f_n'')(y)|\, \mathrm{d}y = \|f'' - f_n''\|_{X_1}\,,\end{aligned}$$

from which we deduce that $f(0) = 0$. Consequently, $f \in \mathrm{dom}(A_{0,m})$ and $\lambda f - A_{0,m} f = g$. Since $\lambda - A_{0,m}$ is one-to-one by (3.5), we have thus shown for any $\lambda > 0$ that

$$\mathrm{rg}(\lambda - A_{0,m}) = X_{1,m} \ \text{ and } \ (\lambda - A_{0,m})^{-1} g = (\lambda - A_{0,1})^{-1} g \ \text{ for all } \ g \in X_{1,m}\,. \tag{3.13}$$

Step 3. For $m \in \{1\} \cup [3, \infty)$, we infer from Step 1, Step 2, and the Lumer-Phillips theorem [30, Theorem 1.4.3] that $A_{0,m} - \omega_m$ belongs to $\mathcal{G}(X_{1,m}, 1, 0)$. Hence, $A_{0,m} \in \mathcal{G}(X_{1,m}, 1, \omega_m)$. Also, it readily follows from (3.13) that $(0, \infty) \subset \rho(A_{0,m})$ and

$$(\lambda - A_{0,m})^{-1} = (\lambda - A_{0,1})^{-1}|_{X_{1,m}}\,.$$

In particular, the latter, along with the exponential formula [30, Theorem 1.8.3], entails that $e^{tA_{0,m}} = e^{tA_{0,1}}|_{X_{1,m}}$ for all $t \geqslant 0$.

Step 4. We next derive the representation formulation for $(e^{tA_{0,1}})_{t\geqslant 0}$. To this end, we first note that, for $t > 0$, the operator $W(t)$ defined in (1.3) is a bounded operator on X_1. Indeed, since $|x - y| \leqslant |x + y| = x + y$ for $(x, y) \in (0, \infty)^2$,

$$k(t, x - y) - k(t, x + y) = \frac{1}{\sqrt{4\pi t}} \left(e^{-|x-y|^2/4t} - e^{-|x+y|^2/4t} \right) \geqslant 0 \tag{3.14}$$

for $(x, y) \in (0, \infty)^2$. By Fubini–Tonelli's theorem and (3.14),

$$\begin{aligned}\|W(t)f\|_{X_1} &\leqslant \int_0^\infty x \int_0^\infty [k(t, x - y) - k(t, x + y)]|f(y)|\, \mathrm{d}y\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty |f(y)| \int_{-y/2\sqrt{t}}^\infty (y + 2z\sqrt{t})e^{-z^2}\, \mathrm{d}z\mathrm{d}y \\ &\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty |f(y)| \int_{-\infty}^{-y/2\sqrt{t}} (-y - 2z\sqrt{t})e^{-z^2}\, \mathrm{d}z\mathrm{d}y \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty |f(y)| \left(\int_{-\infty}^\infty (y + 2z\sqrt{t})e^{-z^2}\, \mathrm{d}z \right) \mathrm{d}y = \|f\|_{X_1}\,.\end{aligned}$$

Thus, $W(t)$ is a contraction on X_1. Since $(\lambda - G)^{-1} = (\lambda - A_{0,1})^{-1}$ on $L_1(0, \infty) \cap X_1$ for all $\lambda > 0$, it follows from the exponential formula, see [30, Theorem 1.8.3], that $e^{tG} = e^{tA_{0,1}}$ on $L_1(0, \infty) \cap X_1$ for all $t \geqslant 0$. Therefore, $W(t) = e^{tA_{0,1}}$ on $L_1(0, \infty) \cap X_1$ for all $t \geqslant 0$. Since $L_1(0, \infty) \cap X_1$ is dense in X_1 by Lemma 2.2 and $e^{tA_{0,1}}$ and $W(t)$ are both bounded operators on X_1, we conclude that $e^{tA_{0,1}} = W(t)$ on X_1 for all $t \geqslant 0$. Together with the outcome of Step 3, this identity proves (3.4). In particular, $(e^{tA_{0,m}})_{t\geqslant 0}$ is a positive semigroup according to (3.14).

Step 5. We are left with showing the analyticity of $(e^{tA_{0,m}})_{t\geqslant 0}$ on $X_{1,m}$. To this end, let $f\in X_{1,m}$. Clearly,

$$t\longmapsto \int_0^\infty [k(t,x-y)-k(t,x+y)]f(y)\,\mathrm{d}y$$

is differentiable on $(0,\infty)$ and, since

$$2t\partial_t k(t,x)=\left(-1+\frac{|x|^2}{2t}\right)k(t,x)\,,\qquad (t,x)\in(0,\infty)\times\mathbb{R}\,,$$

we derive for $(t,x)\in(0,\infty)^2$ that

$$\begin{aligned}
2t\frac{\mathrm{d}}{\mathrm{d}t}e^{tA_{0,m}}f(x) &= -e^{tA_{0,m}}f(x)\\
&\quad +2\int_0^\infty\left(\frac{|x-y|^2}{4t}k(t,x-y)-\frac{|x+y|^2}{4t}k(t,x+y)\right)f(y)\,\mathrm{d}y\\
&= -3e^{tA_{0,m}}f(x)\\
&\quad +2\int_0^\infty\left(1+\frac{|x-y|^2}{4t}\right)k(t,x-y)f(y)\,\mathrm{d}y\\
&\quad -2\int_0^\infty\left(1+\frac{|x+y|^2}{4t}\right)k(t,x+y)f(y)\,\mathrm{d}y\,.
\end{aligned}$$

Let $t>0$. Since $z\mapsto(1+z)e^{-z}$ is non-increasing on $(0,\infty)$ and $|x-y|\leqslant x+y$ for $(x,y)\in(0,\infty)^2$, we see that

$$\left(1+\frac{|x-y|^2}{4t}\right)k(t,x-y)-\left(1+\frac{|x+y|^2}{4t}\right)k(t,x+y)\geqslant 0 \tag{3.15}$$

for $(x,y)\in(0,\infty)^2$ and infer from Fubini–Tonelli's theorem that

$$\begin{aligned}
&\int_0^\infty x^m\left|\int_0^\infty\left[\left(1+\frac{|x-y|^2}{4t}\right)k(t,x-y)-\left(1+\frac{|x+y|^2}{4t}\right)k(t,x+y)\right]f(y)\,\mathrm{d}y\right|\,\mathrm{d}x\\
&\leqslant \int_0^\infty x^m\int_0^\infty\left[\left(1+\frac{|x-y|^2}{4t}\right)k(t,x-y)-\left(1+\frac{|x+y|^2}{4t}\right)k(t,x+y)\right]|f(y)|\,\mathrm{d}y\mathrm{d}x\\
&=\frac{1}{\sqrt{\pi}}\int_0^\infty|f(y)|\left[\int_{-y/2\sqrt{t}}^\infty\left(y+2z\sqrt{t}\right)^m(1+z^2)e^{-z^2}\,\mathrm{d}z\right.\\
&\qquad\qquad\left.-\int_{-\infty}^{-y/2\sqrt{t}}\left(-y-2z\sqrt{t}\right)^m(1+z^2)e^{-z^2}\,\mathrm{d}z\right]\,\mathrm{d}y\\
&=\frac{1}{\sqrt{\pi}}\int_0^\infty|f(y)|\int_{-\infty}^\infty\left|y+2z\sqrt{t}\right|^{m-1}\left(y+2z\sqrt{t}\right)(1+z^2)e^{-z^2}\,\mathrm{d}z\mathrm{d}y\,.
\end{aligned}$$

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Consequently,

$$\begin{aligned} 2t \left\| \frac{\mathrm{d}}{\mathrm{d}t} e^{tA_{0,m}} f \right\|_{X_{1,m}} & \\ \leqslant 3 \left\| e^{tA_{0,m}} f \right\|_{X_{1,m}} &+ \frac{2}{\sqrt{\pi}} \int_0^\infty |f(y)| \int_{-\infty}^\infty \left(y + 2z\sqrt{t}\right) (1+z^2) e^{-z^2} \,\mathrm{d}z\mathrm{d}y \\ &+ \frac{2}{\sqrt{\pi}} \int_0^\infty |f(y)| \int_{-\infty}^\infty \left|y + 2z\sqrt{t}\right|^{m-1} \left(y + 2z\sqrt{t}\right) (1+z^2) e^{-z^2} \,\mathrm{d}z\,\mathrm{d}y \,, \end{aligned}$$

so that

$$\limsup_{t\to 0} t \left\| \frac{\mathrm{d}}{\mathrm{d}t} e^{tA_{0,m}} f \right\|_{X_{1,m}} \leqslant 3 \|f\|_{X_{1,m}}$$

by Lebesgue's convergence theorem. It then follows from [14, Theorem II.4.6 (c)] that this property implies the analyticity of $(e^{tA_{0,m}})_{t\geqslant 0}$ on $X_{1,m}$. Thus, the proof is complete. □

The proof of Proposition 3.1 is now a consequence of the previous lemma and an interpolation argument as shown next.

Proof of Proposition 3.1. We only have to consider the case $m \in (1,3)$. Since $A_{0,1} \in \mathcal{H}(X_{1,1})$ and $A_{0,3} \in \mathcal{H}(X_{1,3})$ according to Lemma 3.2, Hille's characterization implies that there are $\lambda_0 > 0$ and $\kappa \geqslant 1$ such that

$$\|(\lambda - A_{0,1})^{-1}\|_{\mathcal{L}(X_{1,1})} + \|(\lambda - A_{0,3})^{-1}\|_{\mathcal{L}(X_{1,3})} \leqslant \frac{\kappa}{|\lambda - \lambda_0|} \,, \qquad \mathrm{Re}\,\lambda > \lambda_0 \,,$$

see [14, Theorem II.4.6 (d)] for instance. Since $(\lambda - A_{0,3})^{-1} = (\lambda - A_{0,1})^{-1}|_{X_{1,3}}$, it readily follows by interpolation that

$$\|(\lambda - A_{0,1})^{-1}|_{X_{1,m}}\|_{\mathcal{L}(X_{1,m})} \leqslant \frac{\kappa}{|\lambda - \lambda_0|} \,, \qquad \mathrm{Re}\,\lambda > \lambda_0 \,. \tag{3.16}$$

Obviously, $A_{0,m}$ is closed and densely defined in $X_{1,m}$. Let $g \in X_{1,m}$ be arbitrary and $\mathrm{Re}\,\lambda > \lambda_0$. There is a unique $f \in \mathrm{dom}(A_{0,1})$ such that $(\lambda - A_{0,1})f = g$. By (3.16), $f = (\lambda - A_{0,1})^{-1}|_{X_{1,m}} g \in X_{1,m}$ and $f'' = A_{0,m} f = \lambda f - g \in X_{1,m}$. From this identity, we deduce $f \in \mathrm{dom}(A_{0,m})$ with $A_{0,m} f = A_{0,1} f$ and $(\lambda - A_{0,m}) f = g$. Since f is unique, we conclude that $\lambda \in \rho(A_{0,m})$ and

$$(\lambda - A_{0,m})^{-1} g = f = (\lambda - A_{0,1})^{-1}|_{X_{1,m}} g \,;$$

that is, $(\lambda - A_{0,m})^{-1} = (\lambda - A_{0,1})^{-1}|_{X_{1,m}}$. Invoking (3.16) we derive that

$$\|(\lambda - A_{0,m})^{-1}\|_{\mathcal{L}(X_{1,m})} \leqslant \frac{\kappa}{|\lambda - \lambda_0|} \,, \qquad \mathrm{Re}\,\lambda > \lambda_0 \,,$$

and thus conclude that $A_{0,m} \in \mathcal{H}(X_{1,m})$ with $e^{tA_{0,m}} = e^{tA_{0,1}}|_{X_{1,m}}$ for $t \geqslant 0$. The fact that $A_{0,m} \in \mathcal{G}(X_{1,m}, 1, \omega_m)$ follows again by interpolation using Lemma 3.2. □

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4 The absorption semigroup

Let $a \in L_{\infty,loc}([0,\infty))$ be a nonnegative function and $m \geqslant 1$. We recall the definition (1.4) of the Schrödinger operator $A_{a,m}$ on $X_{1,m}$ given by

$$\begin{aligned} \mathrm{dom}(A_{a,m}) &= \{f \in X_{1,m} \ : \ f \in \mathrm{dom}(A_{0,m})\,,\ af \in X_{1,m}\}\,,\\ A_{a,m}f &= f'' - af\,, \qquad f \in \mathrm{dom}(A_{a,m})\,. \end{aligned}$$

The main result of this section is that $A_{a,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m) \cap \mathcal{H}(X_{1,m})$, the parameter ω_m being defined in Proposition 3.1. The proof relies on [3].

Proposition 4.1 *Assume that a satisfies* (1.2) *and let* $m \geqslant 1$. *Then*

$$A_{a,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m) \cap \mathcal{H}(X_{1,m})\,.$$

Moreover, $e^{tA_{a,m}} = e^{tA_{a,1}}|_{X_{1,m}}$ *for* $t \geqslant 0$.

Proof. Let us recall that $X_{1,m}$ is a Banach lattice with order-continuous norm, see [1, Chapter 4] for a precise definition and a complete list of properties, and introduce

$$Y := L_1((0,\infty),(x+x^m)a(x)\mathrm{d}x)\,.$$

We first observe that $\big(\mathrm{dom}(A_{0,m}) \cap Y\big)^{\perp} = \{0\}$, where the disjoint complement $F^{\perp}$ of a subset F of the vector lattice $X_{1,m}$ is given by

$$F^{\perp} := \{g \in X_{1,m} \ : \ \min\{|f|,|g|\} = 0 \ \text{ for all } f \in F\}\,.$$

Indeed, since $C_c^{\infty}((0,\infty))$ is a subset of $\mathrm{dom}(A_{0,m}) \cap Y$, we readily deduce that $g \equiv 0$ for $g \in \big(\mathrm{dom}(A_{0,m}) \cap Y\big)^{\perp}$. Consequently, we are in a position to apply [3, Proposition 4.3] and conclude that there is an extension $\hat{A}_{a,m} \in \mathcal{G}_+(X_{1,m})$ of $A_{a,m}$ with domain $\mathrm{dom}(\hat{A}_{a,m})$ defined as follows: $f \in \mathrm{dom}(\hat{A}_{a,m})$ if and only if $f \in X_{1,m}$ and there exist $(f_n)_{n\geqslant 1}$ in $\mathrm{dom}(A_{0,m})$ and $g \in X_{1,m}$ such that

$$\lim_{n\to\infty} \big(\|f_n - f\|_{X_{1,m}} + \|A_{0,m}f_n - (a \wedge n)f_n + g\|_{X_{1,m}}\big) = 0\,.$$

It first follows from Lemma 4.2 below that $\mathrm{dom}(\hat{A}_{a,m}) = \mathrm{dom}(A_{a,m})$ and therefore $\hat{A}_{a,m} = A_{a,m}$. Moreover, $0 \leqslant e^{tA_{a,m}} = e^{t\hat{A}_{a,m}} \leqslant e^{tA_{0,m}}$ for $t \geqslant 0$ by [3, p. 432]. Since $A_{0,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m)$ due to Proposition 3.1, this ordering property, along with [7, Remark 2.68], implies

$$\|e^{t\hat{A}_{a,m}}\|_{\mathcal{L}(X_{1,m})} \leqslant \|e^{tA_{0,m}}\|_{\mathcal{L}(X_{1,m})} \leqslant e^{\omega_m t}\,, \qquad t \geqslant 0\,.$$

Hence $\hat{A}_{a,m} \in \mathcal{G}_+(X_{1,m},1,\omega_m)$. Finally, recalling that

$$(-\omega_m + A_{0,m}) \in \mathcal{G}_+(X_{1,m},1,0) \cap \mathcal{H}(X_{1,m})$$

by Proposition 3.1, we infer from [3, Theorem 6.1] that $A_{a,m} \in \mathcal{H}(X_{1,m})$. □

It remains to check that $\mathrm{dom}(\hat{A}_{a,m}) = \mathrm{dom}(A_{a,m})$. This property actually follows from the monotonicity of the Laplace operator $A_{0,m}$ and the multiplication $f \mapsto af$.

Lemma 4.2 *Assume that a satisfies* (1.2) *and let* $m \geqslant 1$. *Then* $\mathrm{dom}(\hat{A}_{a,m}) = \mathrm{dom}(A_{a,m})$.

Proof. Pick $f \in \mathrm{dom}(\hat{A}_{a,m})$. Then there are a sequence $(f_n)_{n\geqslant 1}$ in $\mathrm{dom}(A_{0,m})$ and $g \in X_{1,m}$ such that

$$\lim_{n\to\infty} \left(\|f_n - f\|_{X_{1,m}} + \|g_n - g\|_{X_{1,m}} \right) = 0 \tag{4.1}$$

with $g_n := -A_{0,m} f_n + (a \wedge n) f_n$ for $n \geqslant 1$. In particular,

$$\kappa := \sup_{n\geqslant 1} \left\{ \|f_n\|_{X_{1,m}} + \|g_n\|_{X_{1,m}} \right\} < \infty .$$

Step 1. Let us first prove that $af \in X_{1,m}$. We infer from Lemma 2.3 that, for $n \geqslant 1$,

$$\begin{aligned} \|g_n\|_{X_m} &\geqslant \int_0^\infty x^m \,\mathrm{sign}(f_n(x))\, g_n(x)\, \mathrm{d}x \\ &\geqslant m \int_0^\infty x^{m-1}\, |f_n|'(x)\, \mathrm{d}x + \int_0^\infty x^m\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x \\ &= -m(m-1)\|f_n\|_{X_{m-2}} + \int_0^\infty x^m\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x . \end{aligned}$$

In particular, we derive

$$\int_0^\infty x\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x \leqslant \|g_n\|_{X_1} \leqslant \kappa . \tag{4.2}$$

Next, if $m \geqslant 3$, then it follows from Young's inequality that

$$\begin{aligned} \int_0^\infty x^m\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x &\leqslant m(m-1)\left[\frac{m-3}{m-1}\|f_n\|_{X_m} + \frac{2}{m-1}\|f_n\|_{X_1} \right] + \|g_n\|_{X_m} \\ &\leqslant [m(m-1)+1]\kappa . \end{aligned}$$

Likewise, if $m \in (1,3)$, then Lemma 2.1 implies that

$$\begin{aligned} \int_0^\infty x^m\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x &\leqslant m(m-1)\left[\frac{1}{m-1}\|f_n''\|_{X_1} + \|f_n\|_{X_1} \right] + \|g_n\|_{X_m} \\ &\leqslant m\|(a\wedge n)f_n - g_n\|_{X_1} + [m(m-1)+1]\kappa \\ &\leqslant m \sup_{l\geqslant 1}\{\|(a\wedge l)f_l\|_{X_1}\} + (m^2+1)\kappa \\ &\leqslant (m^2+m+1)\kappa , \end{aligned}$$

where the last inequality is due to (4.2). Thus, in all cases for $m \geqslant 1$, we have shown that

$$\int_0^\infty (x + x^m)\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x \leqslant (m^2+m+2)\kappa .$$

Fixing $N \geqslant 1$, we deduce from the previous estimate that, for all $n \geqslant N$,

$$\int_0^\infty (x + x^m)\, (a(x)\wedge N)|f_n(x)|\, \mathrm{d}x \leqslant \int_0^\infty x^m\, (a(x)\wedge n)|f_n(x)|\, \mathrm{d}x \leqslant (m^2+m+2)\kappa .$$

We then let $n \to \infty$ and infer from (4.1) that

$$\int_0^\infty (x + x^m)\, (a(x)\wedge N)|f(x)|\, \mathrm{d}x \leqslant (m^2+m+2)\kappa .$$

Using Fatou's lemma to let $N \to \infty$, we conclude that $af \in X_{1,m}$ with

$$\|af\|_{X_{1,m}} \leqslant (m^2+m+2)\kappa \,. \tag{4.3}$$

Step 2. We next show that $((a\wedge n)f_n)_{n\geqslant 1}$ converges to af in $X_{1,m}$. Let $\chi \in C^\infty((0,\infty))$ be such that $\chi(x)=1$ for $x>2$, $\chi(x)=0$ for $x\in(0,1)$, and $\chi(x)\in[0,1]$ for $x\in[1,2]$. Introducing $\chi_R(x):=\chi(x/R)$ for $x\in(0,\infty)$ and $R>1$, we deduce from Lemma 2.3 (with $\ell(x)=x^m\chi_R(x)$) that

$$\begin{aligned}\int_0^\infty x^m\,(a(x)\wedge n)\chi_R(x)|f_n(x)|\,\mathrm{d}x + \int_0^\infty [x^m\chi_R'(x)+mx^{m-1}\chi_R(x)]|f_n|'(x)\,\mathrm{d}x \\ \leqslant \int_0^\infty x\chi_R(x)\mathrm{sign}(f_n(x))g_n(x)\,\mathrm{d}x \\ \leqslant \int_R^\infty x\,|g_n(x)|\,\mathrm{d}x\,.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^\infty [x^m\chi_R'(x)+mx^{m-1}\chi_R(x)]|f_n|'(x)\,\mathrm{d}x\\ &=-\int_0^\infty [x^m\chi_R''(x)+2mx^{m-1}\chi_R'(x)+m(m-1)x^{m-2}\chi_R(x)]|f_n(x)|\,\mathrm{d}x\\ &=-\int_0^\infty x^m\,|f_n(x)|\left[\frac{1}{R^2}\chi''\left(\frac{x}{R}\right)+\frac{2m}{Rx}\chi'\left(\frac{x}{R}\right)+\frac{m(m-1)}{x^2}\chi\left(\frac{x}{R}\right)\right]\,\mathrm{d}x\end{aligned}$$

and

$$\left|\chi''(y)+\frac{2m}{y}\chi'(y)+\frac{m(m-1)}{y^2}\chi(y)\right| \leqslant 3m^2\|\chi''\|_{L_\infty(0,\infty)}\,, \qquad y\in(0,\infty)\,,$$

we further obtain

$$\begin{aligned}\int_{2R}^\infty x^m\,(a(x)\wedge n)|f_n(x)|\,\mathrm{d}x &\leqslant \int_0^\infty x^m\,(a(x)\wedge n)\chi_R(x)|f_n(x)|\,\mathrm{d}x\\ &\leqslant \sup_{l\geqslant 1}\left\{\int_R^\infty x^m\,|g_l(x)|\,\mathrm{d}x\right\}+\frac{3m^2}{R^2}\|\chi''\|_{L_\infty(0,\infty)}\|f_n\|_{X_m}\\ &\leqslant \sup_{l\geqslant 1}\left\{\int_R^\infty x^m\,|g_l(x)|\,\mathrm{d}x\right\}+\frac{3m^2\kappa}{R^2}\|\chi''\|_{L_\infty(0,\infty)}\end{aligned}$$

for $n\geqslant 1$. Since

$$\lim_{R\to\infty}\sup_{l\geqslant 1}\left\{\int_R^\infty (x+x^m)\,|g_l(x)|\,\mathrm{d}x\right\}=0$$

by (4.1), we conclude

$$\lim_{R\to\infty}\sup_{n\geqslant 1}\left\{\int_{2R}^\infty (x+x^m)\,(a(x)\wedge n)|f_n(x)|\,\mathrm{d}x\right\}=0\,. \tag{4.4}$$

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Now, let $R>1$. Since $a\in L_\infty(0,2R)$ by (1.2), there is $n_R\geqslant 1$ such that $a(x)\wedge n=a(x)$ for $x\in(0,2R)$ and $n\geqslant n_R$. Consequently, for $n\geqslant n_R$,

$$\begin{aligned}\|(a\wedge n)f_n-af\|_{X_{1,m}} &\leqslant \int_0^{2R}(x+x^m)\,a(x)|(f_n-f)(x)|\,\mathrm{d}x\\ &\quad+\int_{2R}^\infty (x+x^m)\,(a(x)\wedge n)|f_n(x)|\,\mathrm{d}x\\ &\quad+\int_{2R}^\infty (x+x^m)\,a(x)|f(x)|\,\mathrm{d}x\\ &\leqslant \|a\|_{L_\infty(0,2R)}\|f_n-f\|_{X_{1,m}}\\ &\quad+\sup_{l\geqslant 1}\left\{\int_{2R}^\infty (x+x^m)\,(a(x)\wedge l)|f_l(x)|\,\mathrm{d}x\right\}\\ &\quad+\int_{2R}^\infty (x+x^m)\,a(x)|f(x)|\,\mathrm{d}x\,.\end{aligned}$$

We then pass to the limit as $n\to\infty$ and infer from (4.1) that

$$\begin{aligned}\limsup_{n\to\infty}\|(a\wedge n)f_n-af\|_{X_{1,m}}\leqslant \sup_{l\geqslant 1}&\left\{\int_{2R}^\infty (x+x^m)\,(a(x)\wedge l)|f_l(x)|\,\mathrm{d}x\right\}\\ &+\int_{2R}^\infty (x+x^m)\,a(x)|f(x)|\,\mathrm{d}x\,.\end{aligned}$$

We finally let $R\to\infty$ with the help of (4.3) and (4.4) and end up with

$$\lim_{n\to\infty}\|(a\wedge n)f_n-af\|_{X_{1,m}}=0\,. \tag{4.5}$$

Step 3. We finally show that $f\in\mathrm{dom}(A_{0,m})$. Indeed, it readily follows from (4.1) that $(f_n'')_{n\geqslant 1}$ converges to f'' in the sense of distributions, while (4.1), (4.3), and (4.5) guarantee that the sequence $(f_n'')_{n\geqslant 1}=((a\wedge n)f_n-g_n)_{n\geqslant 1}$ converges to $af-g$ in $X_{1,m}$. Therefore, f'' belongs to $X_{1,m}$ with $f''=af-g$ and $(f_n'')_{n\geqslant 1}$ converges to f'' in $X_{1,m}$. Since $f_n(0)=0$ for $n\geqslant 1$, this convergence along with Lemma 2.1 ensures that $f(0)=0$ and we have proved that $f\in\mathrm{dom}(A_{0,m})$. □

For further use, we show that the graph norm of $A_{a,m}$ in $X_{1,m}$ controls independently the diffusive and absorption terms in $X_{1,m}$.

Lemma 4.3 *Assume* (1.2). *For* $f\in\mathrm{dom}(A_{a,1})$,

$$\frac{1}{3}\left(\|A_{0,1}f\|_{X_1}+\|af\|_{X_1}\right)\leqslant\|A_{a,1}f\|_{X_1}\,. \tag{4.6}$$

Let $m>1$. *For* $f\in\mathrm{dom}(A_{a,m})$,

$$\frac{1}{4(m+1)}\left(\|A_{0,m}f\|_{X_{1,m}}+\|af\|_{X_{1,m}}\right)-m\|f\|_{X_{1,m}}\leqslant\|A_{a,m}f\|_{X_{1,m}}\,. \tag{4.7}$$

Proof. Let $f\in\mathrm{dom}(A_{a,1})$ and set $g:=-A_{a,1}f=-f''+af$. It follows from Lemma 2.3 (with $\ell(x)=x$) that

$$\|g\|_{X_1}\geqslant\int_0^\infty x\,\mathrm{sign}(f(x))g(x)\,\mathrm{d}x\geqslant\int_0^\infty |f|'(x)\,\mathrm{d}x+\|af\|_{X_1}=\|af\|_{X_1}\,.$$

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Consequently,

$$\|af\|_{X_1} \leqslant \|A_{a,1}f\|_{X_1} \quad \text{and} \quad \|A_{0,1}f\|_{X_1} = \|A_{a,1}f + af\|_{X_1} \leqslant 2\|A_{a,1}f\|_{X_1}\,, \tag{4.8}$$

from which we deduce (4.6).

Next, let $m > 1$ and consider $f \in \mathrm{dom}(A_{a,m})$. We set $g := -A_{a,m}f = -f'' + af$ and infer from Lemma 2.3 (with $\ell(x) = x^m$) that

$$\begin{aligned}\|g\|_{X_m} &\geqslant \int_0^\infty x^m \,\mathrm{sign}(f(x))g(x)\,\mathrm{d}x \geqslant m\int_0^\infty x^{m-1}|f|'(x)\,\mathrm{d}x + \|af\|_{X_m}\\ &= \|af\|_{X_m} - m(m-1)\|f\|_{X_{m-2}}\,.\end{aligned}$$

Either $m \geqslant 3$ and it follows from Young's inequality and the above inequality that

$$\begin{aligned}\|af\|_{X_m} &\leqslant \|g\|_{X_m} + m(m-1)\left(\frac{m-3}{m-1}\|f\|_{X_m} + \frac{2}{m-1}\|f\|_{X_1}\right)\\ &\leqslant \|g\|_{X_m} + m(m-3)\|f\|_{X_m} + 2m\|f\|_{X_1}\\ &\leqslant \|g\|_{X_m} + m^2\|f\|_{X_{1,m}}\,. \end{aligned} \tag{4.9a}$$

Or $m \in (1,3)$ and we infer from (2.2) and (4.8) that

$$\begin{aligned}\|af\|_{X_m} &\leqslant \|g\|_{X_m} + m(m-1)\left(\frac{1}{m-1}\|f''\|_{X_1} + \|f\|_{X_1}\right)\\ &\leqslant \|g\|_{X_m} + m\|A_{0,1}f\|_{X_1} + m(m-1)\|f\|_{X_1}\\ &\leqslant \|g\|_{X_m} + 2m\|A_{a,1}f\|_{X_1} + m^2\|f\|_{X_1}\,. \end{aligned} \tag{4.9b}$$

Collecting (4.8) and (4.9) leads us to

$$\|af\|_{X_{1,m}} \leqslant (1+2m)\|A_{a,m}f\|_{X_{1,m}} + m^2\|f\|_{X_{1,m}}\,,$$

which in turn gives

$$\|A_{0,m}f\|_{X_{1,m}} \leqslant \|A_{a,m}f\|_{X_{1,m}} + \|af\|_{X_{1,m}} \leqslant 2(1+m)\|A_{a,m}f\|_{X_{1,m}} + m^2\|f\|_{X_{1,m}}\,.$$

Consequently,

$$\frac{1}{4(1+m)}\left(\|A_{0,m}f\|_{X_{1,m}} + \|af\|_{X_{1,m}}\right) \leqslant \|A_{a,m}f\|_{X_{1,m}} + \frac{m^2}{2(m+1)}\|f\|_{X_{1,m}}\,,$$

from which (4.7) follows. □

5 The fragmentation-diffusion semigroup

We now consider the operator $\mathbb{A}_m = A_{a,m} + B_m$, where we recall that the nonlocal operator B_m on $X_{1,m}$ is defined by

$$\begin{aligned}\mathrm{dom}(B_m) &= \{f \in X_{1,m} \;:\; af \in X_{1,m}\}\,,\\ B_mf(x) &= \int_x^\infty a(y)b(x,y)f(y)\,\mathrm{d}y\,, \qquad x \in (0,\infty)\,, \qquad f \in \mathrm{dom}(B_m)\,.\end{aligned}$$

We first show that B_m is $A_{a,m}$-bounded in $X_{1,m}$.

Lemma 5.1 *Assume* (1.2) *and* (1.3). *Let* $m \geqslant 1$ *and consider a measurable function* f *on* $(0,\infty)$ *such that* $af \in X_m$. *Then*

$$\int_0^\infty x^m \left| \int_x^\infty a(y)b(x,y)f(y)\, dy \right| \, dx \leqslant \|af\|_{X_m} . \tag{5.1}$$

In addition,

$$M_1(B_m f) = M_1(af) , \qquad f \in \mathrm{dom}(B_m) , \tag{5.2}$$

and B_m *is* $A_{a,m}$*-bounded in* $X_{1,m}$.

Proof. We infer from (1.3) and Fubini–Tonelli's theorem that

$$\begin{aligned}\int_0^\infty x^m \int_x^\infty a(y)b(x,y)|f(y)|\, dydx &= \int_0^\infty a(y)|f(y)| \int_0^y x^m b(x,y)\, dxdy \\ &\leqslant \int_0^\infty y^{m-1}a(y)|f(y)| \int_0^y xb(x,y)\, dxdy = \|af\|_{X_m} ,\end{aligned}$$

from which (5.1) readily follows. Next, (5.2) is a straightforward consequence of (1.3) and Fubini's theorem.

Finally, let $f \in \mathrm{dom}(A_{a,m}) \subset \mathrm{dom}(B_m)$. By (4.7) and (5.1),

$$\begin{aligned}\|B_m f\|_{X_{1,m}} &\leqslant \int_0^\infty (x+x^m) \left| \int_x^\infty a(y)b(x,y)f(y)\, dy \right| \, dx \leqslant \|af\|_{X_{1,m}} \\ &\leqslant 4(m+1)\|A_{a,m}f\|_{X_{1,m}} + 4m(m+1)\|f\|_{X_{1,m}} ,\end{aligned}$$

so that B_m is $A_{a,m}$-bounded. □

As already observed in the literature, see, e.g., [8, Theorem 5.1.47 (c)], the inequality (1.8) implies that, for each $m > 1$, there is $\delta_m \in (0,1)$ such that

$$(1-\delta_m)y^m \geqslant \int_0^y x^m b(x,y)\, dx , \qquad y \in (0,\infty) . \tag{5.3}$$

An immediate consequence of (5.3) is a strict domination of af over $B_m f$ in X_m.

Lemma 5.2 *Assume* (1.2), (1.3), *and* (1.8). *Let* $m > 1$ *and consider* $f \in \mathrm{dom}(B_m)$. *Then*

$$\|B_m f\|_{X_m} \leqslant (1-\delta_m)\|af\|_{X_m} .$$

Proof. It readily follows from (5.3) and Fubini's theorem that

$$\begin{aligned}\|B_m f\|_{X_m} &\leqslant \int_0^\infty x^m \int_x^\infty a(y)b(x,y)|f(y)|\, dydx = \int_0^\infty a(y)|f(y)| \int_0^y x^m b(x,y)\, dxdy \\ &\leqslant (1-\delta_m) \int_0^\infty y^m a(y)|f(y)| dy = (1-\delta_m)\|af\|_{X_m} .\end{aligned}$$

□

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We shall see next that the property (5.3) ensures that B_m is a Miyadera perturbation of $A_{a,m}$. Recall that a similar result is available for the fragmentation equation without diffusion [6].

Proposition 5.3 *Let $m > 1$ and assume (1.2), (1.3), and (1.8). Then there are $q_m \in (0,1)$ and $t_m > 0$ such that*

$$\int_0^{t_m} \|B_m e^{sA_{a,m}} f\|_{X_{1,m}}\, \mathrm{d}s \leqslant q_m \|f\|_{X_{1,m}}\,, \qquad f \in \mathrm{dom}(A_{a,m})\,.$$

In particular, B_m is a Miyadera perturbation of $A_{a,m}$.

Proof. Consider $f \in \mathrm{dom}(A_{a,m})$ and set $F(t) := e^{tA_{a,m}} f$ for $t \geqslant 0$. Owing to Proposition 4.1, we have

$$\|F(t)\|_{X_1} \leqslant \|f\|_{X_1}\,, \quad \|F(t)\|_{X_{1,m}} \leqslant e^{\omega_m t} \|f\|_{X_{1,m}}\,, \qquad t \geqslant 0\,. \tag{5.4}$$

In addition, F is a classical solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} F - A_{a,m} F = 0\,, \quad t > 0\,, \qquad F(0) = f\,,$$

and we deduce from Lemma 2.3 (with $\ell(x) = x^m$) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|F\|_{X_m} + \|aF\|_{X_m} \leqslant m(m-1) \|F\|_{X_{m-2}}\,, \qquad t \geqslant 0\,.$$

Hence, after integration with respect to time,

$$\int_0^t \|aF(s)\|_{X_m}\, \mathrm{d}s \leqslant \|f\|_{X_m} + m(m-1) \int_0^t \|F(s)\|_{X_{m-2}}\, \mathrm{d}s\,, \qquad t \geqslant 0\,. \tag{5.5}$$

Now, let $t > 0$. It follows from (1.3), (5.1) (with $m = 1$), and Lemma 5.2 that

$$\int_0^t \|B_m F(s)\|_{X_{1,m}}\, \mathrm{d}s \leqslant \int_0^t \|aF(s)\|_{X_1}\, \mathrm{d}s + (1-\delta_m) \int_0^t \|aF(s)\|_{X_m}\, \mathrm{d}s\,. \tag{5.6}$$

For $R > 1$, we infer from (1.2) and (5.4) that

$$\begin{aligned}
\int_0^t \|aF(s)\|_{X_1}\, \mathrm{d}s &\leqslant \|a\|_{L_\infty(0,R)} \int_0^t \int_0^R x|F(s,x)|\, \mathrm{d}x\mathrm{d}s \\
&\qquad + R^{1-m} \int_0^t \int_R^\infty x^m a(x) |F(s,x)|\, \mathrm{d}x\mathrm{d}s \\
&\leqslant \|a\|_{L_\infty(0,R)} \int_0^t \|F(s)\|_{X_1}\, \mathrm{d}s + R^{1-m} \int_0^t \int_0^\infty x^m a(x) |F(s,x)|\, \mathrm{d}x\mathrm{d}s \\
&\leqslant \|a\|_{L_\infty(0,R)} \|f\|_{X_1} t + R^{1-m} \int_0^t \|aF(s)\|_{X_m}\, \mathrm{d}s\,.
\end{aligned}$$

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Combining (5.5), (5.6), and the above estimate with $R = R_m := (\delta_m/2)^{1/(1-m)}$ gives

$$\begin{aligned}
\int_0^t \|B_m F(s)\|_{X_{1,m}} \,\mathrm{d}s &\leqslant \|a\|_{L_\infty(0,R_m)} \|f\|_{X_1} t + \left(1 - \frac{\delta_m}{2}\right) \int_0^t \|aF(s)\|_{X_m} \,\mathrm{d}s \\
&\leqslant \|a\|_{L_\infty(0,R_m)} \|f\|_{X_1} t + \left(1 - \frac{\delta_m}{2}\right) \|f\|_{X_m} \\
&\quad + m(m-1)\left(1 - \frac{\delta_m}{2}\right) \int_0^t \|F(s)\|_{X_{m-2}} \,\mathrm{d}s \\
&\leqslant \|a\|_{L_\infty(0,R_m)} \|f\|_{X_1} t + \left(1 - \frac{\delta_m}{2}\right) \|f\|_{X_m} \\
&\quad + m(m-1) \int_0^t \|F(s)\|_{X_{m-2}} \,\mathrm{d}s \,. \qquad (5.7)
\end{aligned}$$

At this point, we handle the cases $m \geqslant 3$ and $m \in (1,3)$ in a different way. We first consider $m \geqslant 3$. We use Young's inequality, along with (5.4), to obtain

$$\begin{aligned}
m(m-1) \int_0^t \|F(s)\|_{X_{m-2}} \,\mathrm{d}s &\leqslant m(m-1) \int_0^t \left[\frac{m-3}{m-1} \|F(s)\|_{X_m} + \frac{2}{m-1} \|F(s)\|_{X_1}\right] \mathrm{d}s \\
&\leqslant \frac{m(m-3)}{1+\omega_m} \|f\|_{X_{1,m}} \left(e^{(1+\omega_m)t} - 1\right) + 2m\|f\|_{X_1} t \,. \qquad (5.8)
\end{aligned}$$

Collecting (5.7) and (5.8) leads us to

$$\begin{aligned}
\int_0^t \|B_m F(s)\|_{X_{1,m}} \,\mathrm{d}s &\leqslant \left[\|a\|_{L_\infty(0,R_m)} t + \frac{m(m-3)}{1+\omega_m} \left(e^{(1+\omega_m)t} - 1\right) + 2mt\right] \|f\|_{X_1} \\
&\quad + \left[1 - \frac{\delta_m}{2} + \frac{m(m-3)}{1+\omega_m} \left(e^{(1+\omega_m)t} - 1\right)\right] \|f\|_{X_m} \,.
\end{aligned}$$

We now pick $t_m > 0$ such that

$$\left(\|a\|_{L_\infty(0,R_m)} + 2m\right) t_m \leqslant 1 - \frac{\delta_m}{2} \quad \text{and} \quad \frac{m(m-3)}{1+\omega_m} \left(e^{(1+\omega_m)t_m} - 1\right) \leqslant \frac{\delta_m}{4}$$

and infer from the previous estimate (with $t = t_m$) that

$$\int_0^{t_m} \|B_m F(s)\|_{X_{1,m}} \,\mathrm{d}s \leqslant \left(1 - \frac{\delta_m}{4}\right) \|f\|_{X_{1,m}} \,.$$

Recalling that B_m is $A_{a,m}$-bounded by Lemma 5.1, we have thus established that B_m is a Miyadera perturbation of $A_{a,m}$ for $m \geqslant 3$.

Let us now consider $m \in (1,3)$. In that case, $m-2 \in (-1,1)$ and it follows from Lemmas 2.1 and 4.3, and (5.4) that, for $s \in (0,t)$,

$$\begin{aligned}
\|F(s)\|_{X_{m-2}} &\leqslant \frac{2(3-m)^{(m-3)/2}}{m-1} \|F''(s)\|_{X_1}^{(3-m)/2} \|F(s)\|_{X_1}^{(m-1)/2} \\
&\leqslant \frac{6(3-m)^{(m-3)/2}}{m-1} \|A_{a,1} F(s)\|_{X_1}^{(3-m)/2} \|f\|_{X_1}^{(m-1)/2} \,.
\end{aligned}$$

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Owing to the analyticity of $\left(e^{tA_{a,m}}\right)_{t\geqslant 0}$, see Proposition 4.1, we further infer from [30, Theorem 2.5.2] that there is $C>0$ such that

$$\|A_{a,1}e^{sA_{a,1}}\|_{\mathcal{L}(X_1)} \leqslant C\frac{e^s}{s} \leqslant C\frac{e^t}{s}\,, \qquad s\in(0,t)\,.$$

Combining the above two estimates gives

$$\|F(s)\|_{X_{m-2}} \leqslant C(m)\|f\|_{X_1}e^{(3-m)t/2}s^{(m-3)/2}\,, \qquad s\in(0,t)\,.$$

Hence, recalling (5.7),

$$\int_0^t \|B_mF(s)\|_{X_{1,m}}\,\mathrm{d}s \leqslant \left[\|a\|_{L_\infty(0,R_m)}t + C(m)e^{(3-m)t/2}t^{(m-1)/2}\right]\|f\|_{X_1} + \left(1-\frac{\delta_m}{2}\right)\|f\|_{X_m}\,.$$

We now choose $t_m>0$ such that

$$\|a\|_{L_\infty(0,R_m)}t_m + C(m)e^{(3-m)t_m/2}t_m^{(m-1)/2} \leqslant 1-\frac{\delta_m}{2}$$

and deduce from the previous inequality (with $t=t_m$) that

$$\int_0^{t_m} \|B_mF(s)\|_{X_{1,m}}\,\mathrm{d}s \leqslant \left(1-\frac{\delta_m}{2}\right)\|f\|_{X_{1,m}}\,.$$

Consequently, using again Lemma 5.1, B_m is also a Miyadera perturbation of $A_{a,m}$ when $m\in(1,3)$. □

We are now in a position to prove the first two statements in Theorem 1.1 for the operator $\mathbb{A}_m = A_{a,m}+B_m$:

Proof of Theorem 1.1(a)-(b). We handle the cases $m=1$ and $m>1$ separately.

(a) If $m=1$, then $A_{a,1}\in\mathcal{G}_+(X_1,1,0)$ by Proposition 4.1, so that it generates a substochastic semigroup in X_1. Moreover, $\mathrm{dom}(A_{a,1})\subset\mathrm{dom}(B_1)$ and B_1 is obviously positive due to the nonnegativity of a and b. Also, for $f\in\mathrm{dom}(A_{a,1})$,

$$M_1(A_{a,1}f+B_1f) = -\int_0^\infty f'(x)\,\mathrm{d}x - M_1(af) + M_1(B_1f) = 0$$

by Lemma 2.1, (5.2), and the Dirichlet boundary condition. Consequently, we infer from [31] and [8, Theorem 4.9.16] that there is an extension $\tilde{\mathbb{A}}_1\in\mathcal{G}_+(X_1,1,0)$ of $\mathbb{A}_1$.

(b) Let $m>1$. Since $A_{a,m}\in\mathcal{H}(X_{1,m})$ by Proposition 4.1 and B_m is a Miyadera perturbation of $A_{a,m}$ by Proposition 5.3, it follows from [14, Corollary III.3.16 & Exercise III.3.17] that $\mathbb{A}_m = A_{a,m}+B_m\in\mathcal{H}(X_{1,m})$ with $\mathrm{dom}(\mathbb{A}_m)=\mathrm{dom}(A_{a,m})$. Note that $D(A_{a,m})$ and $D(\mathbb{A}_m)$ are both Banach spaces and that $D(A_{a,m})$ is continuously embedded in $D(\mathbb{A}_m)$, since B_m is $A_{a,m}$-bounded in $X_{1,m}$ according to Lemma 5.1. Consequently, $D(\mathbb{A}_m)\doteq D(A_{a,m})$ by the open mapping theorem.

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We now check the positivity of $\left(e^{t\mathbb{A}_m}\right)_{t\geqslant 0}$, bearing in mind that we already know from Proposition 4.1 that $A_{a,m}$ is resolvent positive. Pick $\lambda > 0$ sufficiently large. Then $\lambda - \mathbb{A}_m$ is invertible with inverse given by

$$\begin{aligned}(\lambda-\mathbb{A}_m)^{-1} &= (\lambda - A_{a,m} - B_m)^{-1} = (\lambda - A_{a,m})^{-1}\left(1 - B_m(\lambda - A_{a,m})^{-1}\right)^{-1} \\ &= (\lambda - A_{a,m})^{-1}\sum_{j=0}^{\infty}\left[B_m(\lambda - A_{a,m})^{-1}\right]^j ,\end{aligned}$$

where the Neumann series converges since B_m is a Miyadera perturbation of $A_{a,m}$, see the proof of [14, Theorem III.3.14]. Now, B_m is obviously a positive operator on $X_{1,m}$ due to the nonnegativity of a and b, and the positivity of $(\lambda - \mathbb{A}_m)^{-1}$ directly follows from the above identity.

Finally, as in the proof of (a), we have $M_1(\mathbb{A}_m f) = 0$ for any $f \in \mathrm{dom}(A_{a,m})$ by Lemma 2.1, (5.2), and the Dirichlet boundary condition, so that (1.9) immediately follows. □

Proof of Proposition 1.2(a). Let $m \geqslant 1$. The operator $A_{a,m}$ belongs to $\mathcal{G}_+(X_{1,m}) \cap \mathcal{H}(X_{1,m})$ by Proposition 4.1. Since $a \in L_\infty(0,\infty)$ and b satisfies (1.3), the operator B_m is a positive bounded operator on $X_{1,m}$. On the one hand, it now follows from well known perturbation results that $\mathbb{A}_m = A_{a,m} + B_m$ belongs to $\mathcal{H}(X_{1,m})$, see [30, Theorem 3.2.1]. On the other hand, the same argument as in the proof of Theorem 1.1(b) ensures the positivity of $\left(e^{t\mathbb{A}_m}\right)_{t\geqslant 0}$. Finally, for $m = 1$, it readily follows from [8, Proposition 4.9.16] that $\mathbb{A}_1 = \tilde{\mathbb{A}}_1 \in \mathcal{G}_+(X_1, 1, 0)$, thereby completing the proof. □

6 Immediate compactness of the semigroup

We now turn to compactness properties of the semigroup $(e^{t\mathbb{A}_m})_{t\geqslant 0}$ for $m > 1$ as stated in Theorem 1.1 (c). To avoid loss of compactness for large sizes, we further require a to diverge to infinity for large sizes, thus excluding bounded overall fragmentation rates.

Lemma 6.1 *Let $m \geqslant 1$ and assume that a satisfies* (1.2) *and* (1.10). *Then $D(A_{a,m}) \doteq D(\mathbb{A}_m)$ is compactly embedded in $X_{1,m}$.*

Proof. Recall that the relation $D(A_{a,m}) \doteq D(\mathbb{A}_m)$ is established in the proof of Theorem 1.1 (b). Let $(f_n)_{n\geqslant 1}$ be a bounded sequence in $D(A_{a,m})$. According to Lemmas 2.1 and 4.3, there is $C > 0$ such that

$$\sup_{x\geqslant 0}\left\{|f_n(x)| + x|f_n'(x)|\right\} \leqslant C\,, \qquad n \geqslant 1\,, \tag{6.1a}$$

$$\|f_n\|_{X_{1,m}} + \|f_n''\|_{X_1} + \|af_n\|_{X_m} \leqslant C\,, \qquad n \geqslant 1\,. \tag{6.1b}$$

On the one hand, we infer from (6.1a) and Arzelà–Ascoli's theorem that $(f_n)_{n\geqslant 1}$ is relatively compact in $C([1/R,R])$ for each $R > 1$. There are thus a subsequence $(f_{n_j})_{j\geqslant 1}$ and $f \in C((0,\infty))$ such that

$$\lim_{j\to\infty} f_{n_j}(x) = f(x)\,, \qquad x \in (0,\infty)\,. \tag{6.2}$$

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On the other hand, it follows from (6.1) that, if $R > 1$ and E is a measurable subset of $(0,\infty)$, then, for $n \geqslant 1$,

$$\begin{aligned}\int_E (x+x^m)|f_n(x)|\,\mathrm{d}x &\leqslant \int_{E\cap(0,R)} (x+x^m)|f_n(x)|\,\mathrm{d}x + \int_R^\infty (x+x^m)|f_n(x)|\,\mathrm{d}x \\ &\leqslant (R+R^m)\|f_n\|_{L_\infty(0,\infty)}|E\cap(0,R)| \\ &\quad + \frac{2}{\inf_{x\geqslant R}\{a(x)\}}\int_R^\infty x^m\, a(x)|f_n(x)|\,\mathrm{d}x \\ &\leqslant 2CR^m|E\cap(0,R)| + \frac{2C}{\inf_{x\geqslant R}\{a(x)\}}\,. \end{aligned} \tag{6.3}$$

A first consequence of (6.3) with $E=(R,\infty)$ is that

$$\sup_{n\geqslant 1}\int_R^\infty (x+x^m)|f_n(x)|\,\mathrm{d}x \leqslant \frac{2C}{\inf_{x\geqslant R}\{a(x)\}}\,,$$

from which we deduce by (1.10) that

$$\lim_{R\to\infty}\sup_{n\geqslant 1}\int_R^\infty (x+x^m)|f_n(x)|\,\mathrm{d}x = 0\,. \tag{6.4}$$

We next infer from (6.3) that, for $\delta > 0$,

$$\eta(\delta) := \sup\left\{\int_E (x+x^m)|f_n(x)|\,\mathrm{d}x \;:\; n\geqslant 1\,,\; E\in\mathcal{B}((0,\infty))\,,\; |E|\leqslant\delta\right\}$$

satisfies

$$0\leqslant \eta(\delta)\leqslant 2CR^m\delta + \frac{2C}{\inf_{x\geqslant R}\{a(x)\}}$$

for all $R > 1$. Therefore,

$$\limsup_{\delta\to 0}\eta(\delta)\leqslant \frac{2C}{\inf_{x\geqslant R}\{a(x)\}}$$

for all $R > 1$ and we use once more (1.10) to conclude that

$$\lim_{\delta\to 0}\eta(\delta) = 0\,. \tag{6.5}$$

Gathering (6.4) and (6.5) implies that the sequence $(f_n)_{n\geqslant 1}$ is uniformly integrable in $X_{1,m}$ and thus weakly compact in $X_{1,m}$ by Dunford–Pettis' theorem. This just established weak compactness in $X_{1,m}$, along with the pointwise convergence (6.2) and Vitali's theorem, see [19, Theorem 2.24] for instance, entails that $(f_{n_j})_{j\geqslant 1}$ converges to f in $X_{1,m}$, thereby completing the proof. □

We are now in a position to finish off the proof of Theorem 1.1.

Proof of Theorem 1.1(c). Let $m > 1$. By Lemma 6.1, $(\lambda - \mathbb{A}_m)^{-1}$ is compact for $\lambda > 0$ large enough and, since $m > 1$, the analyticity of $\left(e^{t\mathbb{A}_m}\right)_{t\geqslant 0}$ implies that it is continuous with respect to the operator norm for positive times [30, Lemma 2.4.2]. We now may apply [30, Theorem 2.3.3] to conclude that $\left(e^{t\mathbb{A}_m}\right)_{t\geqslant 0}$ is immediately compact. □

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The compactness result of Lemma 6.1 is not valid under the sole assumption (1.2) on a. In particular, we show that it fails when a is bounded.

Lemma 6.2 *Let $m \geqslant 1$ and $a \in L_\infty(0,\infty)$. Then the embedding of $D(A_{a,m}) \doteq D(\mathbb{A}_m)$ in $X_{1,m}$ is not compact.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $0 \leqslant \varphi \leqslant 1$, supp $\varphi \subset [-1,1]$, and $\|\varphi\|_{L_1(\mathbb{R})} = 1$. We fix $m \geqslant 1$ and set

$$\varphi_n(x) := \frac{1}{n+n^m}\varphi(x-n)\,, \qquad x \in (0,\infty)\,, \quad n \geqslant 1\,.$$

Straightforward computations show that

$$\begin{aligned}\|\varphi_n\|_{X_{1,m}} + \|a\varphi_n\|_{X_{1,m}} &\leqslant \left(1+\|a\|_{L_\infty(0,\infty)}\right)\|\varphi_n\|_{X_{1,m}} \\ &\leqslant \left(1+\|a\|_{L_\infty(0,\infty)}\right)\left(1+\int_{-1}^1 \frac{1+m2^m n^{m-1}}{n+n^m}\varphi(y)\,\mathrm{d}y\right) \\ &\leqslant (2+m2^m)\left(1+\|a\|_{L_\infty(0,\infty)}\right)\end{aligned}$$

and

$$\|\varphi_n''\|_{X_{1,m}} \leqslant \frac{n+1+(n+1)^m}{n+n^m}\int_{-1}^1 |\varphi''(y)|\,\mathrm{d}y \leqslant 2^m\|\varphi''\|_{L_1(\mathbb{R})}\,,$$

as well as

$$\lim_{n\to\infty}\|\varphi_n\|_{X_{1,m}} = 1\,, \qquad \lim_{n\to\infty}\|\varphi_n\|_{L_\infty(0,\infty)} = 0\,. \tag{6.6}$$

Therefore, the sequence $(\varphi_n)_{n\geq 1}$ is bounded in $D(A_{a,m})$ but has no cluster point in $X_{1,m}$ due to (6.6). □

Proof of Proposition 1.2(b). Let $m \geqslant 1$ and $a \in L_\infty(0,\infty)$. Since $D(\mathbb{A}_m)$ is not compactly embedded in $X_{1,m}$ by Lemma 6.2, the resolvent $(\lambda - \mathbb{A}_m)^{-1}$ is not compact. Hence, [30, Theorem 2.3.3] implies that the semigroup $(e^{t\mathbb{A}_m})_{t\geqslant 0}$ is not compact. □

7 Steady states and convergence

Throughout this section, we assume that a and b satisfy (1.2), (1.3), (1.8), and (1.10) and that $a > 0$ and $b > 0$.

We begin with the construction of a stationary solution with the help of Schauder's fixed point theorem.

Lemma 7.1 *There is a unique nonnegative*

$$\psi_1 \in \bigcap_{r\geqslant 1}\mathrm{dom}(\mathbb{A}_r)$$

such that $M_1(\psi_1) = 1$ and $\ker(\mathbb{A}_m) = \ker(\mathbb{A}_m^2) = \mathbb{R}\psi_1 = \{r\psi_1 \,:\, r \in \mathbb{R}\}$ for all $m \geqslant 1$.

Proof. We split the proof into three steps.

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Step 1. The uniqueness of a solution $\psi \in \mathrm{dom}(\mathbb{A}_1)$ to $\mathbb{A}_1\psi = 0$ satisfying $M_1(\psi) = 1$ relies on the dissipativity properties of $\mathbb{A}_1$ in X_1 and can be shown exactly as in the proofs of [15, Lemma 3.5] and [25, Proposition 3], to which we refer.

Step 2. We now turn to the existence part. Let $m \geqslant 3$ and consider $f \in X_{1,m}^+$ satisfying $M_1(f) = 1$. Setting $F(t) := e^{t\mathbb{A}_m} f$ for $t \geqslant 0$, it readily follows from Theorem 1.1 (b) that

$$F(t) \geqslant 0 \text{ and } M_1(F(t)) = 1\,, \qquad t \geqslant 0\,. \tag{7.1}$$

Next, by (5.3) and Fubini's theorem,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) &= -m \int_0^\infty x^{m-1} \partial_x F(t,x)\,\mathrm{d}x - \int_0^\infty x^m a(x) F(t,x)\,\mathrm{d}x \\ &\quad + \int_0^\infty a(y) F(t,y) \int_0^y x^m b(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\leqslant m(m-1) M_{m-2}(F(t)) - \delta_m M_m(aF(t))\,.\end{aligned}$$

Owing to (1.10), there is $x_* > 0$ such that $a(x) \geqslant 1$ for $x \geqslant x_*$. Consequently, using (7.1),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) + \delta_m M_m(F(t)) \\ &\quad\leqslant \frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) + \delta_m \int_{x_*}^\infty x^m F(t,x)\,\mathrm{d}x + \delta_m \int_0^{x_*} x^m F(t,x)\,\mathrm{d}x \\ &\quad\leqslant \frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) + \delta_m M_m(aF(t)) + \delta_m x_*^{m-1} \int_0^{x_*} x F(t,x)\,\mathrm{d}x \\ &\quad\leqslant m(m-1) M_{m-2}(F(t)) + \delta_m x_*^{m-1}\,.\end{aligned}$$

Since $m \geqslant 3$, we now deduce from Young's inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) + \delta_m M_m(F(t)) &\leqslant \frac{\delta_m}{2} M_m(F(t)) + 2m \left(\frac{2m(m-3)}{\delta_m}\right)^{(m-3)/2} M_1(F(t)) \\ &\quad + \delta_m x_*^{m-1}\,.\end{aligned}$$

Hence, by (7.1),

$$\frac{\mathrm{d}}{\mathrm{d}t} M_m(F(t)) + \frac{\delta_m}{2} M_m(F(t)) \leqslant 2m \left(\frac{2m(m-3)}{\delta_m}\right)^{(m-3)/2} + \delta_m x_*^{m-1} =: \frac{\delta_m \mu_m}{2}$$

for $t \geqslant 0$. After integration with respect to time, we conclude that

$$M_m(F(t)) \leqslant \max\{M_m(f), \mu_m\}\,, \qquad t \geqslant 0\,. \tag{7.2}$$

Now, introducing

$$\mathcal{C}_m := \{f \in X_{1,m}^+ \,:\, M_1(f) = 1\,,\ M_m(f) \leqslant \mu_m\}\,,$$

which is a closed convex subset of $X_{1,m}$, an immediate consequence of (7.1) and (7.2) is that

$$e^{t\mathbb{A}_m} \mathcal{C}_m \subset \mathcal{C}_m\,, \qquad t \geqslant 0\,.$$

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Owing to the compactness of $e^{t\mathbb{A}_m}$ in $X_{1,m}$ for all $t>0$, see Theorem 1.1(c), we argue as in the proofs of [2, Theorem 22.13] and [20, Theorem 5.2] to deduce from Schauder's fixed point theorem that there is $\psi_m \in \mathcal{C}_m$ such that

$$e^{t\mathbb{A}_m}\psi_m = \psi_m\,, \qquad t \geqslant 0\,.$$

Equivalently, $\mathbb{A}_m\psi_m = 0$ and we have thus shown the existence of a stationary solution to (1.1a) for $m \geqslant 3$. Obviously, ψ_3 also belongs to $\mathrm{dom}(\mathbb{A}_m)$ and satisfies $\mathbb{A}_m\psi_3 = 0$ for any $m \in [1,3)$. Thus, there is at least one stationary solution ψ_m to (1.1) for any $m \geqslant 1$. Obviously, $\psi_m \in \mathrm{dom}(\mathbb{A}_1)$ solves $\mathbb{A}_1\psi_m = 0$ for every $m \geqslant 1$, and we infer from Step 1 that $\psi_m = \psi_1$ for every $m \geqslant 1$.

Step 3. We finally identify $\ker(\mathbb{A}_m^2)$. To this end, let $f \in \ker(\mathbb{A}_m^2)$. Then $\mathbb{A}_m f$ belongs to $\ker(\mathbb{A}_m)$, so that Step 2 implies that there is $\mu \in \mathbb{C}$ such that $\mathbb{A}_m f = \mu\psi_1$. Therefore,

$$\mu = \mu M_1(\psi_1) = M_1(\mathbb{A}_m f) = 0\,.$$

Hence, $f \in \ker(\mathbb{A}_m)$. □

We now supply refined information on the spectrum of $\mathbb{A}_m$ for $m > 1$.

Lemma 7.2 *Let $m > 1$. The spectrum $\sigma(\mathbb{A}_m)$ of $\mathbb{A}_m$ only consists of isolated eigenvalues and satisfies*

$$\sigma(\mathbb{A}_m) \subset \{0\} \cup \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda < -\varepsilon_m\} \tag{7.3}$$

for some $\varepsilon_m > 0$. Moreover, the spectral bound $s(\mathbb{A}_m) = 0$ is a simple eigenvalue of $\mathbb{A}_m$.

Proof. Owing to the immediate compactness of $(e^{t\mathbb{A}_m})_{t\geqslant 0}$, see Theorem 1.1 (c), and [14, Corollary V.3.2], the spectrum $\sigma(\mathbb{A}_m)$ only consists of isolated eigenvalues which are poles of the resolvent with finite algebraic multiplicity. Moreover, for any $r \in \mathbb{R}$,

$$\#\{\lambda \in \sigma(\mathbb{A}_m) \,:\, \mathrm{Re}\,\lambda \geqslant r\} < \infty\,. \tag{7.4}$$

We next claim that $s(\mathbb{A}_m) = 0$. Indeed, since $\mathbb{A}_m \subset \mathbb{A}_1 \subset \tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{A}}_1 \in \mathcal{G}(X_1, 1, 0)$ by Theorem 1.1, any eigenvalue of $\mathbb{A}_m$ is also an eigenvalue of $\tilde{\mathbb{A}}_1$ and it follows from [30, Corollary 1.3.6] that

$$\{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda > 0\} \subset \rho(\tilde{\mathbb{A}}_1)\,.$$

Consequently, any eigenvalue of $\mathbb{A}_m$ has a non-positive real part. Thus,

$$\sigma(\mathbb{A}_m) \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda \leqslant 0\}\,. \tag{7.5}$$

Since zero belongs to the spectrum of $\mathbb{A}_m$ by Lemma 7.1, we deduce from (7.5) that $s(\mathbb{A}_m) = 0$ is a pole of the resolvent of $\mathbb{A}_m$. Recalling that $(e^{t\mathbb{A}_m})_{t\geqslant 0}$ is a positive semigroup on the Banach lattice $X_{1,m}$, it follows from [12, Theorem 8.14] that $\sigma(\mathbb{A}_m) \cap i\mathbb{R}$ is either reduced to $\{0\}$ or contains infinitely many elements. The latter being ruled out by (7.4), we conclude that $\sigma(\mathbb{A}_m) \cap i\mathbb{R} = \{0\}$. Since all eigenvalues are isolated, this last property ensures that there is $\varepsilon_m > 0$ such that (7.3) holds true.

Finally, since $\ker(\mathbb{A}_m) = \ker(\mathbb{A}_m^2) = \mathbb{R}\psi_1$ by Lemma 7.1, zero is a simple eigenvalue of $\mathbb{A}_m$ according to [14, Section IV.1.17]. □

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Proof of Theorem 1.5. Let $m > 1$. From Lemma 7.1 we obtain the existence of a unique nonnegative

$$\psi_1 \in \bigcap_{r\geqslant 1} \mathrm{dom}(\mathbb{A}_r)$$

such that $M_1(\psi_1) = 1$ and $\ker(\mathbb{A}_m) = \mathbb{R}\psi_1$. We next infer from Lemma 7.2 that zero is a dominant eigenvalue of $\mathbb{A}_m$ and a first-order pole of its resolvent with residue P, where $P \in \mathcal{L}(X_{1,m})$ denotes the spectral projection onto $\ker(\mathbb{A}_m)$ and is given by

$$Pf = \lim_{\lambda\to 0} \lambda(\lambda - \mathbb{A}_m)^{-1} f\,, \qquad f \in X_{1,m}\,, \tag{7.6}$$

see, e.g., [14, Section IV.1.17]. It then follows from [14, Corollary V.3.3] that there are $N_m \geqslant 1$ and $\nu_m > 0$ such that

$$\| e^{t\mathbb{A}_m} - P\|_{\mathcal{L}(X_{1,m})} \leqslant N_m e^{-\nu_m t}\,, \qquad t \geqslant 0\,. \tag{7.7}$$

It only remains to identify the spectral projection P. Introducing $g_\lambda := \lambda(\lambda - \mathbb{A}_m)^{-1} f$ for $f \in X_{1,m}$, we have

$$\lambda f = \lambda g_\lambda - \mathbb{A}_m g_\lambda\,,$$

from which we readily deduce that $M_1(f) = M_1(g_\lambda)$. Therefore, (7.6) implies $M_1(Pf) = M_1(f)$. Since $Pf \in \mathbb{R}\psi_1$ and $M_1(\psi_1) = 1$, we conclude that

$$Pf = M_1(f)\psi_1\,, \quad f \in X_{1,m}\,.$$

Recalling (7.7), the above identity completes the proof of Theorem 1.5. □

Remark 7.3 Since $\mathbb{A}_m \in \mathcal{H}(X_{1,m})$ and since $s(\mathbb{A}_m) = 0$, we infer from [14, Corollary IV.3.12] that there exists $\kappa \geqslant 1$ such that $\mathbb{A}_m \in \mathcal{G}(X_{1,m}, \kappa, 0)$.

8 Stationary solutions revisited

We now prove the existence of a stationary solution to (1.1) when the overall fragmentation rate a may be bounded for large sizes but does not decay to zero. Specifically, we assume that a satisfies (1.12); that is,

$$\alpha := \frac{1}{2} \liminf_{x\to\infty} a(x) \in (0, \infty)\,.$$

Proof of Proposition 1.6. As in Lemma 7.1, the proof of the uniqueness assertion in Proposition 1.6 relies on the dissipativity properties of $\mathbb{A}_1$ in X_1 and can be shown exactly as in the proofs of [15, Lemma 3.5] and [25, Proposition 3], to which we refer.

As for the existence assertion, we employ a compactness method. Let $n \geqslant 1$. We set $a_n(x) := a(x) + x/n$ for $x > 0$ and, for $m \geqslant 1$, we denote the operators B_m and $\mathbb{A}_m$ with a_n instead of a by $B_{m,n}$ and $\mathbb{A}_{m,n}$, respectively. Since $a_n(x) \to \infty$ as $x \to \infty$, we infer from Lemma 7.1 that there is a unique nonnegative

$$\psi_{1,n} \in \bigcap_{r\geqslant 1} \mathrm{dom}(\mathbb{A}_{r,n})$$

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such that $M_1(\psi_{1,n}) = 1$ and $\ker(\mathbb{A}_{m,n}) = \mathbb{R}\psi_{1,n}$ for all $m \geqslant 1$. In particular, given $m > 3$, the function $\psi_{1,n}$ belongs to $\mathrm{dom}(A_{0,m})$ with $a_n\psi_{1,n} \in X_{1,m}$ and solves

$$-\psi''_{1,n} + a_n\psi_{1,n} = B_{m,n}\psi_{1,n} \ \text{ in } (0,\infty)\,, \qquad \psi_{1,n}(0) = 0\,. \tag{8.1}$$

It follows from (8.1), Lemma 5.2, and Young's inequality that, for $\varepsilon > 0$,

$$\begin{aligned} M_m(a_n\psi_{1,n}) &= M_m(B_{m,n}\psi_{1,n}) - m\int_0^\infty x^{m-1}\psi'_{1,n}(x)\,\mathrm{d}x \\ &\leqslant (1-\delta_m)M_m(a_n\psi_{1,n}) + m(m-1)M_{m-2}(\psi_{1,n}) \\ &\leqslant (1-\delta_m)M_m(a_n\psi_{1,n}) + m(m-3)\varepsilon M_m(\psi_{1,n}) + 2m\varepsilon^{(3-m)/2}M_1(\psi_{1,n})\,. \end{aligned}$$

Hence,

$$\delta_m M_m(a_n\psi_{1,n}) \leqslant m(m-3)\varepsilon M_m(\psi_{1,n}) + 2m\varepsilon^{(3-m)/2}\,. \tag{8.2}$$

Owing to (1.12), there is $x_* > 0$ such that

$$a(x) \geqslant \alpha\,, \qquad x \geqslant x_*\,. \tag{8.3}$$

In view of (8.2) and (8.3), we obtain

$$\begin{aligned} \alpha\delta_m M_m(\psi_{1,n}) &\leqslant \alpha\delta_m x_*^{m-1}\int_0^{x_*} x\psi_{1,n}(x)\,\mathrm{d}x + \delta_m\int_{x_*}^\infty x^m a_n(x)\psi_{1,n}(x)\,\mathrm{d}x \\ &\leqslant \alpha\delta_m x_*^{m-1}M_1(\psi_{1,n}) + m(m-3)\varepsilon M_m(\psi_{1,n}) + 2m\varepsilon^{(3-m)/2}\,. \end{aligned}$$

Choosing $\varepsilon = \alpha\delta_m/2m(m-3)$ in the above inequality gives

$$\frac{\alpha\delta_m}{2}M_m(\psi_{1,n}) \leqslant \alpha\delta_m x_*^{m-1} + 2m\left(\frac{\alpha\delta_m}{2m(m-3)}\right)^{(3-m)/2}\,.$$

Therefore, there is a positive constant $c_1(m)$ depending only on a and m such that

$$M_m(\psi_{1,n}) \leqslant c_1(m)\,, \qquad n \geqslant 1\,. \tag{8.4}$$

Several additional estimates can now be derived from (8.4). Indeed, it readily follows from (8.1), (8.2) (with $\varepsilon = 1$), and Lemma 5.2 that, for $n \geqslant 1$,

$$\begin{aligned} \|\psi''_{1,n}\|_{X_m} + \|a_n\psi_{1,n}\|_{X_m} + \|B_{m,n}\psi_{1,n}\|_{X_m} &\leqslant 2\left(\|a_n\psi_{1,n}\|_{X_m} + \|B_{m,n}\psi_{1,n}\|_{X_m}\right) \\ &\leqslant 2(2-\delta_m)\|a_n\psi_{1,n}\|_{X_m} \leqslant 4M_m(a_n\psi_{1,n}) \\ &\leqslant \frac{4m(m-3)c_1(m) + 8m}{\delta_m} =: c_2(m)\,. \end{aligned} \tag{8.5}$$

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Similarly, by (1.2), (8.1), (8.4) (with $m=4$), and Lemma 5.1,

$$\begin{aligned}
&\|\psi_{1,n}''\|_{X_1} + \|a_n\psi_{1,n}\|_{X_1} + \|B_{m,n}\psi_{1,n}\|_{X_1} \\
&\qquad \leqslant 2\left(\|a_n\psi_{1,n}\|_{X_1} + \|B_{m,n}\psi_{1,n}\|_{X_1}\right) \leqslant 4\|a_n\psi_{1,n}\|_{X_m} \\
&\qquad \leqslant 4\left(1+\|a\|_{L_\infty(0,1)}\right)\int_0^1 x\psi_{1,n}(x)\,\mathrm{d}x + 4\int_1^\infty x^4 a_n(x)\psi_{1,n}(x)\,\mathrm{d}x \\
&\qquad \leqslant 4\left(1+\|a\|_{L_\infty(0,1)}\right) + 4c_2(4) =: c_3
\end{aligned} \tag{8.6}$$

for $n \geqslant 1$. Moreover, a straightforward consequence of (8.5), (8.6), and Hölder's inequality is that (8.5) is also true for $m \in (1,3]$ with a suitable constant $c_2(m)$.

We next claim that $(\psi_{1,n})_{n\geqslant 1}$ is relatively compact in $X_{1,m}$ for any $m \geqslant 1$. To this end, we note that, thanks to (8.5), (8.6), Lemma 2.1, and Fubini's theorem,

$$\|\psi_{1,n}\|_{X_0} \leqslant \frac{1}{\sqrt{2}}\|\psi_{1,n}''\|_{X_1}^{1/2}\,\|\psi_{1,n}\|_{X_1}^{1/2} \leqslant \sqrt{\frac{c_3}{2}}$$

and

$$\|\psi_{1,n}\|_{X_1} + \|\psi_{1,n}'\|_{X_1} \leqslant 1 + \int_0^\infty x\int_x^\infty |\psi_{1,n}''(y)|\,\mathrm{d}y\mathrm{d}x \leqslant 1 + \frac{\|\psi_{1,n}''\|_{X_2}}{2} \leqslant 1 + c_2(2)$$

for $n \geqslant 1$. Now, let $m \geqslant 2$. In view of (8.5) and the above estimates, $(\psi_{1,n})_{n\geqslant 1}$ is a bounded sequence in $X_0 \cap X_m \cap W_1^1((0,\infty), x\mathrm{d}x)$ and it follows from [9, Proposition 7.2.2] that $(\psi_{1,n})_{n\geqslant 1}$ is relatively compact in X_r for any $r \in (0,m)$. As $m \geqslant 2$ is arbitrary, we conclude that there are $\psi_1 \in \bigcap_{r>0} X_r$ and a subsequence $(\psi_{1,n_j})_{j\geqslant 1}$ of $(\psi_{1,n})_{n\geqslant 1}$ such that

$$\lim_{j\to\infty} \|\psi_{1,n_j} - \psi_1\|_{X_r} = 0 \quad \text{for all } r > 0. \tag{8.7}$$

An immediate consequence of (8.7) and the properties of $(\psi_{1,n})_{n\geqslant 1}$ is that

$$\psi_1 \in X_1^+ \text{ and } M_1(\psi_1) = 1.$$

Finally, let $m \geqslant 1$. We observe that $\mathrm{dom}(\mathbb{A}_{m,n}) \subset \mathrm{dom}(\mathbb{A}_m)$ for $n \geqslant 1$ (as $a_n \geqslant a$) and that (8.1) also reads

$$\mathbb{A}_m\psi_{1,n} = \mathcal{R}_n\,, \tag{8.8}$$

where

$$\mathcal{R}_n(x) := -\frac{x}{n}\psi_{1,n}(x) + \frac{1}{n}\int_x^\infty y b(x,y)\psi_{1,n}(y)\,\mathrm{d}y\,, \qquad x > 0\,.$$

It readily follows from (8.5) that

$$\|\mathcal{R}_n\|_{X_{1,m}} \leqslant \frac{2}{n}\left(\|\psi_{1,n}\|_{X_2} + \|\psi_{1,n}\|_{X_{m+1}}\right) \leqslant \frac{2}{n}\left(c_2(2) + c_2(m+1)\right)\,,$$

so that

$$\lim_{n\to\infty} \|\mathcal{R}_n\|_{X_{1,m}} = 0\,. \tag{8.9}$$

In view of (8.8) and (8.9), the sequence $(\mathbb{A}_m\psi_{1,n_j})_{j\geqslant 1}$ converges to zero in $X_{1,m}$ as $j \to \infty$ and, since $\mathbb{A}_m$ is closed on $X_{1,m}$, we readily deduce from (8.7) that $\psi_1 \in \mathrm{dom}(\mathbb{A}_m)$ and $\mathbb{A}_m\psi_1 = 0$. □

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Conflicts of interest

There is no conflicts of interest.

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