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# AN IMPROVED LOWER BOUND FOR THE CRITICAL PARAMETER OF STAVSKAYA'S PROCESS

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**Abstract**

We consider Stavskaya's process, which is a two-state probabilistic cellular automaton defined on a one-dimensional lattice. The state of any vertex depends only on itself and on the state of its right-adjacent neighbour. This process was one of the first multicomponent systems with local interaction for which the existence of a kind of phase transition has been rigorously proved. However, the exact localisation of its critical value remains as an open problem. We provide a new lower bound for the critical value.

2010 *Mathematics subject classification*: primary 60K35.

*Keywords and phrases*: particle random process, one-dimensional local interaction, phase transition, Stavskaya's process, probabilistic cellular automata.

## 1. Introduction

The development of the local interaction theory of stochastic processes began in the mid-twentieth century and is now better known as the theory of interacting particle systems. Stavskaya's process contributed to this development. This process is a discrete-time version of the well-known contact process [3, 4] and may be described as a $\{0, 1\}$-state probabilistic cellular automaton (PCA) defined on a one-dimensional lattice. We assume that the state of any vertex depends only on itself and on the state of its right-adjacent neighbour. Each time-step of the process may be subdivided into two stages. In the first stage, each vertex of the lattice stays at (or becomes) state 1 provided it or its right-adjacent neighbour is at state 1. In the second stage, each vertex at state 1 turns into 0 with probability $\alpha$, independently of the other transitions. The constant $\alpha$ is the parameter of the model associated with the randomness of the underlying stochastic process. It is not difficult to see that the continuous-time version of this process is the classical contact process. Although the contact process has been extensively studied in the literature, Stavskaya's process has received (much) less attention and today it is an

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interesting source of open problems. For some recent work on existing open questions or generalisations of Stavskaya's process we refer to [1, 6, 7].

Stavskaya's process is one of the first interacting particle systems for which the existence of a phase transition has been rigorously proved [8–11]. More specifically, it was proved that there is $\alpha^* \in (0, 1)$ such that for all $\alpha > \alpha^*$, the process is ergodic, that is, the process starting from any initial measure converges towards $\delta_0$, where $\delta_0$ denotes the Dirac measure concentrated at the configuration with all 0 states. On the other hand, if $\alpha < \alpha^*$, then the process starting from the Dirac measure concentrated at the configuration with all 1s, denoted by $\delta_1$, does not converge to $\delta_0$. The exact value of $\alpha^*$ is not known and only theoretical lower and upper bounds or estimates from computer simulations are available. Toom [11] proved that $\alpha^* \in (0.09, 0.323)$ and Mendonça [5], through computer simulations, estimated $\alpha^* \approx 0.29450(5)$. We revisit the method used in [11] to improve the lower threshold for $\alpha^*$ and show that $\alpha^* > 0.11$.

We now give a formal definition of Stavskaya's process. Let $\mathbb{Z}$ denote the set of integer numbers and call $\{0, 1\}^{\mathbb{Z}}$ the configuration space. Every configuration $x$ is determined by its components $x_i \in \{1, 0\}$, where $i \in \mathbb{Z}$. We shall consider a sequence of probabilistic measures enumerated by $t \in \{0, 1, 2, \ldots\}$, which we call *Stavskaya's process*. We assume that initially all the components are 1 and that at each time-step two transformations occur. The first is denoted by $\mathsf{D}$ and the second is denoted by $\mathsf{R}_\alpha$, where $\alpha \in [0, 1]$. We define Stavskaya's transformation by $\mathsf{Stav} = \mathsf{D} \circ \mathsf{R}_\alpha$ and the state after $t$ time-steps starting from $\delta_1$ by $\delta_1\mathsf{Stav}^t$. Speaking informally, when $\mathsf{D}$ is applied to a configuration $x$, it turns into a configuration $y$ such that $y_i = \max(x_i, x_{i+1})$ for any $i \in \mathbb{Z}$, and when $\mathsf{R}_\alpha$ is applied it turns any 1 to 0 with probability $\alpha$, independently of what happens to other components. See Figure 1 for an illustration of a possible realisation of this process.

We can now declare our goal in this work.

THEOREM 1.1. *There is $\alpha^* > 0.11$ such that, if $\alpha < \alpha^*$, then $\delta_1\mathsf{Stav}^t$ does not converge to $\delta_0$ when $t$ tends to infinity.*

## 2. Proof of Theorem 1.1

**2.1. A coupling.** It is well known that Stavskaya's process can be represented using oriented percolation. For a technical reason, we shall define this oriented graph in a slightly different way than usual. The vertices of our planar graph are given by $(i, t)$,where $i \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$. From every vertex $(i, t + 1)$ oriented bonds come from the vertices $(i, t)$ and $(i + 1, t)$. Every bond from $t$ to $t + 1$ is open and closed in the opposite direction, that is, from $t + 1$ to $t$. We assume that all the vertices $(i, 0)$ are open, and the other vertices are closed with probability $\alpha$ and open with probability $1 - \alpha$ independently of what occurs on the other vertices. A vertex $(i, t)$ has a particle only if there is an open path (formed by open bonds and vertices) on the set

$$\Delta_{(i,t)} = \{(j, s) : 0 \le s \le t \text{ and } i \le j \le i + t - s\},$$

connecting some vertex $(j, 0)$ for $i \le j \le i + t$ with the vertex $(i, t)$. For a fixed vertex $(i, t)$, the set $\Delta_{(i,t)}$ has $(t + 2)(t + 1)/2$ vertices. To get a better graphical representation

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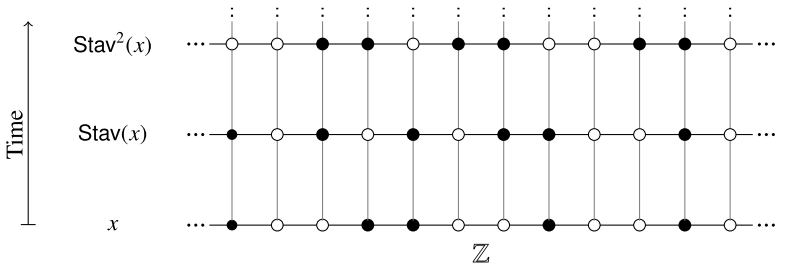
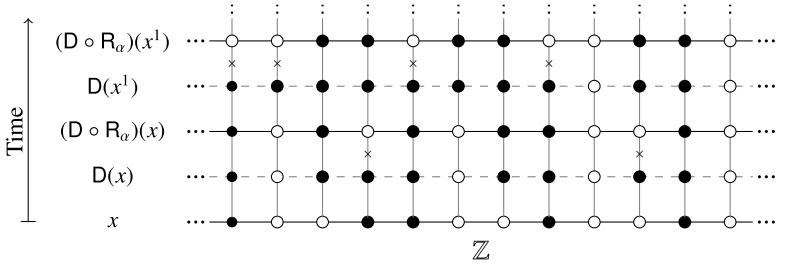



(a)




(b)

FIGURE 1. Graphical space-time representation of Stavskaya's process. Black and white particles represent vertices in state 1 and 0, respectively. (a) Realisation of Stavskaya's transformation with initial configuration $x$. The intermediate stages are represented in Figure 1(b). (b) Realisation of Stavskaya's process stage by stage. The $\times$ marks are used to represent transitions from 1 to 0 coming from the $\mathsf{R}_\alpha$ operator. The process starts from a configuration $x$ and for simplicity we let $x^1 := (\mathsf{D} \circ \mathsf{R}_\alpha)(x)$.

with a triangle, the vertical bonds are inclined to the right. See Figure 2(a), for an illustration of a fragment of the percolation graph for the triangle $\Delta_{(0,8)}$.

In [11] it has been proved that in Stavskaya's process, under the assumption adopted here, there is a particle at position $i$ at time $t$ if and only if there is an open path from some vertex $(j, 0)$ to the vertex $(i, t)$ in the oriented percolation. This result is proved by means of the coupling between Stavskaya's process and the oriented percolation model. Namely, one can consider the states of the initial vertices all open, and the states of the configuration where the particles are (that is, where the initial measure is concentrated) in state 1. The other vertices assume the state, open or closed, according to the action of the operator $\mathsf{R}_\alpha$. The inclined bonds are associated with the action of the operator $\mathsf{D}$.

We shall change our oriented site-bond percolation to an oriented bond percolation. For this we replace all the vertices by a vertical bond orientated up, which will be open or closed following the same assumption as before from its corresponding vertex. Each bond corresponding to an initial vertex $(i, 0)$ is a fount. In this context a fount means a possible starting point of a cluster of oriented paths. Thus, we may have many founts. To avoid this, we establish one fount, denoted by $F$, which will be connected with open

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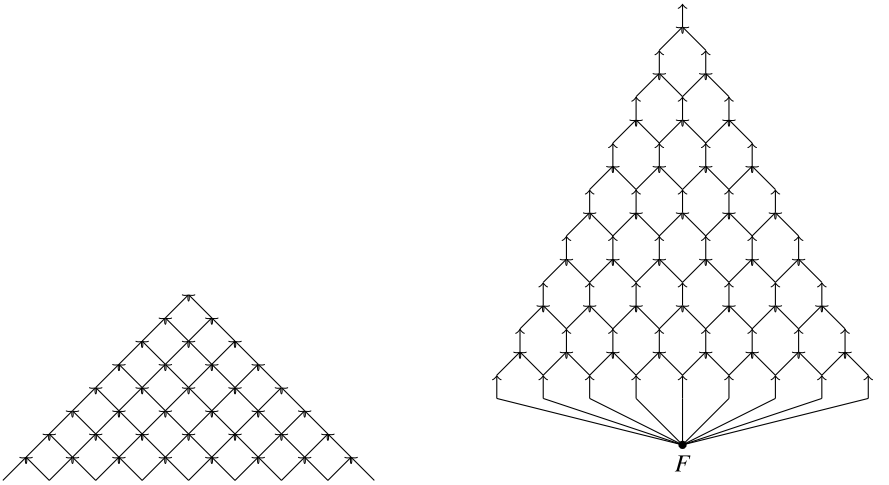


FIGURE 2. Illustration of the oriented percolation graph. (a) Representation of $\Delta_{(0,8)}$. (b) The oriented bond percolation graph, for $\Delta_{(0,8)}$, after the replacement of vertices by vertical edges.

bonds in both directions to all initial bonds. In Figure 2(b), we exhibit the fragment of the percolation graph for $\Delta_{(0,8)}$ after this replacement and insertion of the fount.

As usual, we can define the dual graph, illustrated in Figure 3. In order to do this, we start with our oriented percolation graph, and we construct the dual graph by assuming that the directed bonds $\swarrow$ and $\nwarrow$ are always open and closed in the opposite direction, while the directed bonds $\longrightarrow$ are open with probability $\alpha$ and closed in the opposite direction.

It is a well-known fact that there is no percolation in the original graph if there is an open contour in the counterclockwise direction in the dual graph, surrounding the vertical bond corresponding to the 'peak' vertex $(i, t)$. of $\Delta_{(i,t)}$. Through the coupling between Stavskaya's process and the oriented percolation model, if the probability that there is such a contour is less than 1, then $\delta_1 \mathsf{Stav}^t$ does not converge to $\delta_0$ when $t \to \infty$. This is how we will approach the proof of Theorem 1.1.

**2.2. The recurrent method.** Let $C_k$ denote the number of contours with $k$ horizontal bonds on the dual graph, which start from the left border and end at the right border of the trapezoid (see Figure 3(b)). It was proved in [11] that

$$\delta_1 \mathsf{Stav}^t(x_1 = 0, \ldots, x_m = 0) \leq \sum_{k=1}^{m} C_k \alpha^k. \tag{2.1}$$

When the quantity on the left-hand side of (2.1) is equal to 1, there is a barrier in the corresponding oriented percolation graph which will not permit the percolation. So, to prove Theorem 1.1 it is sufficient to determine when the right-hand side of (2.1) is less than 1, and this is what we will do.

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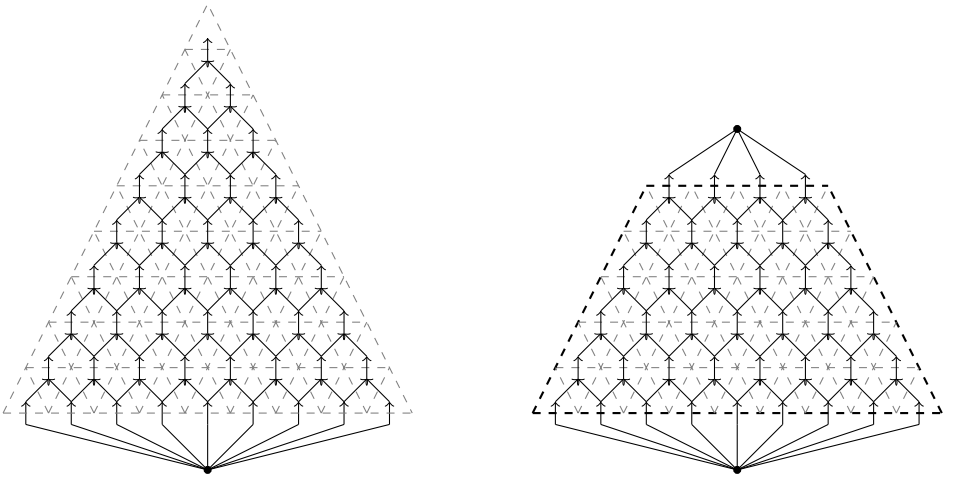

(a) (b)

FIGURE 3. Representation for the oriented percolation graph associated to $\Delta_{(0,8)}$ and its dual graph. (a) The oriented percolation graph of $\Delta_{(0,8)}$ and its dual graph. (b) The subgraph of $\Delta_{(0,8)}$ which is a trapezoid.

TABLE 1. Some elements of the dual graph.

| Bond on the dual graph | Type | Probability of being open | Shift |
|---|---|---|---|
| $\swarrow$ | 1 | 1 | $(-1,-1)$ |
| $\longrightarrow$ | 2 | $\alpha$ | $(2,\ 0)$ |
| $\nwarrow$ | 3 | 1 | $(-1,\ 1)$ |

Let us consider a coordinate system in Figure 3(b), with the origin at the upper left corner of the trapezoid. Given a vertex in the graph, the *shift* is a two-dimensional vector which takes us to the next vertex along the contour. In Table 1, we give the shift corresponding to each oriented bond and we also assign a type to each oriented bond.

A *nice path* is a path starting at the origin, passing several bonds in the directions of the arrows, loopless and without the entries 13 and 31 occurring in the list of types of the successive bonds forming the path. In particular, as noted in [11], 123 and 321 cannot occur as successive types. Each nice path has a *weight*, given by $\alpha^k$ where $k$ is the number of shifts of type 2 in the path. For $r \in \{1,2,3\}$, we denote the sum of weights of all the nice paths with $n$ bonds which end in the vertex $(i,t)$ and have the last bond $r$, by $S_r(i,t,n)$. From (2.1),

$$\delta_1 \mathsf{Stav}^t(x_1 = 0, \ldots, x_m = 0) \le \sum_{n=1}^{\infty} \sum_{r\in\{1,2,3\}} S_r(2m, 0, n). \tag{2.2}$$

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From the definition of a nice path, the numbers $S_r(i,t,n)$ satisfy the initial conditions

$$S_r(i,t,1) = \begin{cases} 1 & \text{if } i=-1, t=-1 \text{ and } r=1, \\ 0 & \text{in all other cases,} \end{cases}$$

and the transition equations

$$\begin{cases} S_1(i,t,n+1) = S_1(i+1,t+1,n) + S_2(i+1,t+1,n), \\ S_2(i,t,n+1) = \alpha(S_1(i-2,t,n) + S_2(i-2,t,n) + S_3(i-2,t,n)), \\ S_3(i,t,n+1) = S_2(i+1,t-1,n) + S_3(i+1,t-1,n), \\ S_1(i,t,n+2) = S_1(i+1,t+1,n+1) + \alpha(S_1(i,t+1,n) + S_2(i,t+1,n)), \\ S_2(i,t,n+2) = \alpha(S_1(i-2,t,n+1) + S_2(i-2,t,n+1) + S_3(i-2,t,n+1)), \\ S_3(i,t,n+2) = \alpha(S_2(i,t-1,n) + S_3(i,t-1,n)) + S_3(i+1,t-1,n+1). \end{cases}$$

Let us define sums

$$S_r(n) = \sum_{i=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} p^i q^t S_r(i,t,n), \quad \text{for } r \in \{1,2,3\},$$

where $p$ and $q$ take positive real values. These quantities satisfy the initial conditions

$$S_1(0) = p^{-1}q^{-1}, \quad S_2(0) = S_3(0) = 0,$$

and the recurrence relations, for $n \geq 1$,

$$\begin{cases} S_1(n+2) = (p^{-2}q^{-2} + \alpha q^{-1})(S_1(n) + S_2(n)), \\ S_2(n+2) = (\alpha pq^{-1} + \alpha^2 p^4) S_1(n) \\ \qquad\qquad + (\alpha pq^{-1} + \alpha^2 p^4 + \alpha pq) S_2(n) + (\alpha^2 p^4 + \alpha pq) S_3(n), \\ S_3(n+2) = (\alpha q + p^{-2}q^2)(S_2(n) + S_3(n)). \end{cases}$$

In matrix notation, $S(n+2) = S(0)M^n$, where $S(n) = (S_1(n), S_2(n), S_3(n))$ and

$$M = \begin{pmatrix} p^{-2}q^{-2} + \alpha q^{-1} & \alpha pq^{-1} + \alpha^2 p^4 & 0 \\ p^{-2}q^{-2} + \alpha q^{-1} & \alpha pq^{-1} + \alpha^2 p^4 + \alpha pq & \alpha q + p^{-2}q^2 \\ 0 & \alpha^2 p^4 + \alpha pq & \alpha q + p^{-2}q^2 \end{pmatrix}.$$

From (2.2) and the definition of the $S_r(n)$,

$$\delta_1 \mathsf{Stav}^t(x_1 = 0, \ldots, x_m = 0) \leq p^{-2m} \sum_{n=1}^{\infty} (S_1(n) + S_2(n) + S_3(n)). \tag{2.3}$$

If the sum on the right-hand side of (2.3) is convergent and $p > 1$, we can choose $m$ such that the expression on the right-hand side of (2.3) is less than 1.

**2.3. Choice of $p$ and $q$.** By the Perron–Frobenius theorem, the convergence of (2.3) occurs when the maximal eigenvalue, $\lambda$, of $M$ is less than 1. Using a corollary of this theorem (see [2]), a necessary and sufficient condition for $\lambda \leq 1$ is that all three dominant minors of the matrix $I - M$ are positive. In our case, verifying all three conditions is too hard, but we are able to verify one of them:

$$1 - p^{-2}q^{-2} - \alpha p^{-1} > 0 \iff \alpha < \frac{1 - p^{-2}q^{-2}}{p^{-1}}. \tag{2.4}$$

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Our task is to maximise the right-hand side of (2.4). Since $\alpha \le 1$,

$$\frac{1-p^{-2}q^{-2}}{p^{-1}} < 1 \Longrightarrow -\frac{1}{\sqrt{p(p-1)}} < q < \frac{1}{\sqrt{p(p-1)}} \Longrightarrow 1 \le q < \frac{1}{\sqrt{p(p-1)}}.$$

The last implication is a consequence of the fact that $p > 1$ and $q \ge 1$. The inequality $1 \le 1/\sqrt{p(p-1)}$ is satisfied for $p \in (1, \frac{1}{2}(1+\sqrt{5})]$.

Now let us define $f(p,q) = (1-p^{-2}q^{-2})/p^{-1}$. Given $p > 1$, the function $f(p,q)$ is increasing as a function of $q \ge 1$. So,

$$f(p,q) \le f\Big(p, \frac{1}{\sqrt{p(p-1)}}\Big) = 1.$$

Summing up, for $p \in (1, \frac{1}{2}(1+\sqrt{5})]$, the maximum of $f(p,q)$ occurs when

$$q = \frac{1}{\sqrt{p(p-1)}}. \tag{2.5}$$

Considering (2.5) and guided by some numerical studies for $\lambda$, we take $p = \frac{1}{2}(1+\sqrt{5})$.

Substituting these values of $p$ and $q$ into the matrix $M$ and then directly computing the maximal eigenvalue gives

$$\lambda = \frac{1}{4}(2\alpha\sqrt{5} + 3\alpha^2\sqrt{5} + 7\alpha^2 + 4\alpha + 3 - \sqrt{5}) + \frac{\sqrt{f(\alpha)}}{4},$$

where

$$\begin{aligned} f(\alpha) = {} & 14 + 28\alpha^2\sqrt{5} + 72\alpha^2 + 94\alpha^4 + 4\alpha\sqrt{5} + 4\alpha - 6\sqrt{5} + 172\alpha^3 \\ & + 76\alpha^3\sqrt{5} + 42\alpha^4\sqrt{5}. \end{aligned}$$

For $\alpha \in (0,1)$, we find $\lambda < 1$ when $\alpha < 0.1142$. This concludes the proof of Theorem 1.1.

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## References

[1] J. Depoorter and C. Maes, 'Stavskaya's measure is weakly Gibbsian', *Markov Process. Related Fields* **12** (2006), 791–804.
[2] F. R. Gantmacher, *Applications of the Theory of Matrices* (Interscience Publishers, New York, 1959).
[3] T. E. Harris, 'Contact interactions on a lattice', *Ann. Probab.* **2**(6) (1974), 969–988.
[4] T. M. Liggett, *Interacting Particle Systems* (Springer, Berlin, 1985).
[5] J. Mendonça, 'Monte Carlo investigation of the critical behavior of Stavskaya's probabilistic cellular automaton', *Phys. Rev.* E **83**(1) (2011), 012102.

https://doi.org/10.1017/S0004972720000404 Published online by Cambridge University Press

[6] L. Ponselet, *Phase Transitions in Probabilistic Cellular Automata*, PhD Thesis, Université catholique de Louvain, 2013.
[7] L. Taggi, 'Critical probabilities and convergence time of percolation probabilistic cellular automata', *J. Stat. Phys.* **159**(4) (2015), 853–892.
[8] A. L. Toom, 'A family of uniform nets of formal neurons', *Sov. Math. Dokl.* **9**(6) (1968).
[9] A. Toom, *Contornos, Conjuntos Convexos e Autômatos Celulares* (in Portuguese), $23^o$ Colóquio Brasileiro de Matemática (IMPA, Rio de Janeiro, 2001).
[10] A. Toom, *Ergodicity of Cellular Automata*, Tartu University, Estonia, 2013. Available online at http://math.ut.ee/emsdk/intensiivkursused/TOOM-TARTU-3.pdf.
[11] A. Toom, N. Vasilyev, O. Stavskaya, L. Mityushin, G. Kurdyumov and S. Pirogov, 'Discrete local Markov systems', in: *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis*, Nonlinear Science: Theory and Applications (eds. R. Dobrushin, V. Kryukov and A. Toom) (Manchester University Press, Manchester, 1990).

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