

GRADIENT FLOWS OF HIGHER ORDER YANG–MILLS–HIGGS FUNCTIONALS

PAN ZHANG

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Abstract

In this paper, we define a family of functionals generalizing the Yang–Mills–Higgs functionals on a closed Riemannian manifold. Then we prove the short-time existence of the corresponding gradient flow by a gauge-fixing technique. The lack of a maximum principle for the higher order operator brings us a lot of inconvenience during the estimates for the Higgs field. We observe that the L^2 -bound of the Higgs field is enough for energy estimates in four dimensions and we show that, provided the order of derivatives appearing in the higher order Yang–Mills–Higgs functionals is strictly greater than one, solutions to the gradient flow do not hit any finite-time singularities. As for the Yang–Mills–Higgs k -functional with Higgs self-interaction, we show that, provided $\dim(M) < 2(k + 1)$, for every smooth initial data the associated gradient flow admits long-time existence. The proof depends on local L^2 -derivative estimates, energy estimates and blow-up analysis.

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1. Introduction

Let (M, g) be a closed Riemannian manifold of real dimension n and let E be a vector bundle over M with structure group G , where G is a compact Lie group. The Yang–Mills functional, defined on the space of connections of E , is given by

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M |F_\nabla|^2 d\text{vol}_g,$$

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where ∇ is a metric compatible connection, F_∇ denotes the curvature and the pointwise norm $|\cdot|$ is given by g and the Killing form of $\text{Lie}(G)$.

The connection ∇ is called a Yang–Mills connection of E if it satisfies the Yang–Mills equation

$$D_\nabla^* F_\nabla = 0.$$

A solution of the Yang–Mills flow is given by a family of connections $\nabla_t := \nabla(x, t)$ such that

$$\frac{\partial \nabla_t}{\partial t} = -D_{\nabla_t}^* F_{\nabla_t} \quad \text{in } M \times [0, T).$$

The Yang–Mills flow was initially studied by Atiyah and Bott [2] and was suggested to understand the topology of the space of connections by infinite-dimensional Morse theory.

In the case that the ∇_t are compatible connections on a holomorphic bundle over a closed Kähler manifold, due to Donaldson [9] and Simpson [40], the Yang–Mills flow exists smoothly for all time and converges to a Hermitian Yang–Mills connection on stable bundles. This results in a correspondence, known as the Hitchin–Kobayashi correspondence [14, 29] or the Donaldson–Uhlenbeck–Yau theorem [10, 48]. Natural generalizations to the unstable case have been obtained by Daskalopoulos and Wentworth [6, 7], Wilkin [51], Jacob [18], Sibley [38], Li *et al.* [25, 26], Nie and Zhang [32] and so on.

For the general Riemannian context, the behavior of Yang–Mills flow is strongly influenced by the dimension of the base manifold. It was proved by Daskalopoulos [5] over a compact Riemann surface, and by Råde [35] in dimensions two and three, that the flow exists for all time and converges. A finite-time blow-up phenomenon is known to occur in supercritical dimensions ($\dim \geq 5$) [31]. Work on characterizing the behavior of the flow in supercritical dimensions has been done by Tao and Tian [45] and more recent developments have been made by Petrache and Rivière [34] in the case of fixed boundary connections. Following the analogy with harmonic map heat flow in dimension two [43], the foundational work of Struwe [44] gives a global weak solution for the Yang–Mills flow over a closed 4-manifold, without excluding the possibility that point singularities will form in finite time. Later, Schlatter *et al.* [37] showed that Yang–Mills flow of $\text{SO}(4)$ -equivariant connections on an $\text{SU}(2)$ bundle over a ball in \mathbb{R}^4 admits a smooth solution for all time. This led them to conjecture that long-time existence holds for solutions of Yang–Mills flow in general. Recently, Waldron [49] confirmed this conjecture. He proved that finite-time singularities do not occur in four-dimensional Yang–Mills flow, which is very different from the two-dimensional harmonic map heat flow [4].

The study of Yang–Mills–Higgs flow has aroused a lot of attention in the present century (see [1, 17, 27, 28, 41, 46, 52, 53] and so on). In the following, we introduce the higher order Yang–Mills–Higgs flow that will be called Yang–Mills–Higgs k -flow.

For each $k \in \mathbb{N} \cup \{0\}$, the Yang–Mills–Higgs k -functional (or Yang–Mills–Higgs k -energy) is defined through a connection ∇ and a section u of a vector bundle E :

$$\mathcal{MH}_k(\nabla, u) = \frac{1}{2} \int_M [|\nabla^{(k)} F_\nabla|^2 + |\nabla^{(k+1)} u|^2] d\text{vol}_g. \tag{1-1}$$

When $k = 0$, (1-1) is nothing but the Yang–Mills–Higgs functional with vanishing Higgs self-interaction [19, page 4].

The Yang–Mills–Higgs k -system, that is, the corresponding Euler–Lagrange equations of (1-1), is

$$\left\{ \begin{aligned} &(-1)^k D_\nabla^* \Delta_\nabla^{(k)} F_\nabla + \sum_{v=0}^{2k-1} P_1^{(v)} [F_\nabla] + P_2^{(2k-1)} [F_\nabla] \\ &+ \sum_{i=0}^k \nabla^{*(i)} (\nabla^{(k+1)} u * \nabla^{(k-i)} u) = 0, \\ &\nabla^{*(k+1)} \nabla^{(k+1)} u = 0, \end{aligned} \right. \tag{1-2}$$

where $\Delta_\nabla^{(k)}$ denotes k iterations of the Bochner Laplacian $-\nabla^* \nabla$ and the notation P is defined in (2-1).

A solution of the Yang–Mills–Higgs k -flow is given by a family of pairs $(\nabla(x, t), u(x, t)) := (\nabla_t, u_t)$ such that

$$\left\{ \begin{aligned} &\frac{\partial \nabla_t}{\partial t} = (-1)^{(k+1)} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{\nabla_t}] \\ &+ P_2^{(2k-1)} [F_{\nabla_t}] + \sum_{i=0}^k \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \\ &\frac{\partial u_t}{\partial t} = -\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t \quad \text{in } M \times [0, T]. \end{aligned} \right. \tag{1-3}$$

When $k = 0$, the flow (1-3) is a Yang–Mills–Higgs flow [13].

Now we state our main result in this paper.

THEOREM 1.1. *Let E be a vector bundle over a closed Riemannian 4-manifold (M, g) . Assume an integer $k > 1$. Then, for every smooth initial value (∇_0, u_0) , there exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k -flow (1-3) in $M \times [0, +\infty)$.*

REMARK 1.2. To prove the long-time existence of the Yang–Mills flow ($k = 0$), coupled with an extra structure (Higgs field [15] or spinor field [16]), one powerful tool is the maximum principle. One can obtain a C^0 -bound of the Higgs field (or spinor field) immediately. This brings us a lot of convenience in the analysis. When $k > 0$, the order of $\nabla_t^{*(k+1)} \nabla_t^{(k+1)}$ is bigger than two and the maximum principle fails. When $k > 0$,

we show that the L^2 -bound of the Higgs field is enough for energy estimates in four dimensions.

It is not surprising to consider such higher order flow. Just recently, in [21], Kelleher studied higher order Yang–Mills flow (with vanishing Higgs field) and, in [36], Saratchandran studied higher order Seiberg–Witten flow (flow of connections coupled with spinor fields). The study of higher order flow also has a long history. In De Giorgi’s program [8] to approximate singular geometric flows with sequences of smooth ones, he conjectured that any compact hypersurfaces in Euclidean space, evolving by the gradient flow of certain functionals with sufficiently high derivatives, do not hit singularities. Similar to the ones proposed by De Giorgi [8], Mantegazza [30] studied higher order generalizations of the mean curvature flow and proved that the flows do not hit singularities provided the order of the derivatives is sufficiently large. Very recently, Jia and Wang [20] extended some results due to Mantegazza to a more general ambient manifold. Actually, there have been many other important works on higher order flow such as Escher *et al.* [12] and Wheeler [50] for surface diffusion flow, Kuwert and Schätzle [23, 24] and Simonett [39] for Willmore flow of surfaces, Streets [42] for a certain flow of Riemannian curvatures, Bahuaud and Helliwell [3] and Kotschwar [22] for a certain flow of Riemannian metrics, Novaga and Okabe [33] for steepest descent flow and so on.

Now we outline the structure of this paper. In Section 2, we give some basic notation. In Section 3, we derive the Euler–Lagrange equations for the Yang–Mills–Higgs k -functional and prove the local existence of the flow. In Section 4, we obtain L^2 -derivative estimates of Bernstein–Bando–Shi type and use these to derive a basic obstruction to long-time existence. In Section 5, we address the blow-up analysis that can be used to derive an L^∞ -bound from an L^p -bound. In Section 6, we prove that both the Yang–Mills–Higgs energy and the Yang–Mills–Higgs k -energy are bounded along the flow in four dimensions. In Section 7, we complete the proof of Theorem 1.1. In Section 8, we show that the long-time existence of Yang–Mills–Higgs 1-flow in dimension four is obstructed by the possibility of concentration of the curvature in smaller and smaller balls. In Section 9, we show that provided $\dim(M) < 2(k + 1)$, for every smooth initial data the associated negative gradient flow of the Yang–Mills–Higgs k -functional with Higgs self-interaction admits long-time existence.

2. Preliminaries

To meet the requirements in the next sections, here, in this short section, the setup and notation are briefly presented. We use some of Kelleher’s notation in [21] and Saratchandran’s in [36].

Let E be a vector bundle over a smooth closed manifold (M, g) of real dimension n . The set of all smooth unitary connections on E is denoted by \mathcal{A}_E . For a given connection $\nabla \in \mathcal{A}_E$, it can be extended to other tensor bundles by coupling with the corresponding Levi-Civita connection ∇_M on (M, g) .

Let D_∇ be the exterior derivative or skew symmetrization of ∇ . The curvature tensor of E is denoted by

$$F_\nabla = D_\nabla \circ D_\nabla.$$

We set ∇^*, D_∇^* to be the formal L^2 -adjoints of ∇, D_∇ , respectively. The Bochner and Hodge Laplacians are given respectively by

$$\Delta_\nabla = -\nabla^*\nabla, \quad \Delta_{D_\nabla} = D_\nabla D_\nabla^* + D_\nabla^* D_\nabla.$$

Let ξ, η be p -forms valued in E or $\text{End}(E)$. Let $\xi * \eta$ denote any multilinear form obtained from a tensor product $\xi \otimes \eta$ in a universal way. That is to say, $\xi * \eta$ is obtained by starting with $\xi \otimes \eta$, taking any linear combination of this tensor, taking any number of metric contractions with respect to g or h and switching any number of factors in the product. We then have

$$|\xi * \eta| \leq C|\xi||\eta|.$$

Denote

$$\nabla^{(i)} = \underbrace{\nabla \cdots \nabla}_i.$$

We also use the P notation, as introduced in [24]. Given a tensor ξ , we denote

$$P_\nabla^{(k)}[\xi] := \sum_{w_1 + \cdots + w_v = k} (\nabla^{(w_1)}\xi) * \cdots * (\nabla^{(w_v)}\xi) * T, \tag{2-1}$$

where $k, v \in \mathbb{N}$ and T is a generic background tensor dependent only on g .

2.1. Commutation formulas for connections. We collect some lemmas appearing in [21, 36]. During the study of the higher order flow, there will be times when we need to switch derivatives, leading us to need the following lemmas.

LEMMA 2.1 (Weitzenböck formula). *Let E be a vector bundle over a Riemannian manifold (M, g) with compatible metric connection ∇ . Let $\Delta_{D_\nabla} = D_\nabla D_\nabla^* + D_\nabla^* D_\nabla$ denote the Hodge Laplacian and $\Delta_\nabla = -\nabla^*\nabla$ denote the Bochner Laplacian. For $\phi \in \Omega^p(M; E)$,*

$$\Delta_{D_\nabla}\phi = -\Delta_\nabla\phi + (\text{Rm} + F_\nabla) * \phi,$$

where Rm denotes the Riemannian curvature of g .

LEMMA 2.2. *Let E be a vector bundle over a Riemannian manifold (M, g) with compatible metric connection ∇ . Let ϕ be a section of E ; then*

$$\begin{aligned} \nabla_{i_k}\nabla_{i_{k-1}}\cdots\nabla_{i_1}\nabla_{j_1}\nabla_{j_2}\cdots\nabla_{j_k}\phi &= \nabla_{i_k}\nabla_{j_k}\cdots\nabla_{i_1}\nabla_{j_1}\phi \\ &+ \sum_{l=0}^{2k-2} [(\nabla_M^{(l)}\text{Rm} + \nabla^{(l)}F_\nabla) * \nabla^{(2k-2-l)}\phi]. \end{aligned}$$

LEMMA 2.3. *Let E be a vector bundle over a Riemannian manifold (M, g) with compatible metric connection ∇ . Let ϕ be a section of E ; then*

$$\nabla^{(n)} \Delta_{\nabla}^{(k)} \phi = \Delta_{\nabla}^{(k)} \nabla^{(n)} \phi + \sum_{j=0}^{2k+n-2} [(\nabla_M^{(j)} \text{Rm} + \nabla^{(j)} F_{\nabla}) * \nabla^{(2k+n-j-2)} \phi].$$

LEMMA 2.4. *Let E be a vector bundle over a Riemannian manifold (M, g) with compatible metric connection ∇ . Let ξ and ζ be sections of E ; then, for $k \in \mathbb{N}$,*

$$\begin{aligned} \int_M \langle \nabla^{(k)} \xi, \nabla^{(k)} \zeta \rangle d\text{vol}_g &= \int_M (-1)^k \langle \xi, \Delta_{\nabla}^{(k)} \zeta \rangle d\text{vol}_g \\ &+ \int_M \left\langle \xi, \sum_{v=0}^{2k-2} ((\nabla_M^{(v)} \text{Rm} + \nabla^{(v)} F_{\nabla}) * \nabla^{(2k-2-v)} \zeta) \right\rangle d\text{vol}_g. \end{aligned}$$

2.2. Interpolation inequalities. The following interpolation results are used in Section 4 when proving local derivative estimates.

LEMMA 2.5 see [21, Lemma 5.3], analogue of [24, Corollary 5.5]. *Let E be a vector bundle over a Riemannian manifold (M, g) with connection ∇ . Let ϕ be a section of E and γ a bump function on M . For $k \in \mathbb{N}$, if $1 \leq i_1, \dots, i_r \leq k$, $i_1 + i_2 \dots + i_r = 2k$ and $s \geq 2k$,*

$$\begin{aligned} \int_M \gamma^s \nabla^{(i_1)} \phi * \dots * \nabla^{(i_r)} \phi d\text{vol}_g \\ \leq C(\dim(M), \text{rk}(E), k, r, s, g, h, \gamma) \|\phi\|_{L^\infty}^{r-2} \left(\int_M |\nabla^{(k)} \phi|^2 \gamma^s d\text{vol}_g + \|\phi\|_{L^2, \gamma > 0}^2 \right), \end{aligned}$$

where the subscript $\gamma > 0$ means $\{x \in M | \gamma(x) > 0\}$.

LEMMA 2.6 [21, Corollary 5.2]. *Let E be a vector bundle over a Riemannian manifold (M, g) with connection ∇ and γ a bump function on M . For $2 \leq p < +\infty$, $l \in \mathbb{N}$ and $s \geq lp$, there exists $C(\varepsilon^{-1}) > 0$ depending on ε^{-1} , $\dim(M)$, $\text{rk}(E)$, p, l, s, g, h, γ such that for a section ϕ of E ,*

$$\|\gamma^{s/p} \nabla^{(l)} \phi\|_{L^p} \leq \varepsilon \|\gamma^{(s+lp)/p} \nabla^{(l+p)} \phi\|_{L^p} + C(\varepsilon^{-1}) \|\phi\|_{L^p, \gamma > 0}.$$

For $p = 2$ and some $K \geq 1$,

$$K \|\gamma^{s/2} \nabla^{(l)} \phi\|_{L^2}^2 \leq \varepsilon \|\gamma^{(s+2j)/2} \nabla^{(l+j)} \phi\|_{L^2}^2 + C(\varepsilon^{-1}) K^2 \|\phi\|_{L^2, \gamma > 0}^2.$$

3. The higher order Yang–Mills–Higgs flow

We first compute the Euler–Lagrange equations of the Yang–Mills–Higgs k -functional to determine the corresponding Yang–Mills–Higgs k -flow. We then prove the local existence of this flow.

LEMMA 3.1. *The Euler–Lagrange equations associated to the Yang–Mills–Higgs k -functional (1-1) are given by (1-2).*

PROOF. Let ∇_t be a time-dependent path of connections with initial value $\nabla_0 = \nabla$. Let u_t be a time-dependent path of Higgs fields with initial value $u_0 = u$. Then

$$\frac{\partial}{\partial t} \Big|_{t=0} \frac{1}{2} \int_M \langle \nabla^{(k+1)} u_t, \nabla^{(k+1)} u_t \rangle = \int_M \left\langle \frac{\partial u_t}{\partial t}, \nabla^{*(k+1)} \nabla^{(k+1)} u_t \right\rangle \Big|_{t=0} \tag{3-1}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \frac{1}{2} \int_M \langle \nabla_t^{(k+1)} u, \nabla_t^{(k+1)} u \rangle &= \int_M \left\langle \frac{\partial(\nabla_t^{(k+1)} u)}{\partial t}, \nabla_t^{(k+1)} u \right\rangle \Big|_{t=0} \\ &= \int_M \left\langle \sum_{i=0}^k \nabla_t^{(i)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(k-i)} u, \nabla_t^{(k+1)} u \right\rangle \Big|_{t=0} \\ &= \int_M \left\langle \frac{\partial \nabla_t}{\partial t}, \sum_{i=0}^k \nabla_t^{*(i)} (\nabla_t^{(k+1)} u * \nabla_t^{(k-i)} u) \right\rangle \Big|_{t=0}, \end{aligned} \tag{3-2}$$

where we used the following variation formula that can be proved by induction on k :

$$\frac{\partial}{\partial t} (\nabla_t^{(k+1)} u_t) = \nabla_t^{(k+1)} \frac{\partial u_t}{\partial t} + \sum_{i=0}^k \left(\nabla_t^{(i)} \frac{\partial \nabla_t}{\partial t} \right) * \left(\nabla_t^{(k-i)} u_t \right). \tag{3-3}$$

Finally, we compute

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \frac{1}{2} \int_M \langle \nabla_t^{(k)} F_{\nabla_t}, \nabla_t^{(k)} F_{\nabla_t} \rangle &= \int_M \left\langle \frac{\partial(\nabla_t^{(k)} F_{\nabla_t})}{\partial t}, \nabla_t^{(k)} F_{\nabla_t} \right\rangle \Big|_{t=0} \\ &= \int_M \left\langle \nabla_t^{(k)} \frac{\partial F_{\nabla_t}}{\partial t} + \sum_{i=0}^{k-1} \nabla_t^{(i)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(k-i-1)} F_{\nabla_t}, \nabla_t^{(k)} F_{\nabla_t} \right\rangle \Big|_{t=0} \\ &= \int_M \left\langle \frac{\partial \nabla_t}{\partial t}, (-1)^k D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{\nu=0}^{2k-1} P_1^{(\nu)} [F_{\nabla_t}] + P_2^{(2k-1)} [F_{\nabla_t}] \right\rangle \Big|_{t=0}, \end{aligned} \tag{3-4}$$

where we use [21, Corollary 2.2]

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla_t^{(k)} F_{\nabla_t}) &= \nabla_t^{(k)} \frac{\partial F_{\nabla_t}}{\partial t} + \sum_{i=0}^{k-1} \nabla_t^{(i)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(k-i-1)} F_{\nabla_t}, \\ \frac{\partial F_{\nabla_t}}{\partial t} &= D_{\nabla_t} \frac{\partial \nabla_t}{\partial t} \end{aligned} \tag{3-5}$$

and Lemma 2.4.

Hence, we prove the lemma by combining (3-1), (3-2) and (3-4). □

Given one-parameter pairs (∇_t, u_t) , we can define Yang–Mills–Higgs k -flow by (1-3). Then we use De Turck’s trick to establish the local existence of the Yang–Mills–Higgs k -flow. We refer to [21] for more details. The proof is standard; we outline the procedures.

THEOREM 3.2. *Let E be a vector bundle over a closed Riemannian manifold (M, g) . There exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k -flow (1-3) in $M \times [0, \epsilon)$ with smooth initial value $(\nabla(0), u(0))$.*

PROOF. (Local existence) Consider one-parameter pairs $(\widetilde{\nabla}_t, \widetilde{u}_t)$ satisfying the following system:

$$\left\{ \begin{aligned} \frac{\partial \widetilde{\nabla}_t}{\partial t} &= (-1)^{k+1} D_{\widetilde{\nabla}_t}^* \Delta_{\widetilde{\nabla}_t}^{(k)} F_{\widetilde{\nabla}_t} + (-1)^k D_{\widetilde{\nabla}_t} \Delta_{\widetilde{\nabla}_t}^{(k)} D_{\widetilde{\nabla}_t}^* (\widetilde{\nabla}_t - \nabla(0)) \\ &\quad + \sum_{\nu=0}^{2k-1} P_1^{(\nu)} [F_{\widetilde{\nabla}_t}] + P_2^{(2k-1)} [F_{\widetilde{\nabla}_t}] + \sum_{i=0}^k \widetilde{\nabla}_t^{*(i)} (\widetilde{\nabla}_t^{(k+1)} u_t * \widetilde{\nabla}_t^{(k-i)} u_t), \\ \frac{\partial \widetilde{u}_t}{\partial t} &= -\widetilde{\nabla}_t^{*(k+1)} \widetilde{\nabla}_t^{(k+1)} \widetilde{u}_t - (-1)^k (\Delta_{\widetilde{\nabla}_t}^{(k)} D_{\widetilde{\nabla}_t}^* (\widetilde{\nabla}_t - \nabla(0))) \widetilde{u}_t, \\ \widetilde{\nabla}(0) &= \nabla(0), \\ \widetilde{u}(0) &= u(0). \end{aligned} \right. \tag{3-6}$$

We show that the system is parabolic and has short-time existence.

Define the operator $\Phi_k := \Phi_k(\cdot, \nabla(0)) : \mathcal{A}_E \rightarrow \Omega^1(\text{End}E)$ by

$$\Phi_k(\widetilde{\nabla}_t, \nabla(0)) = (-1)^{k+1} D_{\widetilde{\nabla}_t}^* \Delta_{\widetilde{\nabla}_t}^{(k)} F_{\widetilde{\nabla}_t} + (-1)^k D_{\widetilde{\nabla}_t} \Delta_{\widetilde{\nabla}_t}^{(k)} D_{\widetilde{\nabla}_t}^* (\widetilde{\nabla}_t - \nabla(0)).$$

For nonzero $B \in \Omega^1(\text{End}E)$, the symbol of Φ_k is given by [29, Page 223]

$$(\sigma[\Phi_k](B))_{r\alpha}^\beta = (-1)^k \partial_q \partial_{i_1 i_1 \dots i_k i_k} (\partial_q B_{r\alpha}^\beta - \partial_r B_{q\alpha}^\beta) + (-1)^{k+1} \partial_r \partial_{i_1 i_1 \dots i_k i_k} \partial_q B_{q\alpha}^\beta.$$

Therefore, for a nonzero cotangent vector ξ on M ,

$$(\mathcal{L}_{\Phi_k}^\xi(B))_{r\alpha}^\beta := (-1)^k |\xi|^{2k+2} B_{r\alpha}^\beta,$$

$$\langle \mathcal{L}_{\Phi_k}^\xi(B), B \rangle = (-1)^{k+1} |\xi|^{2k+2} |B|^2.$$

Thus, $\langle \mathcal{L}_{\Phi_k}^\xi(\cdot), \cdot \rangle$ is either strictly positive definite or negative definite depending on the parity of k . We conclude that Φ_k is an elliptic operator.

Using Lemma 2.2,

$$\widetilde{\nabla}_t^{*(k+1)} \widetilde{\nabla}_t^{(k+1)} \widetilde{u}_t = (-1)^{k+1} \Delta_{\widetilde{\nabla}_t}^{(k+1)} \widetilde{u}_t + \sum_{i=0}^{2k} (\nabla_M^{(i)} \text{Rm} + \widetilde{\nabla}_t^{(i)} F_{\widetilde{\nabla}_t}) * \widetilde{\nabla}_t^{(2k-i)} \widetilde{u}_t.$$

From [21, Lemma 3.5],

$$(\Delta_{\widetilde{\nabla}_t}^{(k)} D_{\widetilde{\nabla}_t}^* (\widetilde{\nabla}_t - \nabla(0))) \widetilde{u}_t = -\Delta_{\widetilde{\nabla}_t}^{(k+1)} \widetilde{u}_t + \delta(\widetilde{\nabla}_t, \widetilde{u}_t),$$

where $\delta(\widetilde{\nabla}_t, \widetilde{u}_t)$ is of lower order than $\Delta_{\widetilde{\nabla}_t}^{(k+1)} \widetilde{u}_t$. Hence, ellipticity of the highest order term in the system (3-6) follows. Therefore, the system (3-6) is parabolic and has short-time existence.

Define a gauge $g(t)$ as

$$\begin{cases} \frac{\partial g(t)}{\partial t} = (-1)^{k+1} \Delta_{\widetilde{\nabla}_t}^{(k)} D_{\widetilde{\nabla}_t}^* (\widetilde{\nabla}_t - \nabla(0))g(t), \\ g(0) = \text{id}. \end{cases}$$

One can check that $(g(t)^* \widetilde{\nabla}_t, g(t)^* \widetilde{u}_t)$ satisfies the Yang–Mills–Higgs k -flow (1-3) with initial condition $(g(0)^* \widetilde{\nabla}_0, g(0)^* \widetilde{u}_0) = (\nabla_0, u_0)$. This proves the short-time existence of (1-3).

(Uniqueness) If we have two solutions to the Yang–Mills–Higgs k -flow (1-3), $(\nabla_1(t), u_1(t))$ and $(\nabla_2(t), u_2(t))$, with the same initial value $(\nabla(0), u(0))$, then we can define two gauges g_1 and g_2 that satisfy the above gauge transformation equations, with ∇_1 and ∇_2 , respectively. We then find that $((g_1^{-1})^* \nabla_1, (g_1^{-1})^* u_1)$ and $((g_2^{-1})^* \nabla_2, (g_2^{-1})^* u_2)$ both solve the parabolic system (3-6) with the same initial value $(\nabla(0), u(0))$. Uniqueness of this system implies that

$$((g_1^{-1})^* \nabla_1, (g_1^{-1})^* u_1) = ((g_2^{-1})^* \nabla_2, (g_2^{-1})^* u_2),$$

which means that

$$(\nabla_1, u_1) = ((g_2^{-1} g_1)^* \nabla_2, (g_2^{-1} g_1)^* u_2).$$

Define a new gauge $g_3 = g_2^{-1} g_1$; a direct calculation yields

$$\begin{cases} \frac{\partial g_3}{\partial t} = g_3 (-1)^{k+1} \Delta_{g_3^* \nabla_2}^{(k)} D_{g_3^* \nabla_2}^* (g_3^* \nabla_2 - \nabla(0)) \\ \quad - (-1)^{k+1} \Delta_{\nabla_2}^{(k)} D_{\nabla_2}^* (\nabla_2 - \nabla(0)) g_3, \\ g_3(0) = \text{id}. \end{cases}$$

Clearly, id is a solution to the above ordinary differential equation (ODE) with fixed initial value. The basic existence–uniqueness theorem for ODEs implies that $g_3(t) = \text{id}$. □

4. Smoothing estimates

In this section, our goal is to obtain derivative estimates of F_{∇_t} and u_t . To accomplish this, we first compute the necessary evolution equations.

4.1. Evolution equations.

LEMMA 4.1. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Then*

$$\begin{aligned} \frac{\partial F_{\nabla_t}}{\partial t} &= (-1)^k \Delta_{\nabla_t}^{(k+1)} F_{\nabla_t} + \sum_{v=0}^{2k} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k)} [F_{\nabla_t}] \\ &\quad + \sum_{i=0}^k D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \end{aligned} \tag{4-1}$$

and, for $l \in \mathbb{N}$,

$$\begin{aligned} \frac{\partial}{\partial t} [\nabla_t^{(l)} F_{\nabla_t}] &= (-1)^k \Delta_t^{(k+1)} \nabla_t^{(l)} F_{\nabla_t} + \sum_{v=0}^{2k+l} (P_1^{(v)} [F_{\nabla_t}] + P_2^{(v)} [F_{\nabla_t}]) \\ &\quad + P_3^{(2k+l-2)} [F_{\nabla_t}] + \sum_{i=0}^k \nabla_t^{(l)} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \\ &\quad + \sum_{j=0}^{l-1} \sum_{i=0}^k [\nabla_t^{(j)} (\nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t))] * \nabla_t^{(l-j-1)} F_{\nabla_t}. \end{aligned} \tag{4-2}$$

PROOF. From (1-3) and the Weitzenböck formula (Lemma 2.1),

$$\begin{aligned} \frac{\partial F_{\nabla_t}}{\partial t} &= D_{\nabla_t} \frac{\partial \nabla_t}{\partial t} = (-1)^{k+1} D_{\nabla_t} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k} P_1^{(v)} [F_{\nabla_t}] \\ &\quad + P_2^{(2k)} [F_{\nabla_t}] + \sum_{i=0}^k D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \\ &= (-1)^{k+1} \Delta_{D_{\nabla_t}} \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k)} [F_{\nabla_t}] \\ &\quad + \sum_{i=0}^k D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \\ &= (-1)^k \Delta_{\nabla_t}^{(k+1)} F_{\nabla_t} + (\text{Rm} + F_{\nabla_t}) * (\Delta_{\nabla_t}^{(k)} F_{\nabla_t}) + \sum_{v=0}^{2k} P_1^{(v)} [F_{\nabla_t}] \\ &\quad + P_2^{(2k)} [F_{\nabla_t}] + \sum_{i=0}^k D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \end{aligned} \tag{4-3}$$

which implies (4-1).

From (3-5),

$$\frac{\partial}{\partial t} [\nabla_t^{(l)} F_{\nabla_t}] = \left[\nabla_t^{(l)} D_{\nabla_t} \frac{\partial \nabla_t}{\partial t} \right]_{T_1} + \left[\sum_{j=0}^{l-1} \left(\nabla_t^{(j)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(l-j-1)} F_{\nabla_t} \right) \right]_{T_2}. \tag{4-4}$$

We manipulate T_1 first. Using the Weitzenböck formula (Lemma 2.1) and Lemma 2.3 yields

$$\begin{aligned} T_1 &= (-1)^k \nabla_t^{(l)} \Delta_{\nabla_t}^{(k+1)} F_{\nabla_t} + \nabla_t^{(l)} [(Rm + F_{\nabla_t}) * \Delta_{\nabla_t}^{(k)} F_{\nabla_t}] \\ &\quad + \sum_{v=0}^{2k+l} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k+l)} [F_{\nabla_t}] + \sum_{i=0}^k \nabla_t^{(l)} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \\ &= (-1)^k \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} F_{\nabla_t} + \sum_{v=0}^{2k+l} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k+l)} [F_{\nabla_t}] \\ &\quad + \sum_{i=0}^k \nabla_t^{(l)} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t). \end{aligned}$$

Next, we manipulate T_2 .

$$\begin{aligned} T_2 &= \sum_{j=0}^{l-1} \left[\nabla_t^{(j)} \left((-1)^{k+1} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k-1)} [F_{\nabla_t}] \right. \right. \\ &\quad \left. \left. + \sum_{i=0}^k \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \right) * \nabla_t^{(l-j-1)} F_{\nabla_t} \right] \\ &= P_2^{(2k+l)} [F_{\nabla_t}] + \sum_{v=0}^{2k+l-2} P_2^{(v)} [F_{\nabla_t}] + P_3^{(2k+l-2)} [F_{\nabla_t}] \\ &\quad + \sum_{j=0}^{l-1} \sum_{i=0}^k [\nabla_t^{(j)} (\nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t))] * \nabla_t^{(l-j-1)} F_{\nabla_t}. \end{aligned}$$

Combining T_1 and T_2 yields (4-2). □

LEMMA 4.2. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Then*

$$\begin{aligned} \frac{\partial}{\partial t} [\nabla_t^{(l)} u_t] &= (-1)^k \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} u_t + \sum_{j=0}^{2k+l} (\nabla_M^{(j)} Rm + \nabla_t^{(j)} F_{\nabla_t}) * \nabla_t^{(2k+l-j)} u_t \\ &\quad + \sum_{j=0}^{l-1} \nabla_t^{(j)} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} * \nabla_t^{(l-j-1)} u_t \\ &\quad + \sum_{j=0}^{l-1} \sum_{v=0}^{2k+j-1} P_1^{(v)} [F_{\nabla_t}] * \nabla_t^{(l-j-1)} u_t \\ &\quad + \sum_{j=0}^{l-1} P_2^{(2k+j-1)} [F_{\nabla_t}] * \nabla_t^{(l-j-1)} u_t \\ &\quad + \sum_{j=0}^{l-1} \sum_{i=0}^k [\nabla_t^{(j)} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t)] * \nabla_t^{(l-j-1)} u_t. \end{aligned} \tag{4-5}$$

PROOF. From (3-3) and (1-3),

$$\begin{aligned} \frac{\partial}{\partial t} [\nabla_t^{(l)} u_t] &= \nabla_t^{(l)} \frac{\partial u_t}{\partial t} + \sum_{j=0}^{l-1} \nabla_t^{(j)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(l-j-1)} u_t \\ &= \nabla_t^{(l)} (-\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t) \\ &\quad + \sum_{j=0}^{l-1} \nabla_t^{(j)} \left[(-1)^{k+1} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{\nabla_t}] \right. \\ &\quad \left. + P_2^{(2k-1)} [F_{\nabla_t}] + \sum_{i=0}^k \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \right] * \nabla_t^{(l-j-1)} u_t. \end{aligned}$$

Then using Lemmas 2.2 and 2.3 yields the desired result. □

4.2. Estimates for derivatives of the Higgs field. In this subsection, we prove local L^2 -derivative estimates for the Higgs field.

The following proposition is a direct consequence of Lemma 4.2.

PROPOSITION 4.3. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Then*

$$\begin{aligned} &\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_t^{(l)} u_t\|_{L^2}^2 \\ &= 2(-1)^k \int_M \langle \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \\ &\quad + \int_M \sum_{j=0}^{2k+1} \langle (\nabla_M^{(j)} \text{Rm} + \nabla_t^{(j)} F_{\nabla_t}) * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \\ &\quad + \int_M \sum_{j=0}^{l-1} \langle \nabla_t^{(j)} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} * \nabla_t^{(l-j-1)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \\ &\quad + \int_M \sum_{j=0}^{l-1} \left\langle \left(\sum_{v=0}^{2k+j-1} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k+j-1)} [F_{\nabla_t}] \right) * \nabla_t^{(l-j-1)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \\ &\quad + \int_M \sum_{j=0}^{l-1} \sum_{i=0}^k \langle [\nabla_t^{(j)} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t)] * \nabla_t^{(l-j-1)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle. \end{aligned} \tag{4-6}$$

We estimate each term on the right-hand side of the above equality. We first introduce the bump function that is highly necessary in the smooth estimates.

Let $\mathfrak{B} := \{\gamma \in C_c^\infty(M) : 0 \leq \gamma \leq 1\}$, that is, the family of bump functions. For $l \in \mathbb{N}$, we denote

$$J_\gamma^{(l)} := \sum_{j=0}^l \|\nabla^{(j)}\gamma\|_{L^\infty(M)}.$$

We also need the following lemma. This can be proved by integration by parts and by an induction method.

LEMMA 4.4 [21, Lemma 3.10]. *Let $p, q, r, s \in \mathbb{N}$, $\nabla \in \mathcal{A}_E$ and $\gamma \in \mathfrak{B}$. If $s \in \mathbb{N} \setminus \{1\}$, then*

$$\begin{aligned} & \int_M (P_1^{(p)}[\phi] * P_1^{(q+r)}[\phi])\gamma^s d\text{vol}_g \\ & \leq \int_M (P_1^{(p+r)}[\phi] * P_1^{(q)}[\phi])\gamma^s d\text{vol}_g \\ & \quad + \sum_{j=0}^{r-1} J_\gamma^{(1)} \int_M (P_1^{(p+j)}[\phi] * P_1^{(q+r-j-1)}[\phi])\gamma^{s-1} d\text{vol}_g, \end{aligned}$$

where ϕ is in some tensor product of TM, E and their corresponding duals.

Now we are ready to handle the right-hand side of (4-6).

LEMMA 4.5. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$, that $Q = \max\{1, \sup_{t \in [0, T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t \in [0, T)} |u_t|\}$ and γ is a bump function. Then, for $s \geq 2(k+l+1)$, there exist $\lambda \in [1, 2)$ and $C := C(\dim(M), \text{rk}(E), s, k, l, g, h, \gamma)$ such that*

$$\begin{aligned} 2(-1)^k \int_M \langle \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle & \leq -\lambda \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 \\ & \quad + CQK^2 \|u_t\|_{L^2, \gamma > 0}^2. \end{aligned}$$

PROOF. From Lemma 2.4,

$$\begin{aligned} & 2(-1)^k \int_M \langle \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \\ & = \left[-2 \int_M \langle \nabla_t^{(k+1)} \nabla_t^{(l)} u_t, \nabla_t^{(k+1)} (\gamma^s \nabla_t^{(l)} u_t) \rangle \right]_{T_1} \\ & \quad + \left[\int_M \sum_{j=0}^{2k} \langle \nabla_t^{(j)} \text{Rm} * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \right]_{T_2} \\ & \quad + \left[\int_M \sum_{j=0}^{2k} \langle \nabla_t^{(j)} F_{\nabla_t} * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \rangle \right]_{T_3}. \end{aligned} \tag{4-7}$$

We manipulate T_1 first. Direct computation yields

$$\begin{aligned}
 T_1 &= -2\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 + \int_M \sum_{j=1}^{k+1} \nabla^{(j)}\gamma^s * \langle \nabla_t^{(k+l+1)}u_t, \nabla_t^{(k+l+1-j)}u_t \rangle \\
 &\leq -2\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 + \int_M \sum_{j=1}^{k+1} C|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t|\gamma^{(s-2j)/2}\nabla_t^{(k+l+1-j)}u_t| \\
 &\leq -2\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 + C\varepsilon_1\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 \\
 &\quad + \sum_{j=1}^{k+1} \frac{C}{\varepsilon_1}\|\gamma^{(s-2j)/2}\nabla_t^{(k+l+1-j)}u_t\|_{L^2}^2 \\
 &\leq (-2 + C(\varepsilon_1 + \varepsilon^{-1}\varepsilon_2))\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 + C\varepsilon_1^{-1}\varepsilon_2^{-1}K^2\|u_t\|_{L^2, \gamma>0}^2,
 \end{aligned}$$

where we use the following identity ([36, Lemma 6.2]) in the second inequality:

$$\nabla^{(j)}\gamma^s = \sum_{p_1+\dots+p_j=j} C_{p_1,\dots,p_j}(\gamma, s)\gamma^{s-j}\nabla^{(p_1)}\gamma * \dots * \nabla^{(p_j)}\gamma \tag{4-8}$$

and we use Lemma 2.6 in the last inequality.

Next, we manipulate T_2 . We divide up the summation into cases when j is either odd or even and apply Lemma 4.4,

$$\begin{aligned}
 T_2 &= \int_M \sum_{j \in 2\mathbb{N} \cup \{0\}}^{2k} P_2^{(2k+2l-j)}[u_t]\gamma^s + \int_M \sum_{j \in 2\mathbb{N}-1}^{2k-1} P_2^{(2k+2l-j)}[u_t]\gamma^s \\
 &\leq \left[\int_M \sum_{j \in 2\mathbb{N} \cup \{0\}}^{2k} P_2^{(2k+2l-j)}[u_t]J_\gamma^{(1)}\gamma^{s-1} \right]_{T_2, \text{Even}} \\
 &\quad + \left[\int_M \sum_{j \in 2\mathbb{N}-1}^{2k-1} P_1^{(\lceil(2k+2l-j)/2\rceil)}[u_t] * P_1^{\lfloor(2k+2l-j)/2\rfloor}[u_t]\gamma^s \right]_{T_2, \text{Odd}}.
 \end{aligned}$$

For the even part of T_2 , applying Lemmas 2.5 and 2.6,

$$\begin{aligned}
 \int_M P_2^{(2k+2l-j)}[u_t]J_\gamma^{(1)}\gamma^{s-1} &\leq C(\|\gamma^{(s-1)/2}\nabla_t^{(k+l-j/2)}u_t\|_{L^2}^2 + \|u_t\|_{L^2, \gamma>0}^2) \\
 &= C(\|\gamma^{(s-1)/2}\nabla_t^{(k+l+1-j/2)}u_t\|_{L^2}^2 + \|u_t\|_{L^2, \gamma>0}^2) \\
 &\leq \varepsilon\|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2 + CK^2\|u_t\|_{L^2, \gamma>0}^2.
 \end{aligned}$$

For the odd part of T_2 , applying the Hölder inequality and Lemmas 2.5 and 2.6,

$$\begin{aligned} & \int_M P_1^{(\lceil(2k+2l-j)/2\rceil)}[u_t] * P_1^{(\lfloor(2k+2l-j)/2\rfloor)}[u_t] \gamma^s \\ & \leq 2 \int_M P_2^{(2\lceil(2k+2l-j)/2\rceil)}[u_t] \gamma^s + 2 \int_M P_2^{(2\lfloor(2k+2l-j)/2\rfloor)}[u_t] \gamma^s \\ & \leq C(\|\gamma^{s/2} \nabla_t^{(\lceil(2k+2l-j)/2\rceil)} u_t\|_{L^2}^2 + \|u_t\|_{L^2, \gamma > 0}^2) \\ & \quad + C(\|\gamma^{s/2} \nabla_t^{(\lfloor(2k+2l-j)/2\rfloor)} u_t\|_{L^2}^2 + \|u_t\|_{L^2, \gamma > 0}^2) \\ & \leq \varepsilon \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + CK^2 \|u_t\|_{L^2, \gamma > 0}^2. \end{aligned}$$

Therefore,

$$T_2 \leq \varepsilon \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + CK^2 \|u_t\|_{L^2, \gamma > 0}^2.$$

Finally, we manipulate T_3 .

$$\begin{aligned} T_3 &= \int_M \sum_{j=0}^{2k} \sum_{i=0}^j \langle \nabla_t^{(i)} (F_{\nabla_t} * \nabla_t^{(2k+l-i)} u_t), \gamma^s \nabla_t^{(l)} u_t \rangle \\ &= \int_M \sum_{j=0}^{2k} \sum_{i=0}^j \langle F_{\nabla_t} * \nabla_t^{(2k+l-i)} u_t, P_1^{(i)} [\gamma^s \nabla_t^{(l)} u_t] \rangle \\ &= \int_M \sum_{j=0}^{2k} \sum_{i=0}^j \left\langle F_{\nabla_t} * \nabla_t^{(2k+l-i)} u_t, \sum_{v=0}^i \nabla_t^{(v)} \gamma^s * \nabla_t^{(l+i-v)} u_t \right\rangle \\ &\leq CQ \int_M \sum_{v=0}^{2k} \gamma^{s-v} P_2^{(2k+2l-v)}[u_t], \end{aligned}$$

where we use (4-8). We divide up the summation into cases when v is either odd or even. Similar to T_2 ,

$$T_3 \leq \varepsilon \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + CQK^2 \|u_t\|_{L^2, \gamma > 0}^2.$$

Combining T_1, T_2 and T_3 , we complete the proof. The constraints on s can be easily checked and we omit this here. □

Similar to the proof of Lemma 4.5, we can derive the estimates for the rest of the terms of (4-6). Despite the term involving $P_2^{(2k+j-1)}[F_{\nabla_t}]$, the rest of the terms of (4-6) are very similar to the ones appearing in Saratchandran’s paper [36, Proposition 6.7].

As for $P_2^{(2k+j-1)}[F_{\nabla_t}]$, we can write it as

$$\begin{aligned}
 P_2^{(2k+j-1)}[F_{\nabla_t}] &= \sum_{\mu=0}^{2k+j-1} P_1^{(\mu)}[F_{\nabla_t}] * P_1^{(2k+j-1-\mu)}[F_{\nabla_t}] \\
 &= \sum_{\mu=0}^{2k+j-1} \sum_{\nu=0}^{\mu} \nabla_t^{(\nu)}(F_{\nabla_t} * \nabla_t^{(2k+j-1-\nu)}[F_{\nabla_t}]).
 \end{aligned}$$

Then, integrating by parts, there are no derivatives of F_{∇_t} appearing in the equation. Noting that F_{∇_t} is bounded, it remains to control $\nabla_t^{(2k+j-1-\nu)}[F_{\nabla_t}]$. After integrating by parts again, we have the following local L^2 -derivative estimate for the Higgs fields.

PROPOSITION 4.6. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Assume that $Q = \max\{1, \sup_{t \in [0, T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t \in [0, T)} |u_t|\}$ and γ is a bump function. Then, for $s \geq 2(k + l + 1)$, there exist $\lambda \in [1, 2)$ and $C := C(\dim(M), \text{rk}(E), s, k, l, g, h, \gamma)$ such that*

$$\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_t^{(l)} u_t\|_{L^2}^2 \leq -\lambda \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + CQ^2 K^4 \|u_t\|_{L^2, \gamma > 0}^2.$$

4.3. Estimates for derivatives of the curvature. Similar to the former subsection, we present local L^2 -derivative estimates for the curvature F_{∇_t} .

From the evolution equation (4-2), we have the following result.

PROPOSITION 4.7. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Then*

$$\begin{aligned}
 &\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_t^{(l)} F_{\nabla_t}\|_{L^2}^2 \\
 &= 2(-1)^k \int_M \langle \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} F_{\nabla_t}, \gamma^s \nabla_t^{(l)} F_{\nabla_t} \rangle \\
 &\quad + \int_M \left\langle \sum_{\nu=0}^{2k+l} (P_1^{(\nu)}[F_{\nabla_t}] + P_2^{(\nu)}[F_{\nabla_t}] + P_3^{(2k+l-2)}[F_{\nabla_t}]), \gamma^s \nabla_t^{(l)} F_{\nabla_t} \right\rangle \\
 &\quad + \int_M \left\langle \sum_{i=0}^k \nabla_t^{(l)} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \gamma^s \nabla_t^{(l)} F_{\nabla_t} \right\rangle \\
 &\quad + \int_M \left\langle \sum_{j=0}^{l-1} \sum_{i=0}^k [\nabla_t^{(j)} (\nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t))] * \nabla_t^{(l-j-1)} F_{\nabla_t}, \gamma^s \nabla_t^{(l)} F_{\nabla_t} \right\rangle.
 \end{aligned}$$

Similar to the proof of Lemma 4.5, we have the following local L^2 -derivative estimate for the curvature.

PROPOSITION 4.8. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Assume that $Q = \max\{1, \sup_{t \in [0, T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t \in [0, T)} |u_t|\}$ and γ is a bump function. Then, for $s \geq 2(k + l + 1)$, there exist $\lambda \in [1, 2)$ and $C := C(\dim(M), \text{rk}(E), s, k, l, g, h, \gamma)$ such that*

$$\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_t^{(l)} F_{\nabla_t}\|_{L^2}^2 \leq -\lambda \|\gamma^{s/2} \nabla_t^{(k+l+1)} F_{\nabla_t}\|_{L^2}^2 + CQ^4 K^2 \|F_{\nabla_t}\|_{L^2, \gamma > 0}^2.$$

4.4. Coupled estimates for the curvature and the Higgs field. Since the Yang–Mills–Higgs k -flow is a coupled system, we cannot obtain a local estimate for the curvature or the Higgs field alone. From Propositions 4.6 and 4.8, we have the following proposition.

PROPOSITION 4.9. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. Assume that $Q = \max\{1, \sup_{t \in [0, T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t \in [0, T)} |u_t|\}$ and γ is a bump function. Then, for $s \geq 2(k + l + 1)$, there exist $\lambda \in [1, 2)$ and $C := C(\dim(M), \text{rk}(E), s, k, l, g, h, \gamma)$ such that*

$$\begin{aligned} & \frac{\partial}{\partial t} (\|\gamma^{s/2} \nabla_t^{(l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^{s/2} \nabla_t^{(l)} u_t\|_{L^2}^2) \\ & \leq -\lambda (\|\gamma^{s/2} \nabla_t^{(k+l+1)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2) \\ & \quad + CQ^4 K^4 (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2). \end{aligned}$$

Using the above proposition and following [21, 36], we can derive estimates of Bernstein–Bando–Shi type.

PROPOSITION 4.10. *Let $q \in \mathbb{N}$ and γ be a bump function. Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times I$. Assume that $Q = \max\{1, \sup_{t \in I} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t \in I} |u_t|\}$ and choose $s \geq (k + 1)(q + 1)$. Then, for $t \in [0, T) \subset I$ with $T < 1/((QK)^4)$, there exists a positive constant $C_q := C_q(\dim(M), \text{rk}(E), q, k, s, g, h, \gamma) \in \mathbb{R}_{>0}$ such that*

$$\|\gamma^s \nabla_t^{(q)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(q)} u_t\|_{L^2}^2 \leq C_q t^{-q/(k+1)} \sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2). \tag{4-9}$$

PROOF. Set $a_q := 1$ and let $\{a_l\}_{l=0}^{q-1} \subset \mathbb{R}$ be coefficients to be determined. Define

$$\Phi(t) := \sum_{l=0}^q a_l t^l (\|\gamma^s \nabla_t^{((k+1)l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)l)} u_t\|_{L^2}^2).$$

Differentiating Φ and applying Proposition 4.9,

$$\begin{aligned} \frac{\partial}{\partial t} \Phi(t) &= \sum_{l=1}^q l a_l t^{l-1} (\|\gamma^s \nabla_t^{((k+1)l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)l)} u_t\|_{L^2}^2) \\ &\quad + \sum_{l=0}^q a_l t^l \frac{\partial}{\partial t} (\|\gamma^s \nabla_t^{((k+1)l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)l)} u_t\|_{L^2}^2) \\ &\leq \sum_{l=0}^{q-1} (l+1) a_{l+1} t^l (\|\gamma^s \nabla_t^{((k+1)(l+1))} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)(l+1))} u_t\|_{L^2}^2) \\ &\quad + \sum_{l=0}^q a_l t^l [-\|\gamma^s \nabla_t^{((k+1)(l+1))} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)(l+1))} u_t\|_{L^2}^2) \\ &\quad + CQ^4 K^4 (\|F_{\nabla_t}\|_{L^2, \gamma>0}^2 + \|u_t\|_{L^2, \gamma>0}^2) \\ &= -t^q (\|\gamma^s \nabla_t^{((k+1)(q+1))} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)(q+1))} u_t\|_{L^2}^2) \\ &\quad + \sum_{l=0}^{q-1} [a_{l+1}(l+1) - a_l] t^l (\|\gamma^s \nabla_t^{((k+1)(l+1))} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)(l+1))} u_t\|_{L^2}^2) \\ &\quad + CQ^4 K^4 \sum_{l=0}^q a_l t^l (\|F_{\nabla_t}\|_{L^2, \gamma>0}^2 + \|u_t\|_{L^2, \gamma>0}^2). \end{aligned}$$

Using the initial condition $\alpha_q = 1$, we choose constants satisfying the recursion relation

$$a_{l+1}(l+1) - a_l \leq 0,$$

and also satisfying $a_l \geq \frac{q!}{l!}$.

Noting that $T < 1/((QK)^4)$ and choosing $C_{(k+1)q} \geq C(\sum_{l=0}^q a_l)$,

$$\frac{\partial}{\partial t} \Phi(t) \leq C_{(k+1)q} Q^4 K^4 (\|F_{\nabla_t}\|_{L^2, \gamma>0}^2 + \|u_t\|_{L^2, \gamma>0}^2),$$

which means that

$$\Phi(t) - \Phi(0) \leq C_{(k+1)q} Q^4 K^4 \int_0^t (\|F_{\nabla_\tau}\|_{L^2, \gamma>0}^2 + \|u_\tau\|_{L^2, \gamma>0}^2) d\tau.$$

Therefore,

$$\begin{aligned} &t^q (\|\gamma^s \nabla_t^{((k+1)q)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)q)} u_t\|_{L^2}^2) \\ &\leq C_{(k+1)q} T Q^4 K^4 \sup_{t \in [0, T]} (\|F_{\nabla_t}\|_{L^2, \gamma>0}^2 + \|u_t\|_{L^2, \gamma>0}^2) + \Phi(0) \\ &\leq C_{(k+1)q} \sup_{t \in [0, T]} (\|F_{\nabla_t}\|_{L^2, \gamma>0}^2 + \|u_t\|_{L^2, \gamma>0}^2) + q! (\|F_{\nabla_0}\|_{L^2, \gamma>0}^2 + \|u_0\|_{L^2, \gamma>0}^2), \end{aligned}$$

which means that

$$\begin{aligned} & \|\gamma^s \nabla_t^{(k+1)q} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(k+1)q} u_t\|_{L^2}^2 \\ & \leq C_{(k+1)q} t^{-q} \sup_{t \in [0, T]} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2). \end{aligned}$$

To complete the proof, it remains to consider $\|\gamma^s \nabla_t^{(k+1)l+w} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(k+1)l+w} u_t\|_{L^2}^2$, where $l \in \mathbb{N} \cup \{0\}$ and $w \in [1, k] \cap \mathbb{N}$. From Lemma 2.6, combined with $T < 1/(Q^4 K^4)$,

$$\begin{aligned} & \|\nabla_t^{(k+1)l+w} F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)l+w} u_t\|_{L^2}^2 \\ & \leq \varepsilon (\|\nabla_t^{(k+1)(l+1)} F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)(l+1)} u_t\|_{L^2}^2) \\ & \quad + C_\varepsilon (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2) \\ & \leq C_{(k+1)(l+1)} t^{-l-1} \sup_{t \in [0, T]} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2) \\ & \quad + C_\varepsilon t^{-l-1} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2) \\ & \leq C t^{-((k+1)(l+1)+w)/(k+1)} \sup_{t \in [0, T]} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2). \end{aligned}$$

Therefore, we have established (4-9) for all q . □

The following corollary is a direct consequence of the above inequality that is used in the blow-up analysis. The proof relies on embedding $W^{p,2} \subset C^0$ provided $p > n/2$ and then uses Kato’s inequality $|d|u_t| \leq |\nabla_t u_t|$. More details can be found in Kelleher’s paper [21, Corollary 3.14].

COROLLARY 4.11. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, \tau]$. Set $\bar{\tau} := \min\{\tau, 1\}$. Assume that $Q = \max\{1, \sup_{t \in [0, \bar{\tau}]} |F_{\nabla_t}|\}$ and $K = \max\{1, \sup_{t \in [0, \bar{\tau}]} |u_t|\}$. Suppose that γ is a bump function. For $s, l \in \mathbb{N}$ with $s \geq (k + 1)(l + 1)$, there exists $C_l > 0$ depending on $\dim(M)$, $\text{rk}(E)$, K , Q , $s, k, l, \tau, g, h, \gamma$ such that*

$$\sup_M (|\gamma^s \nabla_{\bar{\tau}}^{(l)} F_{\nabla_{\bar{\tau}}}|^2 + |\gamma^s \nabla_{\bar{\tau}}^{(l)} u_{\bar{\tau}}|^2) \leq C_l \sup_{M \times [0, \bar{\tau}]} (\|F_{\nabla_t}\|_{L^2, \gamma > 0}^2 + \|u_t\|_{L^2, \gamma > 0}^2).$$

REMARK 4.12. Corollary 4.11 has no dependence on the initial data (∇_0, u_0) .

Using Corollary 4.11, we have the following corollary that can be used for finding obstructions to long-time existence.

COROLLARY 4.13. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$ for $T \in [0, +\infty)$. Assume that*

$$Q = \max\{1, \sup_{t \in [0, T)} |F_{\nabla_t}|, \sup_{t \in [0, T)} \|F_{\nabla_t}\|_{L^2}\}$$

and

$$K = \max\{1, \sup_{t \in [0, T)} |u_t|, \sup_{t \in [0, T)} \|u_t\|_{L^2}\}$$

are finite. Suppose that γ is a bump function. Then, for $t \in [0, T)$, $s, l \in \mathbb{N}$ with $s \geq (k + 1)(l + 1)$, there exists a constant $C_l > 0$ depending on $\nabla_0, u_0, \dim(M), \text{rk}(E), K, Q, s, k, l, g, h, \gamma$ such that

$$\sup_{M \times [0, T)} (|\gamma^s \nabla_t^{(l)} F_{\nabla_t}|^2 + |\gamma^s \nabla_t^{(l)} u_t|^2) \leq C_l.$$

4.5. Long-time existence obstruction. In this section, we use Corollary 4.13 to show that the only obstruction to long-time existence of the Yang–Mills–Higgs k -flow (1-3) is a lack of a supremal bound on $|F_{\nabla_t}| + |u_t|$.

We first recall Kelleher’s lemma.

LEMMA 4.14 [21, Lemma 3.17]. Let $\nabla, \tilde{\nabla} \in \mathcal{A}_E$ and set $\Upsilon := \tilde{\nabla} - \nabla$. Then, for all ξ in some tensor product of TM, E and their corresponding duals,

$$\tilde{\nabla}^{(l)} \xi = \nabla^{(l)} \xi + \sum_{j=0}^{l-1} \sum_{i=0}^j (\tilde{P}_{l-1-i}^{(i)}[\Upsilon] * \tilde{P}_1^{(j-i)}[\xi]).$$

For later use, given a one-parameter family (∇_t, u_t) over $M \times [0, T)$ with $T < +\infty$, set

$$\Upsilon_s := \int_0^s \frac{\partial \nabla_t}{\partial t} dt, \quad \Psi_s := \int_0^s \frac{\partial u_t}{\partial t} dt.$$

PROPOSITION 4.15. Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$ for $T \in [0, +\infty)$. Suppose that for all $l \in \mathbb{N} \cup \{0\}$ there exists $C_l \in \mathbb{R}_{>0}$ such that

$$\max \left\{ \sup_{M \times [0, T)} \left| \nabla_t^{(l)} \left[\frac{\partial \nabla_t}{\partial t} \right] \right|, \sup_{M \times [0, T)} \left| \nabla_t^{(l)} \left[\frac{\partial u_t}{\partial t} \right] \right| \right\} \leq C_l.$$

Then $\lim_{t \rightarrow T} (\nabla_t, u_t) = (\nabla_T, u_T)$ exists and is smooth.

PROOF. For all $s \leq T$,

$$|\Upsilon_s| = \left| \int_0^s \frac{\partial \nabla_t}{\partial t} dt \right| \leq TC_0, \quad |\Psi_s| = \left| \int_0^s \frac{\partial u_t}{\partial t} dt \right| \leq TC_0,$$

which means that (∇_T, u_T) is continuous.

Next, we demonstrate that (∇_T, u_T) is smooth. The proof proceeds by induction on l satisfying $|\nabla_0^{(l)}[\Psi_T]| + |\nabla_0^{(l)}[\Upsilon_T]| < +\infty$. For the base case,

$$\begin{aligned} |\nabla_0[\Psi_s]| &= \left| \int_0^s \nabla_0 \left[\frac{\partial u_t}{\partial t} \right] dt \right| \leq \int_0^s \left(\left| \nabla_t \left[\frac{\partial u_t}{\partial t} \right] \right| + C \left| \Upsilon_t \right| \left| \frac{\partial u_t}{\partial t} \right| \right) dt \\ &\leq TC_1 + CT^2 C_0^2 < +\infty. \end{aligned}$$

We also have

$$|\nabla_0[\Upsilon_s]| < +\infty.$$

Now suppose that the induction hypothesis is satisfied for $\{1, \dots, l - 1\}$. Expanding $\nabla_0^{(l)}[\Psi_s]$, applying Lemma 4.14 and then by assumption,

$$|\nabla_0^{(l)}[\Psi_s]| = \int_0^s \left(\left| \nabla_t^{(l)} \left[\frac{\partial u_t}{\partial t} \right] \right| + \sum_{j=0}^{l-1} \sum_{i=0}^j \left(P_{l-i-1}^{(i)}[\Upsilon_t] * P_1^{(j-i)} \left[\frac{\partial u_t}{\partial t} \right] \right) \right) dt < +\infty,$$

where the notation P is taken with respect to ∇_t . Similarly,

$$|\nabla_0^{(l)}[\Upsilon_s]| < +\infty.$$

Since the bounds are uniform for all $t \in [0, T)$ and Υ_s, Φ_s are continuous,

$$|\nabla_0^{(l)}[\Psi_T]| + |\nabla_0^{(l)}[\Upsilon_T]| < +\infty.$$

Thus, Υ_T, Φ_T are smooth. This completes the proof. □

Using Proposition 4.15, we are ready to prove the main result in this subsection.

THEOREM 4.16. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) for some maximal $T < +\infty$. Then*

$$\sup_{M \times [0, T)} (|F_{\nabla_t}| + |u_t|) = +\infty.$$

PROOF. Suppose to the contrary that

$$\sup_{M \times [0, T)} (|F_{\nabla_t}| + |u_t|) < +\infty.$$

By Corollary 4.13, for all $t \in [0, T)$ and $l \in \mathbb{N} \cup \{0\}$, we have that $\sup_M (|\nabla_t^{(l)} F_{\nabla_t}|^2 + |\nabla_t^{(l)} u_t|^2)$ is uniformly bounded and so, by Proposition 4.15, $\lim_{t \rightarrow T} (\nabla_t, u_t) = (\nabla_T, u_T)$ exists and is smooth. However, by local existence (Theorem 3.2), there exists $\epsilon > 0$ such that (∇_t, u_t) exists over the extended domain $[0, T + \epsilon)$, which contradicts the assumption that T was maximal. □

5. Blow-up analysis

In this section, we address the possibility of Yang–Mills–Higgs k -flow singularities given no bound on $|F_{\nabla_t}| + |u_t|$. To begin with, we establish some preliminary scaling laws for the Yang–Mills–Higgs k -flow.

PROPOSITION 5.1. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. We define the one-parameter family ∇_t^p with local coefficient matrices given by*

$$\Gamma_t^p(x) := \rho \Gamma_{\rho^{2(k+1)}_t}(\rho x),$$

where $\Gamma_t(x)$ are local coefficient matrices of ∇_t . We define the ρ -scaled Higgs field u_t^p by

$$u_t^p(x) := \rho u_{\rho^{2(k+1)}_t}(\rho x).$$

Then $(\nabla_t^\rho, u_t^\rho)$ is also a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $[0, 1/(\rho^{2(k+1)})T)$.

PROOF. We start by computing time derivatives of the scaled connection and Higgs field:

$$\begin{aligned} \frac{\partial \nabla^\rho}{\partial t}(x, t) &= \rho^{2k+3} \frac{\partial \nabla}{\partial t}(\rho x, \rho^{2(k+1)}t), \\ \frac{\partial u^\rho}{\partial t}(x, t) &= \rho^{2k+3} \frac{\partial u}{\partial t}(\rho x, \rho^{2(k+1)}t). \end{aligned}$$

Thus, the desired scaling law holds through the Yang–Mills–Higgs k -flow. □

Next, we show that in the case that the curvature coupled with the Higgs field is blowing up, as one approaches the maximal time, one can extract a blow-up limit. The proof closely follows the arguments in [21, Proposition 3.25].

THEOREM 5.2. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on some maximal time interval $[0, T)$ with $T < +\infty$. Then there exists a blow-up sequence $(\nabla_{t_i}^i, u_{t_i}^i)$ and it converges pointwise to a smooth solution $(\nabla_t^\infty, u_t^\infty)$ to the Yang–Mills–Higgs k -flow (1-3) defined on the domain $\mathbb{R}^n \times \mathbb{R}_{<0}$.*

PROOF. From Theorem 4.16,

$$\limsup_{t \rightarrow T} \sup_M (|F_{\nabla_t}| + \langle u_t, u_t \rangle) = +\infty.$$

Therefore, we can choose a sequence of times $t_i \nearrow T$ within $[0, T)$, and a sequence of points x_i , such that

$$|F_{\nabla_{t_i}}(x_i)| + \langle u_{t_i}(x_i), u_{t_i}(x_i) \rangle = \sup_{M \times [0, t_i]} (|F_{\nabla_t}| + \langle u_t, u_t \rangle).$$

Let $\{\rho_i\} \subset \mathbb{R}_{>0}$ be constants to be determined. Define $\nabla_{t_i}^i(x)$ by

$$\Gamma_{t_i}^i(x) = \rho_i^{1/(2(k+1))} \Gamma_{\rho_i t + t_i}(\rho_i^{1/(2(k+1))} x + x_i)$$

and

$$u_{t_i}^i(x) = \rho_i^{1/(2(k+1))} u_{\rho_i t + t_i}(\rho_i^{1/(2(k+1))} x + x_i).$$

By Proposition 5.1, $(\nabla_{t_i}^i, u_{t_i}^i)$ are also solutions to the Yang–Mills–Higgs k -flow (1-3) and the domain for each $(\nabla_{t_i}^i, u_{t_i}^i)$ is $B_0(\rho_i^{-1/(2(k+1))}) \times [-(t_i/\rho_i), (T - t_i/\rho_i))$. We observe that

$$F_{t_i}^i(x) := F_{\nabla_{t_i}^i}(x) = \rho_i^{1/(k+1)} F_{\nabla_{\rho_i t + t_i}}(\rho_i^{1/(2(k+1))} x + x_i),$$

which means that

$$\begin{aligned} & \sup_{t \in [-t_i/\rho_i, (T-t_i)/\rho_i]} (|F_t^i(x)| + |u_t^i(x)|^2) \\ &= \rho_i^{1/(k+1)} \sup_{t \in [-t_i/\rho_i, (T-t_i)/\rho_i]} (|F_{\nabla_{\rho_i t+t_i}}(\rho_i^{1/(2(k+1))}x + x_i)| + |u_{\rho_i t+t_i}(\rho_i^{1/(2(k+1))}x + x_i)|^2) \\ &= \rho_i^{1/(k+1)} \sup_{t \in [0, t_i]} (|F_{\nabla_t}(x)| + |u_t(x)|^2) \\ &= \rho_i^{1/(k+1)} (|F_{\nabla_{t_i}}(x_i)| + |u_{t_i}(x_i)|^2). \end{aligned}$$

Therefore, setting

$$\rho_i = (|F_{\nabla_{t_i}}(x_i)| + |u_{t_i}(x_i)|^2)^{-(k+1)}$$

gives

$$1 = |F_0^i(0)| + |u_0^i(0)|^2 = \sup_{t \in [-t_i/\rho_i, 0]} (|F_t^i(x)| + |u_t^i(x)|^2). \tag{5-1}$$

Now we are ready to construct smoothing estimates for the sequence (∇_t^i, u_t^i) . Let $y \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_{\leq 0}$. For any $s \in \mathbb{N}$,

$$\sup_{t \in [\tau-1, \tau]} (|\gamma_y^s F_t^i(x)| + |\gamma_y^s u_t^i(x)|^2) \leq 1.$$

By Corollary 4.11, for all $q \in \mathbb{N}$, one may choose $s \geq (k + 1)(q + 1)$ so that there exists a positive constant C_q such that

$$\begin{aligned} & \sup_{x \in B_y(\frac{1}{2})} (|(\nabla_\tau^i)^{(q)} F_\tau^i(x)| + |(\nabla_\tau^i)^{(q)} u_\tau^i(x)|) \\ & \leq \sup_{x \in B_y(1)} (|\gamma_y^s (\nabla_\tau^i)^{(q)} F_\tau^i(x)| + |\gamma_y^s (\nabla_\tau^i)^{(q)} u_\tau^i(x)|) \\ & \leq C_q. \end{aligned}$$

Then, by the Coulomb gauge theorem of Uhlenbeck [47, Theorem 1.3] (also see [17]) and the gauge patching theorem [11, Corollary 4.4.8], passing to a subsequence (without changing notation) and in an appropriate gauge, $(\nabla_t^i, u_t^i) \rightarrow (\nabla_t^\infty, u_t^\infty)$ in C^∞ . □

6. Energy estimates

In this section, we prove that both the Yang–Mills–Higgs k -energy and the Yang–Mills–Higgs energy are bounded along the Yang–Mills–Higgs k -flow.

We first show that the Yang–Mills–Higgs k -energy is bounded.

PROPOSITION 6.1. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. The Yang–Mills–Higgs k -energy (1-1) is decreasing along the flow (1-3).*

PROOF. Direct calculation yields

$$\frac{\partial}{\partial t} \mathcal{YMH}_k(\nabla_t, u_t) = -\left(\left\| \frac{\partial \nabla_t}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial u_t}{\partial t} \right\|_{L^2}^2 \right) \leq 0. \quad \square$$

For later use, we first prove an L^2 -bound for the Higgs field u_t .

LEMMA 6.2. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M \times [0, T)$. We have*

$$\sup_{t \in [0, T)} \|u_t\|_{L^2} < +\infty.$$

PROOF. Direct calculation yields

$$\begin{aligned} \frac{\partial}{\partial t} \int_M \langle u_t, u_t \rangle &= 2 \int_M \langle u_t, -\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t \rangle \\ &= -2 \int_M |\nabla_t^{(k+1)} u_t|^2 \leq 0. \end{aligned} \quad \square$$

Using the above lemma, we can show that the Yang–Mills–Higgs energy is bounded along the Yang–Mills–Higgs k -flow.

PROPOSITION 6.3. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M^4 \times [0, T)$ with $T < +\infty$. Then the Yang–Mills–Higgs energy*

$$\mathcal{YMH}(\nabla_t, u_t) = \frac{1}{2} \int_M [|F_{\nabla_t}|^2 + |\nabla_t u_t|^2] \, d\text{vol}_g$$

is bounded along the flow (1-3).

PROOF. Direct calculation yields

$$\begin{aligned} &\frac{\partial}{\partial t} \mathcal{YMH}(\nabla_t, u_t) \\ &= \int_M \left\langle D_{\nabla_t}^* F_{\nabla_t} + \nabla_t u_t \otimes u_t^*, \frac{\partial \nabla_t}{\partial t} \right\rangle + \int_M \left\langle \frac{\partial u_t}{\partial t}, \nabla_t^* \nabla_t u_t \right\rangle \\ &\leq \int_M \left[\left| \frac{\partial \nabla_t}{\partial t} \right|^2 + \left| \frac{\partial u_t}{\partial t} \right|^2 + C(|\nabla_t F_{\nabla_t}|^2 + |\nabla_t^{(2)} u_t|^2 + |\nabla_t u_t|^2 |u_t|^2) \right] \\ &\leq -\frac{\partial}{\partial t} \mathcal{YMH}_k(\nabla_t, u_t) + C(\|\nabla_t^{(k)} F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)} u_t\|_{L^2}^2) \\ &\quad + \varepsilon(\|F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t u_t\|_{L^2}^2) + C(\|\nabla_t^{(k+1)} u_t\|_{L^2}^4 + \|u_t\|_{L^2}^4), \end{aligned}$$

where we use Lemma 2.6, the Hölder inequality and the following Sobolev inequalities:

$$\begin{aligned} \|u_t\|_{L^4}^2 &\leq C(\|\nabla_t u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2), \\ \|\nabla_t u_t\|_{L^4}^2 &\leq C(\|\nabla_t^{(2)} u_t\|_{L^2}^2 + \|\nabla_t u_t\|_{L^2}^2); \end{aligned}$$

here C is a constant independent of $t \in [0, T]$. Therefore,

$$\begin{aligned} & \mathcal{YMH}(\nabla_t, u_t) - \mathcal{YMH}(\nabla_0, u_0) \\ & \leq CT(\mathcal{YMH}_k(\nabla_0, u_0) + \mathcal{YMH}_k^2(\nabla_0, u_0) + \|u_t\|_{L^2}^4) \\ & \quad + \varepsilon T \sup_{t \in [0, T]} \mathcal{YMH}(\nabla_t, u_t). \end{aligned} \tag{6-1}$$

Next, we borrow an argument in Saratchandran’s paper [36, Theorem 5.3]. Suppose that there exists $t_m \rightarrow T$ such that

$$\lim_{m \rightarrow +\infty} \mathcal{YMH}(\nabla_{t_m}, u_{t_m}) \rightarrow +\infty.$$

By discarding some of the t_m , we can assume that

$$\mathcal{YMH}(\nabla_{t_m}, u_{t_m}) > \mathcal{YMH}(\nabla_{t_{m'}}, u_{t_{m'}})$$

for $m \geq m'$ and that $t_m \geq t_{m'}$ when $m \geq m'$. Partition $[0, T) = [t_0, t_1] \cup [t_1, t_2] \cup \dots \cup [t_k, t_{k+1}] \cup \dots$ with $t_0 = 0$. Define $s_i \in [t_i, t_{i+1}]$ by

$$\sup_{t \in [t_i, t_{i+1}]} \mathcal{YMH}(\nabla_t, u_t) = \mathcal{YMH}(\nabla_{s_i}, u_{s_i}).$$

It is easy to see that $s_i \rightarrow T$ and $\mathcal{YMH}(\nabla_{s_i}, u_{s_i}) \rightarrow +\infty$ as $i \rightarrow +\infty$. Furthermore, $\mathcal{YMH}(\nabla_{s_j}, u_{s_j}) \leq \mathcal{YMH}(\nabla_{s_i}, u_{s_i})$ when $j \leq i$. Then, substituting s_i for t in (6-1),

$$\begin{aligned} & \mathcal{YMH}(\nabla_{s_i}, u_{s_i}) - \mathcal{YMH}(\nabla_0, u_0) - \varepsilon T \mathcal{YMH}(\nabla_{s_i}, u_{s_i}) \\ & \leq CT(\mathcal{YMH}_k(\nabla_0, u_0) + \mathcal{YMH}_k^2(\nabla_0, u_0) + \|u_t\|_{L^2}^4), \end{aligned}$$

which means that

$$\begin{aligned} \mathcal{YMH}(\nabla_{s_i}, u_{s_i}) & \leq \frac{1}{1 - \varepsilon T} CT(\mathcal{YMH}_k(\nabla_0, u_0) + \mathcal{YMH}_k^2(\nabla_0, u_0) \\ & \quad + \|u_t\|_{L^2}^4 + \mathcal{YMH}(\nabla_0, u_0)). \end{aligned}$$

The right-hand side of the above inequality is finite and it is independent of i . After taking $i \rightarrow +\infty$ on the left, we reach a contradiction. Thus, no such $\{t_m\}$ exists and the result follows. □

7. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. To accomplish this, we first show that the L^p -norm controls the L^∞ -norm by blow-up analysis.

PROPOSITION 7.1. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs k -flow (1-3) defined on $M^n \times [0, T)$ and*

$$\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^p} + \|\langle u_t, u_t \rangle\|_{L^p}) < +\infty.$$

If $\dim(M) < 2p$, then

$$\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\langle u_t, u_t \rangle\|_{L^\infty}) < +\infty.$$

PROOF. In order to obtain a contradiction, assume that

$$\sup_{t \in [0, T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\langle u_t, u_t \rangle\|_{L^\infty}) = +\infty.$$

As we did in Theorem 5.2, we can construct a blow-up sequence (∇_t^i, u_t^i) with blow-up limit $(\nabla_t^\infty, u_t^\infty)$. Noting that (5-1), by Fatou’s lemma and the natural scaling law,

$$\begin{aligned} \|F_{\nabla_t^\infty}\|_{L^p}^p + \|\langle u_t^\infty, u_t^\infty \rangle\|_{L^p}^p &\leq \liminf_{i \rightarrow +\infty} (\|F_{\nabla_t^i}\|_{L^p}^p + \|\langle u_t^i, u_t^i \rangle\|_{L^p}^p) \\ &\leq \lim_{i \rightarrow +\infty} \rho_i^{(2p-n)/(2k+2)} (\|F_{\nabla_t^i}\|_{L^p}^p + \|\langle u_t, u_t \rangle\|_{L^p}^p). \end{aligned}$$

Since $\lim_{i \rightarrow +\infty} \rho_i^{(2p-n)/(2k+2)} = 0$ when $2p > n$, the right-hand side of the above inequality tends to zero, which is a contradiction since the blow-up limit has nonvanishing curvature. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $\dim(M) = 4$, in order to use Proposition 7.1, we need $p > 2$. By the Sobolev embedding theorem, we solve for p such that $W^{k,2} \subset L^{2p}$ and then $k > 1$. In this case, using Lemma 2.6,

$$\begin{aligned} &\|F_{\nabla_t}\|_{L^p} + \|\langle u_t, u_t \rangle\|_{L^{2p}} \\ &\leq CS_{k,p} \sum_{j=0}^k (\|\nabla_t^{(j)} F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(j)} u_t\|_{L^2}^2 + 1) \\ &\leq CS_{k,p} (\|\nabla_t^{(k)} F_{\nabla_t}\|_{L^2}^2 + \|F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)} u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + 1) \\ &\leq CS_{k,p} (\mathcal{YMH}_k(\nabla_t, u_t) + \mathcal{YMH}(\nabla_t, u_t) + \|u_t\|_{L^2}^2 + 1). \end{aligned}$$

Noting that both $\mathcal{YMH}_k(\nabla_t, u_t)$ and $\mathcal{YMH}(\nabla_t, u_t)$ are bounded along the Yang–Mills–Higgs k -flow (1-3) (see Propositions 6.1 and 6.3), we conclude that the flow exists smoothly for a long time.

REMARK 7.2. Since the blow-up analysis is not valid at $t = +\infty$, we cannot obtain a definite property at infinity in the present paper.

REMARK 7.3. Since $W^{k+1,2} \subset C^0$ when $k > 1$, the C^0 -bound of u_t can be controlled by $\mathcal{YMH}_k(\nabla_t, u_t)$ and $\|u_t\|_{L^2}^2$ via Lemma 2.6. Then the energy estimate and the blow-up become much more easy. We also address this issue in Section 9.

8. Concentration phenomenon for Yang–Mills–Higgs 1-flow

In this section, we show that the long-time existence of Yang–Mills–Higgs 1-flow in dimension four is obstructed by the possibility of concentration of the curvature in smaller and smaller balls.

PROPOSITION 8.1. *Suppose that (∇_t, u_t) is a solution to the Yang–Mills–Higgs 1-flow (1-3) defined on $M^4 \times [0, T)$ with T maximal. Then there exists some $\epsilon > 0$ such that if $\{(x_i, t_i)\} \subset M \times [0, T)$ with $(x_i, t_i) \rightarrow (X, T)$ has the property that*

$$\lim_{i \rightarrow +\infty} (|F_{\nabla_{t_i}}(x_i)| + |u_{t_i}(x_i)|) = +\infty,$$

then, for all $r > 0$,

$$\limsup_{i \rightarrow +\infty} (\|F_{\nabla_{t_i}}\|_{L^2(B_X(r))}^2 + \|\langle u_{t_i}, u_{t_i} \rangle\|_{L^2(B_X(r))}^2) \geq \epsilon,$$

where $B_X(r)$ denotes the geodesic ball of radius r centered at X .

PROOF. Choose a corresponding blow-up sequence (∇_t^i, u_t^i) as described in Proposition 5.2 with limit $(\nabla_t^\infty, u_t^\infty)$. Then, by (5-1),

$$|F_0^\infty(0)| + |u_0^\infty(0)|^2 = 1.$$

By the smoothness of $(\nabla_t^\infty, u_t^\infty)$, for $(y, t) \in B_0(\delta) \times (-\delta, 0]$,

$$|F_t^\infty(y)| + |u_t^\infty(y)|^2 \geq \frac{1}{2}.$$

Therefore,

$$\limsup_{t \nearrow 0} (\|F_t^\infty\|_{L^2(B_0(\delta))}^2 + \|\langle u_t^\infty, u_t^\infty \rangle\|_{L^2(B_0(\delta))}^2) \geq \frac{1}{8} \text{Vol}[B_0(\delta)].$$

Conversely, using the computations in Theorem 7.1,

$$\begin{aligned} & \|F_t^\infty\|_{L^2(B_0(\delta))}^2 + \|\langle u_t^\infty, u_t^\infty \rangle\|_{L^2(B_0(\delta))}^2 \\ &= \int_{B_0(\delta)} \lim_{i \rightarrow +\infty} (|F_{\nabla_{t_i}}|^2 + |\langle u_{t_i}^i, u_{t_i}^i \rangle|^2) \, d\text{vol}_g \\ &= \lim_{i \rightarrow +\infty} \rho_i^{(2 \times 2 - 4)/(2 \times 1 + 2)} (\|F_{\nabla_{t_i}}\|_{L^2(B_{x_i}(\delta \rho_i^{1/4}))}^2 + \|\langle u_{t_i}, u_{t_i} \rangle\|_{L^2(B_{x_i}(\delta \rho_i^{1/4}))}^2) \\ &= \lim_{i \rightarrow +\infty} (\|F_{\nabla_{t_i}}\|_{L^2(B_{x_i}(\delta \rho_i^{1/4}))}^2 + \|\langle u_{t_i}, u_{t_i} \rangle\|_{L^2(B_{x_i}(\delta \rho_i^{1/4}))}^2). \end{aligned}$$

Since $\lim_{i \rightarrow +\infty} \rho_i^{(1/4)} = 0$, then, for any $r > 0$ and i large enough so that $\max\{|T - t_i|\} < \delta$,

$$\frac{1}{8} \text{Vol}[B_0(\delta)] \leq \limsup_{i \rightarrow +\infty} (\|F_{\nabla_{t_i}}\|_{L^2(B_X(r))}^2 + \|\langle u_{t_i}, u_{t_i} \rangle\|_{L^2(B_X(r))}^2).$$

Taking $\epsilon = \frac{1}{8} \text{Vol}[B_0(\delta)]$ yields the result. □

Note that the lower bound given by ϵ is independent of the point about which the blow-up procedure occurred. From Proposition 8.1, we have the following theorem.

THEOREM 8.2. *Let E be a vector bundle over a closed Riemannian 4-manifold (M, g) . For every smooth initial value (∇_0, u_0) , there exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs 1-flow (1-3) existing on $[0, T)$ for some maximal $T \in \mathbb{R}_{>0} \cup \{+\infty\}$. If $T < +\infty$, then there exists a sequence $\{(x_i, t_i)\} \subset M \times [0, T)$ with $(x_i, t_i) \rightarrow (X, T)$ and, for all $r > 0$,*

$$\limsup_{i \rightarrow +\infty} (\|F_{\nabla_{t_i}}\|_{L^2(B_X(r))}^2 + \|\langle u_{t_i}, u_{t_i} \rangle\|_{L^2(B_X(r))}^2) \geq \epsilon.$$

9. Higher order Yang–Mills–Higgs functional with Higgs self-interaction

In [19], Jaffe and Taubes studied the following Yang–Mills–Higgs functional:

$$\mathcal{YMH}(\nabla, u) = \frac{1}{2} \int_M [|F_\nabla|^2 + |\nabla u|^2 + \frac{\lambda}{4} (|u|^2 - 1)^2] d\text{vol}_g, \tag{9-1}$$

where the constant $\lambda \geq 0$. The term $(\lambda/8)(|u|^2 - 1)^2$ is the Higgs self-interaction.

Following the former sections, we consider the following Yang–Mills–Higgs k -functional with Higgs self-interaction:

$$\mathcal{YMH}_k(\nabla, u) = \frac{1}{2} \int_M [|\nabla^{(k)} F_\nabla|^2 + |\nabla^{(k+1)} u|^2 + \frac{\lambda}{4} (|u|^2 - 1)^2] d\text{vol}_g. \tag{9-2}$$

The associated negative gradient flow of (9-2) is the following system:

$$\left\{ \begin{aligned} \frac{\partial \nabla_t}{\partial t} &= (-1)^{(k+1)} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{\nabla_t}] \\ &\quad + P_2^{(2k-1)} [F_{\nabla_t}] + \sum_{i=0}^k \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \\ \frac{\partial u_t}{\partial t} &= -\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t + \frac{\lambda}{2} (1 - |u_t|^2) u_t. \end{aligned} \right. \tag{9-3}$$

Now we can follow the line of the study of the flow (1-3).

- (1) First of all, the local existence and smoothing estimates can be achieved in a similar way and we have the same obstruction (Theorem 4.16) for the long-time existence.
- (2) It is easy to check that (9-2) is decreasing along the flow (9-3).
- (3) One can check that $\|u_t\|_{L^2}$ is bounded along the flow (9-3).
- (4) After setting $\dim(M) < 2(k + 1)$, we have the Sobolev embedding $W^{k+1,2} \subset C^0$. Note that we have $\|\nabla^{(k+1)} u\|_{L^2}^2$ in (9-2). So, the C^0 -bound of u_t can be controlled by (9-2) by using Lemma 2.6.
- (5) Once we have the C^0 -bound of u_t , it is easy to prove that (9-1) is bounded along the flow (9-3).
- (6) Since u_t is bounded along the flow (9-3) when $\dim(M) < 2(k + 1)$, the obstruction to the long-time existence only depends on F_{∇_t} . Choose a corresponding blow-up

- sequence (∇_t^i, u_t^i) as described in Theorem 5.2. Thus, (∇_t^i, u_t^i) converges to $(\nabla_t^\infty, 0)$ smoothly. Also, $u_t^\infty = 0$ since u_t is bounded.
- (7) Using the blow-up analysis, we can control the C^0 -bound of F_{∇_t} by its L^p -bound with $\dim(M) < 2p$.
- (8) Finally, when $\dim(M) < 2(k+2)$, the C^0 -bound of F_{∇_t} can be derived by Lemma 2.6, the blow-up analysis and the Sobolev embedding $W^{k,2} \subset L^p$ with $\dim(M) < 2p$. Therefore, we obtain the following theorem.

THEOREM 9.1. *Let E be a vector bundle over a closed Riemannian manifold (M, g) . Assume that an integer k satisfies $\dim(M) < 2(k+1)$. Then, for every smooth initial value (∇_0, u_0) , there exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k -flow (9-3) in $M \times [0, +\infty)$.*

REMARK 9.2. The critical dimension $2(k+1)$ may not be optimal when the Higgs self-interaction is not vanishing. In this case, we have $\|u_t\|_{L^4}^4$ in $\mathcal{YMH}_k(\nabla_t, u_t)$, which is better than $\|u_t\|_{L^2}^2$. Maybe one can lower the order of derivative of u_t in $\mathcal{YMH}_k(\nabla_t, u_t)$ to obtain a C^0 -bound for u_t .

REMARK 9.3. If $\dim(M) < 2(k+1)$, the long-time existence of the higher order Seiberg–Witten flow studied in [36] can be proved as in Theorem 9.1.

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PAN ZHANG, School of Mathematical Sciences,
Anhui University, Hefei 230601, PR China
e-mail: panzhang20100@ahu.edu.cn