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LOGICS FOR PROPOSITIONAL CONTINGENTISM

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Abstract. Robert Stalnaker has recently advocated propositional contingentism, the claim that it is contingent what propositions there are. He has proposed a philosophical theory of contingency in what propositions there are and sketched a possible worlds model theory for it. In this paper, such models are used to interpret two propositional modal languages: one containing an existential propositional quantifier, and one containing an existential propositional operator. It is shown that the resulting logic containing an existential quantifier is not recursively axiomatizable, as it is recursively isomorphic to second-order logic, and a natural candidate axiomatization for the resulting logic containing an existential operator is shown to be incomplete.

§1. Introduction. Many philosophers hold that it is contingent what there is, on the basis of examples such as myself: many of them think that had I not been born, there would have been no such thing as me. Had there been no such thing as me, would there have been propositions about myself, such as the proposition that I am human? Many philosophers who have considered it have also given a negative answer to this question, and so advocated *propositional contingentism*, the view that it is contingent what propositions there are; see Fritz (2016, p. 123) for references. Taking this kind of aboutness of propositions seriously might motivate one to adopt a finer-grained understanding of propositions than the one assumed in possible world semantics, where necessarily equivalent propositions are taken to be identical. Assuming such a more fine-grained approach, Fine (1980) investigates propositional contingentism in the form of a number of first-order modal theories of propositions. But propositional contingentism is also compatible with more coarse-grained conceptions of propositions; such versions of propositional contingentism are discussed in Stalnaker (2012) and Williamson (2013, chap. 6).

Stalnaker (2012, Appendix A) develops models for propositional contingentism along familiar lines of possible world semantics, identifying propositions with sets of worlds. These models are developed further in Fritz (2016), where they are called *equivalence systems*. They dispense with an accessibility relation, understanding necessity simply as truth in *all* worlds. They further assume that for each world, the propositions there are at this world form a complete atomic Boolean algebra. Consequently, the propositional domain at each world can be specified using an equivalence relation on the set of worlds; the propositions in the propositional domain of a world are the members of the algebra of propositions generated by the equivalence classes of the equivalence relation associated with the world. This is the set of unions of such equivalence classes, so, for every equivalence relation $\sim$ on a set W, let $\mathcal{A}(\sim) = \{\bigcup X : X \subseteq W/{\sim}\}$. Note that for all $P \subseteq W$, $P \in \mathcal{A}(\sim)$ just in case for all $w, v \in W$, $w \sim v$ only if $w \in P$ iff $v \in P$.

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For reasons discussed in Fritz (2016), the equivalence relation associated with a world w can be understood as a relation of indistinguishability among worlds (from the perspective of w). Formally, Stalnaker's models can be defined as follows:

DEFINITION 1.1. *Let W be a set.*

(i) *An* equivalence system on W *is a function $\approx$ mapping every $w \in W$ to an equivalence relation $\approx_w$ on W.*
(ii) *For every equivalence system $\approx$ on W, the* domain function of $\approx$ *is the function $D^{\approx}_w : W \to \mathcal{P}(\mathcal{P}(W))$ such that for all $w \in W$, $D^{\approx}_w = \mathcal{A}(\approx_w)$.*

Stalnaker imposes further constraints on equivalence systems, which are refined and motivated in Fritz (2016). To define the resulting condition, let an *automorphism* of an equivalence system $\approx$ on a set W be a permutation f of W such that $v \approx_w u$ iff $f(v) \approx_{f(w)} f(u)$ for all $w, v, u \in W$. For $w \in W$, let aut$(\approx)$ be the set of automorphisms of $\approx$ and aut$(\approx)_w$ the subset of members of aut$(\approx)$ mapping w to itself (the first forms a group, and the second is sometimes called the stabilizer subgroup of w). When convenient, functions and binary relations will be understood as sets of pairs.

DEFINITION 1.2. *An equivalence system $\approx$ on a set W* coheres *if for all $w, v, u \in W$ such that $v \approx_w u$, there is an $f \in \text{aut}(\approx)_w$ such that $f(v) = u$ and $f \subseteq \approx_w$.*

It is straightforward to interpret a propositional modal language with an existential propositional quantifier $\exists$ on equivalence systems, interpreting $\Box$ as truth in all worlds, and using the propositional domain function as derived above to interpret propositional quantifiers. Alternatively, the existential propositional quantifier may be replaced by an existential propositional operator E, with $E\varphi$ interpreted as expressing that the proposition expressed by φ exists. Investigating the validities in such languages on classes of equivalence systems is a way of investigating the commitments of the theories of propositional contingency encoded in these classes.

§2 investigates the propositionally quantified logic of the class of all equivalence systems and shows that it is not recursively axiomatizable, by showing it to be recursively isomorphic to full second-order logic. This result is extended to the logic of the class of coherent equivalence systems, and the condition of coherence is compared to the condition of validating a comprehension principle stating that every proposition definable using existing parameters exists. §3 investigates logics with an existential operator, providing an axiomatization of the logic of all equivalence system, and showing that the extension of this axiomatization by a version of the comprehension principle just mentioned does not completely axiomatize the logic of coherent equivalence systems. The concluding §4 mentions some further applications and open questions. An appendix systematically develops several notions of congruences and reductions of equivalence systems, which are used in establishing the results of the main text.

§2. Existential quantifiers.

The formal language used in this section adds propositional quantifiers to a standard propositional modal language:

DEFINITION 2.1. *Let $L_\exists$ be the set of formulas built up from a countably infinite set of proposition letters Φ using Boolean operators $\neg$ and $\wedge$, a unary modal operator $\Box$ and an existential quantifier $\exists$ binding proposition letters. Other Boolean operators, $\Diamond$ and $\forall$ will be used as abbreviations as usual.*

The domain functions derived from equivalence systems above lead to the following straightforward way of interpreting $L_\exists$ on equivalence systems:

DEFINITION 2.2. *An $L_\exists$-formula being* true *relative to an equivalence system* $\approx$ *on a set* W*, a world* $w \in W$*, and an assignment function* $a : \Phi \to \mathcal{P}(W)$ *is defined by the following clauses*:

$$
\begin{array}{lll}
\approx, w, a \vDash p & \textit{iff} & w \in a(p) \\
\approx, w, a \vDash \neg\varphi & \textit{iff} & \textit{not} \approx, w, a \vDash \varphi \\
\approx, w, a \vDash \varphi \wedge \psi & \textit{iff} & \approx, w, a \vDash \varphi \textit{ and} \approx, w, a \vDash \psi \\
\approx, w, a \vDash \Box\varphi & \textit{iff} & \approx, v, a \vDash \varphi \textit{ for all } v \in W \\
\approx, w, a \vDash \exists p\varphi & \textit{iff} & \textit{there is a } P \in D^{\approx}_{w} \textit{ such that} \approx, w, a[P/p] \vDash \varphi.
\end{array}
$$

As usual, $a[P/p]$ *is the assignment function which maps* p *to* P *and every other proposition letter* q *to* $a(q)$.

Dropping one or both of the parameters w *and* a *in* $\approx, w, a \vDash \varphi$ *indicates that* φ *is true relative to all worlds and/or assignment functions*; *e.g.,* $\approx, w \vDash \varphi$ *if* $\approx, w, a \vDash \varphi$ *for all assignment functions* a.

An $L_\exists$*-formula* φ *is* valid on *an equivalence system* $\approx$ (*a class* C *of equivalence systems*) *if* $\approx \vDash \varphi$ (*for every equivalence system* $\approx$ *in* C). *An* $L_\exists$*-formula* φ *is* satisfiable on *an equivalence system* (*a class of equivalence systems*) *if its negation is not valid on it. The* $L_\exists$-logic of *a class of equivalence systems is the set of* $L_\exists$*-formulas valid on it.*

Interpreting a propositionally quantified language on variable domain structures seems not to have been investigated in the literature, apart from some brief remarks by Fine (1970, pp. 344–345). As shown by Fine (1970) and Kaplan (1970), $L_\exists$ interpreted over sets of worlds, with $\Box$ interpreted as above and $\exists$ as ranging over *all* subsets, is recursively axiomatizable, and even decidable. Once accessibility relations are added, the situation changes fundamentally, in many cases leading to a logic recursively isomorphic to full second-order logic; see Kaminski & Tiomkin (1996) for details. It will now be shown that the same sharp increase in complexity occurs when variable domains are introduced as constrained by equivalence systems: the $L_\exists$-logic of (the class of all) equivalence systems is also recursively isomorphic to second-order logic.

2.1. Incompleteness. It turns out that for present purposes, second-order logic can be restricted to binary second-order variables, interpreting second-order quantifiers as restricted to symmetric binary relations. To fix notation, define this as follows:

DEFINITION 2.3. *Let* $L2$ *be a language based on a countably infinite set of first-order variables* $x, y, z \dots$ *and a countably infinite set of binary second-order variables* $X, Y, Z \dots,$ *whose formulas are constructed from atomic formulas of the form* Xyz *using Boolean operators* $\neg$ *and* $\wedge$ *and first- and second-order quantifiers of the form* $\exists x$ *and* $\exists X$.

Define truth *of such a formula relative to a set* D *and an assignment function* a *mapping each first-order variable to a member of* D *and each binary second-order variable* X *to a symmetric binary relation on* D*, using the following clauses*:

$$
\begin{array}{lll}
D, a \vDash Xyz & \textit{iff} & \langle a(y), a(z)\rangle \in a(X) \\
D, a \vDash \exists x A & \textit{iff} & \textit{there is a } d \in D \textit{ such that } D, a[d/x] \vDash A \\
D, a \vDash \exists X A & \textit{iff} & \textit{there is a symmetric binary relation } R \subseteq D^2 \\
 & & \textit{such that } D, a[R/X] \vDash A
\end{array}
$$

and the usual Boolean clauses, with $a[d/x]$ *and* $a[D/X]$ *defined as above.*

Let an $L2$*-formula be* true *in a set* D*, written* $D \vDash A$*, if* $D, a \vDash A$ *for all assignment functions* a*, and let* SB *be the set of* $L2$*-sentences* (*closed formulas*) *true in all sets.*

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Note that D in Definition 2.3 may be empty, as often in model theory; see, e.g., Hodges (1997, p. 2). For present purposes, this is of little significance, since SB is easily seen to be recursively isomorphic to the logic obtained by excluding the empty set in the definition of SB. Although SB is in several ways more restrictive than full second-order logic as it is usually defined, these differences can also be ignored for present purposes:

FACT 2.4. *There is a recursive embedding of full second-order logic in* SB.

This follows directly from a result due to Scott and Rabin, which shows that binary second-order quantifiers may even be restricted to symmetric *irreflexive* relations (although this is not required here); see the presentation of their proof by Nerode & Shore (1980) or the variant construction of Kremer (1997, Appendix). By routine considerations, constructing a recursive embedding of SB in the $L_\exists$-logic of equivalence systems thus suffices to establish that the latter is recursively isomorphic to full second-order logic.

The basic idea of the construction of such an embedding is the following: Consider an equivalence system in which each world contains its singleton proposition and in which some world contains every proposition, and assume that there is a set P of worlds such that for any P-worlds v and u, there is a world w such that the only singletons of P-worlds in w are those of v and u; thus w represents, or *codes*, the set $\{v, u\}$. We simulate $L2$ evaluated on P. First-order quantification is simulated by modalized propositional quantification over singletons of P-worlds. To simulate second-order quantifiers over P restricted to binary symmetric relations, note that symmetric binary relations on P correspond to sets of sets of the form $\{v, u\} \subseteq P$. Thus such second-order quantifiers can be simulated as propositional quantification at the world containing all propositions: a set of worlds Q represents the binary symmetric relation which relates P-worlds v and u just in case there is a $w \in W$ which codes $\{v, u\}$. The remainder of this section makes this way of simulating second-order quantification over symmetric binary relations precise and shows how to turn it into a recursive embedding of SB in the $L_\exists$-logic of equivalence systems.

The simulation will first be carried for equivalence systems in which every world contains its singleton; it will later be shown how to eliminate this assumption. Adapting terminology from Fine (1977) and Fritz & Goodman (2016), define:

DEFINITION 2.5. *An equivalence system* $\approx$ *on a set* W *is* world-selective *if* $\{w\} \in D^{\approx}_{w}$ *for all* $w \in W$.

Given a fixed set P of worlds, the following makes the definition of a world w coding a set $\{u, v\}$ of P-worlds precise, and adds corresponding definitions of a singleton proposition coding its member and a proposition coding a symmetric binary relation on P.

DEFINITION 2.6. *Let* $\approx$ *be an equivalence system on a set* W *and* $P \subseteq W$.

(i) $Q \subseteq W$ codes$_P$ $w \in P$ *iff* $Q = \{w\}$.

(ii) $w \in W$ codes$_P$ $\{v, u\} \subseteq P$ *iff for all* $s \in P$, $\{s\} \in D^{\approx}_{w}$ *iff* $s \in \{v, u\}$. $Q \subseteq W$ codes$_P$ *a symmetric binary relation* $R \subseteq P^2$ *iff* R *is the set of pairs* $\langle w, v\rangle \in P^2$ *such that some element of* Q *codes*$_P$ $\{w, v\}$.

Apart from world-selectiveness, the simulation to be given relies – for a given set P of worlds – on there being a world w coding$_P$ any given P-worlds v, u and there being a world containing every proposition. An equivalence system satisfying these constraints will be called *coding*$_P$:

DEFINITION 2.7. *Let* $\approx$ *be an equivalence system on a set* W, *and* $P \subseteq W$. $\approx$ *is* coding$_P$ *if the following conditions are satisfied*:

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($C0_P$) $\approx$ *is world-selective.*
($C1_P$) *For each* $\{v, u\} \subseteq P$*, there is a* $w \in W$ *such that* w *codes*$_P$ $\{v, u\}$*.*
($C2_P$) *There is a* $w \in W$ *such that* $D^{\approx}_w = \mathcal{P}(W)$*.*

$\approx$ *being* coding$_P$ *will also be phrased as* P *being* coded by $\approx$.

To introduce a number of syntactic abbreviations, fix an injective function which maps each variable ξ of $L2$ (first- or second-order) to a proposition letter p_ξ of $L_\exists$, and a proposition letter p_0 not in the image of this function. Define:

$$\begin{array}{lcl} \varphi = \psi & := & \Box(\varphi \leftrightarrow \psi) \\ E\varphi & := & \exists r(r = \varphi) \\ atom(\varphi) & := & E\varphi \wedge \Diamond\varphi \wedge \forall r(\Box(\varphi \to r) \vee \Box(\varphi \to \neg r)) \\ \downarrow q\varphi & := & \forall q((q \wedge atom(q)) \to \varphi) \\ @q\varphi & := & \Box(q \to \varphi) \\ \Pi q\varphi & := & \downarrow r\Box(p_0 \to \downarrow q@r\varphi) \\ \Sigma q\varphi & := & \neg\Pi q\neg\varphi \\ D(q, s) & := & \Pi r(Er \leftrightarrow (r = q \vee r = s)). \end{array}$$

In these definitions, r is assumed to be an arbitrary proposition letter distinct from any proposition letter occurring on the left hand side. $=$ is intended to express identity, E existence and *atom* being an atom of the algebra of propositions of the world of evaluation. The defined operator $\downarrow q$ is intended to bind the true atom (of the algebra of propositions of the world of evaluation) to q, and, assuming the proposition bound to q is the singleton of some world, $@q$ is intended to evaluate the complement clause at this world. Assuming a world-selective equivalence system, Π and Σ are intended to express quantification, respectively universal and existential, over singletons of worlds in the proposition expressed by p_0; p_0 thus serves the purpose of expressing the set of worlds second-order quantification over which is to be simulated. Assuming that q and s express singleton propositions of worlds v and u, $D(q, s)$ is intended to express that the world of evaluation codes $\{v, u\}$. The following lemma notes more formally that these definitions express the desired conditions:

LEMMA 2.8. *Let* $\approx$ *be an equivalence system on a set* W*,* $P \subseteq W$*, and* a *an assignment function such that* $a(p_0) = P$*.*

(i) $\approx, w, a \vDash \varphi = \psi$ *iff* $\{v \in W : \approx, v, a \vDash \varphi\} = \{v \in W : \approx, v, a \vDash \psi\}$
(ii) $\approx, w, a \vDash E\varphi$ *iff* $\{v \in W : \approx, v, a \vDash \varphi\} \in D^{\approx}_w$
(iii) $\approx, w, a \vDash atom(\varphi)$ *iff* $\{v \in W : \approx, v, a \vDash \varphi\} \in W/{\approx_w}$
(iv) $\approx, w, a \vDash \downarrow q\varphi$ *iff* $\approx, w, a[[w]_{\approx_w}/q] \vDash \varphi$
(v) *If* $a(q) = \{v\}$*, then* $\approx, w, a \vDash @q\varphi$ *iff* $\approx, v, a \vDash \varphi$*.*

For the following, assume further that $\approx$ *is world-selective.*

(vi) $\approx, w, a \vDash \Pi q\varphi$ *iff for all* $v \in P$*,* $\approx, w, a[\{v\}/q] \vDash \varphi$
(vii) $\approx, w, a \vDash \Sigma q\varphi$ *iff for some* $v \in P$*,* $\approx, w, a[\{v\}/q] \vDash \varphi$
(viii) *If* $a(q) = \{v\}$*,* $a(s) = \{u\}$ *and* $v, u \in P$*, then* $\approx, w, a \vDash D(q, s)$ *iff* w *codes*$_P$ $\{v, u\}$

Proof. Routine. □

The simulation of $L2$ in $L_\exists$ can now be introduced formally as a function mapping each $L2$-formula A to an $L_\exists$-formula A^*. Define such a function using the following recursive clauses:

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$$\begin{array}{lll} (Xyz)^* & := & \Diamond(p_X \wedge D(p_y, p_z)) \\ (\neg A)^* & := & \neg A^* \\ (A \wedge B)^* & := & A^* \wedge B^* \\ (\exists x A)^* & := & \Sigma p_x A^* \\ (\exists X A)^* & := & \Diamond \exists p_X A^*. \end{array}$$

In order to show that this has the intended effect, it suffices to consider assignment functions for $L_\exists$ which map p_0 to a given set of worlds P, and p_x, for a first-order variable x, to a singleton of a P-world; each such assignment function naturally determines a corresponding assignment function for $L2$ on P.

DEFINITION 2.9. *Let $\approx$ be an equivalence system on a set W and $P \subseteq W$. An assignment function $a : \Phi \to \mathcal{P}(W)$ is* coding$_P$ *if $a(p_0) = P$ and for every first-order variable x, $a(p_x)$ codes$_P$ a member of P.*

For an assignment function $a : \Phi \to \mathcal{P}(W)$ which is coding$_P$, define an assignment function a^* for $L2$, mapping each first-order variable x to the element of P coded$_P$ by $a(p_x)$, and each second-order variable X to the symmetric binary relation on P coded$_P$ by $a(p_X)$.

The next lemma shows that the mapping $\cdot^*$ functions as intended:

LEMMA 2.10. *If $\approx$ is an equivalence system on a set W and a is an assignment function such that $\approx$ and a are coding$_P$ for some $P \subseteq W$, then for every $L2$-formula A,*

$$P, a^* \vDash A \text{ iff } \approx, a \vDash A^*.$$

Proof. By induction of the complexity of A. The Boolean cases are trivial, leaving the following three:

$P, a^* \vDash Xyz$ iff $\langle a^*(y), a^*(z)\rangle \in a^*(X)$ (by semantics)
iff $a^*(y)$ and $a^*(z)$ are related by the symmetric binary relation on P coded$_P$ by $a(p_X)$ (by definition of $\cdot^*$)
iff some element of $a(p_X)$ codes$_P$ $\{a^*(y), a^*(z)\}$ (by definition of coding$_P$)
iff some element of $a(p_X)$ codes$_P$ $a(p_y) \cup a(p_z)$ (by definition of $\cdot^*$ and coding$_P$)
iff $\approx, w, a \vDash D(p_y, p_z)$ for some $w \in a(p_X)$ (by Lemma 2.8(viii))
iff $\approx, a \vDash \Diamond(p_X \wedge D(p_y, p_z))$ (by semantics)
iff $\approx, a \vDash (Xyz)^*$ (by definition of $\cdot^*$)

$P, a^* \vDash \exists x A$ iff there is a $w \in P$ such that $P, a^*[w/x] \vDash A$ (by semantics)
iff there is a $w \in P$ such that $P, (a[\{w\}/p_x])^* \vDash A$ (by definition of $\cdot^*$ and coding$_P$)
iff there is a $w \in P$ such that $\approx, a[\{w\}/p_x] \vDash A^*$ (by IH)
iff $\approx, a \vDash \Sigma p_x A^*$ (by Lemma 2.8(vii))
iff $\approx, a \vDash (\exists x A)^*$ (by definition of $\cdot^*$)

$P, a^* \vDash \exists X A$ iff there is a symmetric binary relation R on P such that $P, a^*[R/X] \vDash A$ (by semantics)
iff there is a $Q \subseteq W$ such that $P, a^*[R/X] \vDash A$, where R is the symmetric binary relation coded by Q (by $C1_P$ and the definition of coding$_P$)
iff there is a $Q \subseteq W$ such that $P, (a[Q/p_X])^* \vDash A$ (by definition of $\cdot^*$)
iff there is a $Q \subseteq W$ such that $\approx, a[Q/p_X] \vDash A^*$ (by IH)
iff $\approx, a \vDash \Diamond\exists p_X A^*$ (by $C2_P$ and semantics)
iff $\approx, a \vDash (\exists X A)^*$ (by definition of $\cdot^*$) □

The next task is to formulate the model-theoretic condition of being coding$_P$ in $L_\exists$. Assuming a world-selective equivalence system, $C1_P$ and $C2_P$ can be expressed as $T1$ and $T2$, respectively:

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$$T1 := \Pi q \Pi r \Diamond D(q, r)$$
$$T2 := \Diamond \downarrow q \Box \downarrow r @ q E r$$
$$T := T_1 \wedge T_2$$

LEMMA 2.11. *Let $\approx$ be a world-selective equivalence system on a set W, $P \subseteq W$, and a an assignment function for $\approx$ such that $a(p_0) = P$.*

(i) *$\approx, a \vDash T1$ iff $\approx$ satisfies $C1_P$.*

(ii) *$\approx, a \vDash T2$ iff $\approx$ satisfies $C2_P$.*

(iii) *$\approx, a \vDash T$ iff $\approx$ is coding$_P$.*

Proof. (i) is routine using Lemma 2.8, and (iii) follows from (i) and (ii). For (ii), note that since $\approx$ is world-selective, $\approx, w, a \vDash \downarrow q \Box \downarrow r @ q E r$ iff $D_w^{\approx}$ contains all singletons of worlds, which is the case iff $D_w^{\approx} = \mathcal{P}(W)$. □

The free variable p_0 in A^* can now be eliminated by binding it with a necessitated universal quantifier, restricted to T; this provides a way of simulating the evaluation of an $L2$-sentence on the sets coded by an equivalence system using $L_\exists$, interpreted on this system. Note that for any $L2$-sentence A, the only proposition letter free in A^* is p_0, and that formulas like A^*, T and $\Box\forall p_0(T \rightarrow A^*)$ are either true in all or no worlds of an equivalence system.

LEMMA 2.12. *If $\approx$ is a world-selective equivalence system on a set W and A an $L2$-sentence, then*

$$\approx \vDash \Box\forall p_0(T \rightarrow A^*) \text{ iff } A \text{ is true in all sets coded by } \approx.$$

Proof. If there is no $P \subseteq W$ coded by $\approx$, then by Lemma 2.11(iii), $\approx \vDash \neg T$, and so $\approx \vDash \Box\forall p_0(T \rightarrow A^*)$. So assume $\approx$ codes some $P \subseteq W$. Then there is a $w \in W$ such that $D_w^{\approx} = \mathcal{P}(W)$. Using Lemma 2.11(iii) again, it follows that $\approx \vDash \Box\forall p_0(T \rightarrow A^*)$ iff for all $P \subseteq W$ coded by $\approx$ and assignment functions a, $\approx, a[P/p_0] \vDash A^*$, which by Lemma 2.10 is the case iff $P, (a[P/p_0])^* \vDash A$. Since A is closed, $\approx \vDash \Box\forall p_0(T \rightarrow A^*)$ iff $P \vDash A$ for all $P \subseteq W$ such that $\approx$ is coding$_P$, as required. □

The next step is to eliminate the restriction to world-selective equivalence systems. This will be done by defining a weaker condition of being atom-selective: if w can't distinguish itself from v, then w and v must be interchangeable as far as any indistinguishability relation is concerned. On the one hand, this weaker condition can be expressed by an $L_\exists$-formula, and on the other hand, an $L_\exists$-sentence is satisfiable on an atom-selective equivalence system if and only if it is satisfiable on a world-selective equivalence system.

DEFINITION 2.13. *Let an equivalence system $\approx$ on a set W be* atom-selective *if for all $w \in W$ and $v \in [w]_{\approx_w}$,*

(i) *$w \approx_u v$ for all $u \in W$, and*

(ii) *$\approx_w = \approx_v$.*

An $L_\exists$-sentence expressing atom-selectivity can be formulated as follows:

$$S := \Box \downarrow p \Box \forall q((\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)) \wedge (\Box(p \rightarrow Eq) \vee \Box(p \rightarrow \neg Eq))).$$

LEMMA 2.14. *An equivalence system $\approx$ is atom-selective iff $\approx \vDash S$.*

Proof. We show that the two conditions of atom-selectivity correspond to the two conjuncts in S in the following way:

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$\approx, a[[w]_{\approx_w}/p] \vDash \Box\forall q(\Box(p \to q) \vee \Box(p \to \neg q))$
iff for all $u \in W$ and $Q \in D_u^{\approx}$, $[w]_{\approx_w} \subseteq Q$ or $[w]_{\approx_w} \subseteq W \backslash Q$
iff for all $u \in W$ and $v \in [w]_{\approx_w}$, $w \approx_u v$.

$\approx, a[[w]_{\approx_w}/p] \vDash \Box\forall q(\Box(p \to Eq) \vee \Box(p \to \neg Eq))$
iff for all $u \in W$ and $Q \in D_u^{\approx}$, either $Q \in D_v^{\approx}$ for all $v \in [w]_{\approx_w}$ or $Q \notin D_v^{\approx}$ for all $v \in [w]_{\approx_w}$
iff for all $v \in [w]_{\approx_w}$, $\approx_w = \approx_v$.

The claim follows with Lemma 2.8(iv). □

So define a second function mapping each $L2$-*sentence* A to an $L_\exists$-sentence $A^\dagger$ as follows:

$$A^\dagger := S \to \Box\forall p_0(T \to A^*).$$

The following construction provides a way of simplifying equivalence systems, roughly reducing clusters of duplicate worlds to a single world, which on the one hand preserves truth of $L_\exists$-sentences, and on the other hand turns every atom-selective equivalence system into a world-selective one:

DEFINITION 2.15. *For any equivalence system $\approx$ on a set W, let $\sim_\approx$ be the equivalence relation on W such that*

$$w \sim_\approx v \text{ iff } w \approx_u v \text{ for all } u \in W \text{ and } \approx_w = \approx_v.$$

Let the simplification of $\approx$, *written $\approx^s$, be the equivalence system on $W/\sim_\approx$ such that $[v]_{\sim_\approx} \approx^s_{[w]_{\sim_\approx}} [u]_{\sim_\approx}$ iff $v \approx_w u$.*

Simplification is developed in more detail in the appendix as a special case of a general notion of congruences of equivalence systems. Lemmas 5.17 and 5.18 and Proposition 5.20 in the appendix establish the following useful facts:

FACT 2.16. *Let $\approx$ be an equivalence system on a set W.*

(i) *$\approx$ is atom-selective iff $\approx^s$ is world-selective.*
(ii) *If $\approx$ is world-selective, then $\approx$ is isomorphic to $\approx^s$.*
(iii) *For every $L_\exists$-sentence φ, $\approx \vDash \varphi$ iff $\approx^s \vDash \varphi$.*

Simplification permits dropping the restriction to world-selective equivalence systems in Lemma 2.12 in the following way:

LEMMA 2.17. *If $\approx$ is an equivalence system and A an $L2$-sentence, then*

$$\approx \vDash A^\dagger \text{ iff } A \text{ is true in all sets coded by } \approx^s.$$

Proof. Assume first that $\approx$ is not atom-selective. Then by Lemma 2.14, $\approx \vDash \neg S$. By Fact 2.16(i), $\approx^s$ is not world-selective, so there is no set coded by $\approx$. Thus both sides of the claimed biconditional are trivially true. If $\approx$ is atom-selective, then by Lemma 2.14 and Fact 2.16(i), $\approx \vDash S$ and $\approx^s$ is world-selective. So by Fact 2.16(iii), $\approx \vDash A^\dagger$ iff $\approx^s \vDash \Box\forall p_0(T \to A^*)$, which by Lemma 2.12 is the case iff A is true in all sets coded by $\approx^s$. □

With this lemma, a general form of the embedding result can be established, from which the embedding of SB in the L_E-logic of equivalence systems is an easy corollary.

THEOREM 2.18. *Let C be a class of equivalence systems such that for every cardinality κ, there is an equivalence system $\approx$ in C such that $\approx^s$ codes some set of cardinality κ. The $L_\exists$-logic of C is recursively isomorphic to full second-order logic.*

Proof. We first show that $\cdot^{\dagger}$ embeds SB in the $L_{\exists}$-logic of C, in the sense that $A \in$ SB iff $A^{\dagger}$ is valid on C. If $A \in$ SB, then A is true in all sets, and so by Lemma 2.17, $A^{\dagger}$ is valid on all equivalence systems, and thus in particular on C. If $A \notin$ SB, then there is a set of cardinality κ in which it is false. By assumption, C contains an equivalence system $\approx$ such that $\approx^{s}$ codes a set P of cardinality κ. Thus A is false in P, and so by Lemma 2.17, $\approx^{s} \not\models A^{\dagger}$, and hence with Fact 2.16(iii), $\approx \not\models A^{\dagger}$. So $A^{\dagger}$ is not valid on C, as required.

$\cdot^{\dagger}$ is evidently recursive, so it follows with Fact 2.4 that there is a recursive embedding of full second-order logic in the $L_{\exists}$-logic of C. It is routine to derive from this that the two logics are recursively isomorphic (see Kremer (1993) for details). □

COROLLARY 2.19. *The $L_{\exists}$-logic of the class of all equivalence systems is recursively isomorphic to full second-order logic.*

Proof. It is not hard to see that for every cardinality κ, there is an equivalence system whose simplification codes a set of cardinality κ. A detailed construction satisfying further constraints will be given in the proof of Theorem 2.20. □

2.2. Coherence and comprehension. The result just established for the class of all equivalence systems extends to the class of coherent equivalence systems, introduced above. To define the witnesses for this claim, extend any permutation f of a set W to a permutation on $\mathcal{P}(W)$, letting $f.P = \{f(w) : w \in P\}$ for all $P \subseteq W$.

THEOREM 2.20. *The $L_{\exists}$-logic of the class of coherent equivalence systems is recursively isomorphic to full second-order logic.*

Proof. To show that the class of coherent equivalence systems satisfies the conditions stated in Theorem 2.18, let κ be a cardinality and D an infinite set of cardinality $\geq \kappa$. Define $W = \mathcal{P}(D)$, and $\approx$ to be the equivalence system on W such that for all $w, v, u \in W$, $v \approx_{w} u$ iff there is a permutation f of D such that $f.v = u$ and $f(d) = d$ for all $d \in w$.

To show that $\approx$ is coherent, assume $v \approx_{w} u$. Then there is a permutation f of D such that $f.v = u$ and $f(d) = d$ for all $d \in w$; we show that f (extended to W) is the required automorphism. To show that it is an automorphism, assume $y \approx_{x} z$, witnessed by a permutation g. It is routine to show that fgf^{-1} is a permutation which witnesses $f.y \approx_{f.x} f.z$, as required. The remaining conditions are immediate by construction.

Since $\approx$ is world-selective, it follows from Fact 2.16(ii), that $\approx$ and $\approx^{s}$ are isomorphic. It thus suffices to show that $\approx$ codes a set of cardinality κ. Let $P \subseteq D$ of cardinality κ; we show that $\approx$ codes $P' = \{\{d\} : d \in P\}$. $C0_{P'}$: As already noted, $\approx$ is world-selective. $C1_{P'}$: Each $\{\{d\}, \{e\}\} \subseteq P'$ is coded$_{P'}$ by $\{d, e\}$. $C2_{P'}$: $D^{\approx}_{D} = \mathcal{P}(W)$. □

One way of motivating coherence is as a structural criterion guaranteeing that at every world, the existing propositions are not only closed under Boolean operations – which is guaranteed in all equivalence systems by the fact that propositional domains form (complete atomic) Boolean algebras – but also under definability using arbitrary $L_{\exists}$-formulas. In $L_{\exists}$, this can be expressed as the requirement of validating the following schema, where $\mathrm{pl}(\varphi)$ is the set of proposition letters free in φ, and φ is any $L_{\exists}$-formula:

$$(C_{\exists}) \quad \left(\bigwedge_{p \in \mathrm{pl}(\varphi)} Ep\right) \rightarrow E\varphi.$$

For further discussion and motivation of this principle, see Williamson (2013, sec. 6.2) and Fritz & Goodman (2016, sec. 5.1). This comprehension principle supports coherence in the following way:

PROPOSITION 2.21. *$C_{\exists}$ is valid on the class of coherent equivalence systems, but not on the class of all equivalence systems.*

Proof. The validity claim will follow from Proposition 2.25; the invalidity claim can be seen by considering the following system $\approx$ (notation explained below):

Note that $\approx, 1 \vDash \neg E \forall p \Box E p$. □

In this proof and in the following, the conventions for drawing equivalence systems of Fritz (2016) are adopted. Roughly, an equivalence system $\approx$ based on $\{1, \ldots, n\}$ is drawn using n circles of n points, themselves arranged in a circle, with lines connecting the points of the ith circle according to $\approx_i$; labeling of worlds is omitted by adopting the convention of arranging them clockwise with 1 on top. So the equivalence system used in the above proof is based on $\{1, 2\}$, with $\approx_1$ but not $\approx_2$ relating 1 and 2.

Given this result, one might wonder whether all clauses in the condition of coherence are necessary to validate comprehension. This is not so; the condition that the relevant automorphism (see Definition 1.2) maps w to itself is not needed.

DEFINITION 2.22. *Let an equivalence system $\approx$ on a set W be* quasicoherent *if for every $w, v, u \in W$ such that $v \approx_w u$, there is an $f \in \mathrm{aut}(\approx)$ such that $f(v) = u$ and $f \subseteq \approx_w$.*

This is a weaker condition than coherence, as can be demonstrated using a two-world equivalence system:

Note also that from the proof of Theorem 2.20, the following is immediate:

COROLLARY 2.23. *The $L_\exists$-logic of the class of quasicoherent equivalence systems is recursively isomorphic to full second-order logic.*

Returning to the matter of comprehension, quasicoherence suffices for the validity of $C_\exists$. To prove it, the next lemma shows that truth in equivalence systems is invariant under automorphisms. For any automorphism f of an equivalence system $\approx$ and assignment function a, let $f.a$ be the assignment function mapping each $p \in \Phi$ to $f.(a(p))$.

LEMMA 2.24. *Let $\approx$ be an equivalence system on a set W, $w \in W$, and f an automorphism of $\approx$.*

(i) *For any $P \subseteq W$, $P \in D^{\approx}_w$ iff $f.P \in D^{\approx}_{f(w)}$.*
(ii) *For any assignment function a and $\varphi \in L_\exists$, $\approx, w, a \vDash \varphi$ iff $\approx, f(w), f.a \vDash \varphi$.*

Proof. (i) $P \notin D^{\approx}_w$ iff there are $v, u \in W$ such that $v \in P$, $u \notin P$ and $v \approx_w u$. $v \in P$ iff $f(v) \in f.P$; $u \notin P$ iff $f(u) \notin f.P$; and $v \approx_w u$ iff $f(v) \approx_{f(w)} f(u)$. So $P \in D^{\approx}_w$ iff there are $v, u \in W$ such that $f(v) \in f.P$, $f(u) \notin f.P$ and $f(v) \approx_{f(w)} f(u)$; since f is a permutation, this is the case iff there are $v, u \in W$ such that $v \in f.P$, $u \notin f.P$ and $v \approx_{f(w)} u$, i.e., iff $f.P \notin D^{\approx}_{f(w)}$.

(ii) By induction on the structure of φ. Only the case of $\exists$ is of interest:

$\approx, w, a \vDash \exists p \varphi$ iff $\approx, w, a[P/p] \vDash \varphi$ for some $P \in D^{\approx}_w$ (by semantics)
iff $\approx, f(w), f.(a[P/p]) \vDash \varphi$ for some $P \in D^{\approx}_w$ (by IH)
iff $\approx, f(w), (f.a)[f.P/p] \vDash \varphi$ for some $P \subseteq W$ such that $f.P \in D^{\approx}_{f(w)}$ (by (i))
iff $\approx, f(w), f.a \vDash \exists p \varphi$ (by semantics) □

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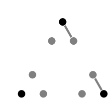
PROPOSITION 2.25. *$C_\exists$ is valid on the class of quasicoherent equivalence systems.*

Proof. Consider any quasicoherent equivalence system $\approx$ on a set W, $w \in W$ and $\varphi \in L_\exists$. It suffices to show that for every assignment function $a : \Phi \to D^{\approx}_w$, $\approx, w, a \vDash E\varphi$, which is the case iff for all $v, u \in W$ such that $v \approx_w u$, $\approx, v, a \vDash \varphi$ iff $\approx, u, a \vDash \varphi$. If $v \approx_w u$, then by quasicoherence, there is an $f \in \text{aut}(\approx)$ such that $f(v) = u$ and $f \subseteq \approx_w$. Since $a : \Phi \to D^{\approx}_w$, $a = f.a$, and so with Lemma 2.24(ii), $\approx, v, a \vDash \varphi$ iff $\approx, u, a \vDash \varphi$, as required. □

Although the structural differences between coherence and quasicoherence are not reflected in matters concerning the validity of $C_\exists$, they are reflected in matters concerning the validity of $L_\exists$-sentences. The following proposition gives a concrete example:

PROPOSITION 2.26. *Coherent systems are world-selective; quasicoherent systems need not be atom-selective. Thus, S (defined above) is valid on the class of coherent equivalence systems, but not on the class of quasicoherent equivalence systems.*

Proof. That coherence entails world-selectivity is immediate. That quasicoherence does not entail atom-selectivity can be seen using the following system:



The conclusions concerning S follow by Lemma 2.14. □

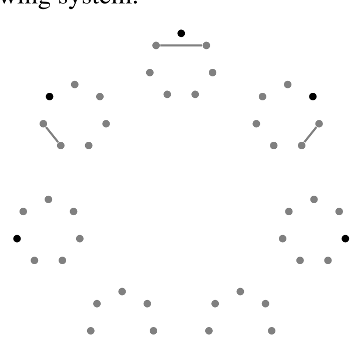
Are all the components of quasicoherence required to ensure the validity of $C_\exists$? It is clear that dropping the requirement for f to map v to u is indispensable, as doing so leads to a condition satisfied by all equivalence systems (f may always be the identity permutation). The only remaining natural candidate is the weakening of quasicoherence which drops the requirement that $f \subseteq \approx_w$. More cautiously, one might wonder whether replacing it by $f(w) = w$ leads to a class of systems which validates $C_\exists$. The following shows that $f \subseteq \approx_w$ is necessary, and cannot be replaced by $f(w) = w$.

PROPOSITION 2.27. *There is an equivalence system $\approx$ such that:*

(i) *If $v \approx_w u$ then there is an $f \in \text{aut}(\approx)_w$ such that $f(v) = u$.*

(ii) *$\approx$ does not validate $C_\exists$.*

Proof. Let $\approx$ be the following system:

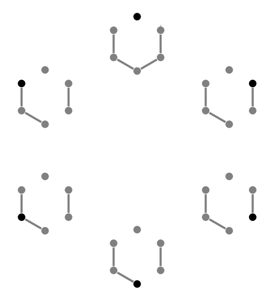


For $7 \approx_1 2$, note that $(72)(63)(54)$ is an automorphism as required, and for the other two cases, the obvious transpositions witness that $\approx$ satisfies the constraint. But $C_\exists$ is not valid, since $\{3\} \in D_1^{\approx}$ and $\{2\} \notin D_1^{\approx}$, even though $\{2\}$ is definable at 1 using $\neg Ep$, assigning $\{3\}$ to p. □

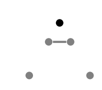
Among a natural range of candidates, quasicoherence is therefore a minimal condition sufficient to guarantee the validity of $C_\exists$. This cannot be strengthened to arbitrary conditions, as $C_\exists$ does not define quasicoherence: there are equivalence systems which validate $C_\exists$ without being quasicoherent. The existence of such equivalence systems follows from results established in Fritz (unpublished). Indeed, the results established there entail that it is even impossible to define quasicoherence in an infinitary higher-order extension of $L_\exists$. The counterexamples derived from these results are infinite, but concerning the question whether $C_\exists$ defines quasicoherence, there are even finite counterexamples:

PROPOSITION 2.28. *There is an equivalence system $\approx$ on a finite set which validates $C_\exists$ without being quasicoherent.*

Proof. Let $\approx$ be the following equivalence system:



This is not quasicoherent, as no automorphism maps 2 to 4, as required by $2 \approx_1 4$. But the simplification of $\approx$ is the following equivalence system:



Since this is coherent, it validates $C_\exists$; by Fact 2.16(iii), so does $\approx$ (note that the antecedent of $C_\exists$ may be replaced by a string of universal quantifiers). □

Given the counterexample of this proof, it is natural to wonder whether a finite equivalence system validates $C_\exists$ just in case its simplification is quasicoherent; this question will be left open here.

§3. Existential operators.

Since the $L_\exists$-logics considered above turn out to be too complex to be recursively axiomatizable, it is natural to consider more restricted languages which still capture some characteristic features of propositional contingentism. One candidate is the language which results from replacing the existential propositional quantifier of $L_\exists$ by an existential propositional operator.

DEFINITION 3.1. *Let L_E be the set of formulas built up from Φ using $\neg$, $\wedge$, $\Box$ and a unary existential operator E.*

Note that E is now a primitive operator, rather than a syntactic abbreviation as above. Nevertheless, L_E can naturally be understood as a syntactic restriction of $L_\exists$, namely as the set of $L_\exists$-formulas in which $\exists$ occurs only in subformulas of the form $\exists p \Box(p \leftrightarrow \varphi)$, with p not free in φ.

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DEFINITION 3.2. Truth *of an L_E-formula relative to an equivalence system $\approx$ on a set W, a world $w \in W$, and an assignment function $a : \Phi \to \mathcal{P}(W)$ is defined like truth of $L_\exists$-formulas, except for the following new clause*:

$$\approx, w, a \vDash E\varphi \quad \textit{iff} \quad \{v \in W : \approx, v, a \vDash \varphi\} \in D^{\approx}_{w}.$$

Validity and other metalogical notions are defined analogous to Definition 2.2.

Note that $\approx, w, a \vDash E\varphi$ iff for all $v, u \in W$, if $v \approx_w u$ then $\approx, v, a \vDash \varphi$ iff $\approx, u, a \vDash \varphi$.

3.1. Completeness. The first main result to be established shows that the logic S5E, axiomatized in the following definition, is the L_E-logic of (the class of all) equivalence systems.

DEFINITION 3.3. *Let an* existential modal logic *be a set of L_E-formulas containing all truth-functional tautologies and the following axioms*:

$K_\Box$:	$\Box(p \to q) \to (\Box p \to \Box q)$	$C\top$:	$E\top$
$T_\Box$:	$\Box p \to p$	$C\neg$:	$Ep \to E\neg p$
$5_\Box$:	$\neg\Box p \to \Box\neg\Box p$	$C\wedge$:	$(Ep \wedge Eq) \to E(p \wedge q)$
$\Box E$:	$\Box(p \leftrightarrow q) \to (Ep \leftrightarrow Eq)$		

and closed under the following schematic rules:

MP : *From φ and $\varphi \to \psi$, derive ψ*
US : *From φ, derive any substitution instance of φ*
Nec : *From φ, derive $\Box\varphi$*

For every set $\Gamma \subseteq L_E$, let S5EΓ*, the* existential modal logic axiomatized by (the members of) Γ*, be the smallest existential modal logic which includes* Γ; *let* S5E = S5E$\emptyset$.

S5EΓ is well-defined for arbitrary $\Gamma \subseteq L_E$, as existential modal logics ordered by inclusion form a complete lattice, which is routine to verify. As the next proposition notes, this lattice can also be understood as a sublattice of the lattice of *classical* or *congruential* bimodal logics, the sets of L_E-formulas containing all tautologies and closed under *MP*, *US* and the congruence rules $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$ and $\varphi \leftrightarrow \psi / E\varphi \leftrightarrow E\psi$ (see Segerberg (1971) and Chellas (1980) for more on such logics).

PROPOSITION 3.4. *For every $\Gamma \subseteq L_E$,* S5EΓ *is the classical modal logic axiomatized by $K_\Box$ – $C\wedge$ (the seven axioms of Definition 3.3) and Γ.*

Proof. It suffices to show that the congruence rules are derivable in S5EΓ and that *Nec* is derivable in the classical modal logic axiomatized by $K_\Box$–$C\wedge$ and Γ. The first claim is routine using *Nec*, $K_\Box$, and $\Box E$. For the second claim, note that Chellas (1980, Theorem 8.15) shows $\Box\top$ to be derivable using $T_\Box$ and $5_\Box$, from which *Nec* follows with the congruence rule for $\Box$. □

The usual syntactic notions in propositional modal logic will be applied to existential modal logics: For a given existential modal logic Λ, $\Gamma \subseteq L_E$ and $\varphi \in L_E$, $\vdash_\Lambda \varphi$ will be used for $\varphi \in \Lambda$ (φ being a *theorem of* Λ), and $\Gamma \vdash_\Lambda \varphi$ for $\vdash_\Lambda \bigwedge \Gamma_0 \to \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$ (φ being a *consequence of* Γ *in* Λ). Γ/φ will be called Λ-*inconsistent* if $\Gamma/\{\varphi\} \vdash_\Lambda \bot$ and Λ-*consistent* otherwise. A set of L_E-formulas will be understood to be Λ *maximal consistent* if it is Λ-consistent and not a proper subset of any Λ-consistent set.

Instead of a more traditional canonical model construction, the proof of the completeness of S5E with respect to equivalence system given below proceeds by an analog of the representation theorem of Jónsson & Tarski (1951). (However, it is routine to derive canonical models from the constructions involved in this representation theorem; cf. Blackburn, de Rijke, & Venema (2001, p. 288).) Consequently, algebraic analogs of equivalence systems are now introduced. They are based on Boolean algebras; the top and bottom elements of such algebras will customarily be called 1 and 0, and the usual entailment order will be called $\leq$. As with equivalence systems, the interpretation of $\Box$ is fixed by the semantics, rather than the specific structure. E is interpreted by a function ε, which can be understood as mapping every proposition to the proposition that it exists. This mapping will be required to satisfy the constraint that for every proposition, the propositions whose existence it entails contain 1 and are closed under negation and conjunction; i.e., such propositions must form a Boolean algebra.

DEFINITION 3.5. *Let an* existential algebra *be a structure* $\langle A, -, \sqcap, \varepsilon\rangle$ *such that* $\langle A, -, \sqcap\rangle$ *is a Boolean algebra and* $\varepsilon : A \rightarrow A$ *is such that for all* $a \in A$, $\varepsilon^{-}(a) = \{b \in A : a \leq \varepsilon b\}$ *is a Boolean subalgebra of* A.

For an existential algebra $\mathcal{A} = \langle A, -, \sqcap, \varepsilon\rangle$, *let an* assignment function *be a function* $a : \Phi \rightarrow A$. *Implicitly extend such functions to* L_E *using the following clauses*:

$$\begin{array}{lcl} a(\neg\varphi) & = & -a(\varphi) \\ a(\varphi \wedge \psi) & = & a(\varphi) \sqcap a(\psi) \\ a(\Box\varphi) & = & \begin{cases} 1 & \text{if } a(\varphi) = 1, \\ 0 & \text{otherwise.} \end{cases} \\ a(E\varphi) & = & \varepsilon a(\varphi) \end{array}$$

An L_E*-formula is* valid on *an existential algebra if it is mapped to the top element by every assignment function*; *it is valid on a class of existential algebras if it is valid on all of them. The* L_E-logic of *a class of existential algebras is the set of* L_E*-formulas valid on it.*

The fixed interpretation of $\Box$ in existential algebras is adapted from algebraic models for S5 described in Lewis & Langford (1932, p. 492), where the construction is attributed to Paul Henle. Just like existential modal logics could be generalized by dropping some of the axioms governing $\Box$, and equivalence systems could be generalized by adding accessibility relations or neighborhood functions to interpret $\Box$, existential algebras could be generalized by adding a function to interpret $\Box$. Such generalizations won't be considered here. The class of existential algebras is adequate for existential modal logics in the following sense:

THEOREM 3.6. *The* L_E*-logic of every class of existential algebras is an existential modal logic, and every* L_E*-logic is the logic of a class of existential algebras.*

Proof. The first claim is routine to verify. For the second, consider any L_E-logic Λ. For every $\Gamma \subseteq L_E$, let $\Box^{-}(\Gamma) = \{\gamma : \Box\gamma \in \Gamma\}$ and $\Box(\Gamma) = \{\Box\gamma : \gamma \in \Gamma\}$.

Let γ be an L_E-formula not in Λ. We construct an existential algebra validating Λ but not γ; this suffices to show that Λ is the L_E-logic of the class of existential algebras validating Λ. As usual, a version of Lindenbaum's lemma establishes that $\neg\gamma$ is contained in some Λ maximal consistent set Γ. Define a property $\Vdash$ of L_E-formulas by letting $\Vdash \varphi$ if $\Box\Box^{-}(\Gamma) \vdash_{\Lambda} \varphi$, and a binary relation $\sim$ on L_E-formulas by letting $\varphi \sim \psi$ if $\Vdash \varphi \leftrightarrow \psi$.

We establish the following two claims:

(i) For all $\varphi \in L_E$, if $\Vdash \varphi$ then $\Vdash \Box\varphi$.

(ii) $\sim$ is a congruence of the term algebra, i.e., an equivalence relation on L_E such that $\varphi \sim \psi$ entails $\neg\varphi \sim \neg\psi$ and analogously for $\wedge$, $\Box$ and E.

For (i), assume $\Vdash \varphi$, so $\vdash_\Lambda \bigwedge_{i<n} \Box\chi_i \to \varphi$ for some $\chi_i \in \Gamma$ ($i < n$). Thus by a routine derivation, $\vdash_\Lambda \bigwedge_{i<n} \Box\Box\chi_i \to \Box\varphi$ and so, by $\vdash_{\mathrm{S5E}} \Box p \to \Box\Box p$, $\vdash_\Lambda \bigwedge_{i<n} \Box\chi_i \to \Box\varphi$, whence $\Vdash \Box\varphi$. The Boolean cases of (ii) are immediate, and the cases of $\Box$ and E follow by (i) and $\vdash_{\mathrm{S5E}} \Box(\varphi \leftrightarrow \psi) \to (\Box\varphi \leftrightarrow \Box\psi)$ and $\Box E$, respectively.

Let $\mathcal{A} = \langle A, -, \sqcap, \varepsilon\rangle$ be the Lindenbaum–Tarski algebra determined by $\sim$, i.e., the algebra based on the equivalence classes of $\sim$, with $-$ such that $-[\varphi]_\sim = [\neg\varphi]_\sim$, and similarly for the other operators; this is well-defined as $\sim$ is a congruence. We show that $\mathcal{A}$ is an existential algebra. That it is based on a Boolean algebra follows from the fact that Λ includes all tautologies; the constraint on ε follows from $C\top$, $C\neg$ and $C\wedge$. So consider $\Box$:

If $[\varphi]_\sim = 1$, then $\Vdash \varphi \leftrightarrow \top$, so $\Vdash \Box\varphi \leftrightarrow \Box\top$, hence $\Box\varphi \sim \top$, and thus $[\Box\varphi]_\sim = 1$.

If $[\varphi]_\sim \neq 1$ then $\nVdash \varphi \leftrightarrow \top$, so (by $T_\Box$ and maximality) $\neg\Box\varphi \in \Gamma$, and thus (by $5_\Box$) $\Box\neg\Box\varphi \in \Gamma$. Therefore $\neg\Box\varphi \sim \top$, and so $[\Box\varphi]_\sim = 0$.

$\mathcal{A}$ is the required existential algebra: It is immediate by construction that it validates Λ. To see that it does not validate γ, consider the canonical assignment function mapping each $p \in \Phi$ to $[p]_\sim$; this maps each formula to its equivalence class. Since Γ is consistent, not $\neg\gamma \sim \bot$, i.e., $[\gamma]_\sim \neq 1$ as required. (Indeed, the image of Γ under the canonical assignment function forms an ultrafilter.) □

Every equivalence system can straightforwardly be turned into an existential algebra:

DEFINITION 3.7. *Let $\approx$ be an equivalence system on a set W. Define the* complex existential algebra of $\approx$*, written $\mathcal{A}^{\approx}$, to be the existential algebra $\langle A, -, \sqcap, \varepsilon^{\approx}\rangle$ such that $\langle A, -, \sqcap\rangle$ is the powerset algebra on W and*

$$\varepsilon^{\approx}(a) = \{w \in W : a \in D^{\approx}_w\}.$$

This preserves truth of L_E-formulas in the following sense:

LEMMA 3.8. *If $\approx$ is an equivalence system on W, $w \in W$, a an assignment function and $\varphi \in L_E$, then $\approx, w, a \vDash \varphi$ iff $w \in a(\varphi)$ (where on the right hand side, a serves as an assignment function for $\mathcal{A}^{\approx}$).*

Proof. By induction on the structure of φ. Consider the case of E: $\approx, w, a \vDash E\varphi$ iff $\{v \in W : \approx, v, a \vDash \varphi\} \in D^{\approx}_w$ iff $a(\varphi) \in D^{\approx}_w$ (by IH) iff $w \in \varepsilon^{\approx}(a(\varphi))$ (by construction of $\varepsilon^{\approx}$) iff $w \in a(E\varphi)$. □

As in Jónsson and Tarski's theorem, an existential algebra can be represented using an equivalence system whose worlds are the ultrafilters of the underlying Boolean algebra. The crucial clause of the equivalence relation associated with an ultrafilter is based on the idea that a world w is unable to distinguish worlds v and u just in case every proposition which exists in w is true in v iff it is true in u.

DEFINITION 3.9. *Let $\mathcal{A} = \langle A, -, \sqcap, \varepsilon\rangle$ be an existential algebra. Define the* ultrafilter equivalence system of $\mathcal{A}$*, written $\approx^{\mathcal{A}}$, to be the equivalence system on the set $Uf(\mathcal{A})$ of ultrafilters of $\langle A, -, \sqcap\rangle$ such that for all ultrafilters w, v, u,*

$$v \approx^{\mathcal{A}}_w u \text{ iff for all } a \in A \text{ such that } \varepsilon a \in w, a \in v \text{ iff } a \in u.$$

An analog of the Jónsson–Tarski theorem can be established using the following result of Makinson (1969); see also Givant & Halmos (2009, chap. 20, Exercise 15).

FACT 3.10. *Let A be a Boolean algebra and B a Boolean subalgebra of A. For any $a \in A$ not in B, there are ultrafilters U and V of A such that $a \in U$, $a \notin V$ and $U \cap B = V \cap B$.*

THEOREM 3.11. *Let $\mathcal{A} = \langle A, -, \sqcap, \varepsilon \rangle$ be an existential algebra, and $r : A \to \mathcal{P}(Uf(\mathcal{A}))$ the function such that for all $a \in A$,*

$$r(a) = \{w \in Uf(\mathcal{A}) : a \in w\}.$$

r embeds $\mathcal{A}$ in $\mathcal{A}^{(\approx^{\mathcal{A}})}$.

Proof. By Stone's theorem, it suffices to prove, for arbitrary $a \in A$, that $r(\varepsilon a) = \varepsilon^{(\approx^{\mathcal{A}})} r(a)$. Since $r(\varepsilon a) = \{w \in Uf(\mathcal{A}) : \varepsilon a \in w\}$ and $\varepsilon^{(\approx^{\mathcal{A}})} r(a) = \{w \in Uf(\mathcal{A}) : r(a) \in D_w^{(\approx^{\mathcal{A}})}\}$, this can be done by showing, for any $w \in Uf(\mathcal{A})$, that $\varepsilon a \in w$ iff $r(a) \in D_w^{(\approx^{\mathcal{A}})}$. By definition, $r(a) \in D_w^{(\approx^{\mathcal{A}})}$ iff for all $u, v \in Uf(\mathcal{A})$, if for all $b \in A$ such that $\varepsilon b \in w$, $b \in v$ iff $b \in u$, then $a \in v$ iff $a \in u$.

Assume that $\varepsilon a \in w$, and consider any $u, v \in Uf(\mathcal{A})$ such that for all $b \in A$ such that $\varepsilon b \in w$, $b \in v$ iff $b \in u$. Since $\varepsilon a \in w$, $a \in v$ iff $a \in u$ as required.

Assume that $\varepsilon a \notin w$. By Fact 3.10, there are $u, v \in Uf(\mathcal{A})$ such that for all $b \in A$ such that $\varepsilon b \in w$, $b \in v$ iff $b \in u$, but $a \in v$ and $a \notin u$ as required. □

It is now easily to derive the completeness of S5E with respect to equivalence systems:

THEOREM 3.12. S5E *is the L_E-logic of the class of all equivalence systems.*

Proof. Soundness is routine. So consider any set $\Gamma \subseteq L_E$ maximal consistent in S5E. Let $\mathcal{A}$ be the Lindenbaum–Tarski algebra of S5E and Γ as defined in the proof of Theorem 3.6, which by Theorem 3.11 can be embedded in $\mathcal{A}^{(\approx^{\mathcal{A}})}$. Let u be the ultrafilter to which Γ is mapped by the canonical assignment function. Extend this canonical assignment function via the embedding to an assignment function a for the embedding algebra $\mathcal{A}^{(\approx^{\mathcal{A}})}$. Then $u \in a(\gamma)$ for all $\gamma \in \Gamma$. By Lemma 3.8, $\approx, u, a \vDash \gamma$ for all $\gamma \in \Gamma$. □

Note that the proof of this result establishes a *strong* completeness result: for every S5E-consistent set of L_E-formulas, there is a world of some equivalence system and assignment function relative to which all of them are true.

3.2. Existential modal logics and equivalence systems. As the results of the previous section demonstrate, existential modal logics relate to equivalence systems in much the same ways as normal modal logics relate to relational frames, with existential algebras corresponding to (normal) Boolean algebras with operators. This section expands this observation, showing that many of the complexities in the study of normal modal logics also occur in the study of existential modal logics. First, a close connection between certain existential modal logics and certain normal bimodal logics will be established, with which a number of results concerning normal modal logics can be transferred to existential modal logics, in particular the existence of a continuum of logics and the existence of logics incomplete with respect to the possible world semantics. The required normal bimodal logics are the following:

DEFINITION 3.13. *Let L_U be like L_E but with U instead of E, i.e., a propositional bimodal language on Φ with modal operators $\Box$ and U. For any $\Gamma \subseteq L_U$, define* UKTΓ *to be the normal modal logic in L_U axiomatized by the members of Γ and the following axioms*:

$$\begin{array}{llll} T_U: & Up \to p & U\Box: & Up \to \Box p \\ 5_U: & \neg Up \to U\neg Up & T_\Box: & \Box p \to p. \end{array}$$

The required existential modal logics are those containing the following axiom:

(Z) $(Ep \wedge p \wedge \Box(p \to q)) \to Eq.$

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The idea behind singling out these two sets of logics is to enable U of L_U and $\Box$ of L_E to play corresponding roles, and likewise for $\Box$ of L_U and E of L_E. The correspondence between the former two is immediate, as both of them play the role of universal modalities. In the case of the latter two, the intended correspondence is for the worlds $\Box$-accessible from a given world w to correspond to those indistinguishable by w from itself. Since indistinguishability is a reflexive relation, $T_\Box$ is included in UKTΓ. In order for $\Box$ to be able to mimic E, the existential facts must be completely determined by which worlds a given world is unable to distinguish from itself; this is guaranteed by the validity of Z, enforcing that two worlds are only indistinguishable from a given world if it can't distinguish them from itself. On these assumptions, L_E and L_U are intertranslateable using the following mappings:

DEFINITION 3.14. *Let* $\tau : L_U \to L_E$ *be the recursive map whose non-trivial clauses are*:

$$\tau(\Box\varphi) = E\tau(\varphi) \wedge \tau(\varphi)$$
$$\tau(U\varphi) = \Box\tau(\varphi)$$

Let $\sigma : L_E \to L_U$ *be the recursive map whose non-trivial clauses are*:

$$\sigma(E\varphi) = \Box\sigma(\varphi) \vee \Box\neg\sigma(\varphi)$$
$$\sigma(\Box\varphi) = U\sigma(\varphi).$$

Implicitly extend such maps to sets of formulas.

The following results establish that the two syntactic mappings are mutual inverses, modulo equivalence in UKT/S5EZ, and preserve and anti-preserve derivability from axioms in the two logics:

LEMMA 3.15. *Let* $\varphi \in L_U$ *and* $\psi \in L_E$.

(i) $\vdash_{\mathrm{UKT}} \varphi \leftrightarrow \sigma\tau(\varphi)$

(ii) $\vdash_{\mathrm{S5EZ}} \psi \leftrightarrow \tau\sigma(\psi)$

Proof. By an induction on the structure of φ and ψ. Only the cases of $\Box$ for L_U and E for L_E are of interest:

(i) $\sigma\tau(\Box\varphi) = (\Box\sigma\tau(\varphi) \vee \Box\neg\sigma\tau(\varphi)) \wedge \sigma\tau(\varphi)$. By IH, it therefore suffices to prove $\vdash_{\mathrm{UKT}} \Box p \leftrightarrow ((\Box p \vee \Box\neg p) \wedge p)$, which is routine using $T_\Box$.

(ii) $\tau\sigma(E\psi) = (E\tau\sigma(\psi) \wedge \tau\sigma(\psi)) \vee (E\neg\tau\sigma(\psi) \wedge \neg\tau\sigma(\psi))$. By IH, it therefore suffices to prove $\vdash_{\mathrm{S5EZ}} Ep \leftrightarrow ((Ep \wedge p) \vee (E\neg p \wedge \neg p))$, which is routine using $C\neg$. □

LEMMA 3.16. *Let* $\varphi \in L_U$, $\Gamma \subseteq L_U$, $\psi \in L_E$, *and* $\Delta \subseteq L_E$.

(i) *If* $\vdash_{\mathrm{UKT}\Gamma} \varphi$ *then* $\vdash_{\mathrm{S5EZ}\tau(\Gamma)} \tau(\varphi)$.

(ii) *If* $\vdash_{\mathrm{S5EZ}\Delta} \psi$ *then* $\vdash_{\mathrm{UKT}\sigma(\Delta)} \sigma(\psi)$.

Proof. By a routine induction on the length of proofs. □

PROPOSITION 3.17. *Let* $\varphi \in L_U$, $\Gamma \subseteq L_U$, $\psi \in L_E$, *and* $\Delta \subseteq L_E$.

(i) $\vdash_{\mathrm{UKT}\Gamma} \varphi$ *iff* $\vdash_{\mathrm{S5EZ}\tau(\Gamma)} \tau(\varphi)$.

(ii) $\vdash_{\mathrm{S5EZ}\Delta} \psi$ *iff* $\vdash_{\mathrm{UKT}\sigma(\Delta)} \sigma(\psi)$.

Proof. By the previous two lemmas. □

This shows that existential modal logics containing Z correspond uniquely to normal extensions of UKT:

COROLLARY 3.18. *The lattices* $\{\mathrm{UKT}\Gamma : \Gamma \subseteq L_U\}$ *and* $\{\mathrm{S5EZ}\Delta : \Delta \subseteq L_E\}$ (*ordered by* $\subseteq$) *are isomorphic.*

Proof. Let f be the function mapping each $\Lambda \in \{\mathrm{UKT}\Gamma : \Gamma \subseteq L_U\}$ to $\mathrm{S5EZ}\tau(\Lambda)$; we show f to be the required isomorphism.

For surjectivity, consider any $\Delta \subseteq L_E$. We show that $f(\mathrm{UKT}\sigma(\Delta)) = \mathrm{S5EZ}\Delta$. Let $\varphi \in L_E$. Then:

$\vdash_{\mathrm{S5EZ}\Delta} \varphi$ iff $\vdash_{\mathrm{UKT}\sigma(\Delta)} \sigma(\varphi)$ (by Lemma 3.17(ii))
iff $\vdash_{\mathrm{UKT}(\mathrm{UKT}\sigma(\Delta))} \sigma(\varphi)$
iff $\vdash_{\mathrm{S5EZ}\tau(\mathrm{UKT}\sigma(\Delta))} \tau\sigma(\varphi)$ (by Lemma 3.17(i))
iff $\vdash_{\mathrm{S5EZ}\tau(\mathrm{UKT}\sigma(\Delta))} \varphi$ (by Lemma 3.15(ii))
iff $\vdash_{f(\mathrm{UKT}\sigma(\Delta))} \varphi$

Let $\Lambda, \Lambda' \in \{\mathrm{UKT}\Gamma : \Gamma \subseteq L_U\}$. Clearly, if $\Lambda \subseteq \Lambda'$, then $f(\Lambda) \subseteq f(\Lambda')$. So, assuming $\Lambda \nsubseteq \Lambda'$, we show $f(\Lambda) \nsubseteq f(\Lambda')$; this also proves that f is injective. Let $\varphi \in \Lambda$ such that $\varphi \notin \Lambda'$. Then $\tau(\varphi) \in f(\Lambda)$, but since $\nvdash_{\mathrm{UKT}\Lambda'} \varphi$, by Lemma 3.17(i), $\nvdash_{\mathrm{S5EZ}\tau(\Lambda')} \tau(\varphi)$ and so $\tau(\varphi) \notin f(\Lambda')$. □

With the following observation, made in Goranko & Passy (1992, p. 16), many known properties of normal unimodal logics containing the axiom $T_\Box$ can be transferred to normal extensions of UKT, and so with the results just established to existential modal logics containing Z. Here, let L be a standard unimodal propositional language based on Φ with the single modality $\Box$; let $KT\Gamma$ be the smallest normal modal logic in L containing the $T_\Box$ axiom and those in $\Gamma \subseteq L$.

FACT 3.19. *For each* $\Gamma \subseteq L$, $\mathrm{UKT}\Gamma$ *is a conservative extension of* $KT\Gamma$.

An example for a result about normal modal logics which can now easily be established for existential modal logics is the existence of a continuum of logics:

COROLLARY 3.20. *There are* $\beth_1$ *existential modal logics.*

Proof. By Corollary 3.18, Fact 3.19 and the fact that there are $\beth_1$ normal extensions of KT (which follows, e.g., from the results established in Fine (1974a)). □

In order for results on the relation the between normal modal logics and relational frames to be transferred to existential modal logics and equivalence systems, a connection has to be established between relational frames and equivalence systems. First, it is noted that Z has the intended semantic effect:

LEMMA 3.21. *Z is valid on an equivalence system $\approx$ on a set W iff for all $w, v, u \in W$, $v \approx_w u$ iff $v = u$ or $v, u \in [w]_{\approx_w}$.*

Proof. Consider any $w \in W$. $\approx, w \vDash Z$ iff for all $P \in D^{\approx}_w$ containing w and $Q \supseteq P$, $Q \in D^{\approx}_w$. This is the case iff $Q \in D^{\approx}_w$ for all $Q \supseteq [w]_{\approx_w}$, which is the case iff for all $v, u \in W$, $v \approx_w u$ only if $v = u$ or $v, u \in [w]_{\approx_w}$. The claim follows, as the right-to-left direction of the required biconditional holds for all equivalence systems. □

Each equivalence system validating Z determines a relational frame, by counting a world as accessible from another world if the second cannot distinguish itself from the first. Although this generality will not be needed, this mapping is well-defined for all equivalence systems:

DEFINITION 3.22. *For any equivalence system $\approx$ on a set W, define a relational frame $F^{\approx} = \langle W, R\rangle$ with Rwv iff $v \in [w]_{\approx_w}$.*

As the next lemma notes, this mapping matches the syntactic mapping τ from L_U to L_E, in the sense that truth of an L_U-formula in the relational frame determined by an equivalence system is equivalent to truth of its image in the equivalence system (interpreting U as truth in all worlds, like $\Box$ of L_E):

LEMMA 3.23. *For any equivalence system $\approx$ on a set W validating Z, $w \in W$, assignment function a and $\varphi \in L_U$,*

$$F^{\approx}, w, a \vDash \varphi \textit{ iff} \approx, w, a \vDash \tau(\varphi).$$

Proof. By induction on the structure of φ; only the case of $\Box$ is of interest:

$\approx, w, a \vDash \tau(\Box\varphi)$ iff $\approx, w, a \vDash E\tau(\varphi) \wedge \tau(\varphi)$ (by definition of τ)
iff $\approx, w, a \vDash \tau(\varphi)$ and for all $v, u \in W$ such that $v \approx_w u$, $\approx, v, a \vDash \tau(\varphi)$ iff $\approx, u, a \vDash \tau(\varphi)$ (by semantics)
iff $\approx, w, a \vDash \tau(\varphi)$ and for all $v, u \in [w]_{\approx_w}$, $\approx, v, a \vDash \tau(\varphi)$ iff $\approx, u, a \vDash \tau(\varphi)$ (by the validity of Z and Lemma 3.21)
iff $\approx, v, a \vDash \tau(\varphi)$ for all $v \in [w]_{\approx_w}$ (since $w \in [w]_{\approx_w}$)
iff $F^{\approx}, v, a \vDash \varphi$ for all $v \in [w]_{\approx_w}$ (by IH)
iff $F^{\approx}, v, a \vDash \varphi$ for all $v \in W$ such that Rwv (by construction of $F^{\approx}$)
iff $F^{\approx}, w, a \vDash \Box\varphi$ (by semantics). □

With this lemma, and the earlier connections between existential modal logics and extensions of KT, the fact that some extensions of KT are not the logic of any class of relational frames can be used to show that some existential modal logics are not the logic of any class of equivalence systems:

THEOREM 3.24. *There is an existential modal logic which is not the logic of any class of equivalence systems.*

Proof. As shown by Fine (1974b) and Thomason (1974), there are sets $\Gamma \subseteq L$ such that $KT\Gamma$ is not the logic of any class of relational frames. By Fact 3.19, UKTΓ is not the logic of any class of relational frames: there is a $\varphi \in L$ such that φ is valid on relational frames validating $KT\Gamma$ but not derivable in UKTΓ. Thus with Proposition 3.17(i), $\nvdash_{\mathrm{S5E}Z\tau(\Gamma)} \tau(\varphi)$. It suffices to show that $\tau(\varphi)$ is valid on any equivalence system $\approx$ validating Z and $\tau(\Gamma)$. So let $\approx$ be such a system; by Lemma 3.23, $F^{\approx}$ validates Γ and so φ; hence $\approx$ validates $\tau(\varphi)$, as required. □

The remainder of this section establishes a few further results on existential modal logics more directly, although mostly following well-known developments in normal modal logics. The first is an analog to a result due to Makinson (1971); to state it, let S5ETriv be the existential modal logic axiomatized by $p \to \Box p$.

PROPOSITION 3.25. *Every consistent existential modal logic is included in* S5ETriv.

Proof. For every existential algebra $\langle A, -, \sqcap, \varepsilon\rangle$, $\varepsilon 0 = \varepsilon 1 = 1$. Thus there is a unique two-element existential algebra, and for every existential algebra a homomorphism from the former to the latter. Thus every L_E-formula not valid on the two-element existential algebra is not valid on any existential algebra. With Theorem 3.6, it follows that every consistent existential modal logic is included in the logic of the two-element existential algebra. As $p \to \Box p$ is valid on the two-element existential algebra and no other existential algebra, it follows from Theorem 3.6 that S5ETriv is the logic of the two-element existential algebra. □

Analogous to the case of normal unimodal logics (but not normal multi-modal logics, as shown in Thomason (1972)) and relational frames, this proof also shows that every existential modal logic is valid on some equivalence system.

The next result shows that each sequence of E operators is a distinct modality in S5E; indeed, this result is shown to extend to the stronger L_E-logic of *coherent* equivalence systems (discussed in more detail below). For the statement and proof of this result, the notation x^n will be used to denote the sequence of n xs.

PROPOSITION 3.26. *There are no $m < n < \omega$ such that $E^n p \rightarrow E^m p$ is valid on the class of coherent equivalence systems.*

Proof. Let $\approx$ be the equivalence system on the set $W = \{0, 1\}^{<\omega}$ of finite sequences of 0s and 1s such that $v \approx_w u$ iff v and u are of equal length and share w as their initial subsequence. For each $x \in W$, let $\hat{x}$ be the permutation of W which maps any element of W of which x is a proper initial subsequence to the result of switching the first element following x (replacing 0 by 1, and *vice versa*), and every other element of W to itself. Note that each such permutation is an automorphism of $\approx$. Further, if $v \approx_w u$, then there are $x_1 \dots x_n \in W$ all of which have w as an initial subsequence such that $\hat{x}_1 \dots \hat{x}_n(v) = u$. $\hat{x}_1 \dots \hat{x}_n$ also maps w to itself, and each element of W to one $\approx_w$-related to it. Thus $\approx$ is coherent.

For each $n < \omega$, let $P_n = W \backslash \{0^m : m < n\}$. Letting $\varepsilon^{\approx}$ be the existence function on $\mathcal{P}(W)$ as defined in the construction of complex existential algebras (Definition 3.7), note that $\varepsilon^{\approx}(P_n) = P_{n-1}$ for all $n > 0$. Given $0 < n < \omega$, let $a(p) = P_n$. Then for all $m \leq n$, $\{\approx, w, a \vDash E^m p\} = P_{n-m}$. For any $m < n$, $P_{n-m} \subsetneq W = P_{n-n}$, so there is some $w \in P_{n-n} \backslash P_{n-m}$, and for any such element, $\approx, w, a \nvDash E^n p \rightarrow E^m p$. □

Finally, the decidability of S5E can be established by showing it to be the logic of finite equivalence systems, using a standard filtration argument.

PROPOSITION 3.27. S5E *is the logic of finite equivalence systems, and so decidable.*

Proof. It suffices to show that every L_E-formula satisfiable on an equivalence system is satisfiable on a finite equivalence system; as usual, this can be done by filtrating any given structure through an arbitrary set of formulas, closed under subformulas. So let $\approx$ be an equivalence system on a set W, a an assignment function and Γ a subformula closed set of L_E-formulas. Let $\sim$ be the equivalence relation on W of verifying, with a, the same members of Γ. Let $\approx'$ be the equivalence system on $W/{\sim}$ given by the following clause:

$$[v]_\sim \approx'_{[w]_\sim} [u]_\sim \text{ iff for all } E\varphi \in \Gamma, \text{ if } \approx, w, a \vDash E\varphi \text{ then } \approx, v, a \vDash \varphi \text{ iff } \approx, u, a \vDash \varphi.$$

Let a' be an assignment function which maps each $p \in \Gamma$ to $\{[w]_\sim : \approx, w, a \vDash p\}$. An induction on the structure of $\varphi \in \Gamma$ shows that $\approx, w, a \vDash \varphi$ iff $\approx', [w]_\sim, a' \vDash \varphi$ for all $w \in W$. The only case of interest is E:

If $\approx, w, a \nvDash E\varphi$, then there are $v, u \in W$ such that $v \approx_w u, \approx, v, a \vDash \varphi$ and $\approx, u, a \nvDash \varphi$. By IH, $\approx', [v]_\sim, a' \vDash \varphi$ and $\approx', [u]_\sim, a' \nvDash \varphi$. By construction of $\approx'$, $v \approx_w u$ entails $[v]_\sim \approx'_{[w]_\sim} [u]_\sim$, so $\approx', [w]_\sim, a' \nvDash E\varphi$.

If $\approx', [w]_\sim, a' \nvDash E\varphi$, then there are $v, u \in W$ such that $[v]_\sim \approx'_{[w]_\sim} [u]_\sim$, $\approx', [v]_\sim, a' \vDash \varphi$ and $\approx', [u]_\sim, a' \nvDash \varphi$, and so by IH $\approx, v, a \vDash \varphi$ and $\approx, u, a \nvDash \varphi$. By construction of $\approx'$, $\approx, w, a \nvDash E\varphi$.

Filtrating a given equivalence system witnessing $\varphi \notin$ S5E, guaranteed to exist by Theorem 3.12, through the subformulas of φ therefore shows φ to be falsifiable on a finite equivalence system, as required. □

3.3. Coherence and comprehension. This section considers the L_E-logic of coherent and quasicoherent equivalence systems, and proves a natural candidate axiomatization to be incomplete with respect to both classes.

Coherent equivalence systems validate all instances of the following schematic comprehension principle, the quantified version of which was discussed above:

$$(C_E)\quad \left(\bigwedge_{p\in \mathrm{pl}(\varphi)} Ep\right) \to E\varphi.$$

Here, φ is an L_E-formula and $\mathrm{pl}(\varphi)$ is the set of proposition letters occurring in φ. A complete axiomatization of the L_E-logic of coherent equivalence systems has to prove all instances of this principle. However, it turns out that to be able to derive all of them, it suffices to add as an axiom one instance:

DEFINITION 3.28. *Let* S5EC *be the existential modal logic axiomatized by the axiom* (CE) $Ep \to EEp$.

PROPOSITION 3.29. *Each instance of* C_E *is derivable in* S5EC.

Proof. It is routine to derive $E\Box p$ in S5E, from which trivially $\vdash_{\mathrm{S5E}} Ep \to E\Box p$. With this, a straightforward induction shows that all instances of C_E are derivable, using $C\neg$, $C\wedge$ and CE. □

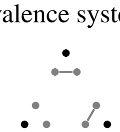
Coherence and quasicoherence are motivated by considerations closely related to the comprehension idea expressed in C_E. S5EC is therefore a natural candidate axiomatization for the logics of coherent and quasicoherent equivalence systems. As will now be shown, S5EC turns out to be too weak. Model-theoretically, it is helpful to consider first the structural differences between equivalence systems validating CE and (quasi)coherent equivalence systems. The first class can be singled out as follows:

DEFINITION 3.30. *An equivalence system* $\approx$ *on a set* W *is* existentially closed *if for all* $w \in W$ *and* $P \in D^{\approx}_{w}$, $\varepsilon^{\approx}(P) \in D^{\approx}_{w}$.

The next results show that existential closure is a non-trivial condition, but one that is weaker than being quasicoherent (which was noted above to be weaker than being coherent):

PROPOSITION 3.31. *There are equivalence systems which are not existentially closed (and so* $\nvdash_{\mathrm{S5E}} CE$*).*

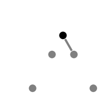
Proof. Consider the following equivalence system:



$\{1\} \in D^{\approx}_{1}$, but $\varepsilon^{\approx}(\{1\}) = \{1, 3\} \notin D^{\approx}_{w}$. □

LEMMA 3.32. *Every quasicoherent equivalence system is existentially closed, but not vice versa.*

Proof. See Proposition 2.25 for the first claim. The second claim is witnessed by the following example:



□

It will now be shown that S5EC is not complete with respect to quasicoherent equivalence systems, and so also not complete with respect to coherent equivalence systems: there are L_E-formulas valid on all quasicoherent equivalence systems which are not theorems of S5EC. The proof strategy is the following: A notion of a canonical description of a finite equivalence system is developed, using a proposition letter for each world. A notion of an equivalence system being an expansion of another is defined, and it is shown that the canonical description of a finite equivalence system is satisfiable on an equivalence system only if the latter is an expansion of the former. Finally, a finite equivalence system $\approx$ is constructed which is shown to be existentially closed, but which cannot be expanded to a quasicoherent equivalent equivalence system. It follows that the canonical description of $\approx$ is not satisfiable on quasicoherent equivalence systems, so its negation is in the logic of quasicoherent equivalence systems. But since *CE* is valid on existentially closed equivalence systems, so is S5EC, and thus any formula satisfiable on existentially closed equivalence systems is not derivable in S5EC. Since the canonical description of $\approx$ is satisfiable on $\approx$, its negation is therefore not derivable in S5EC, and so constitutes the required counterexample to the completeness of S5EC with respect to quasicoherent equivalence systems.

The next definition introduces the notion of a canonical description of a finite equivalence system $\approx$, for simplicity assumed to be based on a set $W = \{1, \ldots, n\}$. This description uses a proposition letter p_i for each $i \in W$; the description states that necessarily, exactly one of p_i, for $i \in W$, is true, and that each of them is possible. Furthermore, each is stated to strictly entail a description $\pi_i(\approx)$ of the existence facts at i. These latter descriptions are formulated as conjunctions of an existential claim or a negation of an existential claim for each proposition (subset of W), using a disjunction of letters p_i to express each proposition.

DEFINITION 3.33. *For every finite equivalence system $\approx$ on $W = \{1, \ldots, n\}$, define the* canonical description of $\approx$, *written $\gamma(\approx)$, as follows*:

$$\gamma(\approx) := \Box \bigvee_{i \leq n} p_i \wedge \bigwedge_{i<j\leq n} \Box(\neg p_i \vee \neg p_j) \wedge \bigwedge_{i\leq n} \Diamond p_i \wedge \bigwedge_{i\leq n} \Box(p_i \rightarrow \pi_i(\approx))$$

where

$$\pi_i(\approx) := \bigwedge_{X\in D_i^{\approx}} E \bigvee_{i\in X} p_i \wedge \bigwedge_{X\in\mathcal{P}(W)\setminus D_i^{\approx}} \neg E \bigvee_{i\in X} p_i.$$

Each canonical description is satisfiable on its equivalence system, and so the negation of every canonical description of a finite existentially closed equivalence system is underivable in S5EC:

LEMMA 3.34. *For each finite equivalence system $\approx$ on $\{1, \ldots, n\}$, $\approx \nvDash \neg\gamma(\approx)$.*

Proof. Let a be an assignment function such that $a(p_i) = \{i\}$ for all $i \leq n$; then $\approx, a \vDash \gamma(\approx)$. □

COROLLARY 3.35. *For each finite existentially closed equivalence system $\approx$ on $\{1, \ldots, n\}$, $\nvdash_{\mathrm{S5EC}} \neg\gamma(\approx)$.*

Proof. By Lemma 3.34 and the validity of CE on existentially closed equivalence systems. □

The required notion of expansions of equivalence system is developed more systematically in the appendix. To state the definition, let, for each function $f : A \rightarrow B$ and $B' \subseteq B$,

$f_{-1}(B')$ be the preimage of B' under f, i.e., the set $\{a \in A : f(a) \in B'\}$; for $b \in B$, let $f_{-1}(b) = f_{-1}(\{b\})$.

DEFINITION 3.36. *Let $\approx, \approx'$ be equivalence systems on sets W, W'. A* reduction from $\approx'$ to $\approx$ *is a surjective total function $f : W' \to W$ such that for all $P \subseteq W$ and $w \in W'$,*

$$P \in D^{\approx}_{f(w)} \text{ iff } f_{-1}(P) \in D^{\approx'}_{w}.$$

f being a reduction from $\approx'$ to $\approx$ is written $f : \approx' \rhd \approx$. In this case, $\approx'$ is called an expansion of $\approx$, *written $\approx \lhd \approx'$.*

Satisfiability of a canonical description entails being an expansion of the described equivalence system:

LEMMA 3.37. *For any finite equivalence system $\approx$ on $\{1, \ldots, n\}$ and equivalence system $\approx'$, if $\gamma(\approx)$ is satisfiable on $\approx'$, then $\approx \lhd \approx'$.*

Proof. Assume that $\gamma(\approx)$ is satisfiable on an equivalence system $\approx'$ on a set W', witnessed by an assignment function a. Let f be the function mapping each $w \in W'$ to the unique $i \leq n$ such that $w \in a(p_i)$; the first three conjuncts of $\gamma(\approx)$ guarantee that this is well-defined and surjective on $\{1, \ldots, n\}$. To show that $f : \approx' \rhd \approx$, let $P \subseteq \{1, \ldots, n\}$. Note that $f_{-1}(P) = \bigcup_{i \in P} a(p_i)$. So for any $j \leq n$ and $w \in a(p_j)$, $f_{-1}(P) \in D^{\approx'}_{w}$ iff $\bigcup_{i \in P} a(p_i) \in D^{\approx'}_{w}$. By $\pi_j(\approx)$, this is the case iff $P \in D^{\approx}_{j}$, i.e., $P \in D^{\approx}_{f(w)}$, as required. □

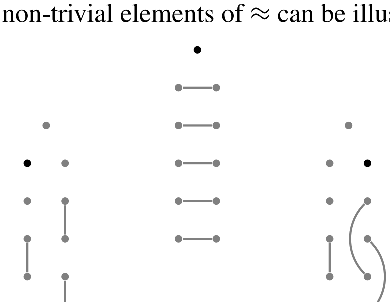
Consider the equivalence system $\approx$ on $W = \{1, \ldots, 11\}$ mapping each $i > 3$ to the identity relation, and $1, 2, 3$ to the equivalence relations corresponding to the following partitions:

$\approx_1$: $\{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}\}$
$\approx_2$: $\{\{1\}, \{2\}, \{3\}, \{4\}, \{10\}, \{5, 7\}, \{6, 8\}, \{9, 11\}\}$
$\approx_3$: $\{\{1\}, \{2\}, \{3\}, \{4\}, \{10\}, \{5, 9\}, \{6, 8\}, \{7, 11\}\}$

To illustrate this, consider the following arrangement of the elements:

1
2 3
4 5
6 7
8 9
10 11

Adapting the conventions for drawing equivalence systems mentioned above to this non-circular arrangement, the non-trivial elements of $\approx$ can be illustrated as follows:



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PROPOSITION 3.38. *$\approx$ is existentially closed.*

Proof. Consider any $i \in W$ and $P \in D_i^{\approx}$.
If $i > 3$, since $D_i^{\approx} = \mathcal{P}(W)$, trivially $\varepsilon^{\approx}(P) \in D_i^{\approx}$.
If $i = 1$, we distinguish two cases: If there are no $w, v \in \{4, \ldots, 11\}$ such that $w \in P$ and $v \notin P$, then $\varepsilon^{\approx}(P) = W \in D_1^{\approx}$. If there are such w, v, then $P \notin D_2^{\approx}$ and $P \notin D_3^{\approx}$, so, $\varepsilon^{\approx}(P) = \{1, 4, \ldots, 11\} \in D_1^{\approx}$.
If $i \in \{2, 3\}$, then for all $w, v \in \{4, \ldots, 11\}$, $P \in D_w^{\approx}$ iff $P \in D_v^{\approx}$, and therefore $\varepsilon^{\approx}(P)$ is a member of both $D_2^{\approx}$ and $D_3^{\approx}$. □

The next proposition shows that $\approx$ cannot be extended to any quasicoherent equivalence system, drawing on the following facts, established in the appendix as Proposition 5.8:

FACT 3.39. *Let $\approx'$ and $\approx''$ be equivalence systems, respectively, and $f : \approx'' \triangleright \approx'$.*

(i) *If $y \approx''_x z$ then $f(y) \approx'_{f(x)} f(z)$.*
(ii) *If $\{v, u\} \in D^{\approx'}_{f(x)}$ and $\{v\} \notin D^{\approx'}_{f(x)}$, then there are $y \in f_{-1}(v)$ and $z \in f_{-1}(u)$ such that $y \approx''_x z$.*

PROPOSITION 3.40. *There is no quasicoherent equivalence system $\approx'$ such that $\approx \triangleleft \approx'$.*

Proof. Assume for contradiction that there is a quasicoherent equivalence system $\approx'$ on a set W' and a function $f : \approx' \triangleright \approx$. Let $w \in f_{-1}(1)$. By Fact 3.39(ii), there are $v \in f_{-1}(2)$ and $u \in f_{-1}(3)$ such that $v \approx'_w u$. By the same result, there are $x \in f_{-1}(5)$ and $y \in f_{-1}(7)$ such that $x \approx'_v y$. By quasicoherence, there is an automorphism g of $\approx'$ such that $g(v) = u$ and $g \subseteq \approx'_w$. So $x \approx'_w g(x)$ and $y \approx'_w g(y)$. Since g is an automorphism, $g(x) \approx'_u g(y)$. By Fact 3.39(i), $fg(x) \approx_{f(u)} fg(y)$, as well as $f(x) \approx_{f(w)} fg(x)$ and $f(y) \approx_{f(w)} fg(y)$; hence $fg(x) \approx_3 fg(y)$, as well as $5 \approx_1 fg(x)$ and $7 \approx_1 fg(y)$. By the latter two, $fg(x) \in \{4, 5\}$ and $fg(y) \in \{6, 7\}$. $\lightning$, as there are no $z \in \{4, 5\}$ and $z' \in \{6, 7\}$ such that $z \approx_3 z'$. □

The main result can now be proven: S5EC is not complete with respect to quasicoherent equivalence systems, and so also not complete with respect to coherent equivalence systems.

THEOREM 3.41. *There is an L_E-formula which is valid on the class of quasicoherent equivalence systems but not a theorem of* S5EC.

Proof. Proposition 3.40 notes that there is no quasicoherent equivalence system $\approx'$ such that $\approx \triangleleft \approx'$. (Recall that $\approx$ is the 11-world system constructed above.) It follows with Lemma 3.37 that $\gamma(\approx)$ is not satisfiable on any quasicoherent system, and so $\neg\gamma(\approx)$ is valid on quasicoherent systems. But by Proposition 3.38, $\approx$ is existentially closed, and so with Corollary 3.35, $\nvdash_{\mathrm{S5E}C} \neg\gamma(\approx)$. □

Although coherence is partly motivated by the idea that propositions definable using existing propositions exist as well, the corresponding comprehension principle C_E does not suffice to axiomatize the L_E-logic of coherent equivalence systems. The counterexample presented here—the negation of the canonical description of the 11-world equivalence system $\approx$—does not suggest any systematic strengthening of the axioms of S5EC which would lead to a recursive axiomatization of coherent or quasicoherent equivalence systems. It will therefore be left open whether the L_E-logics of these classes are recursively axiomatizable, and if so, what natural axiomatizations look like, as well as the question whether the two classes of equivalence systems have the same L_E-logic. In fact, it is also left open here whether S5EC is complete with respect to the class of existentially closed equivalence systems—as shown in Theorem 3.24, there are existential modal logics which are not the logic of the class of equivalence systems on which they are valid.

§4. Conclusion. Equivalence systems naturally serve as models of contingency in what propositions there are. On this interpretation, the present results describe the resulting logics for propositional contingentism: The logics of propositional quantifiers and necessity arising from the class of all equivalence systems and the class of coherent equivalence systems are both recursively isomorphic to full second-order logic, and so not recursively axiomatizable. In the case of logics of an existential operator and a necessity operator, the class of all equivalence systems leads to a decidable logic, axiomatized in Definition 3.3, while the class of coherent equivalence systems leads to a logic which goes beyond the addition of a natural comprehension principle for propositions. The main issue left open here concerns the recursive axiomatizability of the latter logic.

Although equivalence systems are natural models for propositional contingentism, the idea of mapping worlds to equivalence relations on worlds is general enough to admit of useful alternative applications. One such application, tentatively suggested in Fritz & Lederman (2015), concerns the logic of awareness. There, worlds of equivalence systems also represent cognitive states, and the operator E of §3 is read as expressing an agent's awareness of a proposition. Another route to equivalence systems is *via* the idea of using equivalence relations to model subject matters; this route is already taken in Humberstone (2002),[1] building on work by von Kutschera (1994) and Lewis (1988a,b).

In the present terminology, Humberstone works with the class of all equivalence systems, and the fragment of the language L_E which excludes the operator $\Box$. He proves that the resulting logic is the classical (or "congruential") modal logic axiomatized by $C\top$, $C\neg$ and $C\wedge$. Since the classical modal logic axiomatized by $K_\Box$, $T_\Box$, and $5_\Box$ is S5 (see the proof of Proposition 3.4), which is easily seen to be the E-free fragment of S5E, this shows that the axiomatization of S5E given above is nicely modular: for both $\Box$ and E, the axioms only containing the relevant modality axiomatize the relevant fragment, and an axiomatization of the full logic is obtained by adding a single interaction axiom, $\Box E$.

Humberstone's result suggests a number of further questions: One concerns the proof methods used to obtain completeness results. Whereas the completeness proof in §3.1 proceeds by a representation theorem for an algebraic analog of equivalence systems, Humberstone's completeness result proceeds more directly via a relatively standard canonical model construction. It would be interesting to see whether the latter proof method could be extended to cover the former result. In any case, the former construction is of independent interest, as the algebras involved in it will appeal to those sympathetic to the idea that (possible) propositions form a Boolean algebra but skeptical of their satisfying the conditions of completeness and atomicity, as required for equivalence systems.

Another interesting question suggested by Humberstone's result concerns the $\Box$-free fragment of the logic of *coherent* equivalence systems. Despite the negative result of §3.3 concerning the logic of the full language L_E on coherent equivalence systems, it is an open question whether adding the comprehension axiom (CE) (i.e., $Ep \rightarrow EEp$) to Humberstone's axiom system in the $\Box$-free fragment suffices to yield a system which is complete with respect to coherent equivalence systems.

[1] Thanks to a reviewer for bringing this reference to my attention.

§5. Appendix A: Congruences and reductions. This appendix investigates various ways of turning one equivalence system into another by identifying worlds. Such operations will be presented in two guises, on the one hand as an equivalence relation on the set of worlds W of an equivalence system, which gives rise to an equivalence system based on the quotient set of W, and on the other hand as surjective functions mapping the worlds of one equivalence system to the worlds of another. These two guises will first be shown to be equivalent in a general setting, and corresponding further constraints will be imposed on both of them. Concerning the formal languages investigated above, it will be shown that all of these operations preserve validity of L_E-formulas in one direction, and that the analogous claim can only be established for $L_\exists$ by imposing additional restrictions.

5.1. Existential congruences. The first notion of congruences to be investigated imposes the minimal constraint that two worlds may only be identified if they agree on the existence facts concerning all propositions which don't distinguish between worlds that are being identified:

DEFINITION 5.1. *For any equivalence system $\approx$ on W, let an equivalence relation $\sim$ on W be an* existential congruence of $\approx$ *just in case for all $w, v \in W$ and $P \in \mathcal{A}(\sim)$, if $w \sim v$ then $P \in D^{\approx}_w$ iff $P \in D^{\approx}_v$.*

It will be helpful to provide a variant characterization of existential congruences. To state it, note that the equivalence relations on a given set form a complete lattice under the subset order (understanding relations as sets of pairs), in which the least upper bound of equivalence relations E_1 and E_2, written $E_1 \vee E_2$, relates elements x and y iff there is a finite sequence $z_0 \ldots z_n$ such that $z_0 = x$, $z_n = y$ and for all $i < n$, if i is even then $z_i E_1 z_{i+1}$ and if i is odd then $z_i E_2 z_{i+1}$ (cf. Davey & Priestley (2002, p. 139)). With the following lemma, the variant characterization of existential congruences is easily established:

LEMMA 5.2. *Let $\approx$ be an equivalence system on a set W, $\sim$ an equivalence relation on W, $P \in \mathcal{A}(\sim)$ and $w \in W$. There are $v \in P$ and $u \in W \backslash P$ such that $v\ (\approx_w \vee \sim)\ u$ iff there are $v \in P$ and $u \in W \backslash P$ such that $v \approx_w u$.*

Proof. If $v \approx_w u$ then $v\ (\approx_w \vee \sim)\ u$, so the right to left direction is trivial. Assume therefore that there are $v \in P$ and $u \in W \backslash P$ such that $v\ (\approx_w \vee \sim)\ u$. As noted, there is then a finite sequence $z_0 \ldots z_n$ such that $z_0 = v$, $z_n = u$ and for all $i < n$, if i is even then $z_i \approx_w z_{i+1}$ and if i is odd then $z_i \sim z_{i+1}$. Since $P \in \mathcal{A}(\sim)$, for all odd i, if $z_i \in P$ then $z_{i+1} \in P$. So there must be an even i such that $z_i \in P$ and $z_{i+1} \notin P$; since then $z_i \approx_w z_{i+1}$, z_i and z_{i+1} are the required elements. □

PROPOSITION 5.3. *An equivalence relation $\sim$ on a set W is an existential congruence of an equivalence system $\approx$ on W iff for all $w, v \in W$ such that $w \sim v$,*

$$(\approx_w \vee \sim) = (\approx_v \vee \sim).$$

Proof. Consider any $w, v \in W$ such that $w \sim v$; we show that $(\approx_w \vee \sim) = (\approx_v \vee \sim)$ just in case $P \in D^{\approx}_w$ iff $P \in D^{\approx}_v$ for all $P \in \mathcal{A}(\sim)$.

Assume first $(\approx_w \vee \sim) = (\approx_v \vee \sim)$ and let $P \in \mathcal{A}(\sim)$. If $P \notin D^{\approx}_w$ then there are $x \in P$ and $y \in W \backslash P$ such that $x \approx_w y$. So $x\ (\approx_w \vee \sim)\ y$ and therefore $x\ (\approx_v \vee \sim)\ y$. By Lemma 5.2, there are $s \in P$, $t \in W \backslash P$ such that $s \approx_v t$. So $P \notin D^{\approx}_v$. The other direction follows by symmetry, and so $P \in D^{\approx}_w$ iff $P \in D^{\approx}_v$ for all $P \in \mathcal{A}(\sim)$.

Assume now $(\approx_w \vee \sim) \neq (\approx_v \vee \sim)$; by symmetry, we can assume more specifically that there are $x, y \in W$ such that $x\ (\approx_w \vee \sim)\ y$ and not $x\ (\approx_v \vee \sim)\ y$. Then $[x]_{\approx_v \vee \sim}$ is a member of $D_v^{\approx}$ but not a member of $D_w^{\approx}$. Since $[x]_{\approx_v \vee \sim} \in \mathcal{A}(\sim)$, this is a $P \in \mathcal{A}(\sim)$ such that $P \in D_v^{\approx}$ but not $P \in D_w^{\approx}$. □

This variant characterization shows that the following natural way of quotienting equivalence systems by their existential congruences is well-defined:

DEFINITION 5.4. *If $\sim$ is an existential congruence of an equivalence system $\approx$ on a set W, let $\approx/\sim$ be the equivalence system on $W/\sim$ such that for all $w, v, u \in W$,*

$$[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim \textit{ iff } v\ (\approx_w \vee \sim)\ u.$$

For $P \in \mathcal{A}(\sim)$, let $P/\sim = \{[w]_\sim : w \in P\}$. For any assignment function $a : \Phi \to \mathcal{A}(\sim)$, let $a/\sim : \Phi \to \mathcal{P}(W/\sim)$ such that $a/\sim(p) = a(p)/\sim$.

Using the next lemma, the following proposition shows that validity of L_E-formulas is preserved under taking quotients:

LEMMA 5.5. *Let $\approx$ be an equivalence system on a set W and $\sim$ an existential congruence of $\approx$. If $X \subseteq W/\sim$ and $w \in W$ then $\bigcup X \in D_w^{\approx}$ iff $X \in D_{[w]_\sim}^{\approx/\sim}$.*

Proof. $\bigcup X \notin D_w^{\approx}$ iff there are $v \in \bigcup X$ and $u \in W \setminus \bigcup X$ such that $v \approx_w u$. By Lemma 5.2, this is the case iff there are $v \in \bigcup X$ and $u \in W \setminus \bigcup X$ such that $v\ (\approx_w \vee \sim)\ u$. $v \in \bigcup X$ iff $[v]_\sim \in X$; $u \in W \setminus \bigcup X$ iff $[u]_\sim \in (W/\sim) \setminus X$; and by construction of $\approx/\sim$, $v\ (\approx_w \vee \sim)\ u$ iff $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$. Thus $\bigcup X \notin D_w^{\approx}$ iff there are $v, u \in W$ such that $[v]_\sim \in X$, $[u]_\sim \in (W/\sim) \setminus X$ and $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$, which is the case iff $X \notin D_{[w]_\sim}^{\approx/\sim}$. □

PROPOSITION 5.6. *For any existential congruence $\sim$ of an equivalence system $\approx$ on a set W, $w \in W$, assignment function $a : \Phi \to \mathcal{A}(\sim)$ and $\varphi \in L_E$,*

$$\approx, w, a \vDash \varphi \textit{ iff } \approx/\sim, [w]_\sim, a/\sim \vDash \varphi.$$

Therefore, if $\approx \vDash \varphi$ then $\approx/\sim \vDash \varphi$.

Proof. By induction on the structure of φ; only the case of E is of interest.

$$\begin{aligned}
\approx, w, a \vDash E\varphi \ &\text{iff } \{v \in W : \approx, v, a \vDash \varphi\} \in D_w^{\approx} \\
&\text{iff } \{v \in W : \approx/\sim, [v]_\sim, a/\sim \vDash \varphi\} \in D_w^{\approx} \text{ (by IH)} \\
&\text{iff } \textstyle\bigcup \{x \in W/\sim : \approx/\sim, x, a/\sim \vDash \varphi\} \in D_w^{\approx} \\
&\text{iff } \{x \in W/\sim : \approx/\sim, x, a/\sim \vDash \varphi\} \in D_{[w]_\sim}^{\approx/\sim} \text{ (by Lemma 5.5)} \\
&\text{iff } \approx/\sim, [w]_\sim, a/\sim \vDash E\varphi.
\end{aligned}$$

That $\approx \vDash \varphi$ entails $\approx/\sim \vDash \varphi$ follows by the fact that each assignment function for $\approx/\sim$ is identical to $a/\sim$ for some assignment function a for $\approx$: If $\approx/\sim \nvDash \varphi$, then $\approx/\sim, [w]_\sim, a/\sim \nvDash \varphi$ for some w and a, and so $\approx, w, a \nvDash \varphi$, whence $\approx \nvDash \varphi$. □

This lemma can't be strengthened to $L_\exists$-formulas: For every equivalence system, the universal relation is an existential congruence, and the resulting quotient system is the unique (up to isomorphism) one-element system. The one-element system validates $L_\exists$-sentences whose negations are valid on some equivalence systems, such as $\forall p(p \to \Box p)$.

5.2. Reductions. Recall Definition 3.36 of reductions and expansions: A reduction from an equivalence system $\approx'$ on a set W' to an equivalence system $\approx$ on a set W is a surjective total function $f : W' \to W$ such that for all $P \subseteq W$ and $w \in W'$,

$$P \in D_{f(w)}^{\approx} \text{ iff } f_{-1}(P) \in D_w^{\approx'}.$$

In this case, $\approx'$ is called an expansion of $\approx$, symbolized $\approx \lhd \approx'$, and $f : \approx' \rhd \approx$ is used to indicate that f is a reduction from $\approx'$ to $\approx$.

For a function $f : A \to B$, let $\sim_f$ be the equivalence relation on A which relates x and y just in case $f(x) = f(y)$; for $A' \subseteq A$, let $f(A') = \{f(a) : a \in A'\}$. As with existential congruences, it is useful to give a secondary characterization of reductions:

PROPOSITION 5.7. *Let $\approx$ and $\approx'$ be equivalence systems on sets W and W', respectively, and $f : W' \to W$ a surjective function. Then $f : \approx' \rhd \approx$ iff for all $w, v, u \in W'$, $f(v) \approx_{f(w)} f(u)$ iff $v\ (\approx'_w \vee \sim_f)\ u$.*

Proof. (i) Assume first that $f : \approx' \rhd \approx$, and consider $w, v, u \in W'$.

(ia) Assume that not $f(v) \approx_{f(w)} f(u)$. Then $f(u) \notin [f(v)]_{\approx_{f(w)}}$. Let $X = f_{-1}([f(v)]_{\approx_{f(w)}})$. Since $f : \approx' \rhd \approx$, $X \in D^{\approx'}_w$. So there are no $x \in X$ and $y \in W' \backslash X$ such that $x \approx'_w y$. As $X \in \mathcal{A}(\sim_f)$, $v \in X$ and $u \notin X$, it follows with Lemma 5.2 that not $v\ (\approx'_w \vee \sim_f)\ u$.

(ib) Assume not $v\ (\approx'_w \vee \sim_f)\ u$. Let $X = [v]_{\approx'_w \vee \sim_f}$. Then $X \in D^{\approx'}_w$, $v \in X$ and $u \notin X$. Since $X \in \mathcal{A}(\sim_f)$, $X = f_{-1}(f(X))$, and so $f(X) \in D^{\approx}_{f(w)}$. $f(v) \in f(X)$ and $f(u) \notin f(X)$, so not $f(v) \approx_{f(w)} f(u)$.

(ii) Assume now that for all $w, v, u \in W'$, $f(v) \approx_{f(w)} f(u)$ iff $v\ (\approx'_w \vee \sim_f)\ u$, and consider any $P \subseteq W$ and $w \in W'$.

(iia) Assume that $P \notin D^{\approx}_{f(w)}$. By surjectivity of f, there are $v \in f_{-1}(P)$ and $u \in W' \backslash f_{-1}(P)$ such that $f(v) \approx_{f(w)} f(u)$. By assumption $v\ (\approx'_w \vee \sim_f)\ u$. Since $f_{-1}(P) \in \mathcal{A}(\sim_f)$, it follows with Lemma 5.2 that there are $x, y \in W'$ such that $x \approx'_w y$, $w \in f_{-1}(P)$ and $y \notin f_{-1}(P)$. So $f_{-1}(P) \notin D^{\approx'}_w$.

(iib) Assume that $f_{-1}(P) \notin D^{\approx'}_w$. So there are $v \in f_{-1}(P)$ and $u \in W' \backslash f_{-1}(P)$ such that $v \approx'_w u$. By assumption $f(v) \approx_{f(w)} f(u)$. Since $f(v) \in P$ and $f(u) \notin P$, $P \notin D^{\approx}_{f(w)}$. □

The following proposition draws two useful consequences from this result, appealed to in §3.3:

PROPOSITION 5.8. *Let $\approx$ and $\approx'$ be equivalence systems on W and W', respectively, and $f : \approx' \rhd \approx$.*

(i) *If $y \approx'_x z$ then $f(y) \approx_{f(x)} f(z)$.*
(ii) *If $\{v, u\} \in D^{\approx}_{f(x)}$ and $\{v\} \notin D^{\approx}_{f(x)}$, then there are $y \in f_{-1}(v)$ and $z \in f_{-1}(u)$ such that $y \approx'_x z$.*

Proof. (i) is immediate with Proposition 5.7. For (ii), assume $\{v, u\} \in D^{\approx}_{f(x)}$ and $\{v\} \notin D^{\approx}_{f(x)}$. Then $v \approx_{f(x)} u$. By surjectivity of f, there are $y \in f_{-1}(v)$ and $z \in f_{-1}(u)$; by Proposition 5.7, $y\ (\approx'_x \vee \sim_f)\ z$. So there is a finite sequence of elements from y to z alternatingly connected by $\approx'_x$ and $\sim_f$. By (i), $s \approx'_x t$ entails $f(s) \approx_{f(x)} f(t)$. So f maps all elements of this sequence to one of v and u. So there are s, t such that $s \in f_{-1}(v)$, $t \in f_{-1}(u)$ and $s \approx'_x t$. □

With Proposition 5.7, it can also be shown that existential congruences and reductions are interchangeable in the sense that for every existential congruence, the function mapping each world to its equivalence class is a reduction of the equivalence system to its quotient, and that for every reduction, the equivalence relation of being mapped to the same element by the reduction is an existential congruence quotienting by which gives rise to an equivalence system which is isomorphic to the image of the reduction:

PROPOSITION 5.9. *Let* $\approx$ *and* $\approx'$ *be equivalence systems on sets* W *and* W'*, respectively.*

(i) *For any existential congruence* $\sim$ *of* $\approx$, $[\cdot]_\sim : \approx \rhd \approx/\sim$.

(ii) *For any* $f : \approx' \rhd \approx$, $\sim_f$ *is an existential congruence of* $\approx'$ *and* $\approx \cong \approx'/\sim_f$.

Proof. (i) Let $\sim$ be an existential congruence of $\approx$. By construction, for all $w, v, u \in W$, $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$ iff $v\ (\approx_w \vee \sim)\ u$, so with Proposition 5.7, $[\cdot]_\sim : \approx \rhd \approx/\sim$.

(ii) Assume $f : \approx' \rhd \approx$. To show that $\sim_f$ is an existential congruence, we use Proposition 5.3. Consider any $w, v \in W'$ such that $w \sim_f v$ and $x, y \in W'$ such that $x\ (\approx'_w \vee \sim_f)\ y$. Then by Proposition 5.7, $f(x) \approx_{f(w)} f(y)$. Since $f(w) = f(v)$, $f(x) \approx_{f(v)} f(y)$, and so by Proposition 5.7 again $x\ (\approx'_v \vee \sim_f)\ y$. It follows by symmetry that $\sim_f$ is an existential congruence.

Define $f' : W'/\sim_f \to W$ such that $f'([w]_{\sim_f}) = f(w)$ for all $w \in W'$. Since f is surjective, f' is bijective. To show that f' is an isomorphism, consider any $w, v, u \in W'$. By definition of $\approx'/\sim_f$, $[v]_{\sim_f} (\approx'/\sim_f)_{[w]_{\sim_f}} [u]_{\sim_f}$ iff $v\ (\approx'_w \vee \sim_f)\ u$. By Proposition 5.7, this is the case iff $f(v) \approx_{f(w)} f(u)$, which by definition of f' is the case iff $f'([v]_{\sim_f}) \approx_{f'([w]_{\sim_f})} f'([u]_{\sim_f})$, as required. □

Existential congruences and expansions can therefore be seen as different ways of presenting the same structural connection between equivalence systems. On finite equivalence systems, this structural connection turns out to correspond exactly to the condition of preserving validity of L_E-formulas:

THEOREM 5.10. *For any finite equivalence system* $\approx$ *and equivalence system* $\approx'$,

$$\approx \lhd \approx' \text{ iff for all } \varphi \in L_E, \text{ if } \approx' \vDash \varphi \text{ then } \approx \vDash \varphi.$$

Proof. If $\approx \lhd \approx'$, witnessed by a reduction $f : \approx' \rhd \approx$, then by Proposition 5.9(ii), $\approx \cong \approx'/\sim_f$. By Proposition 5.6, if $\approx' \vDash \varphi$ then $\approx'/\sim_f \vDash \varphi$, and so $\approx \vDash \varphi$ by isomorphy.

So assume that for all $\varphi \in L_E$, if $\approx' \vDash \varphi$ then $\approx \vDash \varphi$. Since by Lemma 3.34 $\gamma(\approx)$ is satisfiable on $\approx$ (where $\gamma(\approx)$ is the canonical description of $\approx$, see Definition 3.33), it follows that $\gamma(\approx)$ is satisfiable on $\approx'$. So by Lemma 3.37, $\approx \lhd \approx'$. □

Given the finitary nature of L_E, it is unsurprising that this result cannot be strengthened to arbitrary equivalence systems. Indeed, infinite counterexamples can even be found which validate the same $L_\exists$-formulas:

PROPOSITION 5.11. *There are equivalence systems* $\approx$ *and* $\approx'$ *such that for all* $\varphi \in L_\exists$, $\approx' \vDash \varphi$ *iff* $\approx \vDash \varphi$*, but not* $\approx \lhd \approx'$.

Proof. Let $\approx$ be an equivalence system on an uncountable set W and $\approx'$ an equivalence system on a countably infinite set W' which both map each world to the identity relation. Since there is no surjective function $f : W' \to W$, there is no $f : \approx' \rhd \approx$. But it follows from Fritz (2013, Lemmas 24 and 25) that for all $\varphi \in L_\exists$, $\approx' \vDash \varphi$ iff $\approx \vDash \varphi$. □

5.3. Strict congruences. As noted above, the preservation of validity of L_E-formulas guaranteed by existential congruences and reductions does not extend to $L_\exists$-formulas. This motivates introducing a more restricted notion of congruences; this is the purpose of this section. The relevant notion is based on the idea that two worlds may only be identified if they are exactly alike as far as the facts concerning indistinguishability are concerned:

DEFINITION 5.12. *For any equivalence system* $\approx$ *on* W*, let an equivalence relation* $\sim$ *on* W *be a* strict congruence of $\approx$ *just in case for all* $w, w', v, v', u, u' \in W$,

$$\textit{if } w \sim w',\ v \sim v' \textit{ and } u \sim u',\ \textit{then } v \approx_w u \textit{ iff } v' \approx_{w'} u'.$$

The next lemma establishes some elementary but useful facts about strict congruences:

LEMMA 5.13. *Let* $\sim$ *be a strict congruence of an equivalence system* $\approx$ *on a set* W*, and* $w, v, u \in W$.

(i) $\sim \subseteq \approx_w$.

(ii) *If* $w \sim v$ *then* $\approx_w = \approx_v$.

(iii) $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$ *iff* $v \approx_w u$.

Proof. (i): If $v \sim u$, then as $\sim$ is a strict congruence, $v \approx_w v$ iff $v \approx_w u$, so $v \approx_w u$. (ii): If $w \sim v$, then as $\sim$ is a strict congruence, $u \approx_w u'$ iff $u \approx_v u'$.

(iii): Note that if $w \sim v$, then by (ii), $\approx_w = \approx_v$, so $D^{\approx}_w = D^{\approx}_v$. Thus $\sim$ is an existential congruence, and so $\approx/\sim$ is well-defined. By construction, $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$ iff v $(\approx_w \vee \sim)$ u. By (i), $(\approx_w \vee \sim) = \approx_w$, so v $(\approx_w \vee \sim)$ u iff $v \approx_w u$, as required. □

With these observations, it is easy to see that the condition of strictness is a genuine restriction of existential congruences:

PROPOSITION 5.14. *A strict congruence is an existential congruence, but not necessarily vice versa.*

Proof. The first claim was noted in the proof of Lemma 5.13. For the second claim, consider the universal relation on the following equivalence system:

This is an existential congruence, but by Lemma 5.13(i) not a strict congruence. □

A useful alternative characterization of the strict congruences of an equivalence system $\approx$ on a set W can be given using the equivalence relation $\sim_\approx$, specified in Definition 2.15: for all $w, v \in W$, $w \sim_\approx v$ iff $w \approx_u v$ for all $u \in W$ and $\approx_w = \approx_v$. This turns out to be the coarsest equivalence relation which is a strict congruence, with all of its refinements being strict congruences as well:

PROPOSITION 5.15. *For any equivalence system* $\approx$ *on a set* W*, an equivalence relation* $\sim$ *on* W *is a strict congruence of* $\approx$ *iff* $\sim \subseteq \sim_\approx$.

Proof. If $\sim$ is a strict congruence, then $\sim \subseteq \sim_\approx$ by Lemma 5.19(i) and (ii). So assume $\sim \subseteq \sim_\approx$, $w \sim w'$, $v \sim v'$, $u \sim u'$ and $v \approx_w u$; we show that $v' \approx_{w'} u'$. Since $\sim \subseteq \sim_\approx$, $w \sim w'$ entails that $\approx_w = \approx_{w'}$, and so $v \approx_{w'} u$, and $v \sim v'$ and $u \sim u'$ entail that $v \approx_{w'} v'$ and $u \approx_{w'} u'$. It follows that $v' \approx_{w'} u'$, as required. □

Adapting terminology of Gallin (1975), Definition 2.15 also specifies $\approx^s$, the simplification of $\approx$, to be the equivalence system on $W/\sim_\approx$ such that $[v]_{\sim_\approx} \approx^s_{[w]_{\sim_\approx}} [u]_{\sim_\approx}$ iff $v \approx_w u$. This is simply the quotient of $\approx$ by $\sim_\approx$:

PROPOSITION 5.16. *For any equivalence system* $\approx$*,* $\approx^s = \approx/\sim_\approx$.

Proof. Immediate by Lemma 5.13(iii). □

The next two lemmas establish the connections between simplifications, atom-selectivity and world-selectivity used in §2.1:

LEMMA 5.17. *An equivalence system is atom-selective iff its simplification is world-selective.*

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Proof. Assume $\approx$ is atom-selective, and consider any $w, v \in W$. If $[w]_{\sim_\approx} \approx^s_{[w]_{\sim_\approx}} [v]_{\sim_\approx}$, then $w \approx_w v$. So by atom-selectivity, $w \approx_u v$ for all worlds u and $\approx_w = \approx_v$, so $w \sim_\approx v$. Hence $[w]_{\sim_\approx} = [v]_{\sim_\approx}$. So $\approx^s$ is world-selective.

Assume $\approx^s$ is world-selective, and let $w \approx_w v$. Then $[w]_{\sim_\approx} \approx^s_{[w]_{\sim_\approx}} [v]_{\sim_\approx}$. So by world-selectivity, $w \sim_\approx v$. Hence it follows by the definition of $\sim_\approx$ that $\approx$ satisfies conditions (i) and (ii) of being atom-selective. □

LEMMA 5.18. *If $\approx$ is world-selective, then $\approx$ is isomorphic to $\approx^s$.*

Proof. Assume $\approx$ is world-selective. If $w \sim_\approx v$, then by Lemma 5.13(i), $w \approx_w v$, so $w = v$. Hence $\sim_\approx$ is the identity relation, which means that $[\cdot]_{\sim_\approx}$ is an isomorphism from $\approx$ to $\approx^s$. □

With the next lemma, the following proposition shows, analogous to Proposition 5.6, that strict congruences preserve satisfiability of $L_\exists$-formulas:

LEMMA 5.19. *Let $\sim$ be a strict congruence of an equivalence system $\approx$ on a set W, and $w, v \in W$.*

(i) *If $P \in \mathcal{A}(\sim)$, then $w \in P$ iff $[w]_\sim \in P/\sim$.*

(ii) $D^{\approx/\sim}_{[w]_\sim} = \{P/\sim \,:\, P \in D^{\approx}_w\}$.

Proof. (i): Let $P \in \mathcal{A}(\sim)$. If $w \in P$ then $[w]_\sim \in P/\sim$ by definition. If $[w]_\sim \in P/\sim$, then there is a $v \in [w]_\sim$ such that $v \in P$. Then $w \sim v$, and as $P \in \mathcal{A}(\sim)$, $w \in P$.

(ii): If $P \in D^{\approx}_w$, then by Lemma 5.13(i), $P \in \mathcal{A}(\sim)$, so $P/\sim$ is well-defined. If $[v]_\sim (\approx/\sim)_{[w]_\sim} [u]_\sim$, then $v \approx_w u$, so $v \in P$ iff $u \in P$, and therefore $[v]_\sim \in P/\sim$ iff $[u]_\sim \in P/\sim$ by (i). So $P/\sim \in D^{\approx/\sim}_{[w]_\sim}$.

If $P \in D^{\approx/\sim}_{[w]_\sim}$, let $P' = \{v \in w : [v]_\sim \in P\}$. If $v \approx_w u$, then $[v]_\sim \approx_{[w]_\sim} [u]_\sim$, so $[v]_\sim \in P$ iff $[u]_\sim \in P$, and so by (i), $v \in P'$ iff $u \in P'$; hence $P' \in D^{\approx}_w$. And $P'/\sim = \{[v]_\sim \in W/\sim : v \in P'\} = P$. □

PROPOSITION 5.20. *For any strict congruence $\sim$ of an equivalence system $\approx$ on a set W, $w \in W$, assignment function $a : \Phi \to \mathcal{A}(\sim)$ and $\varphi \in L_\exists$,*

$$\approx, w, a \vDash \varphi \text{ iff } \approx/\sim, [w]_\sim, a/\sim \vDash \varphi.$$

Therefore, if $\approx \vDash \varphi$ then $\approx/\sim \vDash \varphi$.

Proof. By induction on the structure of φ; only the case of the quantifier is of interest. Note that by Lemma 5.13(i), $D^{\approx}_w \subseteq \mathcal{A}(\sim)$ for all $w \in W$.

$\approx, w, a \vDash \exists p\varphi$ iff there is a $P \in D^{\approx}_w$ such that $\approx, w, a[P/p] \vDash \varphi$
iff there is a $P \in D^{\approx}_w$ such that $\approx/\sim, [w]_\sim, (a[P/p])/\sim \vDash \varphi$ (by IH)
iff there is a $P \in D^{\approx}_w$ such that $\approx/\sim, [w]_\sim, a/\sim[(P/\sim)/p] \vDash \varphi$
iff there is a $P \in D^{\approx/\sim}_{[w]_\sim}$ such that $\approx/\sim, [w]_\sim, a/\sim[P/p] \vDash \varphi$ (by Lemma 5.19(ii))
iff $\approx/\sim, [w]_\sim, a/\sim \vDash \exists p\varphi$. □

Note that for $L_\exists$-*sentences*, it immediately follows that $\approx \vDash \varphi$ iff $\approx/\sim \vDash \varphi$.

5.4. Strict reductions. The notion of reductions corresponding to strict congruences is given by the next definition.

DEFINITION 5.21. *Let $f : \approx' \rhd^s \approx$ if $f : W' \to W$ is surjective and such that for all $x, y, z \in W'$, $y \approx'_x z$ iff $f(y) \approx_{f(x)} f(z)$. In this case f is called a* strict reduction*, and $\approx'$ a* strict expansion *of $\approx$, in symbols $\approx \lhd^s \approx'$.*

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That strict congruences do in fact correspond to strict reductions is shown by an extension of Proposition 5.9:

PROPOSITION 5.22. *Let $\approx$ and $\approx'$ be equivalence systems on sets W and W', respectively.*

(i) *For any strict congruence $\sim$ of $\approx$, $[\cdot]_\sim : \approx \rhd^s \approx/\!\sim$.*
(ii) *For any $f : \approx' \rhd^s \approx$, $\sim_f$ is a strict congruence of $\approx'$.*

Proof. (i): Immediate by Lemma 5.13(iii). (ii): Assume $w \sim_f w'$, $v \sim_f v'$ and $u \sim_f u'$. If $v \approx'_w u$, then as f is a strict reduction, $f(v) \approx_{f(w)} f(u)$, and so $v' \approx'_{w'} u'$. The converse direction is similar. □

Analogous to Theorem 5.10, it will now be shown that for finite equivalence systems, being a strict expansion can be characterized in terms of preservation of $L_\exists$-validities. This requires a strengthened notion of a canonical description of a finite equivalence system, making use of the additional expressive power of propositional quantifiers. The following definition treats L_E-formulas as $L_\exists$-formulas, using the definition of E in terms of $\exists$ introduced in §2.1.

DEFINITION 5.23. *For every finite equivalence system $\approx$ on $W = \{1, \ldots, n\}$, define the* strict canonical description of $\approx$*, written $\sigma(\approx)$, as follows*:

$$\sigma(\approx) := \gamma(\approx) \wedge \Box\forall q \Box \bigvee_{X \subseteq W} \Box(q \leftrightarrow \bigvee_{i \in X} p_i).$$

LEMMA 5.24. *For each finite equivalence system $\approx$ on $\{1, \ldots, n\}$, $\approx \not\models \neg\sigma(\approx)$.*

Proof. Let a be an assignment function such that $a(p_i) = \{i\}$ for all $i \leq n$; then $\approx, a \models \sigma(\approx)$. □

That the strict expansions of a finite equivalence system can be characterized in terms of the satisfiability of its strict canonical description will be show using the following variant characterization of strict reductions:

LEMMA 5.25. *For any equivalence systems $\approx, \approx'$ on sets W, W', respectively, a reduction $f : \approx' \rhd \approx$ is strict iff for all $w \in W'$ and $P \in D^{\approx'}_w$, there is a $Q \subseteq W$ such that $P = f_{-1}(Q)$.*

Proof. Assume first that f is strict, and consider any $w \in W'$ and $P \in D^{\approx'}_w$. We show that $P = f_{-1}(f(P))$. Since clearly $P \subseteq f_{-1}(f(P))$, consider any $v \in f_{-1}(f(P))$. Then $f(v) \in f(P)$, so there is a $u \in P$ such that $f(v) = f(u)$. So $f(v) \approx_{f(w)} f(u)$, and thus by the strictness of f, $v \approx'_w u$. Since $u \in P$ and $P \in D^{\approx'}_w$, $v \in P$. Hence $f_{-1}(f(P)) \subseteq P$.

Assume now that for all $w \in W'$ and $P \in D^{\approx'}_w$, there is a $Q \subseteq W$ such that $P = f_{-1}(Q)$. Consider any $x, y, z \in W'$. Since f is a reduction, $y \approx'_x z$ only if $f(y) \approx_{f(x)} f(z)$ by Proposition 5.8(i). So assume $f(y) \approx_{f(x)} f(z)$. Then there is a $Q \subseteq W$ such that $[y]_{\approx'_x} = f_{-1}(Q)$. Since $y \in [y]_{\approx'_x}$, $f(y) \in Q$. As f is a reduction, $Q \in D^{\approx}_{f(x)}$. So $f(z) \in Q$, and therefore $z \in f_{-1}(Q)$. Thus $z \in [y]_{\approx'_x}$ and so $y \approx'_x z$. □

LEMMA 5.26. *For any finite equivalence system $\approx$ on $\{1, \ldots, n\}$ and equivalence system $\approx'$, if $\sigma(\approx)$ is satisfiable on $\approx'$, then $\approx \lhd^s \approx'$.*

Proof. By Lemma 3.37, the mapping f defined in the proof of this lemma is a reduction; with Lemma 5.25, it follows that it is strict. □

THEOREM 5.27. *For any equivalence system $\approx$ on a finite set W and equivalence system $\approx'$ on a set W',*

$$\approx \lhd^s \approx' \textit{ iff for all } \varphi \in L_\exists, \textit{ if } \approx' \vDash \varphi \textit{ then } \approx \vDash \varphi.$$

Proof. If $\approx \lhd^s \approx'$, witnessed by a strict reduction $f : \approx' \rhd^s \approx$, then by Propositions 5.22 and 5.9, $\sim_f$ is a strict congruence of $\approx'$ such that $\approx \cong \approx'/\sim_f$. By Proposition 5.20, if $\approx' \vDash \varphi$ then $\approx'/\sim_f \vDash \varphi$, and so $\approx \vDash \varphi$ by isomorphy.

So assume that for all $\varphi \in L_\exists$, if $\approx' \vDash \varphi$ then $\approx \vDash \varphi$. Since by Lemma 5.24 $\sigma(\approx)$ is satisfiable on $\approx$, it follows that $\sigma(\approx)$ is satisfiable on $\approx'$. So by Lemma 5.26, $\approx \lhd^s \approx'$. □

That this can't be strengthened to infinite equivalence systems is immediate from Proposition 5.11.

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