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Möbius automorphisms of surfaces with many circles

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Abstract. We classify real two-dimensional orbits of conformal subgroups such that the orbits contain two circular arcs through a point. Such surfaces must be toric and admit a Möbius automorphism group of dimension at least two. Our theorem generalizes the classical classification of Dupin cyclides.

1 Introduction

Our main result is Theorem 1, which states a classification of algebraic surfaces that admit many conformal automorphisms. Let us consider some known examples in order to motivate and explain this result. Suppose that a surface $Z \subseteq \mathbb{R}^n$ is a G-orbit for some conformal subgroup G and that Z is not contained in a hyperplane or hypersphere. It follows from Liouville's theorem that the conformal transformations of $\mathbb{R}^n$ for $n \geq 3$ are exactly the Möbius transformations. If $n = 3$, then $Z \subset \mathbb{R}^3$ is Möbius equivalent to either a circular cone, a circular cylinder, or a ring torus, and thus a *Dupin cyclide*. If $\dim G > 2$, then either $Z = \mathbb{R}^2$ or $Z \subset \mathbb{R}^4$ is a stereographic projection of a Veronese surface in the unit-sphere S^4. The considered examples of G-orbits contain at least two circles through each point and motivate us to address the following problem about surfaces that are in a sense "generalized Dupin cyclides":

Problem. *Classify, up to Möbius equivalence, real surfaces that are the orbit of a Möbius subgroup and that contain at least two circles through a point.*

We see in Figure 1 a linear projection of an orbit of a Möbius subgroup in $\mathbb{R}^5$ that contains three circles through each point. This surface is characterized by the third row of Theorem 1.

There has been recent interest in the classification of surfaces that contain at least two circles through each point [12, 15]. Surfaces that contain infinitely many circles through a general point are classified in [7] (see Theorem D). In this article, we consider the Möbius automorphism group as Möbius invariant, and we use methods from [3] (see Theorem B), a classification from [4] (see Theorem C), and results from [9] (see Theorem A).

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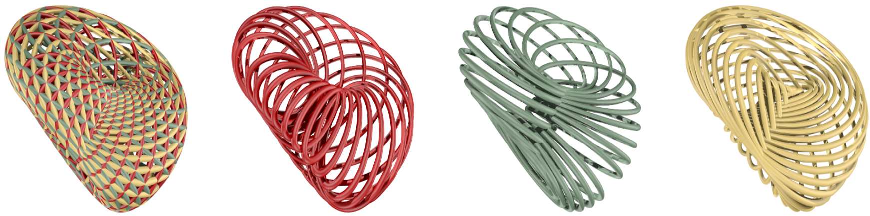

Figure 1: A projection of a smooth surface of degree 6 in $\mathbb{R}^5$ that is an $SO(2) \times SO(2)$-orbit and that contains three circles through each point. The family of Möbius equivalence classes of such surfaces is two-dimensional.

For our classification result, we require Möbius invariants that capture geometric aspects at complex infinity. To uncover this hidden structure, we define a *real variety* X to be a complex variety together with an antiholomorphic involution $\sigma: X \to X$, which is called its *real structure* [14, Section I.1]. We denote the real points of X by $X(\mathbb{R}) := \{p \in X \mid \sigma(p) = p\}$. Real varieties can always be defined by polynomials with real coefficients [13, Section 6.1]. Curves, surfaces, and projective spaces $\mathbb{P}^n$ are real algebraic varieties, and maps between such varieties are compatible with the real structures unless explicitly stated otherwise. Instead of $\mathbb{R}^n$, it is more natural to use the *Möbius quadric* for our space:

$$\mathbb{S}^n := \{x \in \mathbb{P}^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 0\},$$

where $\sigma: \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ sends x to $(\overline{x_0} : \cdots : \overline{x_{n+1}})$. The *Möbius transformations* of $\mathbb{S}^n$ are the biregular automorphisms $\mathrm{Aut}(\mathbb{S}^n)$, and they are linear so that $\mathrm{Aut}(\mathbb{S}^n) \subset \mathrm{Aut}(\mathbb{P}^{n+1})$. We denote a *stereographic projection* from the *unit-sphere* $S^n \subset \mathbb{R}^{n+1}$ by $\pi: S^n \dashrightarrow \mathbb{R}^n$. Notice that $\mathbb{S}^n(\mathbb{R}) \cong S^n$ and that the inverse stereographic projection $\pi^{-1}(\mathbb{R}^n) \subset \mathbb{S}^n$ is an isomorphic copy of $\mathbb{R}^n$ such that the Möbius transformations of $\mathbb{S}^n$ restrict to Möbius transformations of S^n and $\pi^{-1}(\mathbb{R}^n)$. In particular, the Möbius transformations that preserve the projection center of π, correspond to the *Euclidean similarities* of $\mathbb{R}^n$.

A conic $C \subset \mathbb{S}^n$ is called a *circle* if $C(\mathbb{R})$ defines a circle in $S^n \subset \mathbb{R}^{n+1}$. We call a surface in $\mathbb{S}^n$ *λ-circled* if it contains exactly λ circles through a general point. A *celestial surface* is a λ-circled surface $X \subseteq \mathbb{S}^n$ such that $\lambda \geq 2$ and such that X is not contained in a hyperplane section of $\mathbb{S}^n$. If in addition X is of degree d, then its *celestial type* is defined as $\mathbf{T}(X) := (\lambda, d, n)$. If the biregular automorphism group $\mathrm{Aut}(X)$ is a Lie group, then its identity component is denoted by $\mathrm{Aut}_\circ(X)$ and the *Möbius automorphism group* of X is defined as $\mathbf{M}(X) := \mathrm{Aut}_\circ(X) \cap \mathrm{Aut}_\circ(\mathbb{S}^n)$. We denote the *singular locus* of X by $\mathbf{S}(X)$. A complex node, real node, complex tacnode, and real tacnode are denoted by A_1, $\underline{A_1}$, A_3, and $\underline{A_3}$, respectively, and the union of such nodes is written as a formal sum. We write $\mathbf{S}(X) = \emptyset$, if X is smooth.

The *Möbius moduli dimension* $\mathbf{D}(X)$ is defined as the dimension of the space of Möbius equivalence classes of celestial surfaces $Y \subset \mathbb{S}^n$ such that $\mathbf{T}(Y) = \mathbf{T}(X)$, $\mathbf{S}(Y) \cong \mathbf{S}(X)$ as algebraic sets and $\mathbf{M}(Y) \cong \mathbf{M}(X)$ as groups.

We use the following notation for subgroups of $\mathrm{Aut}_\circ(\mathbb{P}^1)$. Let the real structure $\sigma: \mathbb{P}^1 \to \mathbb{P}^1$ be defined by $(x : y) \mapsto (\overline{x} : \overline{y})$ so that $\mathrm{Aut}_\circ(\mathbb{P}^1) \cong \mathrm{PSL}(2, \mathbb{R})$. If $p, q, r \in \mathbb{P}^1$

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such that $p \neq \sigma(p)$, $q = \sigma(q)$, $r = \sigma(r)$, and $q \neq r$, then we denote

$$\begin{aligned} \mathrm{PSO}(2) &:= \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1) \mid \varphi(p) = p,\ \varphi(\sigma(p)) = \sigma(p)\}, \\ \mathrm{PSX}(1) &:= \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1) \mid \varphi(q) = q,\ \varphi(r) = r\}, \\ \mathrm{PSE}(1) &:= \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1) \mid \varphi(x:y) = (x + \alpha\, y : y),\ \alpha \in \mathbb{R}\}, \text{ and} \\ \mathrm{PSA}(1) &:= \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1) \mid \varphi(r) = r\}. \end{aligned}$$

Notice that φ in PSO(2) or PSX(1) maps $(x:y)$ up to choice of coordinates to $(\cos(\alpha)\,x - \sin(\alpha)\,y : \sin(\alpha)\,x + \cos(\alpha)\,y)$ and $(\alpha\,x : y)$, respectively, for some $\alpha \in \mathbb{R}\backslash\{0\}$. The elements in PSA(1) are combinations of elements in PSX(1) and PSE(1) (see also Remark 31).

Theorem 1 (Möbius automorphisms of celestial surfaces) *If $X \subseteq \mathbb{S}^n$ is a celestial surface such that* $\dim \mathbf{M}(X) \geq 2$, *then X is toric and its Möbius invariants* $\mathbf{T}(X)$, $\mathbf{S}(X)$, $\mathbf{M}(X)$ *and* $\mathbf{D}(X)$ *are characterized by a row in the following table:*

$\mathbf{T}(X)$	$\mathbf{S}(X)$	$\mathbf{M}(X)$	$\mathbf{D}(X)$	Name
$(2,8,7)$	$\emptyset$	$\mathrm{PSO}(2) \times \mathrm{PSO}(2)$	3	Double Segre surface
$(2,8,5)$	$\emptyset$	$\mathrm{PSO}(2) \times \mathrm{PSO}(2)$	2	Projected dS
$(3,6,5)$	$\emptyset$	$\mathrm{PSO}(2) \times \mathrm{PSO}(2)$	2	dP6 (see Figure 1)
$(\infty,4,4)$	$\emptyset$	$\mathrm{PSO}(3)$	0	Veronese surface
$(4,4,3)$	$A_1 + A_1 + A_1 + A_1$	$\mathrm{PSO}(2) \times \mathrm{PSO}(2)$	1	Ring cyclide
$(2,4,3)$	$\underline{A_1} + \underline{A_1} + A_1 + A_1$	$\mathrm{PSO}(2) \times \mathrm{PSX}(1)$	0	Spindle cyclide
$(2,4,3)$	$\underline{A_3} + A_1 + A_1$	$\mathrm{PSO}(2) \times \mathrm{PSE}(1)$	0	Horn cyclide
$(\infty,2,2)$	$\emptyset$	$\mathrm{PSO}(3,1)$	0	Two-sphere

Moreover, if $\mathbf{T}(X) \notin \{(2,8,7),\ (2,8,5),\ (\infty,4,4)\}$, *then* $\mathbf{M}(X) = \mathrm{Aut}_\circ(X)$ *and if* $\mathbf{D}(X) = 0$, *then X is unique up to Möbius equivalence.*

If we replace $\mathbb{S}^n$ with $S^n \cong \mathbb{S}^n(\mathbb{R})$, then Theorem 1 holds if we replace PSO(3,1) by SO(3) and remove the remaining P's in the $\mathbf{M}(X)$ column. The case $\mathbf{T}(X) = (\infty,4,4)$ was already known and is revisited in Section 8 (see also [7, Theorem 23] and [1, Section 2.4.3]).

A *smooth model* of a surface $X \subset \mathbb{P}^{n+1}$ is a birational morphism $\widetilde{X} \to X$ from a nonsingular surface $\widetilde{X}$, such that this morphism does not contract (-1)-curves. If $\lambda < \infty$, then the smooth models of the λ-circled surfaces in Theorem 1 are isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ blown-up in either 0, 2, or 4 complex conjugate points (see Notation 11). A smooth model of a Veronese surface is isomorphic to $\mathbb{P}^2$ such that $\sigma{:}\mathbb{P}^2 \to \mathbb{P}^2$ sends x to $(\overline{x_0} : \overline{x_1} : \overline{x_2})$ (see Lemma 12c).

Instead of $\mathbb{S}^n$, one could also consider hyperquadrics of different signature. Although we do not pursue this, we cannot resist to mention the following result, which will come almost for free during our investigations:

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Corollary 2 *If* $Q \subset \mathbb{P}^8$ *is a quadric hypersurface of signature* $(4,5)$ *or* $(3,6)$*, then there exists a unique double Segre surface* $X \subset Q$ *such that* $\mathrm{Aut}_\circ(X) \subset \mathrm{Aut}_\circ(Q)$ *and* X *is isomorphic to* $\mathbb{S}^1 \times \mathbb{S}^1$ *and* $\mathbb{S}^2$*, respectively.*

Our methods are constructive and allow for explicit coordinate description of the moduli space of the celestial surfaces. See [8, `moebius_aut`] for an implementation using [16, Sage].

Definition 4 (names of surfaces) A surface $X \subset \mathbb{S}^3$ is called a *spindle cyclide*, *horn cyclide*, or *ring cyclide* if there exists a stereographic projection $\pi\colon S^3 \dashrightarrow \mathbb{R}^3$ such that $\pi(X(\mathbb{R})) \subset \mathbb{R}^3$ is a circular cone, circular cylinder, and ring torus, respectively. We call $X \subset \mathbb{P}^5$ a *Veronese surface* if there exists a biregular isomorphism $\mathbb{P}^2 \to X$ whose components form a basis of the vector space of degree 2 forms. We call $X \subset \mathbb{P}^8$ a *double Segre surface* (or *dS* for short) if there exists a biregular isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to X$ whose components form a basis of the vector space of bidegree (2,2) forms. A *projected dS* is a surface X that is a degree preserving linear projection of a dS. We call $X \subset \mathbb{P}^6$ a *sextic del Pezzo surface* (or *dP6* for short) if X is an anticanonical model of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in two general complex points.

Remark 5 (overview) Suppose that X is a celestial surface of type $\mathbf{T}(X) = (\lambda, d, n)$ such that $\dim \mathbf{M}(X) \geq 2$ and $\lambda < \infty$.

In Section 2, we classify X under the additional assumption that X is toric.

In Section 3, we give coordinates for a double Segre surface $Y_\star \subset \mathbb{P}^8$ and we investigate actions of real structures on $Y_\star$. This will be needed for finding quadratic forms of signature $(1, n+1)$ in the ideal of $Y_\star$.

We establish in Section 4, that X must be toric. Moreover, there exists a birational linear projection $\rho\colon Y_\star \dashrightarrow X$ and $\mathbf{M}(X)$ is isomorphic to a subgroup of $\mathrm{Aut}_\circ(Y_\star)$ that leaves the center of ρ invariant. We characterize the possible configurations for the center of ρ in $Y_\star$ and for each such configuration, we restrict the possible values for $\mathbf{T}(X)$ and $\mathbf{M}(X)$.

In Section 5, we encode, up to Möbius equivalence, X as an $\mathbf{M}(X)$-invariant quadratic form in the ideal of $Y_\star$. From the Lie algebra of $\mathbf{M}(X)$, we recover the subspace of $\mathbf{M}(X)$-invariant quadratic forms, and each invariant form of signature $(1, n+1)$ in this space encodes a possible Möbius equivalence class for X.

In Section 6, we show how toric real structures act on the Lie algebra of $\mathrm{Aut}_\circ(Y_\star)$, and we recall the classification of Lie algebras of complex subgroups of $\mathrm{Aut}_\circ(Y_\star)$.

In Section 7, we make a case distinction on the established configurations for the center of ρ and Lie algebras of $\mathbf{M}(X)$. If X is not a spindle or horn cyclide, then $\mathbf{M}(X) \cong \mathrm{SO}(2) \times \mathrm{SO}(2)$, and we obtain coordinates for the $\mathbf{D}(X)+1$ generators of all $\mathbf{M}(X)$-invariant quadratic forms of signature $(1, n+1)$ in the ideal of $Y_\star$. This enables us to conclude Section 7 with a proof for Theorem 1 and Corollary 2.

Finally, we present in Section 8 an alternative proof for the known $\lambda = \infty$ case by applying the same methods as before, but with $Y_\star$ replaced with the Veronese surface $Y_\circ \subset \mathbb{P}^5$.

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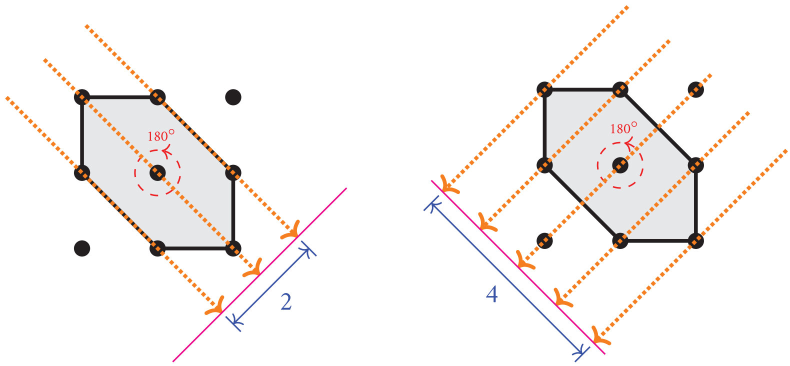

Figure 2: Width of lattice polygon along directions ↘ and ↙.

2 Toric celestial surfaces

In this section, we classify toric celestial surfaces and their real structures.

Suppose that $X \subset \mathbb{P}^n$ is a surface that is not contained in a hyperplane section. The *linear normalization* $X_N \subset \mathbb{P}^m$ of X is defined as the image of its smooth model $\widetilde{X}$ via the map associated to the complete linear series of hyperplane sections of X. Thus, $m \geq n$, X is a linear projection of X_N, and X_N is unique up to $\operatorname{Aut}(\mathbb{P}^m)$.

Let $\mathbb{T}^1 := (\mathbb{C}^*, 1)$ denote the *algebraic torus*. Recall that X is *toric* if there exists an embedding $i\colon \mathbb{T}^2 \hookrightarrow X$ such that $i(\mathbb{T}^2)$ is dense in X and such that the action of $\mathbb{T}^2$ on itself extends to an action on X.

If X is a toric surface, then there exists, up to projective equivalence, a monomial parametrization $\xi\colon \mathbb{T}^2 \to X_N$. The *lattice polygon of* X is defined as the convex hull of the points in the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, whose coordinates are defined by the exponents of the components of ξ. The antiholomorphic involution $\sigma\colon X \to X$ induces an involution $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$. Consequently, σ induces a unimodular involution $\mathbb{Z}^2 \to \mathbb{Z}^2$ that leaves the lattice polygon of X invariant.

Notation 6 *By abuse of notation we denote involutions on algebraic structures, that correspond functorially with the real structure $\sigma\colon X \to X$, by σ as well.*

A lattice projection $\mathbb{Z}^2 \subset \mathbb{R}^2 \to \mathbb{Z}^1 \subset \mathbb{R}^1$ induces a toric map $X_N \to \mathbb{P}^1$. We call a family of curves on X_N *toric* if the family can be defined by the fibers of a toric map. A family of circles on X is called *toric* if it corresponds to a toric family on X_N via a linear projection $X_N \to X$. The toric families of circles that cover a toric surface X that is not covered by complex lines, correspond to the projections of the lattice polygon of X to a line segment that is of minimal width among all such projections [10, Proposition 31].

In Figure 2, we see two examples of lattice projections of a lattice polygon. The width of the polygon in the ↙ direction is 4. The lattice polygon attains its minimal width of 2 in the directions →, ↓, and ↘. Notice that the lines through the origin, along these three directions, are invariant under lattice involution defined by 180° rotation around the central lattice point.

The *lattice type* $\mathbf{L}(X)$ of a toric surface X consists of the following data

(1) The lattice polygon $\Lambda \subset \mathbb{R}^2$ of X.
(2) The unimodular involution $\mathbb{Z}^2 \to \mathbb{Z}^2$ that is induced by the real structure $\sigma\colon X \to X$.
(3) The lattice projections that correspond to toric families of circles. We will represent such projections by arrows ($\downarrow$, $\rightarrow$, $\searrow$, $\swarrow$) pointing in the corresponding direction.

Lattice types $\mathbf{L}(X)$ and $\mathbf{L}(X')$ are equivalent if there exists a unimodular isomorphism between their lattice polygons that is compatible with the unimodular involution. Data 3 are uniquely determined by data 1 and 2. The unimodular involutions $\mathbb{Z}^2 \to \mathbb{Z}^2$, defined by $(x, y) \mapsto (x, y)$, $(x, y) \mapsto (-x, y)$, $(x, y) \mapsto (-x, -y)$ and $(x, y) \mapsto (y, x)$, are represented by their symmetry axes in the lattice polygons.

Proposition 7 (classification of toric celestial surfaces) *If $X \subseteq \mathbb{S}^n$ is a toric surface that is covered by at least two toric families of circles, then its lattice type $\mathbf{L}(X)$, together with $\mathbf{T}(X)$ and the name of X, is up to equivalence characterized by one of the eight cases in* Table 9.

Corollary 8 (classification of toric celestial surfaces)

a) *The antiholomorphic involutions of the double Segre surface—that act as unimodular involutions as in* Table 9a, Table 10a, *and* Table 10b*—are inner automorphic via* $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.
b) *The antiholomorphic involutions of the Veronese surface—that act as unimodular involutions as in Table 9d and Table 10c—are inner automorphic via* $\mathrm{Aut}(\mathbb{P}^2)$.

Before we prove Proposition 7, we state in Lemma 12 and Lemma 13 the known classification of real structures of $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{T}^2$. We include proofs in case we could not find a suitable reference. Theorem A collects results from [9] that we need for Proposition 7, Lemma 16, and Proposition 22.

It will follow from Proposition 22 in Section 4 that Proposition 7 also holds with the following hypothesis: *"If $X \subseteq \mathbb{S}^n$ is a toric celestial surface, then ..."*

Notation 11 *We consider the following normal forms for real structures:*

$$\sigma_+\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad (x : y) \mapsto (\overline{x} : \overline{y}), \qquad \sigma_-\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad (x : y) \mapsto (-\overline{y} : \overline{x}),$$

$$\sigma_s\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad (s : t; u : w) \mapsto (\overline{u} : \overline{w}; \overline{s} : \overline{t}).$$

Notice that $\mathbb{P}^1 \times \mathbb{P}^1$ with real structure $\sigma_+ \times \sigma_+$ or σ_s is isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^2$, respectively.

Lemma 12 (real structures for $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$)

a) *If $\sigma\colon \mathbb{P}^1 \to \mathbb{P}^1$ is an antiholomorphic involution, then there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ such that $(\gamma^{-1} \circ \sigma \circ \gamma)$ is equal to either σ_+ or σ_-.*
b) *If $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is an antiholomorphic involution, then there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$ such that $(\gamma^{-1} \circ \sigma \circ \gamma)$ is equal to either $\sigma_+ \times \sigma_+$, $\sigma_+ \times \sigma_-$, $\sigma_- \times \sigma_-$ or σ_s.*
c) *If $\sigma\colon \mathbb{P}^2 \to \mathbb{P}^2$ is an antiholomorphic involution, then there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ such that $(\gamma^{-1} \circ \sigma \circ \gamma)$ is equal to $\sigma_0 : (s : t : u) \mapsto (\overline{s} : \overline{t} : \overline{u})$.*

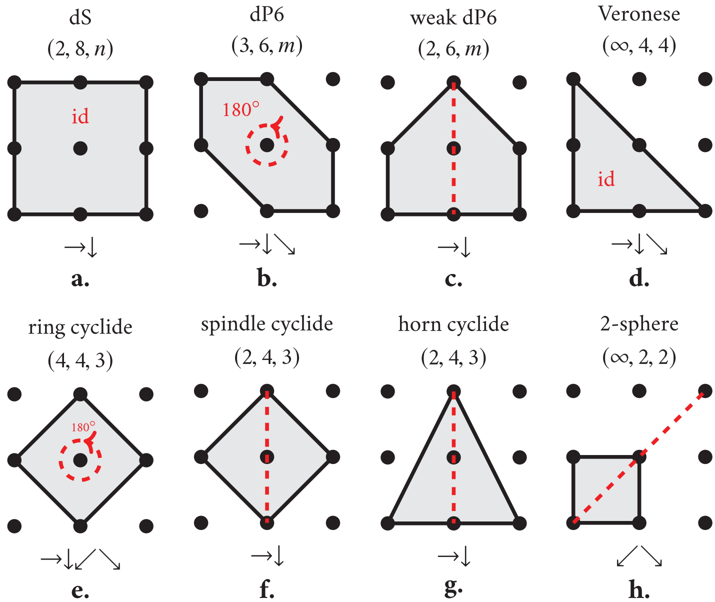

Table 9: See Proposition 7. Lattice types of toric celestial surfaces up to equivalence, together with the corresponding name and possible celestial types. For the celestial types, we have $3 \leq n \leq 7$ and $4 \leq m \leq 5$. The directions correspond to the toric families of circles.

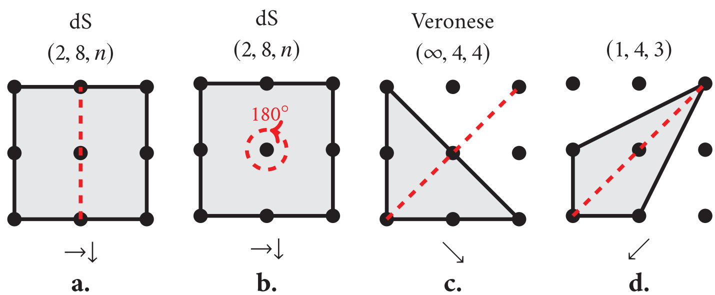

Table 10: See Corollary 8 and the proof of Proposition 7.

Proof *Claim 1. If X is a variety with antiholomorphic involution $\sigma\colon X \to X$ and very ample anticanonical class $-k$, then the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{-k}} & Y \subset \mathbb{P}^{h^0(-k)-1} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma_0} \\ X & \xrightarrow{\varphi_{-k}} & Y \subset \mathbb{P}^{h^0(-k)-1} \end{array}$$

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where $\sigma_0 : (x_0 : \cdots : x_n) \mapsto (\overline{x_0} : \cdots : \overline{x_n})$ *and* Y *is the image of* X *under the birational morphism* φ_{-k} *associated to* $-k$. This claim is a straightforward consequence of [14, I.(1.2) and I.(1.4)].

a) We apply claim 1 with $X = \mathbb{P}^1$ so that $Y \subset \mathbb{P}^2$ is a real conic. We know that Y has signature either $(3, 0)$ or $(2, 1)$. Thus there are, up to inner automorphism, two antiholomorphic involutions of $\mathbb{P}^1$. Moreover, we have that $|\{p \in \mathbb{P}^1 \mid \sigma(p) = p\}| \in \{0, \infty\}$. This concludes the proof, since the σ must be inner automorphic to either σ_+ or σ_-.

b) If σ does not flip the components of $\mathbb{P}^1 \times \mathbb{P}^1$, then it follows from a) that σ is inner automorphic to either $\sigma_+ \times \sigma_+$, $\sigma_+ \times \sigma_-$ or $\sigma_- \times \sigma_-$. Now suppose that σ flips the components of $\mathbb{P}^1 \times \mathbb{P}^1$. Let π_1 and π_2 be the complex first and second projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1_{\mathbb{C}}$, respectively. The composition $\pi_2 \circ \sigma \circ \pi_1^{-1}$ defines an antiholomorphic isomorphism $\tau \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. If σ and σ' are inner automorphic, then there exists complex $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ such that $\pi_2 \circ \sigma \circ \pi_1^{-1}$ is equal to $\beta \circ \pi_2 \circ \sigma' \circ \pi_1^{-1} \circ \alpha$. Conversely, an antiholomorphic isomorphism $\tau \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defines an antiholomorphic involution $(p; q) \mapsto (\tau^{-1}(q); \tau(p))$ that flips the components of $\mathbb{P}^1 \times \mathbb{P}^1$. There exists $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ such that $\beta \circ \tau \circ \alpha$ is defined by $(s : t) \mapsto (\overline{s} : \overline{t})$. We conclude that σ is unique up to inner automorphisms and thus without loss of generality inner automorphic to the real structure σ_s.

c) We apply claim 1 with $X = \mathbb{P}^2$, so that $Y \subset \mathbb{P}^9$ is a surface of degree 9. Since the degree is odd, we obtain infinitely many real points on Y and thus also infinitely many real points on $\mathbb{P}^2$. We know that $-k = 3h$, where $-k$ is the anticanonical class and h is the divisor class of lines in $\mathbb{P}^2$. We can construct two different real lines in $\mathbb{P}^2$, since a line through two real points is real. The linear subseries of $|-k|$ that consists of all cubics that contain these real two lines, is $|-k-2h| = |h|$. Notice that choosing a real subsystem of $|-k|$ is geometrically a real linear projection of Y. It follows that the map $\varphi_h \colon \mathbb{P}^2 \to \mathbb{P}^2$ associated to h is real such that $\sigma_0 \circ \varphi_h = \varphi_h \circ \sigma$. We conclude that σ is inner automorphic to σ_0 as was claimed. ∎

Lemma 13 (real structures for $\mathbb{T}^2$) *If* $\sigma \colon \mathbb{T}^2 \to \mathbb{T}^2$ *is a toric antiholomorphic involution, then there exists* $\gamma \in \mathrm{Aut}(\mathbb{T}^2_{\mathbb{C}})$ *such that* $(\gamma^{-1} \circ \sigma \circ \gamma)$ *is equal to either one of the following:*

$$\sigma_0 : (s, u) \mapsto (\overline{s}, \overline{u}), \quad \sigma_1 : (s, u) \mapsto (\tfrac{1}{\overline{s}}, \overline{u}),$$
$$\sigma_2 : (s, u) \mapsto (\tfrac{1}{\overline{s}}, \tfrac{1}{\overline{u}}), \quad \sigma_3 : (s, u) \mapsto (\overline{u}, \overline{s}),$$

and $\sigma_i \colon \mathbb{T}^2 \to \mathbb{T}^2$ *induces, up to unimodular equivalence, the following unimodular involution* $\sigma_i \colon \mathbb{Z}^2 \to \mathbb{Z}^2$*:*

$$\sigma_0 : (x, y) \mapsto (x, y), \qquad \sigma_1 : (x, y) \mapsto (-x, y),$$
$$\sigma_2 : (x, y) \mapsto (-x, -y), \quad \sigma_3 : (x, y) \mapsto (y, x).$$

The corresponding real points $\Gamma_{\sigma_i} := \{(s, u) \in \mathbb{T}^2 \mid \sigma_i(s, u) = (s, u)\}$ *are:*

$$\begin{aligned}
\Gamma_{\sigma_0} &= \{(s, u) \in \mathbb{T}^2 \mid s = \overline{s},\ u = \overline{u}\} \cong (\mathbb{R}^\star)^2,\\
\Gamma_{\sigma_1} &= \{(s, u) \in \mathbb{T}^2 \mid s\overline{s} = 1,\ u = \overline{u}\} \cong S^1 \times \mathbb{R}^\star,\\
\Gamma_{\sigma_2} &= \{(s, u) \in \mathbb{T}^2 \mid s\overline{s} = 1,\ u\overline{u} = 1\} \cong S^1 \times S^1,\\
\Gamma_{\sigma_3} &= \{(s, u) \in \mathbb{T}^2 \mid s = \overline{u}\} \cong \mathbb{R}^2 \backslash \{(0, 0)\}.
\end{aligned}$$

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Proof Since $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ extends to an antiholomorphic involution of an algebraic surface, we may assume that σ is defined by $(s,u) \mapsto \overline{f(s,u)}$ where f is some bivariate rational function in $\mathbb{C}(s,u)$. From $\sigma(1,1) = (1,1)$, it follows that $f(s,u) = (s^a u^b, s^c u^d)$ with $a,b,c,d \in \mathbb{Z}$. From $(\sigma \circ \sigma)(s,u) = (s,u)$, it follows that $ad - bc = \pm 1$, and thus the induced unimodular involution $(x,y) \mapsto (ax + by, cx + dy)$ is unimodular equivalent to $\sigma_i\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ for some $i \in \{0,1,2,3\}$ as asserted. For σ_1, we find that $f(s,u) = (\frac{1}{s}, u)$, and thus $\overline{f(s,u)} = (s,u)$ if and only if $s\overline{s} = 1$ and $u = \overline{u}$ so that $\Gamma_{\sigma_1} \cong S^1 \times \mathbb{R}^\star$. The proofs for Γ_{σ_0}, Γ_{σ_2}, and Γ_{σ_3} are similar. ∎

For convenience of the reader, Theorem A below extracts result from [9, Theorem 1, Theorem 3, Theorem 4, Corollary 5, Lemma 1, and Lemma 3] that are needed for Proposition 7, Lemma 16, and Proposition 22.

Notation 14 *The Neron–Severi lattice of a surface $X \subset \mathbb{P}^{n+1}$ consists of a unimodular lattice $N(X)$ that is defined by the divisor classes on the smooth model $\widetilde{X}$ up to numerical equivalence. In this article, $N(X)$ will be a sublattice of $\langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle_{\mathbb{Z}}$, where $\ell_0 \cdot \ell_1 = 1$, $\varepsilon_i^2 = -1$ and $\ell_0^2 = \ell_1^2 = \ell_0 \cdot \varepsilon_i = \ell_1 \cdot \varepsilon_i = 0$ for $1 \le i \le 4$. The real structure σ induces a unimodular involution $\sigma\colon N(X) \to N(X)$ such that $\sigma(\ell_0) = \ell_0$, $\sigma(\ell_1) = \ell_1$, $\sigma(\varepsilon_1) = \varepsilon_2$ and $\sigma(\varepsilon_3) = \varepsilon_4$ (see Notation 6). The function $h^0\colon N(X) \to \mathbb{Z}_{\ge 0}$ assigns to a divisor class the dimension of the vector space of its associated global sections. The two distinguished elements $h, k \in N(X)$ correspond to the class of hyperplane sections and the canonical class, respectively. We call a divisor class $c \in N(X)$ indecomposable if $h^0(c) > 0$ and if there do not exist nonzero $a, b \in N(X)$ such that $c = a + b$, $h^0(a) > 0$ and $h^0(b) > 0$. The subset of indecomposable (-2)-classes in $N(X)$ is defined as*

$$B(X) := \{c \in N(X) \mid -k \cdot c = 0,\ c^2 = -2 \text{ and } c \text{ is indecomposable}\}.$$

We use the following shorthand notation for elements in $B(X)$:

$$b_{ij} := \ell_0 - \varepsilon_i - \varepsilon_j, \quad b'_{ij} := \ell_1 - \varepsilon_i - \varepsilon_j, \quad b_1 := \varepsilon_1 - \varepsilon_3, \quad \text{and} \quad b_2 := \varepsilon_2 - \varepsilon_4,$$

and we underline the classes in $\{b \in B(X) \mid \sigma(b) = b\}$. A (projected) weak dP6 is (a degree preserving linear projection of) an anticanonical model of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in two complex points that lie in a fiber of a projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We call $X \subset \mathbb{S}^3$ a CH1 cyclide, if $X(\mathbb{R}) \subset S^3$ is an inverse stereographic projection of a circular hyperboloid of one sheet.

Theorem A (2019) *Suppose that X is a celestial surface of type $\mathbf{T}(X) = (\lambda, d, n)$. We use Notation 14.*

a) *If either $\lambda = \infty$, $d > 4$, $|\mathbf{S}(X_N)| > 2$ or $|B(X)| > 3$, then $\mathbf{T}(X)$, $\mathbf{S}(X_N)$, $B(X)$ and the name of X is characterized by a row in the Table 15, where $3 \le n \le 7$ and $4 \le m \le 5$.*
b) *If $\lambda < \infty$, then the class h of hyperplane sections of X is equal to the anticanonical class $-k$ and without loss of generality equal to either $2\,\ell_0 + 2\,\ell_1$, $2\,\ell_0 + 2\,\ell_1 - \varepsilon_1 - \varepsilon_2$ or $2\,\ell_0 + 2\,\ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$. If $\lambda = \infty$, then either $h = -\frac{2}{3}k$ and $k^2 = 9$, or $h = -\frac{1}{2}k$ and $k^2 = 8$.*
c) *If $\lambda < \infty$, then the smooth model $\widetilde{X}$ is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in either 0, 2, or 4 nonreal complex conjugate points. These points may be infinitely near, but at most two of the noninfinitely near points lie in the same fiber of a projection from*

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$\mathbf{T}(X)$	$\mathbf{S}(X_N)$	$B(X)$	name
$(\infty, 4, 4)$	$\emptyset$	$\emptyset$	Veronese surface
$(\infty, 2, 2)$	$\emptyset$	$\emptyset$	Two-sphere
$(2, 8, n)$	$\emptyset$	$\emptyset$	(Projected) dS
$(3, 6, m)$	$\emptyset$	$\emptyset$	(Projected) dP6
$(2, 6, m)$	$\underline{A_1}$	$\{\underline{b_{12}}\}$	(Projected) weak dP6
$(4, 4, 3)$	$4A_1$	$\{b_{13},\ b_{24},\ b'_{14},\ b'_{23}\}$	Ring cyclide
$(2, 4, 3)$	$2\underline{A_1} + 2A_1$	$\{b_{13},\ b_{24},\ \underline{b'_{12}},\ \underline{b'_{34}}\}$	Spindle cyclide
$(2, 4, 3)$	$\underline{A_3} + 2A_1$	$\{b_{13},\ b_{24},\ \underline{b'_{12}},\ b_1,\ b_2\}$	Horn cyclide
$(3, 4, 3)$	$\underline{A_1} + 2A_1$	$\{b_{13},\ b_{24},\ \underline{b'_{12}}\}$	CH1 cyclide

Table 15: See Theorem A.

$\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. The pullback into $\widetilde{X}$ of a fiber that contains two points is contracted to an isolated singularity of the linear normalization X_N.

Proof of Proposition 7 and Corollary 8 Let $X \subseteq \mathbb{S}^n$ be a λ-circled toric celestial surface of degree d that is covered by at least two toric families of circles. The lattice polygon of X contains i interior and b boundary lattice points.

Claim 1: $(i, b, d) \in \{(0,4,2), (0,6,4), (1,4,4), (1,6,6), (1,8,8)\}$ and $\lambda = \infty$ if $i = 0$. We know from [2, Propositions 10.5.6 and 10.5.8] that i and b are equal to the sectional genus $p_a(h) = \frac{1}{2}(h^2 + k \cdot h) + 1$ and anticanonical degree $-k \cdot h$, respectively (see also [5]). This claim now follows from Theorem Ab.

Claim 2: The lattice polygon of X is, up to unimodular equivalence, preserved by the unimodular involution $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ that is defined by either σ_0, σ_1, σ_2, or σ_3. This claim follows from Lemma 13.

Claim 3: A boundary line segment of the lattice polygon of X, that contains no more and no less than two lattice points, is not left invariant by the unimodular involution $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. Up to unimodular equivalence, we may assume that the two lattice points on a boundary line segment have coordinates $(0,0)$ and $(0,1)$, and that the remaining lattice points of the polygon lie strictly on the right side of these two points. Without loss of generality, the two lattice points correspond to the first two components s^0u^0 and s^0u^1 of a monomial parametrization $\xi\colon \mathbb{T}^2 \to X_N$ so that $\xi(0,u) = (1 : u : 0 : \cdots : 0)$ parametrizes a line in X_N. This line is either linear equivalent or linearly projected to a line in X. We conclude that claim 3 holds, since $X(\mathbb{R}) \subseteq S^n$ does not contain real lines.

Claim 4: If $d \neq 2$, then the lattice polygon of X attains its minimal width along at least two directions that are left invariant by the unimodular involution $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. Indeed, recall that such a direction corresponds to a toric family of circles.

Claim 5: If the lattice polygon of X is contained in a 3×3 grid centered at the origin, then $\mathbf{L}(X)$ together with $\mathbf{T}(X)$ and the name of X is in Table 9 or Table 10. For each real

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structure σ and pair (i, b) listed at claims 1 and 2, we list up to equivalence all lattice polygons in the 3×3 grid that are left invariant by σ and that have i interior and b boundary lattice points. Of these polygons, we discard those that contradict claim 3 or claim 4. For example, we exclude the lattice types in Table 10c,d as they contradict claim 4 and the lattice polygon of Table 10d together with unimodular involution σ_0 would contradict claim 3. We find that a candidate for $\mathbf{L}(X)$ is equivalent to one of Table 9 or Table 10a,b. We recover $|\mathbf{S}(X_N)|$ from the monomial parametrization associated to the lattice polygon. For each lattice type in Table 9 and Table 10a,b, we apply claim 1 and find that either $\lambda = \infty$, $d > 4$, or $|\mathbf{S}(X_N)| > 2$. It follows that the name and celestial type of X correspond to a row of Theorem Aa. If $\mathbf{L}(X)$ is Table 9d or Table 10c, then X is a Veronese surface by Definition 4 as the monomials associated to its lattice polygon span a basis for vector space of degree 2 forms on $\mathbb{P}^2$. If $\mathbf{L}(X)$ is Table 9h, then X is a two-sphere by claim 1. If $\mathbf{L}(X)$ is Table 9e, then there are 4 directions and thus $4 \leq \lambda < \infty$ so that X must be a ring cyclide. If $\mathbf{L}(X)$ is Table 9f or Table 9g, then X is a spindle cyclide and horn cyclide, respectively (see forward Example 35). This concludes the proof of claim 5.

Claim 6: The lattice polygon of X is contained in a 3×3 grid centered at the origin. If $\lambda = \infty$ or $d > 4$, then it follows from Theorem Aa, that X_N is unique up to projective equivalence and thus its lattice type is already realized in Table 9 or Table 10 at claim 5. If $\lambda < \infty$, then the lattice polygon of X must be one of [2, Theorem 8.3.7] such that (i, b) is as in claim 1. It follows that claim 6 holds.

Claim 7: Corollary 8 holds. Notice that X is covered by two families of conics that contain real points. Therefore, the real structure of a celestial double Segre surface must be inner automorphic to $\sigma_+ \times \sigma_+$ by Lemma 12b so that Corollary 8a holds. Corollary 8b is a consequence of Lemma 12c.

We concluded the proof of Proposition 7, since it follows from claims 5, 6, and 7 that $\mathbf{L}(X)$, $\mathbf{T}(X)$ and the name of X is up to equivalence characterized by one of the eight cases in Table 9. ■

3 Embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we give explicit coordinates for double Segre surfaces, which are embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^8$. We describe how real structures and projective automorphisms act on these embeddings.

We denote the vector space of quadratic forms in the ideal $I(X)$ of a surface X as

$$I_2(X) := \langle q \in I(X) \mid \deg q = 2 \rangle_{\mathbb{C}}.$$

Lemma 16 *If X is a toric celestial surface, then $\dim I_2(X_N)$ for the linear normalization X_N is as follows:*

Table 9:	a	b	c	d	e	f	g	h
$\dim I_2(X_N)$:	20	9	9	6	2	2	2	1 .

Proof The dimension of the space U of quadratic forms vanishing on $X_N \subset \mathbb{P}^m$ is equal to $h^0(2h)$, where h is the class of hyperplane sections. The dimension of the

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double Segre surface

(−1, 1)		(0, 1)		(1, 1)
	y_8	y_3	y_5	
(−1, 0)	y_2	y_0	y_1	(1, 0)
	y_6	y_4	y_7	
(−1, −1)		(0, −1)		(1, −1)

Veronese surface

(2, 0)		(2, 1)		(2, 2)
	y_5	y_7	y_8	
(1, 0)	y_3	y_1	y_6	(1, 2)
	y_0	y_2	y_4	
(0, 0)		(0, 1)		(0, 2)

Table 17: Coordinates for lattice points.

space W of quadratic forms in $\mathbb{P}^m$ is equal to $\binom{2+m}{2}$. Thus, we find that

$$\dim I_2(X_N) = \dim W/U = \dim W - \dim U = \binom{2+m}{2} - h^0(2h).$$

We obtain $h^0(2h) = \frac{1}{2}(4\,h^2 - 2\,h \cdot k) + 1$ as a straightforward consequence of Theorem Ac, Riemann–Roch theorem and Kawamata–Viehweg vanishing theorem. The main assertion now follows from Theorem Ab. ■

Let $Y_\star \subset \mathbb{P}^8$ denote the linear normalization of the double Segre surface with lattice polygon as in Table 9a. We consider the left coordinates in Table 17 so that we obtain the parametric map

$$\begin{aligned} \xi\colon \mathbb{T}^2 \to Y_\star \subset \mathbb{P}^8, \quad (s, u) \mapsto \\ (1 : s : s^{-1} : u : u^{-1} : su : s^{-1}u^{-1} : su^{-1} : s^{-1}u) \\ = (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8). \end{aligned}$$

Using ξ, we find the following 20 generators for the vector space of quadratic forms on $Y_\star$ and it follows from Lemma 16 that these form a basis:

$$\begin{aligned} I_2(Y_\star) = \langle\, & y_0^2 - y_1y_2,\ y_0^2 - y_3y_4,\ y_0^2 - y_5y_6,\ y_0^2 - y_7y_8,\ y_1^2 - y_5y_7,\ y_2^2 - y_6y_8, \\ & y_3^2 - y_5y_8,\ y_4^2 - y_6y_7,\ y_0y_1 - y_4y_5,\ y_0y_2 - y_3y_6,\ y_0y_3 - y_2y_5,\ y_0y_4 - y_1y_6, \\ & y_0y_1 - y_3y_7,\ y_0y_2 - y_4y_8,\ y_0y_3 - y_1y_8,\ y_0y_4 - y_2y_7,\ y_0y_5 - y_1y_3, \\ & y_0y_6 - y_2y_4,\ y_0y_7 - y_1y_4,\ y_0y_8 - y_2y_3\,\rangle_{\mathbb{C}}. \end{aligned}$$

Lemma 18 (real structures for $\mathbb{P}^8$) *Let* i *denote the imaginary unit. The maps* $\sigma_i\colon \mathbb{P}^8 \to \mathbb{P}^8$ *and* $\mu_i\colon \mathbb{P}^8 \to \mathbb{P}^8$ *which are defined by*

$$\begin{aligned} &\sigma_0 : y \mapsto (\overline{y_0} : \overline{y_1} : \overline{y_2} : \overline{y_3} : \overline{y_4} : \overline{y_5} : \overline{y_6} : \overline{y_7} : \overline{y_8}), \\ &\sigma_1 : y \mapsto (\overline{y_0} : \overline{y_2} : \overline{y_1} : \overline{y_3} : \overline{y_4} : \overline{y_8} : \overline{y_7} : \overline{y_6} : \overline{y_5}), \\ &\sigma_2 : y \mapsto (\overline{y_0} : \overline{y_2} : \overline{y_1} : \overline{y_4} : \overline{y_3} : \overline{y_6} : \overline{y_5} : \overline{y_8} : \overline{y_7}), \\ &\sigma_3 : y \mapsto (\overline{y_0} : \overline{y_3} : \overline{y_4} : \overline{y_1} : \overline{y_2} : \overline{y_5} : \overline{y_6} : \overline{y_8} : \overline{y_7}), \\ &\mu_0 : x \mapsto x, \\ &\mu_1 : x \mapsto (x_0 : x_1 + \mathrm{i}x_2 : x_1 - \mathrm{i}x_2 : x_3 : x_4 : x_5 + \mathrm{i}x_8 : x_7 - \mathrm{i}x_6 : x_7 + \mathrm{i}x_6 : x_5 - \mathrm{i}x_8), \\ &\mu_2 : x \mapsto (\tfrac{x_0}{2} : x_1 + \mathrm{i}x_2 : x_1 - \mathrm{i}x_2 : x_3 + \mathrm{i}x_4 : x_3 - \mathrm{i}x_4 : x_5 + \mathrm{i}x_6 : x_5 - \mathrm{i}x_6 : x_7 - \mathrm{i}x_8 : x_7 + \mathrm{i}x_8), \\ &\mu_3 : x \mapsto (x_0 : x_3 - \mathrm{i}x_1 : x_2 + \mathrm{i}x_4 : x_3 + \mathrm{i}x_1 : x_2 - \mathrm{i}x_4 : x_5 : x_6 : x_8 - \mathrm{i}x_7 : x_8 + \mathrm{i}x_7), \end{aligned}$$

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make the following diagram commute for all $0 \leq i \leq 3$*:*

$$\begin{array}{ccccc} \mathbb{T}^2 & \xrightarrow{\xi} & Y_\star & \xrightarrow{\mu_i^{-1}} & X_i \\ \sigma_i \downarrow & & \downarrow \sigma_i & & \downarrow \sigma_0 \\ \mathbb{T}^2 & \xrightarrow{\xi} & Y_\star & \xrightarrow{\mu_i^{-1}} & X_i \end{array}$$

where $X_i := \mu_i^{-1}(Y_\star)$ *and real structure* $\sigma_i\colon \mathbb{T}^2 \to \mathbb{T}^2$ *is defined in Lemma 13.*

Proof The specification of $\sigma_i\colon Y_\star \to Y_\star$ for $0 \leq i \leq 3$ follows from the action on the lattice coordinates in Table 17 (recall Notation 6). It is straightforward to verify that μ_i^{-1} makes the diagram commute. ∎

Remark 19 Recall from Corollary 8, Notation 11, and Lemma 12 that the real structures $\sigma_i\colon Y_\star \to Y_\star$ for $0 \leq i \leq 2$ are inner automorphic to $\sigma_+ \times \sigma_+$ via $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ so that $X_i \cong \mathbb{S}^1 \times \mathbb{S}^1$ in these cases. Notice that $\sigma_3\colon Y_\star \to Y_\star$ is via $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ inner automorphic to σ_s so that $X_3 \cong \mathbb{S}^2$.

The surface X_2 from Lemma 18 is contained in $\mathbb{S}^7$. Indeed, if we compose the first four generators of $I_2(Y_\star)$ with μ_2, then

$$(y_0^2 - y_1y_2) \circ \mu_2 = \tfrac{1}{4}\, x_0^2 - x_1^2 - x_2^2, \quad (y_0^2 - y_5y_6) \circ \mu_2 = \tfrac{1}{4}\, x_0^2 - x_5^2 - x_6^2,$$
$$(y_0^2 - y_3y_4) \circ \mu_2 = \tfrac{1}{4}\, x_0^2 - x_3^2 - x_4^2, \quad (y_0^2 - y_7y_8) \circ \mu_2 = \tfrac{1}{4}\, x_0^2 - x_7^2 - x_8^2,$$

and their sum is the equation of $\mathbb{S}^7$.

We extend $\xi\colon \mathbb{T}^2 \to Y_\star$ such that we obtain the biregular isomorphism

$$\begin{aligned} &\tilde{\xi}\colon \mathbb{P}^1 \times \mathbb{P}^1 \to Y_\star, \quad (s:t;u:w) \mapsto \\ &(stuw : s^2uw : t^2uw : stu^2 : stw^2 : s^2u^2 : t^2w^2 : s^2w^2 : t^2u^2) \\ = &(y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8), \end{aligned}$$

and thus $\mathrm{Aut}_\circ(Y_\star) \cong \mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$.

Lemma 20 $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$.

Proof Automorphisms in the identity component $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ act trivially on the Neron–Severi lattice $N(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \ell_0, \ell_1 \rangle_{\mathbb{Z}}$, where generators ℓ_0 and ℓ_1 are the classes of the fibers of the first and second projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. Thus a fiber of π_i is mapped by $\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ as a whole to a fiber of π_i for all $1 \leq i \leq 2$ so that the main assertion is concluded. We remark that $\mathrm{Aut}_\circ(\mathbb{P}^1) \cong \mathrm{PSL}(2,\mathbb{R}) \subsetneq \mathrm{PGL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{P}^1)$. ∎

We associate to $\varphi = (\varphi_1, \varphi_2) \in \mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$ an automorphism

$$\mathcal{S}(\varphi) := \mathrm{Sym}_2(\varphi_1) \otimes \mathrm{Sym}_2(\varphi_2) \in \mathrm{Aut}_\circ(Y_\star) \subset \mathrm{Aut}_\circ(\mathbb{P}^8).$$

We can compute $\mathcal{S}(\varphi)$ via the following specification:

$$\mathcal{S}(\varphi)\colon Y_\star \to Y_\star, \quad \tilde{\xi}(p) \mapsto (\tilde{\xi} \circ \varphi)(p), \tag{1}$$

for all $p \in \mathbb{P}^1 \times \mathbb{P}^1 \cong Y_\star$.

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Example 21 (toric automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$) Since $\varphi \circ \xi \colon \mathbb{T}^2 \hookrightarrow Y_\star$ defines an embedding of the algebraic torus $\mathbb{T}^2$ for all automorphisms $\varphi \in \mathrm{Aut}(Y_\star)$, the double Segre surface $Y_\star$ does not have a unique toric structure. Let $\mathrm{Aut}_\circ^{\mathbb{T}}(Y_\star)$ denote the identity component of the toric automorphisms with respect to ξ. We have the following parametrization:

$$\phi\colon \mathbb{T}^2 \stackrel{\cong}{\to} \mathrm{Aut}_\circ^{\mathbb{T}}(Y_\star), \quad (s,u) \mapsto B(u) \circ A(s), \qquad \text{where}$$
$$A(\alpha) := \mathcal{S}\left(\begin{bmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) \text{ and } B(\alpha) := \mathcal{S}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{bmatrix}\right).$$

Suppose that the real structure of $Y_\star$ is defined by σ_2 in Lemma 18. It follows from Lemma 13 that $\{p \in \mathbb{T}^2 \mid \sigma_2(p) = p\} \cong S^1 \times S^1$ and thus

$$\phi\colon S^1 \times S^1 \stackrel{\cong}{\to} \mathrm{Aut}_\circ^{\mathbb{T}}(Y_\star), \quad \Big((\cos(\alpha), \sin(\alpha)),\ (\cos(\beta), \sin(\beta)) \Big) \mapsto$$
$$\mathcal{S}\left(\begin{bmatrix}\cos(\alpha) + \mathrm{i}\sin(\alpha) & 0\\ 0 & \cos(\alpha) - \mathrm{i}\sin(\alpha)\end{bmatrix}, \begin{bmatrix}\cos(\beta) + \mathrm{i}\sin(\beta) & 0\\ 0 & \cos(\beta) - \mathrm{i}\sin(\beta)\end{bmatrix}\right).$$

Let $\mu_2\colon \mathbb{P}^8 \to \mathbb{P}^8$ be as defined in Lemma 18. From the composition of ϕ with the pullback $\mu_2^*\colon \mathrm{Aut}_\circ^{\mathbb{T}}(Y_\star) \to \mathrm{Aut}_\circ^{\mathbb{T}}(X_2)$ we obtain

$$\mu_2^* \circ \phi\colon S^1 \times S^1 \to \mathrm{Aut}_\circ^{\mathbb{T}}(X_2), \Big((\cos(\alpha), \sin(\alpha)),\ (\cos(\beta), \sin(\beta)) \Big) \mapsto$$
$$\mathcal{S}\left(\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}, \begin{bmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{bmatrix}\right).$$

Notice that the real structure of $X_2 \subset \mathbb{S}^7$ is defined by σ_0 in Lemma 18 and that $\mathrm{Aut}_\circ^{\mathbb{T}}(X_2) \cong \mathrm{PSO}(2) \times \mathrm{PSO}(2)$.

4 Blowups of $\mathbb{P}^1 \times \mathbb{P}^1$

The smooth model of a celestial surface that is not ∞-circled is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in two or four points. Such a blowup is realized by a linear projection of the double Segre surface $Y_\star$ in $\mathbb{P}^8$. The automorphisms of the image surface induce automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ that leave the center of blowup invariant. This allows us to formulate restrictions on the possible Möbius automorphism groups of celestial surfaces. In particular, we find that celestial surfaces with many symmetries must be toric.

Proposition 22 (blowups of $\mathbb{P}^1 \times \mathbb{P}^1$) *If a celestial surface $X \subset \mathbb{S}^n$ is not ∞-circled and* $\dim \mathrm{Aut}_\circ(X) \geq 2$*, then its linear normalization X_N is a toric surface, each family of circles on X is toric, and $\mathrm{Aut}_\circ(X)$ embeds as a subgroup into $\mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$. Moreover, there exists a birational linear projection*

$$\rho\colon Y_\star \subset \mathbb{P}^8 \dashrightarrow X \subset \mathbb{P}^{n+1},$$

whose center of projection is characterized by a row in Table 23 together with $\mathbf{T}(X)$, $\mathbf{S}(X_N)$ *and the projections of* $\mathrm{Aut}_\circ(X)$ *to a subgroup of* $\mathrm{Aut}_\circ(\mathbb{P}^1)$.

Proof By Theorem Ac,b, the smooth model $\widetilde{X}$ is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in a center Λ such that $|\Lambda| \in \{0, 2, 4\}$ and X_N is its anticanonical model. Hence

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Λ	$\mathbf{T}(X)$	$\mathbf{S}(X_N)$	$\pi_1(\mathrm{Aut}_\circ(X))$	$\pi_2(\mathrm{Aut}_\circ(X))$	name
(a)	$(2, 8, n)$	$\emptyset$	PSL(2)	PSL(2)	(projected) dS
(b)	$(3, 6, m)$	$\emptyset$	PSO(2)	PSO(2)	(projected) dP6
(c)	$(2, 6, m)$	A_1	PSO(2)	PSA(1)	(projected) weak dP6
(d)	$(4, 4, 3)$	$4A_1$	PSO(2)	PSO(2)	ring cyclide
(e)	$(2, 4, 3)$	$2\underline{A_1} + 2A_1$	PSO(2)	PSX(1)	spindle cyclide
(f)	$(2, 4, 3)$	$\underline{A_3} + 2A_1$	PSO(2)	PSE(1)	horn cyclide

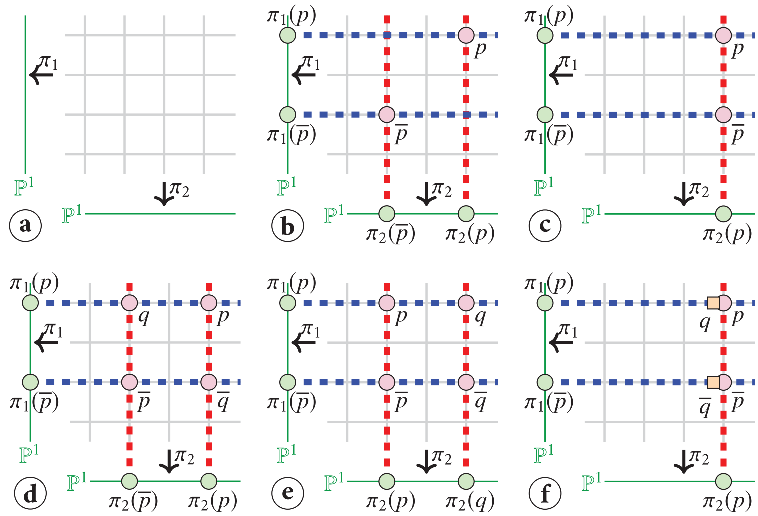

Table 23: See Proposition 22. The possible configurations of the center of blowup $\Lambda \subset \mathbb{P}^1 \times \mathbb{P}^1$ realized by the birational linear projection $\rho\colon Y_\star \dashrightarrow X$ via the isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \cong Y_\star$. At the entries for $\mathbf{T}(X)$ we have $3 \leq n \leq 7$ and $4 \leq m \leq 5$. Since $\mathrm{Aut}_\circ(X)$ embeds into $\mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$, we find that the projection $\pi_i(\mathrm{Aut}_\circ(X))$ is a subgroup of $\mathrm{Aut}_\circ(\mathbb{P}^1)$ for $i \in \{1, 2\}$. An entry for $\pi_1(\mathrm{Aut}_\circ(X))$ and $\pi_2(\mathrm{Aut}_\circ(X))$ denotes the maximal possible subgroup. The vertical and horizontal line segments in the diagrams represent fibers of the projections $\pi_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i \in \{1, 2\}$. The complex conjugate points q and $\overline{q}$ in diagram **(f)** are infinitely near to p and $\overline{p}$, respectively. A fiber that contains two centers of blowup is contracted by ρ to an isolated singularity of X.

$\mathrm{Aut}_\circ(X_N) \cong \mathrm{Aut}_\circ(\widetilde{X})$ and thus $\mathrm{Aut}_\circ(X)$ defines a subgroup of $\mathrm{Aut}_\circ(\widetilde{X})$. Moreover, $\mathrm{Aut}_\circ(\widetilde{X})$ is isomorphic to a subgroup of $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ whose action leaves the blowup center Λ invariant. It now follows from Lemma 20 that $\mathrm{Aut}_\circ(X)$ is isomorphic to a subgroup of

$$\left\{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1) \mid \varphi\big(\pi_1(\Lambda), \pi_2(\Lambda)\big) = \big(\pi_1(\Lambda), \pi_2(\Lambda)\big)\right\},$$

where π_1 and π_2 denote the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to its $\mathbb{P}^1$ factors. We will denote the projections of $\mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$ to its $\mathrm{Aut}_\circ(\mathbb{P}^1)$ factors, by π_1 and π_2 as well.

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We claim that $|\pi_1(\Lambda)| \le 2$ and $|\pi_2(\Lambda)| \le 2$. Suppose by contradiction that $|\pi_1(\Lambda)| > 2$ so that $|\Lambda| = 4$. Let $\pi_1(\mathrm{Aut}_\circ(X))$ and $\pi_2(\mathrm{Aut}_\circ(X))$ denote the subgroups of $\mathrm{Aut}_\circ(\mathbb{P}^1)$ that preserve $\pi_1(\Lambda)$ and $\pi_2(\Lambda)$, respectively. Recall that $\mathrm{Aut}_\circ(\mathbb{P}^1)$ is three-transitive and thus $\dim \pi_1(\mathrm{Aut}_\circ(X)) = 0$. By assumption, $\dim \mathrm{Aut}_\circ(X) \ge 2$, hence $|\pi_2(\Lambda)| = 1$ and $\dim \mathrm{Aut}_\circ(X) = \dim \pi_2(\mathrm{Aut}_\circ(X)) = 2$. By Theorem Ac, at most two noninfinitely near points lie in the same fiber and these points are nonreal. Thus Λ consists of two complex conjugate points and two infinitely near points. We arrived at a contradiction as $\pi_2(\mathrm{Aut}_\circ(X))$ must be a proper subgroup of PSA(1) so that $\pi_2(\mathrm{Aut}_\circ(X)) \cong \mathrm{PSE}(1)$.

As $|\pi_1(\Lambda)| \le 2$, $|\pi_2(\Lambda)| \le 2$ and Λ does not contain real points, it follows that all possible configurations of Λ together with $\pi_1(\mathrm{Aut}_\circ(X))$ and $\pi_2(\mathrm{Aut}_\circ(X))$ are listed in Table 23. Moreover, since the algebraic torus $\mathbb{T}^1$ embeds into $\mathbb{P}^1 \setminus \pi_i(\Lambda)$ such that $\mathrm{Aut}_\circ(\mathbb{T}^1)$ extends to a subgroup of $\pi_i(\mathrm{Aut}_\circ(X))$ for $i \in \{1,2\}$, we deduce that X_N must be toric.

The bidegree $(2,2)$ forms define an isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to Y_\star \subset \mathbb{P}^8$. Since X_N is an anticanonical model, the bidegree $(2,2)$ forms that pass through the blowup center Λ, define a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X_N \subset \mathbb{P}^r$ for some $n+1 \le r \le 8$. Assigning linear conditions to the forms, so that they pass through Λ, corresponds to a linear projection $f\colon Y_\star \subset \mathbb{P}^8 \dashrightarrow X_N \subset \mathbb{P}^r$. It follows from the definition of linear normalization that there exists a degree preserving linear projection $g\colon X_N \to X$. We now define $\rho\colon Y_\star \dashrightarrow X$ as the composition of f with g. Notice that $Y_\star \cong \mathbb{P}^1 \times \mathbb{P}^1$ and thus we may interpret the blowup center Λ instead as the center of projection of ρ. It follows from Theorem Ac that the fibers that contain two points in Λ are contracted via ρ to an isolated singularity of X.

We assume without loss of generality that the generators ℓ_0 and ℓ_1 as defined at Notation 14 are the classes of the pullbacks to $\widetilde{X}$ of the fibers of π_1 and π_2, respectively. The generators ε_1, ε_2, ε_3, and ε_4 are the classes of the pullbacks of (-1)-curves that contract to the points p, $\overline{p}$, q, and $\overline{q}$, respectively. For each configuration of Λ we obtain an explicit description of $B(X)$. For example, if Λ is as in Table 23f, then $b_1 \in B(X)$ since q is infinitely near to p, and $\underline{b_{12}} \in B(X)$ since p and $\overline{p}$ lie in a real fiber of π_2, and so on. Notice that $\mathbf{S}(X_N)$ corresponds to the Dynkin diagram with vertex set $B(X)$ and edge set $\{(a,b) \mid a \cdot b > 0\}$. We find that $|B(X)| > 3$ in each case and thus the values at the $\mathbf{T}(X)$, $\mathbf{S}(X_N)$ and name columns are a direct consequence of Theorem Aa.

If any two choices for the blowup center Λ are characterized by the same row of Table 23, then these choices are equivalent up to $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. It follows that the linear normalization $X_N \subset \mathbb{P}^r$ is up to $\mathrm{Aut}(\mathbb{P}^r)$ uniquely determined by the name of the celestial surface X, and thus $\mathbf{L}(X)$ is up to equivalence determined by this name as well. We now apply Proposition 7 and find by comparing Table 23 with Table 9, that each family of circles on X is realized by some toric family. ∎

Remark 24 (toric projections of the double Segre surface) Recall that a lattice polygon in Table 9 defines, up to projective isomorphism, a monomial parametrization of the linear normalization of a toric celestial surface. The inclusion of lattice polygons with the same unimodular involution defines an arrow reversing projection between the corresponding toric models. The corresponding *toric projection* is defined by omitting components of the monomial parametrization associated to the bigger lattice polygon such that the exponents of the remaining components define the lattice points

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of the smaller lattice polygon. Thus toric surfaces with lattice types b, c, e, f, g in Table 9 are toric projections of b, a, b, a, a in Table 10, respectively. We will use this concept in Example 25, Example 35, and in the proof of Lemma 37.

Example 25 (dP6 as the image of a toric projection) Suppose that $Y_\star \subset \mathbb{P}^8$ and $Z \subset \mathbb{P}^6$ have lattice types as in Table 10b and Table 9b, respectively. Thus the real structure of $Y_\star$ is defined by σ_2 in Lemma 18. We use the left coordinates of Table 17 and omit the monomial components corresponding to y_5 and y_6 coordinates such that

$$\xi_b\colon \mathbb{T}^2 \to Z \subset \mathbb{P}^6, \quad (s,u) \mapsto (1 : s : s^{-1} : u : u^{-1} : su^{-1} : s^{-1}u) \\ = (y_0 : y_1 : y_2 : y_3 : y_4 : y_7 : y_8).$$

Let the projective isomorphism $\mu_2\colon \mathbb{P}^6 \to \mathbb{P}^6$ be a restriction of μ_2 as defined in Lemma 18. We find that $X := \mu_2(Z)$ is contained in $\mathbb{S}^5$ and has celestial type $(3,6,5)$ (see Figure 1). The center of the linear projection $\rho\colon \mathbb{P}^8 \dashrightarrow \mathbb{P}^6$ is a line that intersects $Y_\star$ transversely in $p = (0:0:0:0:0:1:0:0:0)$ and its complex conjugate $\overline{p} = \sigma_2(p) = (0:0:0:0:0:0:1:0:0)$. We remark that $\sigma_2\colon Y_\star \to Y_\star$ is inner automorphic to σ_0 via $\operatorname{Aut}(Y_\star)$ by Corollary 8a. The projection ρ realizes a blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \cong Y_\star$ with centers p and $\overline{p}$ as in Table 23b. Notice that $\operatorname{Aut}_\circ(Z) \cong \{\varphi \in \operatorname{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1) \mid \varphi(p) = p,\ \varphi(\overline{p}) = \overline{p}\}$, since ρ is an isomorphism almost everywhere. Recall that $\operatorname{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1) \cong \operatorname{Aut}_\circ(\mathbb{P}^1) \times \operatorname{Aut}_\circ(\mathbb{P}^1)$ by Lemma 20 and thus $\pi_1(\operatorname{Aut}_\circ(Z)) \cong \pi_2(\operatorname{Aut}_\circ(Z)) \cong \operatorname{PSO}(2)$ as it is stated in Table 23b.

5 Invariant quadratic forms on $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we reformulate the problem of classifying Möbius automorphism groups of celestial surfaces, into the problem of finding invariant quadratic forms of given signature in a vector space.

Suppose that $Y \subset \mathbb{P}^m$ is a surface such that $\operatorname{Aut}(Y) \subset \operatorname{Aut}(\mathbb{P}^m)$. For example, $Y \subset \mathbb{P}^8$ is the double Segre surface or $Y \subset \mathbb{P}^5$ is the Veronese surface. Suppose that we have a birational linear projection with $m \geq n+1 \geq 3$:

$$\rho\colon Y \subset \mathbb{P}^m \dashrightarrow X \subset \mathbb{S}^n \subset \mathbb{P}^{n+1}.$$

The *Möbius pair* of X with respect to ρ is defined as

$$(Y,Q) \quad \text{where} \quad Q \subset \mathbb{P}^m \quad \text{is the Zariski closure of} \quad \rho^{-1}(\mathbb{S}^n).$$

Notice that Q is a hyperquadric of signature $(1, n+1)$ such that $Y \subset Q$ and such that the singular locus of Q coincides with the center of the linear projection ρ. We define the following equivalence relation on Möbius pairs:

$$(Y,Q) \sim (Y,Q') \quad :\Leftrightarrow \quad \exists\, \varphi \in \operatorname{Aut}(\mathbb{P}^m)\colon \quad \varphi(Y) = Y \quad \text{and} \quad \varphi(Q) = Q'.$$

Suppose that $G \subseteq \operatorname{Aut}_\circ(Y)$ is a subgroup. The vector space of *G-invariant quadratic forms* in the ideal $I(Y)$ of Y is defined as

$$I_2^G(Y) := \langle q \in I_2(Y) \mid q \circ \varphi = q \text{ for all } \varphi \in G \rangle_{\mathbb{C}},$$

where we assume that $\varphi \in G \subset \mathrm{PSL}(m+1)$ is normalized to have determinant one. Notice that the real structure $\sigma\colon Y \to Y$ induces an antiholomorphic involution on $I_2^G(Y)$. We denote the *zeroset* of a form $q \in I(Y)$ by $V(q)$.

Proposition 26 (properties of Möbius pairs) *Let (Y, Q) and (Y, Q') be the Möbius pairs of surfaces $X \subset \mathbb{S}^n$ and $X' \subset \mathbb{S}^n$, respectively.*

a) *There exists $\alpha \in \mathrm{Aut}(\mathbb{S}^n)$ with $\alpha(X) = X'$ if and only if (Y, Q) and (Y, Q') are equivalent. In particular, we have that*

$$\mathbf{M}(X) \cong \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^m) \mid \varphi(Y) = Y \text{ and } \varphi(Q) = Q\}.$$

b) *The subgroup $G \subseteq \mathrm{Aut}_\circ(Y)$ is isomorphic to a subgroup of $\mathbf{M}(X)$ if and only if $Q = V(q)$ for some $q \in I_2^G(Y)$.*

c) *If $G, G' \subset \mathrm{Aut}_\circ(Y)$ are inner automorphic subgroups and $q \in I_2^G(Y)$, then there exists $q' \in I_2^{G'}(Y)$ such that $(Y, V(q))$ and $(Y, V(q'))$ are equivalent as Möbius pairs.*

Proof a) Let $\rho\colon \mathbb{P}^m \dashrightarrow \mathbb{P}^{n+1}$ be a birational linear projection such that $\rho(Q) = \mathbb{S}^n$ and $\rho(Y) = X$. Similarly, let $\rho'\colon \mathbb{P}^m \dashrightarrow \mathbb{P}^{n+1}$ be such that $\rho'(Q') = \mathbb{S}^n$ and $\rho'(Y) = X'$.

$\Rightarrow$: We show that there exists $\varphi \in \mathrm{Aut}(\mathbb{P}^m)$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
Y & \hookrightarrow & Q & \xrightarrow{\varphi} & Q' & \hookleftarrow & Y \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle\rho'} \\
X & \hookrightarrow & \mathbb{S}^n & \xrightarrow{\alpha} & \mathbb{S}^n & \hookleftarrow & X'
\end{array}$$

If $m = n+1$, then ρ and ρ' are projective isomorphisms and the claim follows immediately. If $m > n+1$, then the centers of the linear projections ρ and ρ' coincide with the singular loci $\mathbf{S}(Q)$ and $\mathbf{S}(Q')$ of Q and Q', respectively. Let $\Lambda, \Lambda' \subset Y$ be the centers of the projections $\rho|_Y$ and $\rho'|_Y$, respectively. The linear isomorphism α induces via the projections ρ and ρ' the algebraic isomorphisms $\varphi|_Q\colon Q \backslash \mathbf{S}(Q) \to Q' \backslash \mathbf{S}(Q')$ and $\varphi|_Y\colon Y \backslash \Lambda \to Y \backslash \Lambda'$. The automorphism α leaves the union of exceptional curves that contract to points in Λ invariant, and thus we can extend $\varphi|_Y$ so that $\varphi \in \mathrm{Aut}_\circ(Y)$ and $\varphi(\Lambda) = \Lambda'$. Since $\mathrm{Aut}(Y) \subset \mathrm{Aut}(\mathbb{P}^m)$ and since Q contains $\rho^{-1}(\mathbb{S}^n)$ by assumption, we find that $\varphi \in \mathrm{Aut}(\mathbb{P}^m)$ such that $\varphi(Q) = Q'$ and $\varphi(Y) = Y$ as was to be shown.

$\Leftarrow$: For the converse, we need to show that for given $\varphi \in \mathrm{Aut}(\mathbb{P}^m)$, there exists $\alpha \in \mathrm{Aut}(\mathbb{S}^n)$ such that the above diagram commutes. This is immediate, since we define α as the composition $\rho'|_{Q'} \circ \varphi \circ (\rho|_Q)^{-1}$.

The remaining assertion follows if we set $Q' := Q$ and $X' := X$ in the above diagram.

b) We first show the $\Leftarrow$ direction. By assumption $q \circ \varphi = q$ for all $\varphi \in G$. Since $\varphi^{-1}(V(q)) = \{\varphi^{-1}(x) \in \mathbb{P}^m \mid q(x) = 0\} = V(q \circ \varphi)$, we find that $G \subseteq \{\varphi \in \mathrm{Aut}_\circ(\mathbb{P}^m) \mid \varphi(Y) = Y \text{ and } \varphi(Q) = Q\}$. It now follows from a) that G embeds as a subgroup into $\mathbf{M}(X)$. For the $\Rightarrow$ direction, we again apply the characterization of $\mathbf{M}(X)$ in a) and find that $\varphi^{-1}(Q) = Q$ and thus $q \circ \varphi = q$ for all $\varphi \in G$ so that $q \in I_2^G(Y)$.

c) Since $q \in I_2^G(Y)$ the following holds for all $\varphi \in G$ and $\alpha \in \mathrm{Aut}(Y)$:

$$q \circ \varphi = q \quad \Leftrightarrow \quad q \circ \varphi \circ \alpha = q \circ \alpha \quad \Leftrightarrow \quad q \circ \alpha \circ \alpha^{-1} \circ \varphi \circ \alpha = q \circ \alpha.$$

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By assumption $G' = \alpha^{-1} \circ G \circ \alpha$ for some $\alpha \in \mathrm{Aut}(Y)$ and thus for all $\varphi' \in G'$, there exists $\varphi \in G$ such that $\varphi' = \alpha^{-1} \circ \varphi \circ \alpha$. It follows that $q' \circ \varphi' = q'$ for all $\varphi' \in G'$, where $q' := q \circ \alpha$ so that $q' \in I_2^{G'}(Y)$. Thus

$$\alpha^{-1}(V(q)) = \{\alpha^{-1}(x) \in \mathbb{P}^m \mid q(x) = 0\} = V(q \circ \alpha) = V(q'),$$

so that $(Y, V(q))$ is equivalent to $(Y, V(q'))$. ■

The following theorem is essentially [3, Theorem 2.5] and allows us to compute G-invariant quadratic forms via the *Lie algebra* $\mathrm{Lie}(G)$ of G.

Theorem B (DeGraaf–Pílniková–Schicho, 2009) *Suppose that $Y \subset \mathbb{P}^{m-1}$ is a variety such that $\mathrm{Aut}_\circ(Y) \subset \mathrm{Aut}_\circ(\mathbb{P}^{m-1})$. Let the one-parameter subgroup $H \subset \mathrm{Aut}_\circ(Y)$ be represented by an $m \times m$ matrix whose entries are smooth functions in the parameter α such that $\det H(\alpha) = 1$ for all α and such that $H(0)$ is the identity matrix. Let the $m \times m$ matrix D in $\mathrm{Lie}(H)$ be the tangent vector $(\partial_\alpha H)(0)$ of H at the identity. Then the H-invariant quadratic forms are*

$$I_2^H(Y) = \langle q_A \in I_2(Y) \mid D^T \cdot A + A \cdot D = 0 \rangle_{\mathbb{C}}, \tag{2}$$

where q_A denotes the quadratic form $x^\top \cdot A \cdot x$ associated to the symmetric $m \times m$ matrix A.

Proof We observe that $I_2^H(Y) = \langle q_A \in I_2(Y) \mid H^\top \cdot A \cdot H = A \rangle_{\mathbb{C}}$. Let us first assume that $H^\top \cdot A \cdot H = A$. We differentiate both sides of the equation with respect to α and evaluate at 0 so that we obtain the necessary condition $D^\top \cdot A + A \cdot D = 0$. For the converse, we assume that $D^\top \cdot A + A \cdot D = 0$. Thus $G^\top \cdot A \cdot G = B$ for some matrix B where $G = \exp(\alpha D)$ is a one-parameter subgroup. We differentiate both sides of the equivalent equation $G^\top \cdot A \cdot G \cdot \exp(\alpha) = B \cdot \exp(\alpha)$ with respect to α and evaluate at 0 so that we obtain $D^\top \cdot A + A \cdot D + A = B$. It follows that $A = B$ and we know from Lie theory that $H = G$ so that $H^\top \cdot A \cdot H = A$ as is required. ■

Remark 27 (goal) Our goal is to classify subgroups $G \subseteq \mathrm{Aut}_\circ(Y)$ up to inner automorphism such that $\dim G \geq 2$ and $I_2^G(Y)$ contains quadratic forms q of signature $(1, n+1)$ with $n \geq 3$. It follows from Proposition 26 that the Möbius pairs $(Y, V(q))$ for such q, correspond to the celestial surfaces $X \subset \mathbb{S}^n$ such that G is isomorphic to a subgroup of $\mathbf{M}(X)$.

6 Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

Motivated by Remark 27 with Y the double Segre surface $Y_\star \cong \mathbb{P}^1 \times \mathbb{P}^1$, we would like to classify Lie subgroups of $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ up to inner automorphism. By Theorem B, it is sufficient to classify Lie subalgebras of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Let us first investigate real structures of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Consider the toric involutions $\sigma_i\colon \mathbb{T}^2 \to \mathbb{T}^2$ in Lemma 13 with $0 \leq i \leq 3$. By Lemma 18, these toric involutions induce involutions on related algebraic structures (recall Notation 6): $\sigma_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $\sigma_i\colon \mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\sigma_i\colon \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Lemma 28 (real structures for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$) *If*

$$m := \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2,$$

then

$$\sigma_0(m) = \left(\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}, \begin{bmatrix} \overline{e} & \overline{f} \\ \overline{g} & \overline{h} \end{bmatrix}\right), \quad \sigma_1(m) = \left(\begin{bmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} \overline{e} & \overline{f} \\ \overline{g} & \overline{h} \end{bmatrix}\right),$$

$$\sigma_2(m) = \left(\begin{bmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} \overline{h} & \overline{g} \\ \overline{f} & \overline{e} \end{bmatrix}\right), \quad \sigma_3(m) = \left(\begin{bmatrix} \overline{e} & \overline{f} \\ \overline{g} & \overline{h} \end{bmatrix}, \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}\right),$$

where σ_0, σ_1, σ_2, and σ_3 are real structures $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ induced by the corresponding involutions in Lemma 13 and Lemma 18.

Proof Suppose that $M \subset \mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ is a one-parameter subgroup such that m is the tangent vector of M at the identity. The one-parameter subgroup $\sigma_i(M)$ has tangent vector $\sigma_i(m)$ for all $0 \le i \le 3$. We compute the representation $\mathcal{S}(M) \in \mathrm{Aut}(\mathbb{P}^8)$ using (1), where the entries of M are set as indeterminates. Let L_i denote the 9×9 permutation matrix corresponding to the induced antiholomorphic involution $\sigma_i\colon \mathbb{P}^8 \to \mathbb{P}^8$ as stated in Lemma 18. It is immediate to verify that $\overline{L_i^{-1} \circ \mathcal{S}(M) \circ L_i} = \mathcal{S}(\sigma_i(M))$, where σ_i acts on $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$ as an involution and $\overline{\cdot}$ denotes complex conjugation. We conclude that the lemma holds, since the action of σ_i on m is the same as the action of σ_i on M. ■

Remark 29 The real structure $\sigma_2\colon \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is inner automorphic to $\sigma_H := \alpha \circ \sigma_2 \circ \alpha^{-1}$, where

$$\alpha := \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} \text{so that } \sigma_H\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \left(\begin{bmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} \overline{h} & -\overline{g} \\ -\overline{f} & \overline{e} \end{bmatrix}\right).$$

The Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with real structure σ_H is usually denoted by $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. The real elements in $\mathfrak{su}_2$ are skew Hermitian matrices. Similarly, $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with real structure σ_1 can be identified with $\mathfrak{su}_2 \oplus \mathfrak{sl}_2(\mathbb{R})$.

Notation 30 *We consider the following elements in $\mathfrak{sl}_2$:*

$$t := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad q := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad s := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad r := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Recall that $\mathfrak{sl}_2$ over the complex numbers is generated by $\langle t, q, s\rangle$, where the Lie brackets of the generators are $[t,q] = s$, $[t,s] = -2t$, and $[q,s] = 2q$. We shall denote $(g,e) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by g_1 and $(e,g) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by g_2 for all $g \in \mathfrak{sl}_2$. Notice that $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = \langle t_1, q_1, s_1, t_2, q_2, s_2\rangle$ where the Lie bracket acts componentwise.

Remark 31 Suppose that the real structure of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is defined by σ_0 in Lemma 28. In this case,

$$\mathrm{Lie}(\mathrm{PSE}(1)) = \langle t\rangle, \qquad \mathrm{Lie}(\mathrm{PSX}(1)) = \langle s\rangle, \qquad \mathrm{Lie}(\mathrm{PSO}(2)) = \langle r\rangle,$$
$$\mathrm{Lie}(\mathrm{PSA}(1)) = \langle t, s\rangle \qquad \text{and} \qquad \mathrm{Lie}(\mathrm{PSL}(2)) = \langle t, q, s\rangle.$$

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These groups correspond to translations, scalings, rotations, affine transformations, and projective transformations, respectively. Indeed the generators are the tangent vectors at the identity of the following one-parameter subgroups:

$$t \leftrightsquigarrow \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad s \leftrightsquigarrow \begin{bmatrix} \alpha+1 & 0 \\ 0 & (\alpha+1)^{-1} \end{bmatrix}, \quad r \leftrightsquigarrow \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Now suppose that the real structure of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is defined by σ_2 in Lemma 28. In this case, $\mathrm{Lie}(\mathrm{PSO}(2)) = \langle \mathrm{i}s \rangle$, since

$$\mathrm{i}s = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} \leftrightsquigarrow \begin{bmatrix} \cos(\alpha) + \mathrm{i}\sin(\alpha) & 0 \\ 0 & \cos(\alpha) - \mathrm{i}\sin(\alpha) \end{bmatrix}.$$

See also Example 21.

Suppose that F is a Lie group. We call two Lie subalgebras $\mathfrak{g}, \mathfrak{h} \subset \mathrm{Lie}(F)$ *(complex) inner automorphic* if there exists (complex) $M \in F$ such that $\mathfrak{g} = M^{-1} \cdot \mathfrak{h} \cdot M$. Theorem C and thus Corollary 32 follow from [4].

Theorem C (Douglas–Repka, 2016) *A Lie subalgebra $0 \subsetneq \mathfrak{g} \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is, up to flipping the left and right factor, complex inner automorphic to either one of the following with $\alpha \in \mathbb{C}^*$:*

$$\begin{aligned}
&\langle t_1 \rangle,\ \langle s_1 \rangle,\ \langle t_1 + t_2 \rangle,\ \langle t_1 + s_2 \rangle,\ \langle s_1 + \alpha s_2 \rangle,\ \langle t_1, s_1 \rangle,\ \langle t_1, t_2 \rangle,\ \langle t_1, s_2 \rangle,\ \langle s_1, s_2 \rangle,\\
&\langle s_1 + t_2, t_1 \rangle,\ \langle t_1 + t_2, s_1 + s_2 \rangle,\ \langle s_1 + \alpha s_2, t_1 \rangle,\ \langle t_1, q_1, s_1 \rangle,\ \langle t_1, s_1, t_2 \rangle,\ \langle t_1, s_1, s_2 \rangle,\\
&\langle s_1 + \alpha s_2, t_1, t_2 \rangle,\ \langle t_1 + t_2, q_1 + q_2, s_1 + s_2 \rangle,\ \langle t_1, s_1, t_2, s_2 \rangle,\ \langle t_1, q_1, s_1, t_2 \rangle,\\
&\langle t_1, q_1, s_1, s_2 \rangle,\ \langle t_1, q_1, s_1, t_2, s_2 \rangle,\ \langle t_1, q_1, s_1, t_2, q_2, s_2 \rangle.
\end{aligned}$$

Corollary 32 *If $\mathfrak{g} \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is a Lie subalgebra such that $\dim \mathfrak{g} \geq 2$ and $\mathfrak{g}$ is not complex inner automorphic to $\langle s_1, s_2 \rangle$, then $\mathfrak{g}$ contains a subalgebra that is complex inner automorphic to either $\langle t_1 \rangle$, $\langle t_2 \rangle$ or $\langle t_1 + t_2 \rangle$.*

7 The classification of $\mathbb{P}^1 \times \mathbb{P}^1$

In a perfect world, we directly use Theorem B to compute for each Lie subalgebra $\mathrm{Lie}(G) \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, the vector space $I_2^G(Y_\star)$ generated by G-invariant quadratic forms on the double Segre surface $Y_\star \cong \mathbb{P}^1 \times \mathbb{P}^1$. We would then proceed by classifying quadratic forms in $I_2^G(Y_\star)$ of signature $(1, n+1)$ as was suggested in Remark 27.

Unfortunately, there are two problems. Theorem C only provides the classification of subalgebras of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ up to complex inner automorphisms and thus the real structure is not preserved. The second problem is that it is in general difficult to classify quadratic forms in $I_2^G(Y_\star)$ of fixed signature. For example, for what $n \geq 3$, do there exist quadratic forms of signature $(1, n+1)$ in the vector space $I_2^G(X_1)$ at Lemma 33c?

Lemma 36 plays a crucial role in circumventing these two problems using geometric arguments. We are able to prove Lemma 37, since the invariant quadratic forms in Lemma 33a have a particularly nice basis. This section will end with a proof for Theorem 1 and Corollary 2. In particular, we will see that the answer to the question in the previous paragraph is $n \in \{3\}$.

Lemma 33 (invariant quadratic forms for $\mathbb{P}^1 \times \mathbb{P}^1$) *Let the real structures σ_0, σ_1, σ_2, and σ_3 for Lie algebras be as defined in Lemma 28. Let $Y_\star$, X_0, X_1, X_2, and X_3 be the double Segre surfaces in $\mathbb{P}^8$ as defined in Lemma 18. We suppose that G is a Lie subgroup of* $\operatorname{Aut}_\circ(Y_\star)$.

a) *If* $\operatorname{Lie}(G) = \langle \mathfrak{i}s_1, \mathfrak{i}s_2\rangle$ *with real structure σ_2, then* $G \cong \operatorname{PSO}(2) \times \operatorname{PSO}(2)$,

$$I_2^G(Y_\star) = \left\langle y_0^2 - y_1y_2,\ y_0^2 - y_3y_4,\ y_0^2 - y_5y_6,\ y_0^2 - y_7y_8\right\rangle_{\mathbb{C}}, \text{ and}$$
$$I_2^G(X_2) = \left\langle \tfrac{1}{4}x_0^2 - x_1^2 - x_2^2,\ \tfrac{1}{4}x_0^2 - x_3^2 - x_4^2,\ \tfrac{1}{4}x_0^2 - x_5^2 - x_6^2,\ \tfrac{1}{4}x_0^2 - x_7^2 - x_8^2\right\rangle_{\mathbb{C}}.$$

b) *If* $\operatorname{Lie}(G) = \langle \mathfrak{i}s_1, s_2\rangle$ *with real structure σ_1, then* $G \cong \operatorname{PSO}(2) \times \operatorname{PSX}(1)$,

$$I_2^G(Y_\star) = \left\langle y_0^2 - y_1y_2,\ y_0^2 - y_3y_4,\ y_0^2 - y_5y_6,\ y_0^2 - y_7y_8\right\rangle_{\mathbb{C}}, \text{ and}$$
$$I_2^G(X_1) = \left\langle x_0^2 - x_1^2 - x_2^2,\ x_0^2 - x_3x_4,\ x_5x_6 - x_7x_8,\ x_0^2 - x_5x_7 - x_6x_8\right\rangle_{\mathbb{C}}.$$

c) *If* $\operatorname{Lie}(G) = \langle \mathfrak{i}s_1, t_2\rangle$ *with real structure σ_1, then* $G \cong \operatorname{PSO}(2) \times \operatorname{PSE}(1)$,

$$I_2^G(Y_\star) = \left\langle\, y_0^2 - y_3y_4,\ y_4^2 - y_6y_7,\ y_1y_6 - y_2y_7,\ 2y_1y_2 - y_5y_6 - y_7y_8 \,\right\rangle_{\mathbb{C}}, \text{ and}$$
$$I_2^G(X_1) = \left\langle\, x_0^2 - x_3x_4,\ x_4^2 - x_6^2 - x_7^2,\ x_1x_6 - x_2x_7,\ x_1^2 + x_2^2 - x_5x_7 - x_6x_8 \,\right\rangle_{\mathbb{C}}.$$

d) *If* $\operatorname{Lie}(G) = \langle t_1, q_1, s_1, t_2, q_2, s_2\rangle$ *with real structure either σ_0 or σ_3, then* $G \cong \operatorname{PSL}(2) \times \operatorname{PSL}(2)$,

$$I_2^G(Y_\star) = \langle\, 2y_0^2 - 2y_1y_2 - 2y_3y_4 + y_5y_6 + y_7y_8 \,\rangle_{\mathbb{C}},$$
$$I_2^G(X_0) = \langle\, 2x_0^2 - 2x_1x_2 - 2x_3x_4 + x_5x_6 + x_7x_8 \,\rangle_{\mathbb{C}}, \text{ and}$$
$$I_2^G(X_3) = \langle\, 2x_0^2 - 4x_2x_3 - 4x_1x_4 + x_5x_6 + x_7^2 + x_8^2 \,\rangle_{\mathbb{C}}$$

Proof a) We know from Remark 31 that $G \cong \operatorname{PSO}(2) \times \operatorname{PSO}(2)$, since $\mathfrak{i}s_1$ and $\mathfrak{i}s_2$ are the tangent vectors of the following two one-parameter subgroups of $\operatorname{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1)$:

$$\left(\begin{bmatrix}\cos(\alpha) + \mathfrak{i}\sin(\alpha) & 0\\ 0 & \cos(\alpha) - \mathfrak{i}\sin(\alpha)\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) \&$$
$$\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\cos(\alpha) + \mathfrak{i}\sin(\alpha) & 0\\ 0 & \cos(\alpha) - \mathfrak{i}\sin(\alpha)\end{bmatrix}\right).$$

Via the map $\phi\colon S^1 \times S^1 \to \operatorname{Aut}_\circ{}^{\mathbb{T}}(Y_\star) \subset \operatorname{Aut}(\mathbb{P}^8)$ from Example 21, we obtain one-parameter subgroups H_1 and H_2 of $\operatorname{Aut}_\circ(\mathbb{P}^8)$. We use Theorem B to compute the vector spaces $I_2^{H_1}(Y_\star)$ and $I_2^{H_2}(Y_\star)$ of invariant quadratic forms. Since $\operatorname{Lie}(G) = \langle \mathfrak{i}s_1, \mathfrak{i}s_2\rangle$, we have $I_2^G(Y_\star) = I_2^{H_1}(Y_\star) \cap I_2^{H_2}(Y_\star)$. In order to compute $I_2^G(X_2)$, we compose the generators of $I_2^G(Y_\star)$ with μ_2 from Lemma 18. The proofs of b), c), and d) are similar. The invariant quadratic forms can be computed automatically with [8, `moebius-aut`]. ■

Remark 34 It follows from Remark 19 that X_3 is a two-uple embedding of $\mathbb{S}^2$ into $\mathbb{P}^8$. Hence, X_3 cannot be projectively equivalent to a celestial surface. We nevertheless included the real structure $\sigma_3\colon Y_\star \to Y_\star$ at Lemma 33d in order to prove Corollary 2. If $G \cong \operatorname{PSL}(2) \times \operatorname{PSL}(2)$, then $\sigma_3\colon \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and thus $\sigma_3\colon G \to G$, is specified in Lemma 28. Notice that in this case, $\{\varphi \in G \mid \sigma_3(\varphi) = \varphi\}$ is three-dimensional and not six-dimensional.

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Example 35 (spindle cyclide and horn cyclide) Let $Z_s \subset \mathbb{P}^4$ be the image of the monomial parametrization defined by Table 9f (spindle cyclide) using the left coordinates in Table 17. Let $Z_h \subset \mathbb{P}^4$ denote the image of the monomial parametrization defined by Table 9g (horn cyclide). Recall from Remark 24, that both Z_s and Z_h are toric projections of $Y_\star$. It follows from Lemma 16 that $I_2(Z_s)$ is generated by the generators of $I_2(Y_\star)$ that do not contain y_i for $i \in \{5,6,7,8\}$. Similarly, $I_2(Z_h)$ is generated by generators of $I_2(Y_\star)$ that do not contain y_i for $i \in \{1,2,5,8\}$. Thus

$$I_2(Z_s) = \langle y_0^2 - y_1y_2, y_0^2 - y_3y_4\rangle_{\mathbb{C}} \quad \text{and} \quad I_2(Z_h) = \langle y_0^2 - y_3y_4, y_4^2 - y_6y_7\rangle_{\mathbb{C}}.$$

Let μ_s and μ_h be restrictions of $\mu_1\colon \mathbb{P}^8 \to \mathbb{P}^8$ in Lemma 18 as follows:

$$\begin{aligned} \mu_s\colon \mathbb{P}^4 \to \mathbb{P}^4, &\quad (x_0 : x_1 : x_2 : x_3 : x_4) &\mapsto& \quad (x_0 : x_1 + \mathrm{i}x_2 : x_1 - \mathrm{i}x_2 : x_3 : x_4),\\ \mu_h\colon \mathbb{P}^4 \to \mathbb{P}^4, &\quad (x_0 : x_3 : x_4 : x_6 : x_7) &\mapsto& \quad (x_0 : x_3 : x_4 : x_7 - \mathrm{i}x_6 : x_7 + \mathrm{i}x_6).\end{aligned}$$

We set $X_s := (\alpha_s \circ \mu_s)^{-1}(Z_s)$ and $X_h := (\alpha_h \circ \mu_h)^{-1}(Z_h)$, where

$$\begin{aligned} \alpha_s\colon \mathbb{P}^4 \to \mathbb{P}^4, &\ (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_4 : x_1 : x_2 : \tfrac{1}{\sqrt{2}}(x_0 - x_3) : \tfrac{1}{\sqrt{2}}(x_0 + x_3)),\\ \alpha_h\colon \mathbb{P}^4 \to \mathbb{P}^4, &\ (x_0 : x_3 : x_4 : x_6 : x_7) \mapsto (\tfrac{1}{\sqrt{2}}x_4 : x_3 : -x_0 - x_3 : x_6 : x_7).\end{aligned}$$

Notice that $X_s, X_h \subset \mathbb{S}^3$, since $\quad I_2(X_s) = \langle x_1^2 + x_2^2 - x_4^2,\ x_0^2 - x_3^2 - 2x_4^2\rangle_{\mathbb{C}} \quad$ and

$$I_2(X_h) = \langle x_4^2 + 2x_0x_3 + 2x_3^2,\ x_0^2 + 2x_0x_3 + x_3^2 - x_6^2 - x_7^2\rangle_{\mathbb{C}}.$$

Let $\pi^h, \pi^s\colon S^3 \dashrightarrow \mathbb{R}^3$ be stereographic projections so that the projective closures of these projections with the above coordinates are

$$\begin{aligned} \tilde{\pi}^h\colon \mathbb{S}^3 \dashrightarrow \mathbb{P}^3, &\quad (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 - x_3 : x_1 : x_2 : x_4),\\ \tilde{\pi}^s\colon \mathbb{S}^3 \dashrightarrow \mathbb{P}^3, &\quad (x_0 : x_3 : x_4 : x_6 : x_7) \mapsto (x_0 + x_3 : x_4 : x_7 : x_6).\end{aligned}$$

We verify that $\pi^h(X_s(\mathbb{R}))$ is a circular cone and that $\pi^s(X_h)$ is a circular cylinder so that, by Definition 4, X_s and X_h are indeed a spindle cyclide and horn cyclide, respectively. Notice that both π^h and π^s have a real isolated singularity as center of projection. Since these isolated singular points have to be preserved by the Möbius automorphisms, it follows from Section 1 that $\mathbf{M}(X_s)$ and $\mathbf{M}(X_h)$ are subgroups of Euclidean similarities. The circular cone and the circular cylinder are unique up to Euclidean similarities and thus $\mathbf{D}(X_s) = \mathbf{D}(X_h) = 0$. By Lemma 33b, the generators of the vector space $I_2(Z_s)$ are $\mathrm{PSO}(2) \times \mathrm{PSX}(1)$ invariant. Similarly, by Lemma 33c, the generators of the vector space $I_2(Z_h)$ are $\mathrm{PSO}(2) \times \mathrm{PSE}(1)$ invariant. Proposition 22 characterizes the projections from $\mathrm{Aut}_\circ(X_s)$ and $\mathrm{Aut}_\circ(X_h)$ to $\mathrm{Aut}_\circ(\mathbb{P}^1)$. We conclude that $\mathbf{M}(X_s) \cong \mathrm{PSO}(2) \times \mathrm{PSX}(1)$ and $\mathbf{M}(X_h) \cong \mathrm{PSO}(2) \times \mathrm{PSE}(1)$ so that $\mathrm{Aut}_\circ(X_s) = \mathbf{M}(X_s)$ and $\mathrm{Aut}_\circ(X_h) = \mathbf{M}(X_h)$.

Lemma 36 (Möbius automorphism groups) *If $X \subset \mathbb{S}^n$ is a λ-circled celestial surface such that $\lambda < \infty$ and* $\dim \mathbf{M}(X) \geq 2$, *then either*

(1) $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSO}(2)$ *and* $\mathrm{Lie}(\mathbf{M}(X)) \subset \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ *is, up to inner automorphism, equal to* $\langle \mathrm{i}s_1, \mathrm{i}s_2\rangle$ *with real structure* σ_2 *in Lemma 28,*
(2) $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSX}(1)$, $\quad \mathbf{T}(X) = (2,4,3)$, $\quad \mathbf{S}(X) = 2\underline{A_1} + 2A_1$, $\quad \mathbf{D}(X) = 0$, $\mathbf{M}(X) = \mathrm{Aut}_\circ(X)$ *and X is a spindle cyclide, or*

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(3) $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSE}(1)$, $\mathbf{T}(X) = (2,4,3)$, $\mathbf{S}(X) = \underline{A_3} + 2A_1$, $\mathbf{D}(X) = 0$, $\mathbf{M}(X) = \mathrm{Aut}_\circ(X)$ *and* X *is a horn cyclide.*

Proof In Proposition 22, we related $\mathbb{P}^1 \times \mathbb{P}^1$ to X, via a birational linear projection $\rho\colon Y_\star \dashrightarrow X$, where $Y_\star \cong \mathbb{P}^1 \times \mathbb{P}^1$. Recall from Remark 34 that the real structure of $Y_\star$, that is via ρ compatible with the real structure of X, cannot be σ_3. We know from Lemma 20 that $\mathrm{Aut}_\circ(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$. The first and second projection are denoted by $\pi_i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with $i \in \{1,2\}$, and we denote the projections of $\mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$ to $\mathrm{Aut}_\circ(\mathbb{P}^1)$ by π_1 and π_2 as well. The automorphisms of X in the identity component factor via ρ through automorphisms of $Y_\star$ that leave the center of projection Λ invariant. We make a case distinction on the configurations of Λ in Table 23 were we identified $Y_\star$ with $\mathbb{P}^1 \times \mathbb{P}^1$. By Proposition 22, these are all possible configurations for Λ.

We first suppose that Λ is the empty-set as in Table 23a.

We consider the action of subgroups of the Möbius automorphism group $\mathbf{M}(X)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We start by showing that either Lemma 36.1 holds or there exists a one-dimensional subgroup of $\mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves a real fiber L of π_2 invariant and leaves a real point $\hat{c} \in L$ on this fiber invariant. We write $\mathfrak{g} \sim_\mathbb{C} \mathfrak{h}$ and $\mathfrak{g} \sim \mathfrak{h}$ if Lie subalgebras $\mathfrak{g}, \mathfrak{h} \subset \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ are complex and real inner automorphic, respectively. Recall from Corollary 32 that if $\mathrm{Lie}(\mathbf{M}(X)) \not\sim_\mathbb{C} \langle s_1, s_2 \rangle$, then there exists a one-dimensional Lie subgroup $H \subset \mathbf{M}(X)$ such that without loss of generality either $\mathrm{Lie}(H) \sim_\mathbb{C} \langle t_2 \rangle$ or $\mathrm{Lie}(H) \sim_\mathbb{C} \langle t_1 + t_2 \rangle$.

Suppose that $\mathrm{Lie}(\mathbf{M}(X)) \sim_\mathbb{C} \langle s_1, s_2 \rangle$ such that both $\pi_1(\mathbf{M}(X))$ and $\pi_2(\mathbf{M}(X))$ leave complex conjugate basepoints invariant while acting on $\mathbb{P}^1$. By Corollary 8a, we may assume without loss of generality that the real structure of $Y_\star$ is defined by σ_2 in Lemma 18. The induced real structure on $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is as in Lemma 28. It follows from Remark 31 that $\mathrm{Lie}(\mathbf{M}(X)) \sim \langle \mathrm{i}s_1, \mathrm{i}s_2 \rangle$, so that $\pi_i(\mathbf{M}(X))$ consists of all automorphisms in $\mathrm{Aut}_\circ(\mathbb{P}^1)$ that preserve two complex conjugate basepoints for $1 \le i \le 2$. Hence, $\mathbf{M}(X) \subseteq \mathrm{PSO}(2) \times \mathrm{PSO}(2)$ and since $\dim \mathbf{M}(X) \ge 2$ by assumption this must be an inclusion of connected Lie groups of the same dimension, so that $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSO}(2)$. We conclude that Lemma 36.1 holds in this case.

Suppose that $\mathrm{Lie}(\mathbf{M}(X)) \sim_\mathbb{C} \langle s_1, s_2 \rangle$ such that $\pi_2(\mathbf{M}(X))$ leaves real points invariant while acting on $\mathbb{P}^1$. Thus there exists a subgroup $H \subset \mathbf{M}(X)$ such that $\mathrm{Lie}(H) \sim \langle s_2 \rangle$ and the action of H on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves two real fibers L and L' of π_2 pointwise invariant.

Suppose that $\mathrm{Lie}(H) \sim_\mathbb{C} \langle t_2 \rangle$. It follows from Remark 31 that the action of H on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves exactly one fiber $L := \pi_2^{-1}(u)$ pointwise invariant for some point $u \in \mathbb{P}^1$. The number of fibers that are preserved are invariant under complex inner automorphisms, and thus this fiber must be real so that $\mathrm{Lie}(H) \sim \langle t_2 \rangle$.

Suppose that $\mathrm{Lie}(H) \sim_\mathbb{C} \langle t_1 + t_2 \rangle$. Analogously as before we find that $\mathrm{Lie}(H) \sim \langle t_1 + t_2 \rangle$, since the action of H on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves M and L invariant, where M and L are real fibers of π_1 and π_2, respectively. Moreover, the action leaves the real point $\hat{c} \in L$ invariant such that $\{\hat{c}\} = M \cap L$.

Now suppose by contradiction that Lemma 36.1 does not hold. Notice that we are still in the case where $\Lambda = \emptyset$ as in Table 23a. We showed that there exists a one-dimensional subgroup $H \subset \mathbf{M}(X)$ such that either $\mathrm{Lie}(H) \sim \langle s_2 \rangle$, $\mathrm{Lie}(H) \sim \langle t_2 \rangle$ or

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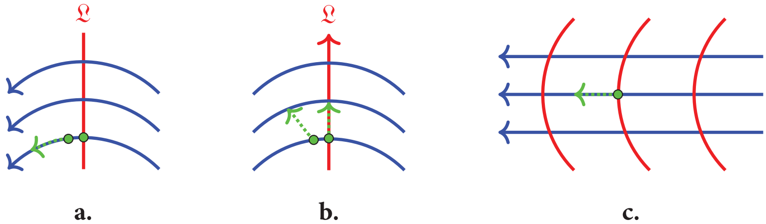

Figure 3: Euclidean similarities acting on a stereographic projection of $X(\mathbb{R})$. A dotted arrow depicts the direction of the orbit of the point at the tail.

$\mathrm{Lie}(H) \sim \langle t_1 + t_2 \rangle$. Moreover, the action of H on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves a real fiber L of π_2 invariant as a whole and leaves a real point $\hat{c} \in L$ invariant. The point $\hat{c}$ corresponds via ρ to a point $c \in X$ such that $c \in X(\mathbb{R}) \subset S^n$. We assume without loss of generality that c is the center of a stereographic projection $\pi : S^n \dashrightarrow \mathbb{R}^n$. Recall from Section 1 that H induces a one-dimensional subgroup of the Euclidean similarities of $\mathbb{R}^n$ that leaves $\pi(X(\mathbb{R}))$ invariant as a whole. We call fibers of π_1 *horizontal* and fibers of π_2 *vertical*, since they correspond to horizontal and vertical line segments in Table 23, respectively. A horizontal/vertical fiber that meets $\hat{c}$ correspond via ρ and π to a *horizontal/vertical line* in $\pi(X(\mathbb{R}))$. The horizontal/vertical fibers that do not meet $\hat{c}$ correspond to *horizontal/vertical circles* in $\pi(X(\mathbb{R}))$. Let $\mathfrak{L} \subset \pi(X(\mathbb{R}))$ be the vertical line corresponding to the vertical fiber L. Thus $\mathfrak{L}$ is the stereographic projection of the set of real points in $\rho(L)$ and the action of H on $\pi(X(\mathbb{R}))$ leaves the line $\mathfrak{L}$ invariant as a whole. The H-orbits of a general point on L and a general point on a horizontal fiber corresponds to the H-orbits of points on $\mathfrak{L}$ and some horizontal circle, respectively. The directions of such orbits in a small neighborhood are illustrated in Figure 3a if $\mathrm{Lie}(H) \sim \langle s_2 \rangle$ or $\mathrm{Lie}(H) \sim \langle t_2 \rangle$, and in Figure 3b if $\mathrm{Lie}(H) \sim \langle t_1 + t_2 \rangle$. Suppose that φ is a general Euclidean similarity in H. Recall that an Euclidean similarity of $\mathbb{R}^n$ factors as a rotation, translation, and/or scaling. Notice that $\pi(X(\mathbb{R}))$ is not covered by lines and thus the scaling component of φ is trivial. It follows from Figure 3a,b that φ has a nontrivial rotational component. We arrived at a contradiction, since the line $\mathfrak{L}$ meets the horizontal circles and thus cannot be left invariant by the action of H. We established that if $\Lambda = \emptyset$ as in Table 23a, then Lemma 36.1 holds.

For the next case, we suppose that Λ is as in Table 23b or Table 23d. It follows from Proposition 22 that $\mathbf{M}(X) \subseteq \mathrm{PSO}(2) \times \mathrm{PSO}(2)$ and since $\dim \mathbf{M}(X) \geq 2$, we find as before that $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSO}(2)$. We know from Proposition 7 and Table 9b,e that X has real structure σ_2. Hence Lemma 36.1 holds as well for these cases.

If Λ is as in Table 23e or Table 23f, then $X \subset \mathbb{S}^3$ is either the spindle cyclide or the horn cyclide. We showed in Example 35 that $\mathbf{M}(X)$ and $\mathbf{D}(X)$ are as asserted in Lemma 36.2 and Lemma 36.3, respectively. The assertions for $\mathbf{T}(X)$ and $\mathbf{S}(X)$ follow from Proposition 22. We remark that the fiber corresponding to L as considered for the case $\Lambda = \emptyset$ is in this case contracted to an isolated singularity $c \in X$ so that $\pi(X(\mathbb{R}))$ is covered by lines.

Finally, we suppose by contradiction that Λ is as in Table 23c.

Let L be the real vertical fiber of π_2 spanned by the complex conjugate points p and $\overline{p}$ as depicted in Table 23c. We first show that there exists a subgroup $H \subset \mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves L and the horizontal fibers invariant.

If $\mathrm{Lie}(\mathbf{M}(X)) \sim_{\mathbb{C}} \langle s_1, s_2 \rangle$, then the action of $\pi_2(\mathbf{M}(X))$ on $\mathbb{P}^1$ leaves $\pi_2(p)$ and some other point $r \in \mathbb{P}^1$ invariant. Thus in this case, there exists a subgroup $H \subset \mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves the vertical fibers $L' := \pi_2^{-1}(r)$ and L pointwise invariant, and leaves each horizontal fiber invariant as a whole. Now suppose that $\mathrm{Lie}(\mathbf{M}(X)) \nsim_{\mathbb{C}} \langle s_1, s_2 \rangle$. It follows from Corollary 32 that there exists a subgroup $H \subset \mathbf{M}(X)$ such that $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_1 + t_2 \rangle$ or $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_i \rangle$ for $i \in \{1, 2\}$. Since automorphisms of $\mathbb{P}^1$ are three-transitive and $|\pi_1(\Lambda)| = 2$, it follows that $\dim \pi_1(\mathbf{M}(X)) \leq 1$ and $\mathrm{Lie}(H) \nsim_{\mathbb{C}} \langle t_1 \rangle$. Since $\dim \mathbf{M}(X) \geq 2$ by assumption, we find that $\dim \pi_2(\mathbf{M}(X)) \geq 1$. Therefore, there exists a subgroup $H \subset \mathbf{M}(X)$ such that $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_2 \rangle$. In this case, the action of H on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves L and the horizontal fibers invariant.

Since $|\Lambda \cap L| = 2$, it follows that $\rho(L)$ is an isolated singularity of X. We assume without loss of generality that this isolated singularity $c \in X(\mathbb{R})$ is the center of stereographic projection $\pi\colon S^n \to \mathbb{R}^n$. We use the same notation as before and find that, except for L, the horizontal fibers and vertical fibers correspond via ρ to horizontal lines and vertical circles in $\pi(X(\mathbb{R})) \subset \mathbb{R}^n$, respectively. We showed that there exists a subgroup $H \subset \mathbf{M}(X)$ of Euclidean similarities whose action on $\pi(X(\mathbb{R}))$ leaves the horizontal lines invariant and sends vertical circles to vertical circles as in Figure 3c. Thus, the orbit of a point in a vertical circle is a horizontal line. If we let the subgroup of scalings or translations act on the spanning plane of a circle contained in $\pi(X(\mathbb{R}))$, then we obtain $\mathbb{R}^3$ so that $X \subset \mathbb{S}^3$. We arrived at a contradiction as $\mathbf{T}(X)$ is equal to $(2, 6, m)$, where $m > 3$ by Proposition 22.

We concluded the proof, as we considered all cases for Λ in Table 23. ∎

Lemma 37 (rotational Möbius automorphism group) *If $X \subset \mathbb{S}^n$ is a λ-circled celestial surface such that $\lambda < \infty$ and such that $\mathbf{M}(X) \cong \mathrm{PSO}(2) \times \mathrm{PSO}(2)$, then Theorem 1 holds for X.*

Proof Let $(Y_\star, Q_c)$ denote the Möbius pair of X, where Q_c is a hyperquadric of signature $(1, n+1)$. The existence of this pair follows from Proposition 22, and we denote the corresponding birational linear projection by $\rho\colon Y_\star \dashrightarrow X$. By Corollary 8, we may assume without loss of generality that the real structure of $Y_\star$ is defined by σ_2 in Lemma 18. We know from Proposition 26 that we may assume up to Möbius equivalence that $Q_c = V(q)$ for some invariant quadratic form $q \in I_2^G(Y_\star)$, where G is isomorphic to $\mathrm{PSO}(2) \times \mathrm{PSO}(2)$. Thus it follows from Lemma 36 and Lemma 33a that

$$Q_c = \left\{ y \in \mathbb{P}^8 \;\middle|\; \sum_{i \in \{1,3,5,7\}} c_i \left(y_0^2 - y_i y_{i+1} \right) = 0 \right\},$$

for some coefficient vector $c = (c_1 : c_3 : c_5 : c_7) \in \mathbb{P}^3$. The singular locus of Q_c is defined by

$$\mathbf{S}(Q_c) = \bigcap_{i \in I} \{ y \in \mathbb{P}^8 \mid y_i = y_{i+1} = 0 \} \quad \text{with} \quad I := \{ i \in \{1,3,5,7\} \mid c_i \neq 0 \}.$$

It follows from Lemma 18 that ρ factors as $\mu_t^{-1} \circ \rho_\ell$ and $\rho_r \circ \mu_2^{-1}$ as in the following commutative diagram

$$\begin{array}{ccc} Y_\star & \xrightarrow{\mu_2^{-1}} & X_2 \\ \rho_\ell \downarrow & & \downarrow \rho_r \\ Z & \xrightarrow{\mu_t^{-1}} & X \end{array}$$

Thus ρ_ℓ and ρ_r are birational linear projections, so that the real structures of Z and X are induced by $\sigma_2\colon Y_\star \to Y_\star$ and $\sigma_0\colon X_2 \to X_2$, respectively. The center of ρ_ℓ coincides with the singular locus of Q_c by the definition of Möbius pair and thus ρ_ℓ is a toric projection (see Remark 24). The vector space $I_2^G(Z)$ is generated by the generators of $I_2^G(Y_\star)$ that do not contain an element in $\{y_i\}_{i\in I} \cup \{y_{i+1}\}_{i\in I}$ as a variable. We obtain the lattice type $\mathbf{L}(Z)$ by taking the convex hull of the lattice polygon that is obtained by removing the lattice points of the polygon in Table 10b that are indexed by $\{y_i\}_{i\in I} \cup \{y_{i+1}\}_{i\in I}$ in Table 17.

We first want to determine the possible values for $\mathbf{T}(X)$, $\mathbf{S}(X)$, $\dim \mathbb{P}(I_2^G(X))$ and whether $\mathbf{M}(X)$ is equal to $\mathrm{Aut}_\circ(X)$. We make a case distinction on $I \subset \{1,3,5,7\}$. Notice that $|I| \leq 2$, otherwise the resulting lattice polygon is one-dimensional.

- If $I = \emptyset$, then $\mathbf{T}(X) = (2,8,7)$, $\mathbf{S}(X) = \emptyset$, $\dim \mathbb{P}(I_2^G(X)) = 3$, and $\mathbf{M}(X) \subsetneq \mathrm{Aut}_\circ(X)$ as a direct consequence of the definitions.
- If $I \in \{\{1\},\{3\}\}$, then $\mathbf{L}(Z)$ is as in Table 10b, $\mathbf{T}(X) = (2,8,5)$ and $\mathbf{M}(X) \subsetneq \mathrm{Aut}_\circ(X)$. Notice that if $I = \{3\}$, then the surface Z is projectively isomorphic to the surface obtained with $I = \{1\}$. If $I = \{1\}$, then, as discussed before, we omit the generators of $I_2^G(Y_\star)$ that contain y_1 or y_2 as variable and find that $I_2^G(Z) = \langle y_0^2 - y_3y_4,\ y_0^2 - y_5y_6,\ y_0^2 - y_7y_8\rangle_\mathbb{C}$ so that $\dim \mathbb{P}(I_2^G(X)) = 2$. We conclude from the monomial parametrization $\rho_\ell \circ \xi\colon \mathbb{T}^2 \to Z$ that $\mathbf{S}(Z) = \emptyset$ and thus $\mathbf{S}(X) = \emptyset$.
- If $I \in \{\{5\},\{7\}\}$, then $\mathbf{T}(X) = (3,6,5)$ and $\mathbf{L}(Z)$ is equivalent to Table 9b. It follows from Proposition 22 that $\mathbf{M}(X) = \mathrm{Aut}_\circ(X)$ and $\mathbf{S}(X) = \emptyset$. As before we verify that $\dim \mathbb{P}(I_2^G(X)) = 2$.
- If $I \in \{\{1,5\},\{1,7\},\{3,5\},\{3,7\},\{5,7\}\}$, then $\mathbf{T}(X) = (4,4,3)$ and $\mathbf{L}(Z)$ is equivalent to Table 9e. It follows from Proposition 22 that $\mathbf{M}(X) = \mathrm{Aut}_\circ(X)$ and $\mathbf{S}(X) = 4A_1$. We verify that $\dim \mathbb{P}(I_2^G(X)) = 1$ as before.
- If $I = \{1,3\}$, then the lattice points corresponding to y_0, y_5, y_6, y_7 and y_8 in Table 17, correspond after the unimodular transformation $(x,y) \mapsto (x-y, y+x)$ to a 2:1 monomial map $\xi_e(s^2,t^2)$ such that the lattice type of the monomial parametrization $\xi_e(s,t)$ is as in Table 9e. Thus $\mathbf{L}(Z)$ is equivalent to Table 9e and we may assume without loss of generality that $I = \{5,7\}$ which we already considered.

We verify that X is in all five cases biregular isomorphic to its linear normalization X_N. We know from Proposition 22 that X_N is toric and thus X is toric as well.

It remains to show that $\mathbf{D}(X) = \dim \mathbb{P}(I_2^G(X))$. It follows from Proposition 26a that $(Y_\star, Q_c)$ and $(Y_\star, Q_{c'})$ correspond to Möbius equivalent celestial surfaces if and only if there exists $\alpha \in \mathrm{Aut}(Y_\star)$ such that $\alpha(Q_c) = Q_{c'}$. Let $\varphi = (\varphi_1, \varphi_2)$ be an indeterminate element of $\mathrm{Aut}_\circ(\mathbb{P}^1) \times \mathrm{Aut}_\circ(\mathbb{P}^1)$. Thus φ_1 and φ_2 are nonsingular $2{\times}2$ matrices in eight indeterminates $\vec{a} = (a_1, \ldots, a_8)$. Recall from (1) that there exist, a value for $\vec{a}$ such that α is defined by the $9{\times}9$-matrix $\mathcal{S}(\varphi)$. We compose, for all $i \in I$,

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the polynomials $y_0^2 - y_i y_{i+1}$ with the map defined by $\mathcal{S}(\varphi)$ so that we obtain quadratic polynomials in y_i and coefficients in $\mathbb{Q}[a_1, \dots, a_8]$. Since $\alpha(Q_c) = Q_{c'}$, we require that coefficients of monomials, that are not of the form y_0^2 or $y_j y_{j+1}$ for some $j > 0$, vanish. We verify with a computer algebra system that the only possible value for $\vec{a}$ such that φ_1 and φ_2 have nonzero determinant, is when $\vec{a}$ defines the identity automorphism. Therefore $(Y_\star, Q_c)$ and $(Y_\star, Q_{c'})$ are equivalent if and only if $c = c'$. We conclude that $\mathbf{D}(X) = \dim \mathbb{P}(I_2^G(X))$ as was left to be shown. ■

Proof of Theorem 1 Suppose that the celestial surface $X \subset \mathbb{S}^n$ is λ-circled. If $\lambda < \infty$, then Theorem 1 follows from Lemma 36 and Lemma 37. If $\lambda = \infty$, then Theorem 1 follows from [7, Section 1] and [1, Section 2.4.3]; we will give an alternative proof at Theorem D. ■

Proof of Corollary 2 Our goal is as in Remark 27, but with signature $(4, 5)$ or $(3, 6)$ instead of $(1, n+1)$. Notice that everything in Section 5 works if we replace $\mathbb{S}^n$ with a hyperquadric Q of different signature. It follows from Lemma 12b that the real structure of $\mathbb{P}^1 \times \mathbb{P}^1$ with real points is either $\sigma_+ \times \sigma_+$ or σ_s. These real structures are compatible with $\sigma_0\colon Y_\star \to Y_\star$ and $\sigma_3\colon Y_\star \to Y_\star$ in Lemma 18, respectively. This corollary is now a direct consequence of Proposition 26 and Lemma 33d. We remark that if Q has signature $(3, 6)$, then the unique double Segre surface in Q is not covered by real conics. ■

8 The classification of $\mathbb{P}^2$ revisited

If $X \subseteq \mathbb{S}^n$ is ∞-circled, then $\mathbf{T}(X)$ is either $(\infty, 4, 4)$ or $(\infty, 2, 2)$. We know from [7, Section 1] and [1, Section 2.4.3] that $\mathbf{M}(X)$ is either PSO(3) or PSO(3,1). Moreover, $X \subseteq \mathbb{S}^n$ is in both cases unique up to Möbius equivalence. We believe it might be instructive to give an alternative proof by using the methods of Section 5. We hope that this convinces the reader that our methods have the potential to be used outside the scope of this paper.

Suppose that $Y_\circ \subset \mathbb{P}^5$ is the Veronese surface with lattice type $\mathbf{L}(Y_\circ)$ as in Table 9d. Indeed, by Corollary 8b, we may assume without loss of generality that the antiholomorphic involution acting on $Y_\circ$ is complex conjugation. We proceed analogously as in Section 3 with the coordinates in Table 17 (right side) so that we obtain the following parametric map

$$\xi_d\colon \mathbb{T}^2 \to Y_\circ \subset \mathbb{P}^5, \quad (s, t) \mapsto (1 : st : s : t : s^2 : t^2) = (y_0 : \dots : y_5),$$

which extends to $\tilde{\xi_d}\colon \mathbb{P}^2 \to Y_\circ \subset \mathbb{P}^5$, $\quad (s : t : u) \mapsto (u^2 : st : su : tu : s^2 : t^2)$. Since $Y_\circ$ is isomorphic to $\mathbb{P}^2$ via $\tilde{\xi_d}$, we have $\mathrm{Aut}_\circ(Y_\circ) \cong \mathrm{PSL}(3)$. Using ξ_d, we find the following six generators for the vector space of quadratic forms on $Y_\circ$ and it follows from Lemma 16 that these form a basis:

$$I_2(Y_\circ) = \langle y_1y_1 - y_4y_5,\ y_0y_1 - y_2y_3,\ y_2y_2 - y_0y_4,\ y_3y_3 - y_0y_5,$$
$$y_1y_2 - y_3y_4,\ y_1y_3 - y_2y_5\rangle_{\mathbb{C}}.$$

Our goal is as in Remark 27 with $Y_\circ$ as Y. Notice that the real structure of $Y_\circ$ acts as complex conjugation on the Lie algebra $\mathfrak{sl}_3$. We consider the following elements in $\mathfrak{sl}_3$:

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$$a_1 := \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix},\ a_2 := \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix},\ a_3 := \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix},\ c_1 := \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix},$$

$$b_1 := \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix},\ b_2 := \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix},\ b_3 := \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix},\ c_2 := \begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{bmatrix}.$$

Lemma 38 (invariant quadratic forms for $\mathbb{P}^2$) *Suppose that $G \subseteq \operatorname{Aut}_\circ(Y_\circ)$ is a Lie subgroup, where $Y_\circ \subset \mathbb{P}^5$ is the Veronese surface. If* $\operatorname{Lie}(G) = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3\rangle$, *then $G \cong \mathrm{PSO}(3)$ and*

$$I_2^G(Y_\circ) = \langle x_1^2 + x_2^2 + x_3^2 - x_0x_4 - x_0x_5 - x_4x_5\rangle_{\mathbb{C}}.$$

If $\operatorname{Lie}(G) = \langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\rangle$, *then $G \cong \mathrm{PSL}(3)$ and $I_2^G(Y_\circ) = \langle 1\rangle_{\mathbb{C}}$.*

Proof The subgroup $\mathrm{PSO}(3) \subset \mathrm{PSL}(3)$ is generated by the three 3×3 rotation matrices and thus $\mathfrak{so}_3 = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3\rangle$. The generators for the Lie algebra $\mathfrak{sl}_3$ can be found for example in [6, Section 6.2]. For the remaining assertions, we apply Theorem B as in the proof of Lemma 33. ∎

Lemma 39 *If $(Y_\circ, Q)$ and $(Y_\circ, Q')$ are Möbius pairs of celestial surfaces in $\mathbb{S}^n$ for $n \geq 3$, then these pairs are equivalent.*

Proof We consider the following group actions

$$\mathcal{A}\colon \mathrm{PSL}(3) \times \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{and} \quad \mathcal{B}\colon \mathrm{PSL}(3) \times Y_\circ \to Y_\circ.$$

As in (1), the group action $\mathcal{B}$ is defined via $\operatorname{Sym}_2(\cdot)$ and can be computed via the isomorphism $\tilde{\xi}_d\colon \mathbb{P}^2 \to Y_\circ \subset \mathbb{P}^5$. These group actions induce group actions on the spaces of quadratic forms $V := \mathbb{P}(I_2(\mathbb{P}^2))$ and $W := \mathbb{P}(I_2(Y_\circ))$:

$$\mathcal{A}_\star\colon \mathrm{PSL}(3) \times V \to V \quad \text{and} \quad \mathcal{B}_\star\colon \mathrm{PSL}(3) \times W \to W.$$

Recall that a quadratic form in V is equivalent via $\mathcal{A}_\star$ to a diagonal form of signature $(1,0)$, $(2,0)$, $(1,1)$, $(3,0)$, or $(2,1)$. It is left to the reader to verify that W contains quadratic forms of signatures $(1,2)$, $(1,3)$, $(1,5)$, $(2,2)$, and $(3,3)$. We can also define a group action $\mathcal{C}_\star\colon \mathrm{PSL}(3) \times W \to W$ via the action $\mathcal{A}_\star$ and an isomorphism $V \to W$. Irreducible representations $\mathrm{PSL}(3) \to \operatorname{Aut}(\mathbb{P}^5)$ are isomorphic and thus $\mathcal{B}_\star$ and $\mathcal{C}_\star$ must be isomorphic group actions. Hence, we can match the orbits of $\mathcal{A}_\star$ with the orbits of $\mathcal{B}_\star$ and thus we identified all possible signatures of quadratic forms in W. Thus, $Q = V(q)$ and $Q' = V(q')$, where the quadratic forms q and q' in W have both signature $(1,5)$. The group action $\mathcal{A}_\star$, and thus also the group action $\mathcal{B}_\star$, acts transitively on quadratic forms of the same signature. It follows that $q' = q \circ \varphi^{-1}$ for some $\varphi \in \operatorname{Aut}_\circ(Y_\circ)$. Therefore $\varphi(Q) = Q'$ so that $(Y_\circ, Q)$ and $(Y_\circ, Q')$ are equivalent as Möbius pairs. ∎

The following theorem is a consequence of [7, Theorem 23]. We give an alternative proof by applying the methods in Section 5.

Theorem D (Kollár, 2018) *If $X \subset \mathbb{S}^n$ is an ∞-circled celestial surface, then Theorem 1 holds for X.*

Proof If $n \leq 2$, then $\mathbf{T}(X) = (\infty, 2, 2)$, $\mathbf{S}(X) = \emptyset$, $\mathbf{M}(X) \cong \mathrm{PSO}(3,1)$ and $\mathbf{D}(X) = 0$ so that Theorem 1 holds. If $n > 2$, then we know from Theorem Aa that $\mathbf{T}(X) =$

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$(\infty, 4, 4)$, $\mathbf{S}(X) = \emptyset$ and X is projectively equivalent to the Veronese surface $Y_\circ$. By assumption there exists a subgroup $G \subseteq \mathbf{M}(X)$ such that $\dim G \geq 2$. We know from Proposition 26b that X has Möbius pair $(Y_\circ, V(q))$ for some invariant quadratic form $q \in I_2^G(Y_\circ)$ of signature $(1,5)$. It follows from Lemma 38 and Proposition 26c that $G \ncong \mathrm{PSL}(3)$ and that if $G \cong \mathrm{PSO}(3)$, then $q \circ \varphi = x_1^2 + x_2^2 + x_3^2 - x_0x_4 - x_0x_5 - x_4x_5$ for some $\varphi \in \mathrm{Aut}(Y_\circ)$. It follows from Lemma 39 and Proposition 26a that X is unique up to Möbius equivalence. Therefore, $\mathrm{PSO}(3) \subseteq \mathbf{M}(X)$ and $\mathbf{D}(X) = 0$. There exists no subgroup G such that $\mathrm{PSO}(3) \subsetneq G \subsetneq \mathrm{PSL}(3)$ and thus we conclude that $\mathbf{M}(X) \cong \mathrm{PSO}(3)$. ∎

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