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On an open problem of Weidmann: essential spectra and square-integrable solutions

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In 1987, Weidmann proved that, for a symmetric differential operator τ and a real λ, if there exist fewer square-integrable solutions of $(\tau - \lambda)y = 0$ than needed and if there is a self-adjoint extension of τ such that λ is not its eigenvalue, then λ belongs to the essential spectrum of τ. However, he posed an open problem of whether the second condition is necessary and it has been conjectured that the second condition can be removed. In this paper, we first set up a formula of the dimensions of null spaces for a closed symmetric operator and its closed symmetric extension at a point outside the essential spectrum. We then establish a formula of the numbers of linearly independent square-integrable solutions on the left and the right subintervals, and on the entire interval for nth-order differential operators. The latter formula ascertains the above conjecture. These results are crucial in criteria of essential spectra in terms of the numbers of square-integrable solutions for real values of the spectral parameter.

1. Introduction

In his book [9], Weidmann considered the nth-order formal symmetric differential expression

$$\tau y = w^{-1}\biggl\{\sum_{j=0}^{[n/2]}(-1)^j(p_j y^{(j)})^{(j)} + \sum_{j=0}^{[(n-1)/2]}(-1)^j[(q_j y^{(j)})^{(j+1)} - (q_j^* y^{(j+1)})^{(j)}]\biggr\} \tag{1.1}$$

on (a, b), where $0 < n \in \mathbb{N}$, $-\infty \leqslant a < b \leqslant +\infty$, $y = y(t)$ is a complex-valued m-vector function, $y^{(j)} = \mathrm{d}^j y/\mathrm{d}t^j$, $p_j(t), q_j(t)$ and $w(t)$ are measurable $m\times m$ matrices, $p_j(t)$, $w(t)$ are Hermitian, and $w(t) > 0$ a.e. $t \in (a, b)$.

If n is even, say $n = 2k$, assume further that p_k is regular, and $|p_k^{-1}|$, $|p_k^{-1}q_{k-1}|$, $|p_{k-1} - q_{k-1}^* p_k^{-1} q_{k-1}|$, $|p_j|$, $|q_j|$ ($j = 0, \dots, k-2$), and $|w|$ are locally integrable on (a, b). If n is odd, say $n = 2k + 1$, assume that q_k is absolutely continuous, $\hat{q}_k := q_k - q_k^*$ is regular on (a, b), and $|\hat{q}_k^{-1}|$, $|\hat{q}_k^{-1}(p_k + q_k')|$, $|\hat{q}_k^{-1} q_{k-1}|$, $|p_j|$, $|q_j|$ ($j = 0, \dots, k-1$) and $|w|$ are locally integrable on (a, b) [9, pp. 27, 31]. Under these assumptions, it was shown in [9, theorem 3.1, p. 43] that the differential expression τ given in (1.1) is well defined and formally symmetric.

By introducing the *quasi-derivatives* $y^{\{j\}}$ of y, the initial-value problem at a point $c \in (a, b)$ of (1.1) is well posed by

$$\tau y = f, \quad y^{\{j\}}(c) = y_j, \quad j = 0, 1, \dots, n-1. \tag{1.2}$$

For details about the definition of quasi-derivatives, see [9, pp. 25–29] or the appendix of this paper.

The form of (1.1) includes two important cases. For $n = 2$, $m = 1$ and $q_0 = 0$, (1.1) reduces to the Sturm–Liouville differential expression:

$$\tau_{\mathrm{SL}} y = w^{-1}(-(p_1 y')' + p_0 y). \tag{1.3}$$

It is known that τ_{SL} is well defined and formally symmetric under the classical assumption: w, p_1, p_0 are real-valued functions, $w(t), p_1(t) > 0$, a.e. $t \in (a, b)$, and w, $1/p_1$, p_0 are locally integrable on (a, b). In this case, the quasi-derivatives are $y^{\{0\}} = y$ and $y^{\{1\}} = p_1 y'$.

For $n = 1$, $m = 2$ and $q_0 = J/2$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

equation (1.1) reduces to the one-dimensional Dirac differential expression:

$$\tau_{\mathrm{D}} y = w^{-1}(Jy' + p_0 y). \tag{1.4}$$

If the 2×2 matrices w and p_0 are Hermitian with $w(t) > 0$ and all components of w and p_0 are locally integrable on (a, b), then τ_{D} is well defined and formally symmetric.

This paper studies the operators associated to τ in the complex Hilbert space $L^2_w(a, b)$ of all weighted square-integrable m-vector functions on (a, b), equipped with the inner product

$$\langle f, g\rangle_w = \int_a^b g^*(t) w(t) f(t)\, \mathrm{d}t \tag{1.5}$$

and the corresponding norm $\|\cdot\|_w$. As usual, $L^2_w(a, b)$ reduces to $L^2(a, b)$ when $w = \mathrm{id}$.

In [10, p. 145], Zettl considered the spectral problem of the Sturm–Liouville operator $\tau_{\mathrm{SL}} y = \lambda y$ on (a, b) and defined the endpoint a to be *regular* if $1/p_1, p_0, w \in L^1_w(a, c)$ for some $c \in (a, b)$. He then proved [10, theorem 2.3.1] that a is regular if and only if the quasi-derivatives $y^{\{0\}}(t) = y(t)$ and $y^{\{1\}}(t) = p_1(t) y'(t)$ of every solution have finite limits as $t \to a+$. Following Zettl, we will call the endpoint a regular if the quasi-derivatives $y^{\{j\}}(t)$, $j = 0, \dots, n-1$, of every solution $y(t)$ of $\tau y = f$ have finite limits at a for every $f \in L^2_w(a, b)$. Otherwise, a is called *singular*. The regularity and singularity of the endpoint b are defined in a similar way. In this paper both the endpoints a and b are allowed to be singular.

Remark 1.1. We note that in much of the literature an infinite endpoint is usually classified as a singular endpoint, but it may be regular by the above definition. In fact, a regular infinite endpoint can be transformed into a regular finite endpoint by means of independent variable transformation of the system. We can also deduce that an endpoint is regular if and only if the initial-value problem (1.2) at this endpoint is well posed and the corresponding fundamental matrix solution is well defined and locally bounded. This fact will be used in the proof of lemma 3.2 in this paper.

For $\lambda \in \mathbb{C}$ and $c \in (a,b)$, denote by $\gamma(\lambda)$ (respectively, $\gamma_a(\lambda)$, $\gamma_b(\lambda)$) the number of linearly independent solutions of $\tau y = \lambda y$ in $L^2_w(a,b)$ (respectively, $L^2_w(a,c]$, $L^2_w[c,b)$). It is known that $\gamma(\lambda)$, $\gamma_a(\lambda)$ and $\gamma_b(\lambda)$ are independent of λ in each of the areas $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$. Set $\gamma_\pm(\tau) = \gamma(\pm \mathrm{i})$, $\gamma_a^\pm = \gamma_a(\pm\mathrm{i})$ and $\gamma_b^\pm = \gamma_b(\pm\mathrm{i})$. The numbers $\gamma_+(\tau)$ and $\gamma_-(\tau)$ are called the positive and negative deficiency indices of τ, respectively. We say τ is *essentially self-adjoint* if

$$\gamma_+(\tau) = \gamma_-(\tau) = 0. \tag{1.6}$$

If τ is essentially self-adjoint, then the associated minimal operator T_0 to τ has only one self-adjoint extension, which is $\bar{T}_0$. If, for some $\lambda_0 \in \mathbb{R}$,

$$\gamma_b(\lambda_0) = mn, \tag{1.7}$$

then we say τ is *limit-circle* at b (LC at b, for short). Otherwise, τ is said to be *limit-point* (LP) at b. The LC and LP cases at a are defined similarly. The limit-circle case of an endpoint is independent of the values of λ_0 (cf. [1, theorem 9.11.2, p. 296]).

For real λ, the number $\gamma(\lambda)$ has close relations with the essential spectrum of τ, denoted by $\sigma_e(\tau)$, which we will elaborate in § 2. Essential spectra of operators have been studied by many authors using various theories such as the oscillation theory, asymptotic analysis, energy-like functions, the perturbation theory, singular sequences and square-integrable solutions for real values of the spectral parameter. Among these methods the last one has attracted lots of attention because it takes advantage of using numerous tools available in the fundamental theory of differential equations. Here we introduce two important theorems of Hartman and Wintner [3]. One of them says that $\lambda \in \sigma_e(\tau)$ if and only if there exists an $f \in L^2_w$ such that $(\tau - \lambda)y = f$ has no any L^2_w-solutions and the other is the following.

THEOREM 1.2 (Hartman and Wintner [3]). *Let τ_{SL} be regular at a, limit-point at b and $w(t) \equiv 1$.*

(i) *If, for some $\lambda \in \mathbb{R}$, $\gamma(\lambda) = 0$, then $\lambda \in \sigma_e(\tau_{\mathrm{SL}})$.*

(ii) *If $\gamma(\lambda) \equiv 1$ on an interval I of $\mathbb{R}$, then for every self-adjoint realization associated to τ_{SL} the continuous spectrum in I is empty and the point spectrum is nowhere dense in I.*

It was conjectured that theorem 1.2(ii) can be improved to get that every self-adjoint realization associated to τ_{SL} has a pure point spectrum in I. This conjecture was disproved by Remling in [5]. In [9, theorems 11.1 and 11.7], Weidmann extended theorem 1.2 to cases where the order $n \geqslant 2$ and both the endpoints a and b are allowed to be singular. Recently, the result of part (ii) has been extended to higher-order differential equations with arbitrary equal deficiency indices in [6]. The following is the extension of theorem 1.2(i) in [9, section 11].

THEOREM 1.3 ([9, theorem 11.1]). *Assume that τ in (1.1) has equal deficiency indices $\gamma_\pm(\tau) =: \gamma(\tau)$.*

(i) *If $\gamma_a(\lambda) + \gamma_b(\lambda) < mn + \gamma(\tau)$, then for every self-adjoint extension T of τ, $\lambda \in \sigma(T)$.*

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(ii) *If in addition there exists a self-adjoint extension T of τ such that λ is not an eigenvalue of T, then $\lambda \in \sigma_e(\tau)$.*

Weidmann posed an open problem of whether the additional assumption in theorem 1.3(ii) can be removed. This naturally gives rise to the following conjecture.

CONJECTURE 1.4. *Assume that τ in (1.1) has equal deficiency indices $\gamma_\pm(\tau) = \gamma(\tau)$. If*

$$\gamma_a(\lambda) + \gamma_b(\lambda) < mn + \gamma(\tau), \tag{1.8}$$

then $\lambda \in \sigma_e(\tau)$.

We will prove this conjecture. In some cases, for example, in the case where the minimal operator associated to τ has no eigenvalues, the extra condition in theorem 1.3(ii) naturally holds and is therefore redundant. It is because the associated minimal operator may have eigenvalues (see examples 2.2 and 2.6) that the proof of the conjecture is non-trivial.

We will first establish more general results for closed symmetric operators. For a closed symmetric operator, theorem 2.1 sets up the relation between the dimension of the null space of the adjoint operator and the deficiency indices at a real λ that is not in the essential spectrum. Theorem 2.7 gives a formula for the dimensions of null spaces of a closed symmetric operator and its symmetric extension. These results are of importance in their own right. Then, for the differential expression in (1.1), we will obtain in theorem 2.9 that

$$\lambda \notin \sigma_e(\tau) \quad \Rightarrow \quad \gamma_a(\lambda) + \gamma_b(\lambda) = mn + \gamma(\tau). \tag{1.9}$$

Clearly, the validity of conjecture 1.4 is a consequence of (1.9).

From the above we see that the basic idea in this paper is to obtain an exact formula for numbers of square-integrable solutions so that we can determine the essential spectrum by finding the numbers of square-integrable solutions alone, without knowing any information about the extensions of the operator. Our method is particularly useful in cases where the endpoints are both singular.

Moreover, by a consequence (corollary 2.11) of our results, if τ has non-trivial self-adjoint extensions for $mn = 2$ or τ is defined on an interval with at least one regular or LC endpoint, then Weyl's essential spectrum has a better expression:

$$\sigma_e(\tau) = \bigcap_{T \in \mathcal{T}} \sigma(T), \tag{1.10}$$

where $\mathcal{T}$ is the set of all self-adjoint extensions of τ. This formula also implies that a subset is contained in the essential spectrum of τ if and only if it is invariant for self-adjoint extensions of τ.

Section 2 will introduce some notation, state the main results and give illustrative examples. The proofs of the results are left to § 3.

2. Main results and their corollaries

In this section, we will state our main results, theorems 2.1, 2.7, 2.9 and 2.10, and leave their proofs to § 3.

Let T be a linear operator in a complex Hilbert space $\mathbb{H}$. The domain of definition, the range and the null space of T are denoted by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. If T is closed, denote the resolvent set of T by $\rho(T)$ and the spectrum of T by $\sigma(T)$. There are various definitions of the essential spectrum of T in the literature. For the purpose of this paper, we introduce only a few of them. For a closed operator T, the essential spectrum of T is defined in [2, p. 1393] as

$$\sigma_e(T) = \{\lambda \in \sigma(T) \colon \mathcal{R}(\lambda - T) \neq \overline{\mathcal{R}(\lambda - T)}\}, \tag{2.1}$$

while Weyl defined the essential spectrum of T as

$$\sigma_{e1}(T) = \bigcap_{K \in \mathcal{K}} \sigma(T + K), \tag{2.2}$$

where $\mathcal{K}$ is the set of all compact operators on $\mathbb{H}$. Of course, $\sigma_{e1}(T)$ is invariant under compact perturbations. If T is self-adjoint, its essential spectrum, denoted by $\sigma_{e2}(T)$, is defined as the union of the set of all accumulation points of $\sigma(T)$ and the set $\sigma_\infty(T)$ of all isolated eigenvalues of infinite multiplicity ([8, p. 202] and [9, p. 162]). In the self-adjoint case, $\sigma_{e1}(T) = \sigma_{e2}(T)$ and we know from [2, theorem 6.5, p. 1395] that

$$\sigma_{e2}(T) = \sigma_e(T) \cup \sigma_\infty(T). \tag{2.3}$$

Furthermore, if T_0 is the minimal (closed) operator associated to the differential expression τ given in (1.1), then $\sigma_\infty(T_0) = \varnothing$, and hence, by (2.3), $\sigma_e(T_0) = \sigma_{e2}(T_0)$. Since every self-adjoint extension of T_0 (if any) has the same essential spectrum $\sigma_e(T_0)$, we will view $\sigma_e(T_0)$ as the essential spectrum of τ, denoted by $\sigma_e(\tau)$.

For a closed symmetric operator T_0, set $T_1 = T_0^*$. It is known that $\mathcal{N}(\lambda - T_1)$ is independent of λ in each of $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$, so we define

$$\mathcal{N}_+(T_0) = \mathcal{N}(\mathrm{i} - T_1) \quad \text{and} \quad \mathcal{N}_-(T_0) = \mathcal{N}(-\mathrm{i} - T_1), \tag{2.4}$$

the positive, negative deficiency spaces of T_0, and define $\gamma_+(T_0) = \dim \mathcal{N}_+(T_0)$, $\gamma_-(T_0) = \dim \mathcal{N}_-(T_0)$ the positive, negative deficiency indices of T_0, respectively.

For $\lambda \in \mathbb{C}$, set $\mathcal{N}_0(\lambda - T_1) = \mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0)$ and

$$\gamma(\lambda, T_0) = \dim \mathcal{N}(\lambda - T_1), \quad \gamma_0(\lambda, T_0) = \dim \mathcal{N}_0(\lambda - T_1). \tag{2.5}$$

If $\operatorname{Im}\lambda \neq 0$, then $\gamma(\lambda, T_0) = \gamma_+(T_0)$ for $\operatorname{Im}\lambda > 0$, $\gamma(\lambda, T_0) = \gamma_-(T_0)$ for $\operatorname{Im}\lambda < 0$ and $\mathcal{N}_0(\lambda - T_1) = \{0\}$. In the case $\operatorname{Im}\lambda = 0$, however, it is possible that $\gamma_0(\lambda) > 0$ and we should pay much more attention.

Theorem 2.1. *Let T_0 be a closed symmetric operator in $\mathbb{H}$ and $T_1 = T_0^*$. If there exists a $\lambda \in \mathbb{R}$ such that $\lambda \notin \sigma_e(T_0)$, then $\gamma_+(T_0) = \gamma_-(T_0) =: \gamma(T_0)$ and*

$$\dim \mathcal{N}(\lambda - T_1) = \gamma(T_0) + \dim \mathcal{N}_0(\lambda - T_1). \tag{2.6}$$

Furthermore, there exists a self-adjoint extension T of T_0 such that

$$[\mathcal{N}(\lambda - T_1) \ominus \mathcal{N}_0(\lambda - T_1)] \cap \mathcal{D}(T) = \{0\}. \tag{2.7}$$

The following is an example where $\dim \mathcal{N}_0(\lambda - T_1) > 0$ for some $\lambda \in \mathbb{R}$.

Example 2.2. Consider the second-order differential expression

$$\tau y = -y''(t) + t^2 y(t) \quad \text{on } (-\infty, +\infty). \tag{2.8}$$

It is known [2, pp. 1399–1400] that $\gamma_+(\tau) = \gamma_-(\tau) = 0$, i.e. τ is essentially self-adjoint. Let τ_1 and τ_2 denote the restrictions of τ to functions on $(-\infty, 0]$ and $[0, +\infty)$, respectively. By the decomposition method we know that

$$\sigma_e(\tau) = \sigma_e(\tau_1) \cup \sigma_e(\tau_2).$$

For details of the decomposition method, the reader is referred to [9, pp. 54–55]. Since $t^2 \to \infty$ as $t \to \pm\infty$, one sees from [4, theorem 10.3.4] that $\sigma_e(\tau_1) = \sigma_e(\tau_2) = \varnothing$, and hence we have from theorem 2.1 that, for $\lambda \in \mathbb{R}$,

$$\gamma(\lambda) = 0 + \gamma_0(\lambda) = \gamma_0(\lambda).$$

Choose $\lambda = 1$. Clearly, $y_0(t) = \mathrm{e}^{-t^2/2}$ is a solution of $\tau y = y$ and $y_0 \in L^2(-\infty, \infty)$, and hence $\gamma(\lambda) = \gamma_0(\lambda) = 1$, which means that $\mathcal{N}_0(1 - T_1) \neq \{0\}$. We note that $\lambda = 1$ is an eigenvalue of T_0, but not an essential spectral point of T_0.

From theorem 2.1 we immediately have the following corollary.

Corollary 2.3. *For a closed symmetric operator* T_0*, if* $\gamma_+(T_0) \neq \gamma_-(T_0)$*, then* $\sigma_e(T_0) = \mathbb{R}$.

Since every self-adjoint extension of a closed symmetric operator T_0 with finite equal deficiency indices is a sum of T_0 and a compact operator by the second formula of von Neumann (lemma 3.1 of this paper), we know that the set $\mathcal{T}$ of all self-adjoint extensions of T_0 is contained in $\{T_0 + K : K \in \mathcal{K}\}$, and hence, by (2.2),

$$\sigma_e(T_0) \subset \bigcap_{T \in \mathcal{T}} \sigma(T).$$

This together with the last assertion of theorem 2.1 yields the following corollary.

Corollary 2.4. *For a closed symmetric operator* T_0*, if* $\gamma_+(T_0) = \gamma_-(T_0) < \infty$ *and* $\mathcal{N}_0(\lambda - T_1) = \{0\}$ *for all* $\lambda \notin \sigma_e(T_0)$*, then*

$$\sigma_e(T_0) = \bigcap_{T \in \mathcal{T}} \sigma(T). \tag{2.9}$$

Remark 2.5. The restriction $\mathcal{N}_0(\lambda - T_1) = \{0\}$ in corollary 2.4 is required to get rid of other spectral points from $\bigcap_{T \in \mathcal{T}} \sigma(T)$. For this, see example 2.2 when T_0 is essentially self-adjoint and see example 2.6 when T_0 has non-trivial self-adjoint extensions. Sufficient conditions for $\mathcal{N}_0(\lambda - T_1) = \{0\}$ will be given at the end of this section.

Example 2.6. Consider the differential expression

$$L := \tau_1 \oplus \tau_2 \quad \text{on } (-\infty, +\infty),$$

where τ_1 and τ_2 are given by

$$\tau_1 y = -y'' + (t^2 - 1)y, \quad \tau_2 y = \mathrm{e}^{t^2}(-y'' + y) \quad \text{on } (-\infty, +\infty).$$

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For $\lambda = 0$, $LY = 0$ has three linearly independent solutions:

$$Y_1 = \begin{pmatrix} e^{-t^2/2} \\ 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 \\ e^{t} \end{pmatrix} \quad \text{and} \quad Y_3 = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

in $L^2_W(-\infty, +\infty)$ with $W(t) = \operatorname{diag}(1, e^{-t^2})$. Thus, $\gamma(0) = 3$. Since $\gamma_\pm(\tau_1) = 0$ and $\gamma_\pm(\tau_2) = 2$, we have

$$\gamma_\pm(L) = \gamma_\pm(\tau_1) + \gamma_\pm(\tau_2) = 2.$$

Noting that L is the orthogonal sum of τ_1 and τ_2, we know that

$$\sigma_e(L) = \sigma_e(\tau_1) \cup \sigma_e(\tau_2) = \varnothing,$$

and hence, from theorem 2.1,

$$3 = \gamma(0) = 2 + \gamma_0(0) \quad \Rightarrow \quad \gamma_0(0) = 1.$$

This implies that $\mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0) \neq \{0\}$ for $\lambda = 0$ and that $\lambda = 0$ is an eigenvalue, but not an essential spectral point of T_0.

The next result gives the relation between dimensions of null spaces of a closed symmetric operator and its extension.

THEOREM 2.7. *Let S_0 be a closed symmetric operator in $\mathbb{H}$ and T_0 be a k-dimensional closed symmetric extension of S_0. Let $S_1 = S_0^*$ and $T_1 = T_0^*$. If $\lambda \in \mathbb{R} \backslash \sigma_e(T_0)$, then*

$$\dim \mathcal{N}(\lambda - S_1) - \dim \mathcal{N}(\lambda - T_1) = k - \dim(\mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0)). \tag{2.10}$$

As a consequence of theorem 2.7, we have the following corollary.

COROLLARY 2.8.

(i) *If, in addition to the conditions in theorem 2.7, $\dim \mathcal{N}(\lambda - S_1) = k$ and $\mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0) = \{0\}$, then λ is not an eigenvalue of T_1.*

(ii) $\dim \mathcal{N}(\lambda - S_1) = \infty$ *if and only if* $\dim \mathcal{N}(\lambda - T_1) = \infty$.

In the following results, τ is defined in (1.1), T_0 and $T_1 = T_0^*$ are the minimal and the maximal operators associated to τ, respectively.

Recalling the definition of $\gamma_0(\lambda) := \gamma_0(\lambda, T_0) = \dim \mathcal{N}_0(\lambda, T_1)$ in (2.6), we have the following theorem.

THEOREM 2.9. *Let T_0 be the minimal operator generated by τ given in (1.1). If there exists a $\lambda_0 \in \mathbb{R} \setminus \sigma_e(T_0)$, then*

$$\gamma_+(T_0) = \gamma_-(T_0) =: \gamma(\tau) = \gamma_a^\pm + \gamma_b^\pm - mn \tag{2.11}$$

and, for every $\lambda \in \mathbb{R} \setminus \sigma_e(T_0)$,

$$\gamma(\lambda) = \gamma(\tau) + \gamma_0(\lambda) \tag{2.12}$$

and

$$\gamma(\tau) = \gamma_a(\lambda) + \gamma_b(\lambda) - mn. \tag{2.13}$$

It is easy to see that (2.13) ascertains conjecture 1.4.

From (2.12) we have $\gamma(\lambda) \geqslant \gamma_{\pm}(T_0)$. Therefore, theorems 2.1 and 2.9 improve [2, lemma 7, p. 1397] and [2, Corollary 8, p. 1398], respectively.

Now we turn to the question of under what conditions $\mathcal{N}_0(\lambda - T_1) = \{0\}$ holds. The following result partly answers this question.

THEOREM 2.10. *Let τ be defined as in (1.1).*

(i) *If one of the endpoints a and b is regular, then $\mathcal{N}_0(\lambda - T_1) = \{0\}$ for all $\lambda \in \mathbb{C}$.*

(ii) *If one of the endpoints a and b is limit circle, then $\mathcal{N}_0(\lambda - T_1) = \{0\}$ for all $\lambda \notin \sigma_e(\tau)$.*

(iii) *If $mn = 2$ in (1.1) and $\gamma_{\pm}(\tau) > 0$, then $\mathcal{N}_0(\lambda - T_1) = \{0\}$ for all $\lambda \notin \sigma_e(\tau)$.*

Theorem 2.10 and corollary 2.4 enable us to simplify the expression of Weyl's definition of essential spectrum for some special differential expressions.

COROLLARY 2.11. *Let τ be defined as in (1.1). If one of the endpoints a and b is regular or LC, or if $mn = 2$ and $\gamma_{\pm}(\tau) > 0$, then*

$$\sigma_e(\tau) = \bigcap_{T \in \mathcal{T}} \sigma(T). \tag{2.14}$$

3. The proofs of the results

In this section we give the proofs of theorems 2.1, 2.7, 2.9 and 2.10. For this, we need the following lemma.

LEMMA 3.1 (the formulae of von Neumann; cf. [8, theorems 8.11–8.13]). *Let T_0 be a closed symmetric operator in a complex Hilbert space. Then*

$$\mathcal{D}(T_0^*) = \mathcal{D}(T_0) \dotplus \mathcal{N}_+ \dotplus \mathcal{N}_-, \tag{3.1}$$

where $\mathcal{N}_{\pm} = \mathcal{N}_{\pm}(T_0)$ are the deficiency spaces of T_0, $\dotplus$ is the direct sum, and

(i) *T is a closed symmetric extension of T_0 if and only if there are closed subspaces $\mathcal{F}_+$ of $\mathcal{N}_+$ and $\mathcal{F}_-$ of $\mathcal{N}_-$ and an isometric mapping V of $\mathcal{F}_+$ onto $\mathcal{F}_-$ such that*

$$\mathcal{D}(T) = \mathcal{D}(T_0) \dotplus \{g + Vg : g \in \mathcal{F}_+\}, \tag{3.2}$$

$$T(f_0 + g + Vg) = T_0 f_0 + \mathrm{i}g - \mathrm{i}Vg, \quad f_0 \in \mathcal{D}(T_0), \quad g \in \mathcal{F}_+. \tag{3.3}$$

(ii) *T is an m-dimensional symmetric extension of T_0 if and only if $\mathcal{F}_+$ is m-dimensional.*

(iii) *T is self-adjoint if and only if $\mathcal{F}_{\pm} = \mathcal{N}_{\pm}$.*

Proof of theorem 2.1. We may assume that $\lambda = 0 \notin \sigma_e(T_0)$, for otherwise replace T_0 with $T_0 - \lambda$.

First, we construct a self-adjoint extension of T_0 to prove $\gamma_+(T_0) = \gamma_-(T_0)$. Set $\mathcal{N}_0(T_1) = \mathcal{N}(T_1) \cap \mathcal{D}(T_0)$. Let T be the restriction of $T_1 = T_0^*$ to

$$\mathcal{D}(T) = \mathcal{D}(T_0) + \mathcal{N}(T_1) = \mathcal{D}(T_0) \dotplus (\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1)). \tag{3.4}$$

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The operator T is symmetric. In fact, for $x, y \in \mathcal{D}(T)$, we have from (3.4) that

$$x = x_0 + x_N, \quad y = y_0 + y_N, \quad x_0,\ y_0 \in \mathcal{D}(T_0), \quad x_N,\ y_N \in \mathcal{N}(T_1).$$

Therefore,

$$\begin{aligned}\langle Tx, y\rangle &= \langle Tx_0 + Tx_N, y\rangle = \langle T_0x_0 + T_1x_N, y\rangle \\ &= \langle T_0x_0, y\rangle = \langle x_0, T_0^*y\rangle = \langle x_0, T_1y\rangle = \langle x_0, T_0y_0\rangle.\end{aligned}$$

Similarly, we have $\langle x, Ty\rangle = \langle T_0x_0, y_0\rangle$, and hence the symmetry of T follows from the symmetry of T_0.

Next we claim that $T^* = T$, i.e. $\mathcal{D}(T^*) \subset \mathcal{D}(T)$. Since $0 \notin \sigma_e(T_0)$ implies $\mathcal{R}(T_0) = \overline{\mathcal{R}(T_0)} = (\mathcal{R}(T_0)^\perp)^\perp$ and $\mathcal{R}(T_0)^\perp = \mathcal{N}(T_0^*) = \mathcal{N}(T_1)$, we have that $\mathcal{R}(T_0) = \mathcal{N}(T_1)^\perp$. Now for $x \in \mathcal{D}(T^*)$ and $y \in \mathcal{N}(T_1)$, since

$$\langle T^*x, y\rangle = \langle x, Ty\rangle = \langle x, T_1y\rangle = 0,$$

we see that $T^*x \in \mathcal{N}(T_1)^\perp = \mathcal{R}(T_0)$. We can then find an $x_0 \in \mathcal{D}(T_0)$ such that $T^*x = T_0x_0$, which means $T_1(x_0 - x) = 0$, i.e. $x_N := x_0 - x \in \mathcal{N}(T_1)$. Now, $x = x_0 + x_N \in \mathcal{D}(T)$ by (3.4) and T is self-adjoint. Thus, $\gamma_+(T_0) = \gamma_-(T_0) =: \gamma(T_0)$.

Since every self-adjoint extension of T_0 is a γ-dimensional extension of T_0 by lemma 3.1(iii), we see from (3.4) that

$$\dim(\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1)) = \gamma(T_0) \quad \Rightarrow \quad \dim \mathcal{N}(T_1) = \gamma(T_0) + \dim(\mathcal{N}(T_1) \cap \mathcal{D}(T_0)),$$

which proves (2.6).

To prove the last assertion of theorem 2.1, for the self-adjoint extension T of T_0, let $V : \mathcal{N}_+ \to \mathcal{N}_-$ be an isometric mapping in lemma 3.1 such that

$$\mathcal{D}(T) = \mathcal{D}(T_0) \dotplus (I + V)\mathcal{N}_+,$$

By (3.4), $\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1) = (I + V)\mathcal{N}_+$. Define a new isometry $\tilde{V} : \mathcal{N}_+ \to \mathcal{N}_-$ by

$$\tilde{V}x = -Vx, \quad x \in \mathcal{N}_+ \tag{3.5}$$

and the self-adjoint restriction $\tilde{T}$ of T_1 by

$$\mathcal{D}(\tilde{T}) = \mathcal{D}(T_0) \dotplus (I + \tilde{V})\mathcal{N}_+.$$

We claim that $(\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1)) \cap \mathcal{D}(\tilde{T}) = \{0\}$. If $x \in (\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1)) \cap \mathcal{D}(\tilde{T})$, then, from $(\mathcal{N}(T_1) \ominus \mathcal{N}_0(T_1)) = (I + V)\mathcal{N}_+$, there exist $x_0 \in \mathcal{D}(T_0)$, $\tilde{g} \in \mathcal{N}_+$ and $g \in \mathcal{N}_+$ such that

$$x = x_0 + (I + \tilde{V})\tilde{g} = (I + V)g,$$

which leads to

$$x_0 + (\tilde{g} - g) + (\tilde{V}\tilde{g} - Vg) = 0.$$

It follows from (3.1) that

$$x_0 = 0, \quad g = \tilde{g}, \quad \tilde{V}\tilde{g} - Vg = 0,$$

and hence $Vg = 0$ by (3.5), and $x = g \in \mathcal{N}_+$ since $x = g + Vg$ and $g \in \mathcal{N}_+$. But $\mathcal{N}_+ \cap \mathcal{N}(T_1) = \{0\}$ implies $x = 0$. This proves $\mathcal{N}_0(T_1) \cap \mathcal{D}(\tilde{T}) = \{0\}$. □

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Proof of theorem 2.7. Since T_0 is a k-dimensional extension of S_0, there exists a k-dimension subspace X of $\mathbb{H}$ such that

$$\mathcal{D}(T_0) = \mathcal{D}(S_0) \dotplus X. \tag{3.6}$$

We may suppose that $\lambda = 0 \notin \sigma_e(T_0)$, or consider $\lambda - T_0$ instead. Set

$$T_1 = T_0^*, \quad \mathcal{N}_0 = \mathcal{N}(T_1) \cap \mathcal{D}(T_0), \quad X_0 = X \ominus \mathcal{N}_0. \tag{3.7}$$

Then (3.6) can be written as

$$\mathcal{D}(T_0) = \mathcal{D}(S_0) \dotplus X_0 \dotplus \mathcal{N}_0. \tag{3.8}$$

Letting T_0 act on both sides of (3.8), we get $\mathcal{R}(T_0) = \mathcal{R}(S_0) + T_0(X_0)$ and then claim

$$\mathcal{R}(T_0) = \mathcal{R}(S_0) \dotplus T_0(X_0). \tag{3.9}$$

If $x \in \mathcal{R}(S_0) \cap T_0(X_0)$, then there exist $x_0 \in X_0$ and $y_0 \in \mathcal{D}(S_0)$ such that $x = T_0x_0 = S_0y_0$. Since $X_0 \subset \mathcal{D}(T_0)$ and

$$S_0 \subset T_0 \subset T_0^* = T_1 \subset S_0^* =: S_1,$$

we have $T_1(x_0 - y_0) = 0$, and hence $x_0 - y_0 \in \mathcal{N}_0$, i.e. $x_0 - y_0 \in (\mathcal{D}(S_0) \dotplus X_0) \cap \mathcal{N}_0$, which means $x_0 = y_0$ by (3.8). Therefore, we have $x_0 \in \mathcal{D}(S_0) \cap X_0 = \{0\}$. As a result, $x = T_0x_0 = 0$ and (3.9) is valid.

We now proceed to assertion of (2.10). Since $0 \notin \sigma_e(T_0)$, we know that $\mathcal{R}(T_0)$ is closed by the definition of essential spectrum. We will prove $\mathcal{R}(S_0)$ is closed.

Let $\tilde{T}$ and $\tilde{S}$ be the restrictions of T_0 and S_0 to

$$\mathcal{D}(\tilde{T}) = \mathcal{D}(T_0) \cap \mathcal{N}(T_0)^\perp \quad \text{and} \quad \mathcal{D}(\tilde{S}) = \mathcal{D}(S_0) \cap \mathcal{N}(S_0)^\perp,$$

respectively. Clearly, $\mathcal{R}(\tilde{T}) = \mathcal{R}(T_0)$ and $\mathcal{R}(\tilde{S}) = \mathcal{R}(S_0)$. We claim that $\tilde{T}$ is closed. If $x_n \in \mathcal{D}(\tilde{T})$ and $\tilde{T}x_n \to y$ and $x_n \to x$, then we have $x \in \mathcal{D}(T_0)$ and $y = T_0x$ since T_0 is closed. Note that $x_n \in \mathcal{N}(T_0)^\perp$, i.e. $(x_n, f) = 0$ for all $f \in \mathcal{N}(T_0)$. Letting $n \to \infty$ in $(x_n, f) = 0$ gives $(x, f) = 0$ for all $f \in \mathcal{N}(T_0)$. Thus, $x \in \mathcal{D}(\tilde{T})$ and $\tilde{T}$ is closed. Since for every $0 \neq x \in \mathcal{D}(\tilde{T})$, $\tilde{T}x = T_0x \neq 0$, one sees that $\tilde{T}$ is invertible. Furthermore, $\tilde{T}^{-1}$ is closed since $\tilde{T}$ is closed, and hence the closedness of $\mathcal{R}(T_0) = \mathcal{R}(\tilde{T})$ implies $\tilde{T}^{-1}$ is bounded by the closed graph theorem.

Similarly, we can prove that $\tilde{S}$ is invertible and $\tilde{S}^{-1}$ is closed. Clearly, $\tilde{S}$ is the restriction of $\tilde{T}$ to $\mathcal{D}(\tilde{S})$. Therefore, $\tilde{S}^{-1}$ is bounded in view of the boundedness of $\tilde{T}^{-1}$. Again, by the closed graph theorem, we then have that $\mathcal{R}(\tilde{S}) = \mathcal{R}(S_0)$ is closed.

It follows from $\mathcal{R}(T_0)^\perp = \mathcal{N}(T_1), \mathcal{R}(S_0)^\perp = \mathcal{N}(S_1)$ that

$$\mathbb{H} = \mathcal{R}(T_0) \dotplus \mathcal{N}(T_1) = \mathcal{R}(S_0) \dotplus \mathcal{N}(S_1).$$

This, together with (3.9), yields that

$$\mathbb{H} = \mathcal{R}(S_0) \dotplus T_0(X_0) \dotplus \mathcal{N}(T_1) = \mathcal{R}(S_0) \dotplus \mathcal{N}(S_1),$$

and hence

$$\dim(\mathcal{N}(S_1) \ominus \mathcal{N}(T_1)) = \dim T_0(X_0). \tag{3.10}$$

From the definition of X_0 in (3.7) we see that

$$\dim T_0(X_0) = \dim X_0 = \dim X - \dim(\mathcal{N}(T_1) \cap \mathcal{D}(T_0)). \tag{3.11}$$

Clearly, (3.10) and (3.11) imply (2.10) since $\dim X = k$. □

In order to prove theorem 2.9, we need the following lemma.

LEMMA 3.2. *Suppose that τ in (1.1) is defined on the interval (a, b) with at least one regular endpoint, T_0 is the minimal operator and $T_1 = T_0^*$ is the maximal operator generated by τ. Then T_0 has no eigenvalues, namely, for all $\lambda \in \mathbb{R}$, it holds that*

$$\mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0) = \{0\}. \tag{3.12}$$

Proof. We will prove this result in a more general formulation, that is, we will prove the corresponding result for the Hamiltonian differential expression:

$$LY := JY' - QY = \lambda WY \quad \text{on } (a, b), \tag{3.13}$$

acting on $L^2_W(a, b)$, the Hilbert space of equivalence classes of Lebesgue measurable mn-vector functions F satisfying $\int_0^\infty F^*(s)W(s)F(s)\,\mathrm{d}s < \infty$ with the semi-norm

$$\|F\|_W^2 = \langle F, F\rangle_W = \int_0^\infty F^*(s)W(s)F(s)\,\mathrm{d}s.$$

Here in (3.13), $Y = Y(t)$ is an mn-vector function, $Q(t)$ and $W(t)$ are $mn \times mn$ Hermitian matrices on (a, b), $W(t) \geqslant 0$, J is the standard symplectic identity matrix, i.e. $J^* = -J$, $J^2 = -I_{mn}$ and I_{mn} is the $mn \times mn$ identity matrix.

By a result of Walker [7], the differential expression $\tau y = f$ can be transformed to an equivalent Hamiltonian differential expression $LY = WF$ in the form given by (3.13), where $F = (f, 0, \dots, 0)^{\mathrm{T}}$, $W = \operatorname{diag}(w, 0, \dots, 0)$.

In this case, the minimal operator T_0 is the closure of the pre-minimal operator T_{00}, whose domain is given by

$$\mathcal{D}(T_{00}) = \{Y \in L^2_W(a, b) : \operatorname{supp} Y \subset (a, b),\ \exists F \in L^2_W(a, b) \text{ s.t. } LY = WF\}$$

and $T_{00}Y = F$ if $LY = WF$ for $Y \in \mathcal{D}(T_{00})$. For $Y \in \mathcal{D}(T_0)$, there exists a sequence $Y_k \in \mathcal{D}(T_{00})$ such that, as $k \to \infty$,

$$Y_k \to Y, \quad T_{00}Y_k =: F_k \to F \quad \text{in } L^2_W(a, b)$$

and $Y = T_0F$. From remark 1.1, without loss of generality, we assume that a is regular, $a > -\infty$ and let $\Phi(t)$ be the fundamental matrix of $Jy'(t) = Q(t)y(t)$ such that $\Phi(a) = I_{mn}$. Since $JY_k' = QY_k + WF_k$ on $[a, b)$ and $Y_k(a) = 0$, the variation formula gives

$$Y_k(t) = \Phi(t)\int_a^t J\Phi^*(s)W(s)F_k(s)\,\mathrm{d}s. \tag{3.14}$$

For a fixed $a_1 \in (a, b)$, there is an $M > 0$ such that $\|\Phi(t)\|^2 \leqslant M$ on $(a, a_1]$ and

$$\int_a^{a_1} \Phi^*(s)W(s)\Phi(s)\,\mathrm{d}s \leqslant M.$$

Set $Y_{ij} = Y_i - Y_j$. We have from (3.14) and the Schwarz inequality that

$$|Y_{ij}(t)|^2 \leqslant M \int_a^t \Phi^*(s)W(s)\Phi(s)\,\mathrm{d}s \int_a^t F_{ij}^*(s)W(s)F_{ij}(s)\,\mathrm{d}s \leqslant M^2 \|F_{ij}\|_W^2 \to 0$$

as $i, j \to \infty$. Then $Y_k(t)$ uniformly converges to $Y(t)$ on $(a, a_1]$, and hence $Y(a) = \lim_{k\to\infty} Y_k(a) = 0$ since $Y_k(a) = 0$. Now if $Y \in \mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0)$, then $Y(t)$ is a solution of (3.13) with the initial condition $Y(a) = 0$, and hence $Y(t) \equiv 0$ on $[a, b)$. This proves (3.12). □

Proof of theorem 2.9. We will apply the decomposition method to the proof. Fix a $c \in (a, b)$. Define an operator S_0 to be the restriction of T_0 to

$$\mathcal{D}(S_0) = \{y \in \mathcal{D}(T_0) : y^{\{j\}}(c) = 0,\ j = 0, \dots, n-1\}.$$

It is shown in [9, theorem 4.2] that S_0 is an mn-dimensional restriction of T_0 and

$$\gamma_{\pm}(T_0) = \gamma_{\pm}(S_0) - mn = \gamma_a^{\pm}(T_0) + \gamma_b^{\pm}(T_0) - mn.$$

This proves (2.11) in theorem 2.9. Since S_0 has no eigenvalues by the definition of $\mathcal{D}(S_0)$ and $\lambda \notin \sigma_e(S_0) = \sigma_e(T_0)$, it follows from theorem 2.1 that

$$\dim \mathcal{N}(\lambda - S_0^*) = \gamma_{\pm}(S_0).$$

Since T_0 is an mn-dimensional extension of S_0, from (2.10) in theorem 2.7, we obtain

$$\dim \mathcal{N}(\lambda - T_1) = \dim \mathcal{N}(\lambda - S_0^*) - mn + \gamma_0(\lambda), \tag{3.15}$$

where $\gamma_0(\lambda) = \dim(\mathcal{N}(\lambda - T_1) \cap \mathcal{D}(T_0))$. From theorem 2.1, we see

$$\gamma(\tau) = \dim \mathcal{N}(\lambda - T_1) - \gamma_0(\lambda) = \dim \mathcal{N}(\lambda - S_0^*) - mn. \tag{3.16}$$

Hence, S_0 is the direct sum of S_{a0} and S_{b0}, where $S_{a0} = \bar{S}_{a00}$ and $S_{b0} = \bar{S}_{b00}$ and S_{a00} and S_{b00} are defined on $\mathcal{D}(S_{a00}) = \{y \in \mathcal{D}(T_0) : \operatorname{supp} y \subset (a, c)\}$ and $\mathcal{D}(S_{b00}) = \{y \in \mathcal{D}(T_0) : \operatorname{supp} y \subset (c, b)\}$, respectively. Then we know that

$$\dim \mathcal{N}(\lambda - S_0^*) = \dim \mathcal{N}(\lambda - S_{a0}^*) + \dim \mathcal{N}(\lambda - S_{b0}^*). \tag{3.17}$$

Clearly, the domains of definition of S_{a0}^* and S_{b0}^* are given by

$$\mathcal{D}(S_{a0}^*) = \{y \in \mathcal{D}(T_1) : \tau y = T_1 y \text{ on } (a, c]\},$$
$$\mathcal{D}(S_{b0}^*) = \{y \in \mathcal{D}(T_1) : \tau y = T_1 y \text{ on } [c, b)\}.$$

Since τ is regular at c, $\dim \mathcal{N}(\lambda - S_{a0}^*)$ (respectively, $\dim \mathcal{N}(\lambda - S_{b0}^*)$) is the number of linearly independent solutions of $\tau y = \lambda y$ in $L_w^2(a, c)$ (respectively, $L_w^2(c, b)$), we have

$$\dim \mathcal{N}(\lambda - S_{a0}^*) = \gamma_a(\lambda), \qquad \dim \mathcal{N}(\lambda - S_{b0}^*) = \gamma_b(\lambda). \tag{3.18}$$

Inserting (3.18) into (3.16) via (3.17) yields

$$\gamma(\tau) = \gamma_a(\lambda) + \gamma_b(\lambda) - mn$$

and then (3.15) becomes

$$\gamma(\lambda) = \gamma(\tau) + \gamma_0(\lambda).$$

The proof of theorem 2.9 is complete. □

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Proof of theorem 2.10. Evidently, (i) follows from lemma 3.2.

(ii) Without loss of generality, we suppose that τ is limit circle at a, i.e. $\gamma_a(\lambda) = mn$ for all $\lambda \in \mathbb{C}$. It is clear that $\gamma(\lambda) = \gamma_b(\lambda)$. By theorem 2.9, for $\lambda \notin \sigma_e(T_0)$, we have

$$\gamma(\lambda) = \gamma_a(\lambda) + \gamma_b(\lambda) - mn + \gamma_0(\lambda) = \gamma_b(\lambda) + \gamma_0(\lambda),$$

and hence $\gamma_0(\lambda) = 0$.

(iii) For $\lambda \notin \sigma_e(T_0)$, it follows from theorems 2.1 and 2.9 that

$$\gamma(\lambda) = \gamma_{\pm}(T_0) + \gamma_0(\lambda) = \gamma_a(\lambda) + \gamma_b(\lambda) - mn + \gamma_0(\lambda),$$

and hence

$$\gamma_a(\lambda) + \gamma_b(\lambda) = \gamma_{\pm}(T_0) + mn.$$

Since $mn = 2$ and $\gamma_{\pm}(T_0) > 0$, we know that at least one of $\gamma_a(\lambda)$ and $\gamma_b(\lambda)$ equals 2, and hence it follows from (ii) that $\gamma_0(\lambda) = 0$. □

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Appendix A. Definition of quasi-derivatives

Case 1 ($n = 2k$). If $k = 1$, then

$$\begin{aligned} y^{\{0\}} &= y, \\ y^{\{1\}} &= p_1 \frac{\mathrm{d}}{\mathrm{d}t} y^{\{0\}} - q_0 y^{\{0\}}, \\ y^{\{2\}} &= -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{1\}} - q_0^* p_1^{-1} y^{\{1\}} + (p_0 - q_0^* p_1^{-1} q_0) y^{\{0\}} = w\tau y. \end{aligned}$$

If $k \geqslant 2$, then

$$\begin{aligned} y^{\{j\}} &= y^{(j)} = \frac{\mathrm{d}^j}{\mathrm{d}t^j} y, \quad j = 0, \dots, k-1, \\ y^{\{k\}} &= p_k \frac{\mathrm{d}}{\mathrm{d}t} y^{\{k-1\}} - q_{k-1} y^{\{k-1\}}, \\ y^{\{k+1\}} &= -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{k\}} + p_{k-1} y^{\{k-1\}} - q_{k-1}^* \frac{\mathrm{d}}{\mathrm{d}t} y^{\{k-1\}} - q_{k-2} y^{\{k-2\}} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{k\}} - q_{k-1}^* p_k^{-1} y^{\{k\}} + (p_{k-1} - q_{k-1}^* p_k^{-1} q_{k-1}) y^{\{k-1\}} - q_{k-2} y^{\{k-2\}}, \\ y^{\{k+j\}} &= -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{k+j-1\}} + p_{k-j} y^{\{k-j\}} - q_{k-j}^* y^{\{k-j+1\}} - q_{k-j-1} y^{\{k-j-1\}}, \\ &\qquad\qquad j = 2, \dots, k-1, \\ y^{\{n\}} &= y^{\{2k\}} = -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{n-1\}} + p_0 y - q_0^* y^{\{1\}} = w\tau y. \end{aligned}$$

Case 2 ($n = 2k+1$). If $k = 0$ ($n = 1$), then

$$y^{\{0\}} = y,$$
$$y^{\{1\}} = \hat{q}_0 \frac{\mathrm{d}}{\mathrm{d}t} y^{\{0\}} + (q_0' + p_0) = w\tau y.$$

If $k \geqslant 1$, then

$$y^{\{j\}} = y^{(j)} = \frac{\mathrm{d}^j}{\mathrm{d}t^j} y, \quad j = 0, \dots, k,$$

$$y^{\{k+1\}} = \hat{q}_k \frac{\mathrm{d}}{\mathrm{d}t} y^{\{k\}} + (p_k + q_k') y^{\{k\}} - q_{k-1} y^{\{k-1\}}, \quad \hat{q}_k := q_k - q_k^*,$$

$$y^{\{k+j\}} = -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{k+j-1\}} - q_{k-j} y^{\{k-j\}} + p_{k-j+1} y^{\{k-j+1\}} - q_{k-j+1}^* y^{\{k-j+2\}}, \qquad j = 2, \dots, k,$$

$$y^{\{n\}} = y^{\{2k+1\}} = -\frac{\mathrm{d}}{\mathrm{d}t} y^{\{2k\}} + p_0 y - q_0^* y^{\{1\}} = w\tau y.$$

One uses these quasi-derivatives to transform the differential equation $(\tau - \lambda)y = f$ into a first-order system so that the initial-value problem has a concise form and is well posed.

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