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COMPUTING MINIMAL SIGNATURE OF COHERENT SYSTEMS THROUGH MATRIX-GEOMETRIC DISTRIBUTIONS

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Abstract

Signatures are useful in analyzing and evaluating coherent systems. However, their computation is a challenging problem, especially for complex coherent structures. In most cases the reliability of a binary coherent system can be linked to a tail probability associated with a properly defined waiting time random variable in a sequence of binary trials. In this paper we present a method for computing the minimal signature of a binary coherent system. Our method is based on matrix-geometric distributions. First, a proper matrix-geometric random variable corresponding to the system structure is found. Second, its probability generating function is obtained. Finally, the companion representation for the distribution of matrix-geometric distribution is used to obtain a matrix-based expression for the minimal signature of the coherent system. The results are also extended to a system with two types of components.

Keywords: Matrix-geometric distribution; minimal signature; probability generating function; reliability; signature

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1. Introduction

The concept of signature is a powerful and useful tool for reliability evaluation of coherent systems. It has been effectively used in various problem setups. Progress in signature-based reliability research appears in two specific dimensions. One is related to calculation of the signature, which is often challenging, and the other is concerned with its use in performance evaluation, including the assessment of ageing characteristics and stochastic comparison. Various types of signatures have been defined after the well-known Samaniego signature, which is defined as an n-dimensional vector whose ith coordinate is the probability that the ith failure of components causes the system to fail (see e.g. [18]). As is well known, the reliability of a coherent system can be written as a mixture using the signature vector when the system has exchangeable components.

Navarro and Rubio [15] obtained an algorithm to compute all the coherent systems with n components and their signatures. Using this algorithm it has been shown that there exist

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180 coherent systems with five components, and they computed their signatures. Gertsbakh, Shpungin, and Spizzichino [11] provided an efficient method to compute the signature of a coherent system by reducing the complexity of the large system via series, parallel and recurrent structures. Da, Xia, and Hu [4] derived basic formulae for computing the signature of a system which can be decomposed into two disjoint modules. Eryilmaz [6] considered the problem studied in [4] and developed a method that is based on combinatorial arguments. Marichal [12] provided conversion formulae between the signature and the reliability function. Marichal, Mathonet, and Spizzichino [13] represented the signature of a system in terms of the signature of its disjoint modules. Franko and Yalcin [10] computed the signatures of series and parallel systems consisting of non-disjoint modules. Da, Chan, and Xu [3] proposed a novel approach to computing the signature of a system consisting of subsystems with shared components. Yi and Cui [21] proposed a Markov-process-based method to compute the signature. Da, Xu, and Chan [5] developed an algorithm to compute the signature of a system with exchangeable components. The algorithm relies only on the information of minimal cut sets or minimal path sets. The signatures of particular coherent systems were computed using various techniques in Triantafyllou and Koutras [19], Eryilmaz and Zuo [8], Triantafyllou and Koutras [20], Eryilmaz and Tuncel [7], and Navarro and Spizzichino [16].

For a coherent system consisting of n independent and identical components with common reliability r, its reliability can be written as

$$R = \sum_{i=1}^{n} \omega_i r^i,$$

where the vector of coefficients $\omega = (\omega_1, \ldots, \omega_n)$ satisfying $\sum_{i=1}^{n} \omega_i = 1$ is called the minimal signature. For a coherent system with lifetime T and exchangeable component lifetimes $T_1, \ldots, T_n$, the survival function can also be written as

$$\mathbb{P}\{T > t\} = \sum_{i=1}^{n} \omega_i \, \mathbb{P}\{T_{1:i} > t\},$$

where $T_{1:i} = \min(T_1, \ldots, T_i)$, $i \geq 1$ (see e.g. [17]).

In this paper we present a method to compute the minimal signature of a coherent system. Our method is based on the probability generating function (p.g.f.) of the waiting time random variable that is concerned with the failure of the system. Let us consider the series system as a starting point. If the lifetimes have a continuous distribution, then the series system fails upon the failure of one of its components. Thus the waiting time random variable can be defined as the number of trials until the first failure in a sequence of Bernoulli trials with two possible outcomes as either success or failure. If the waiting time random variable is denoted by ξ, then its p.g.f. is

$$\mathbb{E}(z^{\xi}) = \frac{(1-r)z}{1-zr},$$

where r is the component reliability. That is, the random variable ξ has a geometric distribution with mean $1/(1-r)$. The reliability of the series system with n components is represented by $\mathbb{P}\{\xi > n\}$. In the general case when we have an arbitrary coherent system to hand, the corresponding waiting time random variable may be defined by a matrix-geometric distribution which has a rational p.g.f. Our approach is as follows. First, a proper matrix-geometric random variable corresponding to the system structure is found. Second, its probability generating

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function is obtained. Finally, the companion representation for the distribution of matrix-geometric distribution is used to obtain a matrix-based expression for the minimal signature of the coherent system. Our method is efficient, especially when the p.g.f. corresponding to the system failure is available. As will be illustrated, for some systems it is more practical to obtain the p.g.f. for the system failure.

The remainder of this paper is structured as follows. In Section 2 the notation and definitions are presented. Section 3 presents the main result on computing the minimal signature. In Section 4 we provide a method to compute the minimal signature of the series system that consists of two disjoint modules based on the minimal signatures of the modules. Section 5 contains results on systems with two types of components. Section 6 concludes the paper with some remarks. Proofs and computational details are presented in the Appendix.

2. Preliminaries

If the random variable ξ has a matrix-geometric distribution, then it has a rational p.g.f. which is given by

$$\psi(z) = \mathbb{E}(z^{\xi}) = \frac{c_1 z + \cdots + c_m z^m}{1 + d_1 z + \cdots + d_m z^m}. \tag{2.1}$$

Using (2.1), the probability mass function (p.m.f.) and survival function of ξ can be represented as

$$\mathbb{P}\{\xi = l\} = \mathbf{a}\mathbf{Q}^{l-1}\mathbf{u}'$$

and

$$\mathbb{P}\{\xi > l\} = \mathbf{a}\mathbf{Q}^{l}(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{u}',$$

where $\mathbf{a} = (1, 0, \ldots, 0)$ and

$$\mathbf{Q} = \begin{bmatrix} -d_1 & 0 & 0 & \cdots & 0 & 1 \\ -d_m & 0 & 0 & \cdots & 0 & 0 \\ -d_{m-1} & 1 & 0 & \cdots & 0 & 0 \\ -d_{m-2} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_2 & 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \mathbf{u}' = \begin{bmatrix} c_1 \\ c_m \\ c_{m-1} \\ c_{m-2} \\ \vdots \\ c_2 \end{bmatrix}$$

(see e.g. [2]).

Consider a coherent system with structure function $\phi(x_1, \ldots, x_n)$ such that $\phi(x_1, \ldots, x_n) = 1$ if the system works and $\phi(x_1, \ldots, x_n) = 0$ if the system has failed, where x_i denotes the state of the ith component with $x_i = 1$ if the ith component works and $x_i = 0$ if it has failed. Let $r = \mathbb{P}\{X_i = 1\}$ be the common reliability of its components, $i = 1, 2, \ldots, n$. Assume that the reliability of the system can be represented by the probability

$$\mathbb{P}\{\phi(X_1, \ldots, X_n) = 1\} = \mathbb{P}\{\xi > n\}, \tag{2.2}$$

where $\xi \sim MG(\mathbf{a}, \mathbf{Q}_r, \mathbf{u}_r)$. Currently we do not know if a random variable ξ satisfying (2.2) and having the distribution $MG(\mathbf{a}, \mathbf{Q}_r, \mathbf{u}_r)$ exists for any coherent system. However, as will be illustrated throughout the paper, there are many well-known and less-known coherent system models following the assumption.

The random variable ξ actually represents the waiting time for the occurrence of the system failure. More explicitly, it is the total number of trials until the occurrence of a specific event in a sequence of binary trials with possible outcomes as either 0 (failure) or 1 (working). Note that all or some of the coefficients $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ depend on component reliability r. That is, the probability (2.2) depends on r through the matrix $\mathbf{Q}_r$ and the vector $\mathbf{u}_r$. Therefore $\mathbf{Q}_r$ and $\mathbf{u}_r$ depend on r.

For an illustration, below we give examples for well-known coherent structures.

Example 2.1. Let $\phi(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$, i.e. the system has a series structure. Then $\psi(z)$ corresponds to the p.g.f. of the random variable ξ that denotes the number of trials until the first failure in a sequence of independent and identical binary trials. Clearly

$$\psi(z) = \frac{(1-r)z}{1-zr}.$$

Therefore $\xi \sim MG(1, r, 1-r)$.

Example 2.2. Let the system have k-out-of-n:F structure. Because the system fails as soon as k failures occur, the random variable ξ denotes the waiting time for a total of k failures in a binary sequence of two possible outcomes as either 0 (failure) or 1 (working). Thus the random variable ξ is the sum of k independent geometric variables, and

$$\psi(z) = \left[\frac{(1-r)z}{1-zr}\right]^k = \frac{(1-r)^k z^k}{1+\sum_{i=1}^{k}(-1)^i\binom{k}{i}r^i z^i}.$$

Therefore $\xi \sim MG(\mathbf{a}, \mathbf{Q}_r, \mathbf{u}_r)$ with $\mathbf{a} = (1, 0, \ldots, 0)$ and

$$\mathbf{Q}_r = \begin{bmatrix} kr & 0 & 0 & \cdots & 0 & 1 \\ (-1)^{k+1}r^k & 0 & 0 & \cdots & 0 & 0 \\ (-1)^k\binom{k}{k-1}r^{k-1} & 1 & 0 & \cdots & 0 & 0 \\ (-1)^{k-1}\binom{k}{k-2}r^{k-2} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\binom{k}{2}r^2 & 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \mathbf{u}_r' = \begin{bmatrix} 0 \\ (1-r)^k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

3. Computing the minimal signature

In this section we develop a method for computing the minimal signature of a coherent system by utilizing matrix-geometric distributions that were mentioned in the previous section.

Theorem 3.1. *Let $\phi(x_1, \ldots, x_n)$ be the structure function of a binary coherent system and let r be the common reliability of its components. Suppose that the reliability of the system is represented by $P\{\xi > n\}$, where ξ is a discrete random variable having a matrix-geometric distribution with p.g.f.*

$$\psi(z) = \frac{c_1 z + \cdots + c_m z^m}{1 + d_1 z + \cdots + d_m z^m}.$$

Then the minimal signature of the system with structure function ϕ can be computed from

$$\omega = \mathbf{A}^{-1}\mathbf{b},$$

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where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{2} & \frac{1}{2^2} & \cdots & \frac{1}{2^n} \\ \vdots & & \ddots & \\ \frac{1}{2^{n-1}} & \frac{1}{2^{2(n-1)}} & \cdots & \frac{1}{2^{n(n-1)}} \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 1 \\ a\Delta_1^n(\mathbf{I} - \Delta_1)^{-1}\mathbf{d}_1' \\ \vdots \\ a\Delta_{n-1}^n(\mathbf{I} - \Delta_{n-1})^{-1}\mathbf{d}_{n-1}' \end{bmatrix},$$

where $\boldsymbol{\Delta}_t$ *and* $\mathbf{d}_t$ *are obtained by replacing* r *in* $\mathbf{Q}_r$ *and* $\mathbf{u}_r$ *with* $1/2^t$, $t = 1, 2, \ldots, n-1$.

Proof. Because the minimal signature only depends on the system structure, without loss of generality, assume that the system with lifetime $T = \phi(T_1, \ldots, T_n)$ consists of independent and identically distributed components with common lifetime distribution $F(t) = 1 - (1-p)^t$, $t = 1, 2, \ldots, n$. That is, the lifetimes of components follow a geometric distribution with mean $1/p$. Then

$$p(t) = \mathbb{P}\{T > t\} = \sum_{i=1}^{n} \omega_i[(1-p)^t]^i = \omega_1(1-p)^t + \omega_2(1-p)^{2t} + \cdots + \omega_n(1-p)^{nt}.$$

Clearly

$$\begin{aligned} p(0) &= 1, \\ p(1) &= \omega_1(1-p) + \omega_2(1-p)^2 + \cdots + \omega_n(1-p)^n \\ &\vdots \\ p(n-1) &= \omega_1(1-p)^{n-1} + \omega_2(1-p)^{2(n-1)} + \cdots + \omega_n(1-p)^{n(n-1)}. \end{aligned}$$

Thus the minimal signature $(\omega_1, \omega_2, \ldots, \omega_n)$ is obtained from

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ (1-p) & (1-p)^2 & \cdots & (1-p)^n \\ \vdots & & \ddots & \\ (1-p)^{n-1} & (1-p)^{2(n-1)} & \cdots & (1-p)^{n(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ p(1) \\ \vdots \\ p(n-1) \end{bmatrix}.$$

Because $r = \bar{F}(t)$, from equation (2.2) we have

$$p(t) = \mathbb{P}\{T > t\} = \mathbf{a}\mathbf{Q}_{\bar{F}(t)}^n(\mathbf{I} - \mathbf{Q}_{\bar{F}(t)})^{-1}\mathbf{u}_{\bar{F}(t)}' = a\boldsymbol{\Delta}_t^n(\mathbf{I} - \boldsymbol{\Delta}_t)^{-1}\mathbf{d}_t',$$

where $\boldsymbol{\Delta}_t$ and $\mathbf{d}_t$ are obtained by replacing r in $\mathbf{Q}_r$ and $\mathbf{u}_r$ with $\bar{F}(t) = (1-p)^t$, $t = 1, 2, \ldots, n-1$. The result now follows upon choosing $p = \frac{1}{2}$. □

Remark 3.1. Theorem 3.1 incorporates an instrumental geometric random variable for modeling lifetimes of components. This is done to obtain simple terms in the matrix **A** and the

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vector $\mathbf{b}$. Any other lifetime distribution can also be used since $\mathbf{A}^{-1}\mathbf{b}$ is independent of the component lifetime distribution.

As is clear from Theorem 3.1, the computation of the minimal signature vector requires the p.g.f. of the random variable ξ corresponding to the system structure. For some systems, the derivation of the p.g.f. for the occurrence of the system failure is simpler than obtaining an expression for the system reliability. Consider the following examples.

Example 3.1. The (n,f,k) system consists of n components ordered in a line, and it fails if and only if there exist at least f failed components or at least k consecutive failed components. The p.g.f. of the waiting time for the occurrence of the $(n,f,2)$ system failure can be obtained from Proposition 1 of Triantafyllou and Koutras [20]. Note that [20, Proposition 1] includes the p.g.f. for the reliability function defined by

$$R(z;r) = \sum_{n=0}^{\infty} z^n \, \mathbb{P}\{\xi > n\}.$$

The p.g.f. for the waiting time for the occurrence of the system failure can be obtained from $\psi(z) = 1 - R(z;r)(1-z)$ (see Appendix A) and is given by

$$\psi(z) = 1 - \frac{(1-rz)^f - ((1-r)z)^2 [\sum_{i=0}^{f-3} (r(1-r)z^2)^i (1-rz)^{f-i-1} + (r(1-r)z^2)^{f-2}]}{(1-rz)^f}.$$

If in particular $f = 3$, then

$$\psi(z) = \frac{(1-r)^2 z^2 - 2r(1-r)^2 z^3 + r(1-r)^2 z^4}{1 - 3rz + 3r^2 z^2 - r^3 z^3}.$$

Thus $\xi \sim MG(\mathbf{a},\mathbf{Q}_r,\mathbf{u}_r)$ with $\mathbf{a} = (1, 0, 0, 0)$ and

$$\mathbf{Q}_r = \begin{bmatrix} 3r & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ r^3 & 1 & 0 & 0 \\ -3r^2 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{u}_r' = \begin{bmatrix} 0 \\ r(1-r)^2 \\ -2r(1-r)^2 \\ (1-r)^2 \end{bmatrix}.$$

Using Theorem 3.1, the minimal signature of the $(n,3,2)$ system can be computed from

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{2} & \frac{1}{2^2} & \cdots & \frac{1}{2^n} \\ \vdots & & \ddots & \\ \frac{1}{2^{n-1}} & \frac{1}{2^{2(n-1)}} & \cdots & \frac{1}{2^{n(n-1)}} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ p(1) \\ \vdots \\ p(n-1) \end{bmatrix},$$

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TABLE 1. Minimal signature of the (*n*,3,2) system.

n	$\boldsymbol{\omega}$
5	(0, 0, 6, −7, 2)
6	(0, 0, 0, 10, −14, 5)
7	(0, 0, 0, 0, 15, −23, 9)
8	(0, 0, 0, 0, 0, 21, −34, 14)
9	(0, 0, 0, 0, 0, 0, 28, −47, 20)
10	(0, 0, 0, 0, 0, 0, 0, 36, −62, 27)

where

$$p(t) = (1, 0, 0, 0) \begin{bmatrix} 3\frac{1}{2^t} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \left(\frac{1}{2^t}\right)^3 & 1 & 0 & 0 \\ -3\left(\frac{1}{2^t}\right)^2 & 0 & 1 & 0 \end{bmatrix}^n$$

$$\times \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3\frac{1}{2^t} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \left(\frac{1}{2^t}\right)^3 & 1 & 0 & 0 \\ -3\left(\frac{1}{2^t}\right)^2 & 0 & 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{2^t}\left(1 - \frac{1}{2^t}\right)^2 \\ -2\frac{1}{2^t}\left(1 - \frac{1}{2^t}\right)^2 \\ \left(1 - \frac{1}{2^t}\right)^2 \end{bmatrix},$$

for $t = 1, 2, \ldots, n-1$. In Table 1 we present $\omega = (\omega_1, \ldots, \omega_n)$ for the $(n, 3, 2)$ system for selected values of n.

Example 3.2. A consecutive-k-out-of-n:F system with sparse d is a system that consists of n linearly ordered components such that the system fails if and only if there exist at least k consecutive failed components with sparse d, i.e. there must be k or more failures and sparse d or less if the system fails [22]. The p.g.f. of the waiting time for the occurrence of the system failure has been obtained as [14]

$$\psi(z) = \frac{((1-r)z)^k(1-(rz)^{d+1})^{k-1}(1-z+(1-r)z(rz)^{d+1})}{(1-rz)((1-z)(1-rz)^{k-1} + ((1-r)z)^k(rz)^{d+1}(1-(rz)^{d+1})^{k-1})}.$$

Let $k = 2$ and $d = 1$. Then

$$\psi(z) = \frac{c_1 z + \cdots + c_7 z^7}{1 + d_1 z + \cdots + d_7 z^7},$$

where

$$c_1 = 0, \quad c_2 = (1-r)^2, \quad c_3 = -(1-r)^2, \quad c_4 = -r^2(1-r)^2,$$
$$c_5 = r^2(1-r)^2(2-r), \quad c_6 = 0, \quad c_7 = -r^4(1-r)^3,$$
$$d_1 = -(1+2r), \quad d_2 = r(r+2), \quad d_3 = -r^2, \quad d_4 = r^2(1-r)^2,$$
$$d_5 = -r^3(1-r)^2, \quad d_6 = -r^4(1-r)^2, \quad d_7 = r^5(1-r)^2.$$

If, for example, $n = 5$, then using Theorem 3.1, the minimal signature of the consecutive-2-out-of-5:F system with sparse $d = 1$ is obtained as $(0, 0, 3, -1, -1)$.

4. Minimal signature of a series system with two modules

Consider a coherent system consisting of $n + m$ components with the index set of components $C = \{1, \ldots, n + m\}$. Suppose that the system with the component index set C consists of two disjoint modules with respective component index sets $\{1, \ldots, n\}$ and $\{n + 1, \ldots, n + m\}$ and structure functions ϕ_1 and ϕ_2. If the overall system has a series structure, i.e. the disjoint modules are serially connected, then the system's reliability is represented as

$$\mathbb{P}\{\phi_1(X_1, \ldots, X_n) = 1, \phi_2(X_{n+1}, \ldots, X_{n+m}) = 1\} = \mathbb{P}\{\xi_1 > n\}\,\mathbb{P}\{\xi_2 > m\},$$

where $\xi_i \sim MG(\mathbf{a}^{(i)}, \mathbf{Q}_r^{(i)}, \mathbf{u}_r^{(i)})$, $i = 1, 2$. The problem is to compute the minimal signature of the series system with structure function $\min(\phi_1(x_1, \ldots, x_n), \phi_2(x_{n+1}, \ldots, x_{n+m}))$ based on the minimal signatures of the systems with structure functions $\phi_1(x_1, \ldots, x_n)$ and $\phi_2(x_{n+1}, \ldots, x_{n+m})$. This problem has been considered in the literature by using a different approach that is based on the number of path sets of the structures (see e.g. [6] and [4]).

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, respectively, denote the minimal signatures of the systems with structure functions ϕ_1 and ϕ_2. Then the reliability of the system with structure function $\min(\phi_1(x_1, \ldots, x_n), \phi_2(x_{n+1}, \ldots, x_{n+m}))$ can be written as

$$\mathbb{P}\{\xi_s > n + m\} = \sum_{i=1}^{n} \alpha_i r^i \sum_{j=1}^{m} \beta_j r^j = (\alpha \otimes \beta)(\mathbf{r}_1' \otimes \mathbf{r}_2') = \sum_{i=1}^{n+m} \omega_i r^i,$$

where $\otimes$ denotes the Kronecker product and $\mathbf{r}_1 = [r, r^2, \ldots, r^n]$, $\mathbf{r}_2 = [r, r^2, \ldots, r^m]$.

For given α and β, the minimal signature $(\omega_1, \omega_2, \ldots, \omega_{n+m})$ can be computed by collecting the coefficients of the terms r^i in the expansion of $(\alpha \otimes \beta)(\mathbf{r}_1' \otimes \mathbf{r}_2')$. This can be done using the computer code presented in Appendix B.

Example 4.1. Let ϕ_1 and ϕ_2 be the structure functions of 2-out-of-3:F and 3-out-of-4:F systems, respectively. Then, from Example 2.2 and Theorem 3.1, the minimal signatures are obtained as $\alpha = (0, 3, -2)$ and $\beta = (0, 0, 4, -3)$. Then

$$\alpha \otimes \beta = (0, 0, 0, 0, 0, 0, 12, -9, 0, 0, -8, 6)$$

and

$$\mathbf{r}_1 \otimes \mathbf{r}_2 = (r^2, r^3, r^4, r^5, r^3, r^4, r^5, r^6, r^4, r^5, r^6, r^7).$$

Using the algorithm presented in Appendix B, the minimal signature of the system which is a series connection of 2-out-of-3:F and 3-out-of-4:F systems is obtained as

$$\omega = (0, 0, 0, 0, 12, -17, 6).$$

5. Systems with two types of components

Consider a coherent system that consists of two types of components. Let n_i denote the number of components of type i, $i = 1, 2$. Then the reliability of the system can be represented by

$$\mathbb{P}\{\xi > n\} = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \omega_{m_1,m_2} r_1^{m_1} r_2^{m_2},$$

where r_i denotes the reliability of components of type i, $i = 1, 2$ and $\xi \sim MG(\mathbf{a},\mathbf{Q}_{r_1,r_2},\mathbf{u}_{r_1,r_2})$. The matrix $\mathbf{Q}_{r_1,r_2}$ and the vector $\mathbf{u}_{r_1,r_2}$ are constructed with the help of the coefficients in the probability generating function of ξ. The vector

$$\omega = (\omega_{0,0}, \omega_{0,1}, \ldots, \omega_{0,n_2}, \ldots, \omega_{n_1,0}, \omega_{n_1,1}, \ldots, \omega_{n_1,n_2})$$

of $(n_1 + 1) \times (n_2 + 1)$ elements represents the minimal signature of the system (see e.g. [9]). Using the method presented in Theorem 3.1, the minimal signature of the system can be computed from

$$\omega = \mathbf{A}^{-1}\mathbf{b}$$

for $i = 1, 2, \ldots, (n_1 + 1) \times (n_2 + 1)$, and the (i,j)th element of the matrix $\mathbf{A}$ is

$$a_{ij} = \begin{cases} (1-p_2)^{(j-1)(i-1)} & \text{if } j = 1, 2, \ldots, n_2+1, \\ (1-p_1)^{i-1}(1-p_2)^{(j-(n_2+2))(i-1)} & \text{if } j = n_2+2, \ldots, 2n_2+2, \\ (1-p_1)^{2(i-1)}(1-p_2)^{(j-(2n_2+3))(i-1)} & \text{if } j = 2n_2+3, \ldots, 3n_2+3, \\ \vdots & \\ (1-p_1)^{n_1(i-1)}(1-p_2)^{(j-(n_1n_2+n_1+1))(i-1)} & \text{if } j = n_1n_2+n_1+1, \ldots, n_1n_2+n_1+n_2+1. \end{cases}$$

The vector $\mathbf{b}$ is given by

$$\mathbf{b} = \begin{bmatrix} 1 \\ p(1) \\ \vdots \\ p(n_1n_2+n_1+n_2) \end{bmatrix},$$

where

$$p(t) = \mathbf{a}\boldsymbol{\Delta}_t^n(\mathbf{I} - \boldsymbol{\Delta}_t)^{-1}\mathbf{d}_t',$$

and $\boldsymbol{\Delta}_t$ and $\mathbf{d}_t$ are obtained by replacing r_1 and r_2 in $\mathbf{Q}_{r_1,r_2}$ and $\mathbf{u}_{r_1,r_2}$ with $(1-p_1)^t$ and $(1-p_2)^t$, $t = 1, 2, \ldots, n_1n_2 + n_1 + n_2$.

Example 5.1. For a 2-out-of-n:F system consisting of n_1 components of type 1 and $n - n_1$ components of type 2, the p.g.f. of the waiting time for the system failure is (see Appendix C for the derivation)

$$\psi(z) = \frac{(1-r_1)^2[r_1z^2 - n_1r_1^{n_1}z^{n_1+1} + (n_1-1)r_1^{n_1+1}z^{n_1+2}]}{r_1(1-zr_1)^2} + \frac{n_1r_1^{n_1-1}(1-r_1)(1-r_2)z^{n_1+1}}{1-zr_2} + \frac{r_1^{n_1}(1-r_2)^2z^{n_1+2}}{(1-zr_2)^2}. \tag{5.1}$$

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Note that if $n_1 = 0$ and $r_1 = r_2 = r$ in (5.1), then

$$\psi(z) = \left[\frac{(1-r)z}{1-zr}\right]^2,$$

which is the p.g.f. corresponding to the 2-out-of-n:F system with a single type of components.

Example 5.2. For a 2-out-of-3:F system, let $n_1 = 1$. Then, from (5.1), we have

$$\psi(z) = \frac{(1-r_1)(1-r_2)z^2 + (1-r_2)(r_1-r_2)z^3}{1-2r_2z+r_2^2z^2}.$$

Thus $\xi \sim MG(\mathbf{a},\mathbf{Q}_{r_1,r_2},\mathbf{u}_{r_1,r_2})$ with $\mathbf{a} = (1, 0, 0)$ and

$$\mathbf{Q}_{r_1,r_2} = \begin{bmatrix} 2r_2 & 0 & 1 \\ 0 & 0 & 0 \\ -r_2^2 & 1 & 0 \end{bmatrix}, \quad \mathbf{u}'_{r_1,r_2} = \begin{bmatrix} 0 \\ (1-r_2)(r_1-r_2) \\ (1-r_1)(1-r_2) \end{bmatrix}.$$

Using the method, the minimal signature of the 2-out-of-3:F system consisting of $n_1 = 1$ component of type 1 and $n - n_1 = 2$ components of type 2 is obtained as

$$\omega = (\omega_{0,0}, \omega_{0,1}, \omega_{0,2}, \omega_{1,0}, \omega_{1,1}, \omega_{1,2}) = (0, 0, 1, 0, 2, -2).$$

Example 5.3. A consecutive-2-out-of-n:F system is a system that consists of n linearly ordered components, and fails if and only if there exist at least two consecutive failed components. Assume that the first n_1 components have common reliability r_1 and the remaining $n - n_1$ components have common reliability r_2. Then the p.g.f. of the random variable ξ which denotes the waiting time until two consecutive failures is given by the following equations when $n_1 = 1$, $n_1 = 2$, and $n_1 = 3$ (see Appendix D for the proofs):

$$\psi(z) = \frac{(1-r_1)(1-r_2)z^2 + (1-r_2)(r_1-r_2)z^3}{1-r_2z+r_2(r_2-1)z^2} \quad \text{for } n_1 = 1, \tag{5.2}$$

$$\psi(z) = \frac{(1-r_1)^2z^2 + (1-r_1)(r_1-r_2)z^3 + (1-r_2)(r_1-r_2)z^4}{1-r_2z+r_2(r_2-1)z^2} \quad \text{for } n_1 = 2, \tag{5.3}$$

and for $n_1 = 3$

$$\begin{aligned}\psi(z) = \frac{1}{1-r_2z+r_2(r_2-1)z^2}&[(1-r_1)^2z^2 + (1-r_1)^2(r_1-r_2)z^3 \\ &+ (1-r_1)(r_1-r_2)(1-r_2(1-r_1))z^4 \\ &+ (r_1(r_1-r_2)(r_1-2)(r_2-1))z^5]\end{aligned} \tag{5.4}$$

The following result is useful for obtaining the p.g.f. corresponding to a series system of two or more modules.

Proposition 5.1. *The p.g.f. of the waiting time for the failure of the two-component series system, when one component is of type 1 with reliability r_1 and the other component is of type 2 with reliability r_2, is*

$$\psi(z) = \frac{(1-r_1)z + (r_1-r_2)z^2}{1-zr_2}. \tag{5.5}$$

Proof. The result is immediate from

$$\psi(z) = z(1 - r_1) + zr_1\varphi(z),$$

where $\varphi(z)$ denotes the p.g.f. of the geometric distribution with probability of success $1 - r_2$, that is,

$$\varphi(z) = \frac{z(1 - r_2)}{1 - zr_2}.$$

□

Example 5.4. Consider the system with structure function

$$\phi(x_1, x_2, x_3, x_4) = \min(x_1, \max(x_2, x_3, x_4)). \tag{5.6}$$

Assume that component 1 has reliability p_1 and components 2, 3, and 4 have common reliability p_2. Taking $r_1 = p_1$ and $r_2 = 1 - (1 - p_2)^3$ in (5.5), the p.g.f. of the waiting time for the failure of the system with structure function (5.6) is obtained as

$$\psi(z) = \frac{(1 - p_1)z + (p_1 - 1 + (1 - p_2)^3)z^2}{1 - z(1 - (1 - p_2)^3)}.$$

As illustrated by Example 5.4, the result presented in Proposition 5.2 is useful for obtaining the p.g.f. of a series system consisting of disjoint modules. There are coherent systems which can be written as a series system of two or more systems which may contain common components. In the following, we obtain the p.g.f. of the waiting time for the failure of the series system consisting of two modules that have one common component. The representation of a coherent system as a series system of two modules considerably reduces the complexity in computing the signature.

Proposition 5.2. *Let ϕ_1 and ϕ_2 be the structure functions of two coherent systems of sizes n and m, respectively. Suppose that component i is a common component of both systems. If $n + m$ components have common reliability value p, then the p.g.f. corresponding to the failure of the series system of the two systems ϕ_1 and ϕ_2 is given by*

$$\psi(z) = \frac{c_1 z + c_2 z^2 + c_3 z^3}{1 + d_1 z + d_2 z^2 + d_3 z^3},$$

where

$$\begin{aligned}
c_1 &= 1 - pr_1 - (1 - p)r_1^*,\\
c_2 &= pr_2^*(r_1 - 1) + p(r_1 - r_2) + (1 - p)[r_1^* - r_2^* + r_2(r_1^* - 1)],\\
c_3 &= pr_2^*(r_2 - r_1) + (1 - p)r_2(r_2^* - r_1^*),\\
d_1 &= -(r_2 + r_2^*),\\
d_2 &= r_2 r_2^*,\\
d_3 &= 0
\end{aligned}$$

for $r_1^*, r_2^* > 0$, where r_j (r_j^*) denotes the reliability of the system ϕ_j when the component i works (fails), $j = 1, 2$.

Proof. Manifestly

$$\psi(z) = \mathbb{E}(z^\xi \mid X_i = 1)\,\mathbb{P}\{X_i = 1\} + \mathbb{E}(z^\xi \mid X_i = 0)\,\mathbb{P}\{X_i = 0\} = p\psi_1(z) + (1 - p)\psi_2(z),$$

where $\psi_1(z) = \mathbb{E}(z^\xi \mid X_i = 1)$ and $\psi_2(z) = \mathbb{E}(z^\xi \mid X_i = 0)$. When the state of the component i is fixed, the entire system becomes a series system with two disjoint modules such that the first

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module has reliability r_1 (r_1^*) and the second module has reliability r_2 (r_2^*) when the component is working (has failed). Thus, from Proposition 5.1,

$$\psi_1(z) = \begin{cases} 1 & \text{if } r_1 = 1 \text{ and } r_2 = 1, \\ \dfrac{(1-r_1)z + (r_1 - r_2)z^2}{1 - zr_2} & \text{otherwise,} \end{cases}$$

and

$$\psi_2(z) = \frac{(1-r_1^*)z + (r_1^* - r_2^*)z^2}{1 - zr_2^*}.$$

Thus the proof is complete. □

Corollary 5.1. *Let ξ denote the waiting time for the failure of the series system of two coherent systems ϕ_1 and ϕ_2. If component i is a common component of systems ϕ_1 and ϕ_2, then $\xi \sim MG(\mathbf{a},\mathbf{Q},\mathbf{u})$ with $\mathbf{a} = (1, 0, 0)$ and*

$$\mathbf{Q} = \begin{bmatrix} (r_2 + r_2^*) & 0 & 1 \\ 0 & 0 & 0 \\ -r_2 r_2^* & 1 & 0 \end{bmatrix}, \quad \mathbf{u}' = \begin{bmatrix} c_1 \\ c_3 \\ c_2 \end{bmatrix},$$

where c_1, c_2, and c_3 are given by Proposition 5.2.

Example 5.5. Consider the system with structure function

$$\phi(x_1, x_2, x_3) = \min(\max(x_1, x_2), \max(x_2, x_3)).$$

In this case $\phi_1(x_1, x_2) = \max(x_1, x_2)$ and $\phi_2(x_2, x_3) = \max(x_2, x_3)$, and component 2 is common to both systems. Assume that the components have common reliability p. Clearly $r_1 = r_2 = 1$ and $r_1^* = r_2^* = p$. Thus, from Corollary 5.1, the p.g.f. of the system is obtained as

$$\psi(z) = p + \frac{z(1-p)}{1-zp}(1-p) = \frac{p + z(1-2p)}{1-zp}.$$

Example 5.6. Consider the system with structure function

$$\phi(x_1, x_2, x_3, x_4) = \min(\max(x_1, x_2), \max(\min(x_2, x_3), x_4)).$$

In this case $\phi_1(x_1, x_2) = \max(x_1, x_2)$ and $\phi_2(x_2, x_3, x_4) = \max(\min(x_2, x_3), x_4)$, and component 2 is shared by two systems. Assume that the components have common reliability p. Clearly $r_1 = 1$, $r_2 = 2p - p^2$, and $r_1^* = r_2^* = p$. Thus, from Corollary 5.1, the p.g.f. of the system is obtained as

$$\psi(z) = \frac{c_1 z + c_2 z^2 + c_3 z^3}{1 + d_1 z + d_2 z^2 + d_3 z^3},$$

where

$$c_1 = (1-p)^2, \quad c_2 = -p(1-p)^3, \quad c_3 = -p^2(1-p)^2,$$

$$d_1 = p^2 - 3p, \quad d_2 = p(2p - p^2), \quad d_3 = 0.$$

Example 5.7. Consider the system with structure function

$$\phi(x_1, x_2, x_3, x_4, x_5) = \min(\min(x_1, \max(x_2, x_3)), \max(\min(x_2, x_4), x_5)).$$

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In this case $\phi_1(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$ and $\phi_2(x_2, x_4, x_5) = \max(\min(x_2, x_4), x_5)$, and component 2 is common to both systems. Assume that the components have common reliability p. Manifestly $r_1 = p$, $r_2 = 2p - p^2$, and $r_1^* = p^2$, $r_2^* = p$. Thus, from Corollary 5.1, the p.g.f. of the system is found to be

$$\psi(z) = \frac{c_1 z + c_2 z^2 + c_3 z^3}{1 + d_1 z + d_2 z^2 + d_3 z^3},$$

where

$$c_1 = 1 - 2p^2 + p^3, \quad c_2 = p(p^4 - 3p^3 + 2p^2 + 3p - 3), \quad c_3 = p^2(2 - 4p + 3p^2 - p^3),$$
$$d_1 = p^2 - 3p, \quad d_2 = p(2p - p^2), \quad d_3 = 0.$$

6. Summary and conclusions

In this paper we have developed a method for computing the minimal signature of a coherent system by using matrix-geometric distributions. The method is mainly based on using the p.g.f. of a properly defined waiting time random variable in a sequence of binary trials. If the p.g.f. is rational, then the minimal signature is computed using distributional properties of the matrix-geometric distributions. In some setups (especially if the system is not defined in terms of its structure function), derivation of the probability generating function corresponding to the system failure is easier than obtaining the reliability function directly (see e.g. Examples 3.1 and 3.2). Indeed, the construction of the minimal path sets representation for the systems given in Examples 3.1 and 3.2 is quite a difficult task because of the complexity of the structures.

The proposed method has been modified to calculate the minimal signature of a system with two types of components. Moreover, in Proposition 5.1 and 5.2, respectively, we obtained the p.g.f. of a series system consisting of disjoint models and the p.g.f. of a series system of two modules containing a common component. As has been illustrated by several examples, the method is useful not only for a system with a single type of components but also for a system consisting of two (and possibly more) types of components.

The modification of the method for multi-state systems will be among our future research problems.

Appendix A

Manifestly

$$\begin{aligned}
\psi(z) &= \sum_{n=1}^{\infty} z^n \, \mathbb{P}\{\xi = n\} \\
&= \sum_{n=1}^{\infty} z^n [\mathbb{P}\{\xi > n-1\} - \mathbb{P}\{\xi > n\}] \\
&= z \sum_{n=1}^{\infty} z^{n-1} \, \mathbb{P}\{\xi > n-1\} - \left[\sum_{n=0}^{\infty} z^n \, \mathbb{P}\{\xi > n\} - 1 \right] \\
&= zR(z;r) - R(z;r) + 1 \\
&= 1 - R(z;r)(1 - z).
\end{aligned}$$

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Appendix B

Inputs

n, m, α, β

Process

```
γ = (α ⊗ β)
r1 = [r, r^2, . . . , r^n]
r2 = [r, r^2, . . . , r^m]
d = r1 ⊗ r2
for j = 2 : n + m
  ω(j) = 0
  for i = 1 : nm
    if d(i) = r^j
    ω(j) = ω(j) + 1
  end
end
end
```

Output

ω

Appendix C

Proof of (5.1). For the 2-out-of-n:F system, equation (5.1) corresponds to the probability generating function of the waiting time for two failed components. That is, $\psi(z) = E(z^{\xi})$, where ξ is the number of trials until two zeros (failures) in a sequence of binary trials with two possible outcomes as either 0 (failure) or 1 (working). If the two failures appear in the n_1th trial or before, then we have a sequence in the form

$$\underbrace{\ldots 0}_{i \text{ trials}},$$

where the first $i-1$ trials include exactly one failure, $i = 2, 3, \ldots, n_1$. The contribution of such a sequence to the p.g.f. is

$$z(1-r_1)z(1-r_1)\sum_{i=2}^{n}(i-1)z^{i-2}r_1^{i-2}. \tag{C.1}$$

On the other hand, if the two failures occur after the n_1th trial, then the contribution to the p.g.f. is

$$n_1 z(1-r_1)(zr_1)^{n_1-1}\left[\frac{z(1-r_2)}{1-zr_2}\right] + (zr_1)^{n_1}\left[\frac{z(1-r_2)}{1-zr_2}\right]^2, \tag{C.2}$$

where the first term is associated with the case when the first n_1 trials include only one failure, and the second term corresponds to the case when the first n_1 trials do not include any failure. Equation (5.1) is now obtained by taking the sums of (C.1) and (C.2). □

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Appendix D

Proofs of (5.2)–(5.4). For the consecutive-2-out-of-n:F system, the random variable ξ represents the number of trials until the occurrence of two consecutive failures (zeros) in a sequence of binary trials. For example, if the outcomes are 10110100, then $\xi = 8$.

For a consecutive-2-out-of-n:F system consisting of $n_1 = 1$ components of type 1 with common reliability r_1 and $n-1$ components of type 2 with common reliability r_2, the typical sequences for the waiting time ξ until failure of the system are

$$00, \quad 0\underbrace{1\ldots00}_{\text{type 2}}, \quad 1\underbrace{\ldots00}_{\text{type 2}}.$$

The contributions of these sequences to the p.g.f. of ξ are $z(1-r_1)z(1-r_2)$, $z(1-r_1)zr_2\lambda(z)$, and $zr_1\lambda(z)$, where $\lambda(z)$ is the p.g.f. of the waiting time until two consecutive failures of type 2. Here $\lambda(z)$ is actually the p.g.f. of the geometric distribution of order 2, and is given by (see e.g. [1])

$$\lambda(z) = \frac{((1-r_2)z)^2 - ((1-r_2)z)^3}{1 - z + r_2(1-r_2)^2 z^3}.$$

Thus, if $n_1 = 1$, then

$$\psi(z) = z(1-r_1)z(1-r_2) + z(1-r_1)zr_2\lambda(z) + zr_1\lambda(z),$$

which gives (5.2). Similarly, if $n_1 = 2$, then the typical sequences for the waiting time ξ until failure of the system are

$$00, \quad 100, \quad 01\underbrace{\ldots00}_{\text{type 2}}, \quad 101\underbrace{\ldots00}_{\text{type 2}}, \quad 11\underbrace{\ldots00}_{\text{type 2}}.$$

Considering each sequence of binary trials, we obtain

$$\psi(z) = z^2(1-r_1)^2 + z^3 r_1(1-r_1)(1-r_2) + z^2 r_1(1-r_1)\lambda(z) + z^3 r_1(1-r_1)r_2\lambda(z) + z^2 r_1^2\lambda(z),$$

which gives (5.3) after some simple manipulations. Equation (5.4) can be obtained similarly by considering different sequences that correspond to system failure. □

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