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LARGE DEVIATIONS FOR THE STOCHASTIC PREDATOR–PREY MODEL WITH NONLINEAR FUNCTIONAL RESPONSE

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Abstract

In this paper we consider a diffusive stochastic predator–prey model with a nonlinear functional response and the randomness is assumed to be of Gaussian nature. A large deviation principle is established for solution processes of the considered model by implementing the weak convergence technique.

Keywords: Predator–prey model; population dynamics; large deviation; stochastic partial differential equation

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1. Introduction

It is significant to study the qualitative properties of biological models so as to have an insight into the persistence, structure, and dynamics of biological communities comprising of a number of species. The analysis of species models mainly contributes to the population dynamics and also to develop strategies for controlling epidemics of infectious diseases. The interactions among the species and their dependence with each other have to be permitted in the modeling process. The pioneering model for the predator–prey interactions was developed by Lotka and Volterra in the 1920s and since then several models have been proposed by considering the possible factors affecting the predator–prey populations, and the qualitative properties have been investigated. For instance, the existence of solutions for a predator–prey model with mixed boundary conditions was established by Shangerganesh and Balachandran [27]. Sambath and Balachandran [24] analyzed the spatiotemporal dynamics of a ratio-dependent predator–prey model with cross diffusion by taking into account the proportion of prey refuge. The stability and bifurcation analyses of diffusive models with different functional responses have also been studied by many authors; see, for example, [25] and [28].

Population models are commonly formed in the deterministic sense, but it is more appropriate to consider the natural stochastic behavior of the system leading to stochastic differential equations (SDEs). Indeed, it can be observed that the predator and prey populations admit a fascinating interplay between the nonlinear dynamical behavior of the evolutionary forces affecting the population dynamics and the stochastic nature of the interactions. The effect of random perturbations on the system is often measured by means of the approximations of the central limit theorem. At instances where the normal approximations using the central limit theorem are not appropriate, large deviation approximations assert to be good as in the case of

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DNA sequencing and in the observation of clusters of mildly similar proteins; see [2] and the references therein.

Large deviation theory is the study of events with extremely small chances of occurrence. Those highly improbable events may have a huge impact during their occurrence and so the study of their qualitative and quantitative properties is indeed essential. A concise introductory study of large deviations and their applications can be found in [10] and [31]. The study of large deviations for distributions of SDEs is of interest to many researchers. The theory is mainly concentrated in estimating the rate at which the occurrence probability becomes negligible. The study of the large deviation principle (LDP) to SDEs was initiated by Varadhan [31] for diffusion processes and it was carried over to a general class of diffusion processes by Freidlin [13]. Because of the different nature of nonlinearities affecting the system, each equation has to be studied individually for the LDP. Considerable work has been carried out concerning the LDP for differential equations and advancements have also been made with delay type equations; see, for example, [21]. In [6], Budhiraja and Dupuis established a variational representation for positive functionals of Brownian motion applicable to the study of large deviations for a variety of differential equations. It is consequential to exert the variational representation technique to study the large deviations for solution processes of SDEs (see [26] and [30]) and indeed the main result to be established in this paper relies on this technique.

The theory of large deviations enables us to quantify the deviation of the SDE from its corresponding deterministic equation. The theory has been successfully applied to problems from many areas ranging from physics to biology. Florens-Landais and Pham [12] established large deviations for an Ornstein–Uhlenbeck model. A moderate deviation principle for the stochastic Lotka–Volterra model was established by Klebaner *et al.* [16]. The studies on large deviations help to describe the phenomena in the rare occurrence of events as in the case of interacting particle models [3] and protein folding [32]. Large deviations for stochastic hybrid systems have been established by Bressloff and Newby [5] using the path-integral representation. The theory can also be applied to a vast range of problems to analyze the asymptotic behavior of solutions, for instance, to find the time at which the solution vanishes or reaches a desired state. In the case of population models, the theory helps in predicting the likely path of extinction of a certain species subject to random disturbances; see, for example, [15], [17], and [22]. Also the long time behavior of the epidemic processes can be predicted and the time at which an epidemic becomes extinct can be estimated. Recently, Kratz *et al.* [18] analyzed the time of exit from the domain of attraction of a stable equilibrium for an SIR epidemic model.

In this paper we study the large deviation problem for the predator–prey model considered by Li [19] with Gaussian random behavior. The problem is studied in its abstract setting and, hence, enables us to implement the result to a wide range of problems having the same abstract structure. The Gaussian randomness is considered in the abstract framework and it is to be mentioned here that the abstract space-valued Brownian motions, Gaussian randomness, or Wiener processes were first introduced by Gross as a tool to investigate the Dirichlet problems; see [14]. To establish the LDP for the considered abstract problem, we use the weak convergence approach by implementing the conclusions of Budhiraja and Dupuis [6]. This method involves the verification of a sequence of solutions of the associated control equation for its compactness and the corresponding perturbed equation for a weak convergence result.

2. Problem formulation

Let $\mathcal{O} \subset \mathbb{R}^2$ be a two dimensional bounded domain on which the prey and predator interactions are modeled as a differential equation and the densities of the two populations are analyzed within the time interval $[0, T]$ for finite T. We consider the stochastic predator–prey model which can be written in the abstract formulation on a suitable function space as

$$\mathrm{d}u + Au\,\mathrm{d}t = f(u)\,\mathrm{d}t + \sigma(t, u)\,\mathrm{d}W(t), \tag{1}$$

with the initial condition $u(0) = u_0$ and Neumann boundary condition $\partial u/\partial \nu = 0$, where ν denotes the outward normal unit vector emanating from the boundary $\partial\mathcal{O}$. Physically interpreting, the Neumann conditions mean that the predator or prey populations do not migrate outside the bounded domain $\mathcal{O}$. In (1), u is a vector representing the predator–prey population densities, A is the spatial diffusion operator, and $f(\cdot)$ a nonlinear functional response. The probabilistic or random factors are approximated to be of Gaussian nature and $W(\cdot)$ represents the Wiener process with its random noise coefficient denoted by $\sigma(t, \cdot)$. The study of qualitative properties in the abstract setting enables us to bring a variety of problems under one shell. Indeed, in our case it is possible to apply the result to problems with functional responses of similar characteristics.

Throughout the paper, we intend to work with the Lebesgue and Sobolev spaces—a precise introduction and significance of these spaces can be gained from [1]. For the Sobolev space $\mathbb{H}^1(\mathcal{O})$, it follows, from Poincare's inequality and the boundedness of $\mathcal{O}$, that the norm $(\int_{\mathcal{O}} |\nabla u(x)|^2\,\mathrm{d}x)^{1/2}$ is equivalent to the predefined $\mathbb{H}^1(\mathcal{O})$ norm. We make use of this norm equivalence in all the proceeding analysis.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an increasing family $\{\mathcal{F}_t\}_{0\le t\le T}$ of sub-sigma fields of $\mathcal{F}$ satisfying the right continuity and probability completeness with respect to $\mathbb{P}$. Let us take

$$W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix},$$

where $W_1(t)$ and $W_2(t)$ are independent $\mathbb{L}^2(\mathcal{O})$-space-valued Wiener processes. For more details on stochastic space settings, Hilbert space-valued Wiener processes, and their integrals, we refer the reader to [8] and [9]; for instance, see [8, Section 3.2] for details on building SDEs on abstract Hilbert spaces. Let the random coefficient be of the form

$$\sigma(t, u) = \begin{pmatrix} \sigma_1(t, u_1, u_2) \\ \sigma_2(t, u_1, u_2) \end{pmatrix},$$

where u_1 and u_2 denote the prey and predator populations respectively.

Let Q be the covariance operator of the Wiener process $W(t)$ with the assumption that it is strictly positive, symmetric, and a trace class operator on $\mathbb{L}^2(\mathcal{O})$. Define the space $\mathcal{H}_0 = Q^{1/2}\mathbb{L}^2(\mathcal{O})$. Then $\mathcal{H}_0$ is a Hilbert space with the inner product

$$(u, v)_0 = (Q^{-1/2}u, Q^{-1/2}v) \quad \text{for all } u, v \in \mathcal{H}_0.$$

The norm in the $\mathcal{H}_0$ space will be $\|u\|_0^2 = (u, u)_0$. Since Q is trace class, the identity mapping from $\mathcal{H}_0$ to $\mathbb{L}^2(\mathcal{O})$ is a Hilbert–Schmidt operator. In the proof of our main result on the LDP, the Hilbert–Schmidt embedding $\mathcal{H}_0$ in $\mathbb{L}^2(\mathcal{O})$ has a major consequence in that it turns weakly converging sequences in $\mathcal{H}_0$ to strong convergence; see [6, Section 2]. In addition, it is to be remarked that the space $\mathcal{H}_0$ is closely associated to the so called Cameron–Martin space (for details, see [4] and the references therein). Let $L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))$ denote the space of all linear bounded operators from $\mathbb{L}^2(\mathcal{O})$ to $\mathbb{L}^2(\mathcal{O})$ and let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert–Schmidt operator from $\mathbb{L}^2(\mathcal{O})$ to $\mathbb{L}^2(\mathcal{O})$.

The predator–prey model considered by Li [19] in the nondimensional form with spatial diffusion and nonlinear Holling type III functional response is given by

$$\frac{\partial u_1}{\partial t} - \eta_1 \Delta u_1 = u_1(\alpha - u_1) - \frac{\beta u_1^2 u_2}{1 + u_1^2}, \tag{2}$$

$$\frac{\partial u_2}{\partial t} - \eta_2 \Delta u_2 = \frac{\gamma u_1^2 u_2}{1 + u_1^2} - \delta u_2, \tag{3}$$

where the initial populations of prey and predator are denoted by $u_{1,0}$ and $u_{2,0}$, and it is assumed that the two populations satisfy the Neumann boundary conditions. Also η_1 and η_2 are positive coefficients describing the spatial diffusion of prey and predator respectively; α is the carrying capacity of the prey. The positive parameters β, γ, and δ are obtained in the nondimensionalizing process of the system and are representations of products and ratios of the intrinsic growth rate and carrying capacity of prey, number of newly born predators for each captured prey, and death rate of the predators; see [19]. The model is featured in the sense that the absence of predator permits the prey population to grow at a positive rate but with the restriction imposed by its carrying capacity. In order to improve the accuracy of the mathematical analysis, the factors of probabilistic occurrence ought to be involved in the model. If we define

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad A = \begin{pmatrix} -\eta_1 \Delta - \alpha & 0 \\ 0 & -\eta_2 \Delta + \delta \end{pmatrix}, \tag{4}$$

$$f(u) = \begin{pmatrix} -u_1^2 - \dfrac{\beta u_1^2 u_2}{1 + u_1^2} \\ \dfrac{\gamma u_1^2 u_2}{1 + u_1^2} \end{pmatrix}, \qquad u(0) = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} = u_0, \tag{5}$$

then the SDE corresponding to the predator–prey system in (2) and (3) can be framed in its abstract formulation as (1) with initial and Neumann boundary conditions. Let u^ε denote the solution of (1) with the noise coefficient perturbed by a small parameter $\varepsilon > 0$ as

$$\mathrm{d}u^\varepsilon + Au^\varepsilon \,\mathrm{d}t = f(u^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma(t, u^\varepsilon)\,\mathrm{d}W(t). \tag{6}$$

We shall consider the abstract equation (6) with A and $f(\cdot)$ defined as in (4) and (5), and study the LDP for the corresponding solution process u^ε. Let us impose the following assumptions on the multiplicative noise coefficient $\sigma\colon [0, T] \times \mathbb{H}^1(\mathcal{O}) \to L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))$.

(H1) The function $\sigma \in \mathbb{C}([0, T] \times \mathbb{H}^1(\mathcal{O}); L_Q(\mathcal{H}_0; \mathbb{L}^2(\mathcal{O})))$.

(H2) For all $t \in (0, T)$, there exists a positive constant C_1 such that, for all $u, v \in \mathbb{H}^1$,

$$\|\sigma(t, u) - \sigma(t, v)\|_{L_Q} \le C_1 \|\nabla(u - v)\|_{\mathbb{L}^2}. \tag{7}$$

(H3) For all $t \in (0, T)$ and $u \in \mathbb{H}^1$, the following linear growth condition holds:

$$\|\sigma(t, u)\|_{L_Q}^2 \le C_2(1 + \|\nabla u\|_{\mathbb{L}^2}^2), \tag{8}$$

where C_2 is a positive constant.

Under the above assumptions, there exists a strong positive solution for the system (6); see [8]. Let $\mathbb{H}^1(\mathcal{O};\mathbb{R}^{2+})$ be the Sobolev space $\mathbb{H}^1(\mathcal{O})$ of all $\mathbb{R}^{2+}$-valued functions, where $\mathbb{R}^{2+}$ denotes the first quadrant of $\mathbb{R}^2$. We need the following preliminary lemmas stating the coercivity and Lipschitz continuity of the linear operator A and nonlinear functional response $f(\cdot)$.

Lemma 1. *The following coercive inequality holds for all* $u=(u_1,u_2)\in\mathbb{H}^1(\mathcal{O};\mathbb{R}^{2+})$:

$$(Au,u)\geq \eta\|\nabla u\|_{\mathbb{L}^2}^2-\alpha\|u_1\|_{\mathbb{L}^2}^2, \tag{9}$$

where $\eta=\eta_1\wedge\eta_2$ *(that is,* $\eta=\min\{\eta_1,\eta_2\}$*).*

Proof. The proof follows at once from the definition of the linear operator A. □

Lemma 2. *For* $u, v\in\mathbb{H}^1(\mathcal{O};\mathbb{R}^{2+})$*, where* $u=(u_1,u_2)$, $v=(v_1,v_2)$*, the following hold.*

(i) *Boundedness:*

$$(f(u),u)\leq\frac{\beta}{2}\|u_1\|_{\mathbb{L}^2}^2+\bigg(\frac{\beta}{2}+\gamma\bigg)\|u_2\|_{\mathbb{L}^2}^2. \tag{10}$$

(ii) *Lipschitz continuity: when* $z=(z_1,z_2):=u-v$,

$$\begin{aligned}2(f(u)-f(v),z)\leq{}&\eta\|\nabla z\|_{\mathbb{L}^2}^2+\frac{4}{\eta}\|z_1\|_{\mathbb{L}^2}^2(1+\beta^2)(\|u\|_{\mathbb{L}^2}^2+\|v\|_{\mathbb{L}^2}^2)+\frac{2\beta^2C^2}{\eta}\|z\|_{\mathbb{L}^2}^2\\&+\frac{\gamma^2}{\eta}\|z\|_{\mathbb{L}^2}^2(\|u_2\|_{\mathbb{L}^2}^2+\|v_2\|_{\mathbb{L}^2}^2)+\frac{8\gamma^2C_a}{\eta}\|z_2\|_{\mathbb{L}^2}^2,\end{aligned} \tag{11}$$

where C_a *is a constant taking the value of* $(\text{area}(\mathcal{O}))^{1/2}$.

Proof. The proof can be found in Appendix A. □

3. The LDP

We implement the theory developed by Budhiraja and Dupuis [6] to establish the Laplace principle for the family of solutions $\{u^\varepsilon:\varepsilon>0\}$ of (6). Indeed the Laplace principle and the LDP are equivalent when the underlying space is Polish; see Theorems 1.2.1 and 1.2.3 in [11]. Let $\mathcal{Z}$ denote the space $\mathbb{C}([0,T];\mathbb{L}^2(\mathcal{O}))\cap\mathbb{L}^2((0,T);\mathbb{H}^1(\mathcal{O}))$. Then $\mathcal{Z}$ is complete and a separable metric space and, hence, Polish; see [1]. The solution u^ε of the stochastic predator–prey model in (6) can be written as $\mathcal{G}^\varepsilon(W(\cdot))$ for a Borel measurable function $\mathcal{G}^\varepsilon\colon\mathbb{C}([0,T];\mathcal{H}_0)\to\mathcal{Z}$; see [20], [30], and the references therein. We are interested in the LDP for the family $u^\varepsilon=\mathcal{G}^\varepsilon(W(\cdot))$.

Let $\mathcal{A}$ denote the class of $\mathcal{H}_0$-valued $\{\mathcal{F}_t\}$-predictable processes ϕ which satisfy

$$\int_0^T\|\phi(s)\|_0^2\,\mathrm{d}s<\infty\quad\text{almost surely.}$$

Also let $S_M=\{k\in\mathbb{L}^2((0,T);\mathcal{H}_0)\colon\int_0^T\|k(s)\|_0^2\,\mathrm{d}s\leq M\}$. Then S_M endowed with the weak topology is a compact Polish space. Define $\mathcal{A}_M=\{\Phi\in\mathcal{A}\colon\Phi(\omega)\in S_M,\text{ almost surely}\}$. The proof of our main result on large deviations relies upon the following theorem formulated by Budhiraja and Dupuis [6, Theorem 4.4].

Theorem 1. *Suppose that there exists a measurable map* $\mathcal{G}^0\colon\mathbb{C}([0,T];\mathcal{H}_0)\to\mathcal{Z}$ *such that the following two conditions hold.*

(i) *Let* $\{k^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$ *for some* $M < \infty$. *If* k^ε *converges to* k *in distribution as* S_M*-valued random elements, then*

$$\mathcal{G}^\varepsilon\bigg(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^{\cdot} k^\varepsilon(s)\,\mathrm{d}s\bigg) \to \mathcal{G}^0\bigg(\int_0^{\cdot} k(s)\,\mathrm{d}s\bigg) \quad \textit{in distribution as } \varepsilon \to 0.$$

(ii) *For each* $M < \infty$, *the set* $K_M = \{\mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s) : k \in S_M\}$ *is a compact subset of* $\mathcal{Z}$.

Then the family $\{u^\varepsilon,\ \varepsilon > 0\}$ *satisfies the Laplace principle in* $\mathcal{Z}$ *with the rate function* I *given by*

$$I(g) = \inf\bigg\{\frac{1}{2}\int_0^T \|k(t)\|^2\,\mathrm{d}t;\ u_k = g \text{ and } k \in L^2((0,T);\mathcal{H}_0)\bigg\}$$

for each $g \in \mathcal{Z}$ *with the convention that the infimum of an empty set is* ∞.

Hence, establishing a Laplace principle is now simplified to that of satisfying assumptions (i) and (ii) for our system. The main theorem of this paper is as follows.

Theorem 2. *Let* $\{u^\varepsilon(\cdot) : \varepsilon > 0\}$ *denote the strong solution of the stochastic system (6). Then with the assumptions (H1)–(H3) on* σ, *the family* $\{u^\varepsilon\}$ *satisfies the LDP in* $\mathcal{Z} = \mathbb{C}([0,T];\mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2((0,T);\mathbb{H}^1(\mathcal{O}))$ *with a good rate function*

$$I(g) = \inf_{\{k \in \mathbb{L}^2((0,T);\,\mathcal{H}_0) :\ g = \mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s)\}} \bigg\{\frac{1}{2}\int_0^T \|k(s)\|_0^2\,\mathrm{d}s\bigg\},$$

where the infimum over an empty set is taken as ∞ *and* $\mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s)$ *denotes the solution* u_k *of the system*

$$\mathrm{d}u_k + Au_k\,\mathrm{d}t = f(u_k)\,\mathrm{d}t + \sigma(t,u_k)k\,\mathrm{d}t, \tag{12}$$

with $u_k(0) = u_0$ *and* $k \in \mathcal{A}_M$ *for some* $M < \infty$.

For the controlled equation (12), the existence of a nonnegative solution $u_k = (u_{1,k}, u_{2,k}) \in \mathcal{Z}$ can be attained using results similar to the ones obtained by Chen and Jungel [7]. The uniqueness of the solution is assured by the Lipschitz continuity of the noise coefficient $\sigma(t,\cdot)$ and the nonlinearity $f(\cdot)$. Indeed, if there were two solutions, say u_k and $\tilde{u}_k$, for (12) with the same initial condition, then taking $w_k = u_k - \tilde{u}_k$ results in

$$\mathrm{d}w_k + Aw_k\,\mathrm{d}t = (f(u_k) - f(\tilde{u}_k))\,\mathrm{d}t + (\sigma(t,u) - \sigma(t,\tilde{u}))k\,\mathrm{d}t,$$

with $w_k(0) = 0$. Taking the inner product of the above equation with w_k and then integrating and simplifying further using the Lipschitz continuity of the noise coefficient $\sigma(t,\cdot)$ given by (7) and that of the nonlinearity $f(\cdot)$ given by (11), we finally obtain

$$\sup_{0\le t\le T} \|w_k(t)\|_{\mathbb{L}^2}^2 + \eta\int_0^T \|\nabla w_k(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \le C\int_0^T \|w_k(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s$$

for some positive constant $C < \infty$. Applying Gronwall's inequality yields the desired uniqueness result. Let us now consider the following controlled stochastic equation associated with (6) with control $k^\varepsilon \in \mathcal{A}_M$, $\varepsilon > 0$:

$$\mathrm{d}u^\varepsilon_{k^\varepsilon} + Au^\varepsilon_{k^\varepsilon}\,\mathrm{d}t = f(u^\varepsilon_{k^\varepsilon})\,\mathrm{d}t + \sigma(t,u^\varepsilon_{k^\varepsilon})k^\varepsilon\,\mathrm{d}t + \sqrt{\varepsilon}\sigma(t,u^\varepsilon_{k^\varepsilon})\,\mathrm{d}W(t). \tag{13}$$

The existence of a unique strong solution to the above equation follows at once from the existence of a solution to (6) by implementing the Girsanov's theorem; see [29] and [30] for a proof of a similar kind.

Lemma 3. *For $k^\varepsilon \in \mathcal{A}_M$, $0 < M < \infty$, and $\varepsilon > 0$, there exists a unique strong solution to the stochastic controlled equation (13) with $u^\varepsilon_{k^\varepsilon}(0) = u_0$.*

Proof. Since $k^\varepsilon \in \mathcal{A}_M$, $0 < M < \infty$, and $\varepsilon > 0$, by virtue of Girsanov's theorem (see [9]), $\tilde{W}(\cdot) := W(\cdot) + (1/\sqrt{\varepsilon})\int_0^\cdot k^\varepsilon(s)\,\mathrm{d}s$ is also a Wiener process with covariance operator Q under the probability measure

$$\mathrm{d}\tilde{\mathbb{P}}^\varepsilon_{k^\varepsilon} := \exp\bigg\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T k^\varepsilon(s)\,\mathrm{d}W(s) - \frac{1}{2\varepsilon}\int_0^T \|k^\varepsilon(s)\|_0^2\,\mathrm{d}s\bigg\}\,\mathrm{d}\mathbb{P}$$

and so there exists a solution to (6) with W replaced by $\tilde{W}$. This, in turn, implies the existence of solutions to the stochastic controlled system (13) under the probability measure $\mathrm{d}\mathbb{P}$. Likewise, the uniqueness of a solution to (13) also follows by making use of the same Girsanov argument. This completes the proof. □

Hence, the solution to (13) can be written as $\mathcal{G}^\varepsilon(W(\cdot)+(1/\sqrt{\varepsilon})\int_0^\cdot k^\varepsilon(s)\,\mathrm{d}s)$. Before proceeding to the verification of assumptions (i) and (ii) of Theorem 1, we first put forth the following preliminary lemma which will aid in estimating the solution processes.

Lemma 4. *For the solution process $u^\varepsilon_{k^\varepsilon} = (u^\varepsilon_{1,k^\varepsilon}, u^\varepsilon_{2,k^\varepsilon})$ of the perturbed stochastic equation (13) with $0 < \varepsilon < \eta/C_2 \wedge 1/8C_2^2 \wedge \eta^2/8C_2^2$ (C_2 being the constant in (8) of (H3)), the following energy estimate holds:*

$$\mathbb{E}\Big\{\sup_{0\le t\le T}\|u^\varepsilon_{k^\varepsilon}(t)\|^2_{\mathbb{L}^2}\Big\} + \eta\mathbb{E}\int_0^T \|\nabla u^\varepsilon_{k^\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s \le K, \tag{14}$$

where K is a positive constant defined by

$$K = C\bigg\{\|u_0\|^2_{\mathbb{L}^2} + \frac{3\eta T}{2}\bigg\}\exp\bigg(\tilde{C}T + \frac{2C_2}{\eta}\int_0^T \|k(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\bigg) \tag{15}$$

and is independent of ε.

Proof. Applying Itô's formula (see [9]) to the function $|u^\varepsilon_{k^\varepsilon}(t)|^2$ and integrating over time from 0 to t, we obtain

$$\begin{aligned}
&\|u^\varepsilon_{k^\varepsilon}(t)\|^2_{\mathbb{L}^2} + 2\int_0^t (Au^\varepsilon_{k^\varepsilon}(s), u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}s \\
&\quad = \|u_0\|^2_{\mathbb{L}^2} + 2\int_0^t (f(u^\varepsilon_{k^\varepsilon}(s)), u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}s + 2\int_0^t (\sigma(s, u^\varepsilon_{k^\varepsilon}(s))k(s), u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}s \\
&\qquad + \varepsilon\int_0^t \mathrm{tr}(\sigma(s, u^\varepsilon_{k^\varepsilon}(s))Q\sigma^*(s, u^\varepsilon_{k^\varepsilon}(s)))\,\mathrm{d}s + 2\sqrt{\varepsilon}\int_0^t (u^\varepsilon_{k^\varepsilon}(s), \sigma(s, u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}W(s)),
\end{aligned}$$

where σ^* denotes the conjugate of the noise coefficient σ, that is, $(\sigma u, v) = (u, \sigma^* v)$ for $u, v \in \mathbb{L}^2(\mathcal{O})$. Define the stopping time $\tau_N = \inf\{t\colon \|u^\varepsilon_{k^\varepsilon}(t)\|^2_{\mathbb{L}^2} + \int_0^t \|u^\varepsilon_{k^\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s > N\}$.

Using (9), (10), the Cauchy–Schwarz and Young inequalities, taking the supremum on both sides, and then taking the expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(t)\|^2_{\mathbb{L}^2}\Big\}+2\eta\mathbb{E}\int_0^{T\wedge\tau_N}\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad\le\|u_0\|^2_{\mathbb{L}^2}+\tilde{C}\mathbb{E}\int_0^{T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{\eta}{2}\int_0^{T\wedge\tau_N}(1+\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s+\frac{2C_2}{\eta}\int_0^{T\wedge\tau_N}\|k(s)\|^2_{\mathbb{L}^2}\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\varepsilon\mathbb{E}\int_0^{T\wedge\tau_N}\operatorname{tr}(\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))Q\sigma^*(s,u^{\varepsilon}_{k^{\varepsilon}}(s)))\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\Big|\int_0^t(u^{\varepsilon}_{k^{\varepsilon}}(s),\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))\,\mathrm{d}W(s))\Big|\Big\},
\end{aligned}\tag{16}$$

where $\tilde{C}=\max\{2\alpha,\gamma\}+\beta/2$. Applying the Burkholder–Davis–Gundy inequality (see [23]) for the stochastic integral term on the right-hand side, we obtain

$$\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\Big|\int_0^t(u^{\varepsilon}_{k^{\varepsilon}}(s),\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))\,\mathrm{d}W(s))\Big|\Big\}\\
&\quad\le 2\sqrt{2\varepsilon}C_2\mathbb{E}\Big\{\Big(\int_0^{T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}(1+\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s\Big)^{1/2}\Big\}\\
&\quad\le\sqrt{2\varepsilon}C_2\Big(\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(t)\|^2_{\mathbb{L}^2}\Big\}+\mathbb{E}\int_0^{T\wedge\tau_N}\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+T\Big),
\end{aligned}$$

where we have also made use of the linear growth property of $\sigma(t,\cdot)$ given by (8) and Young's inequality. Using this in (16) and choosing $\varepsilon<\eta/2C_2\wedge 1/8C_2^2\wedge\eta^2/8C_2^2$, we end up with the estimate

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(t)\|^2_{\mathbb{L}^2}\Big\}+\eta\mathbb{E}\int_0^{T\wedge\tau_N}\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad\le 2\Big\{\|u_0\|^2_{\mathbb{L}^2}+\tilde{C}\mathbb{E}\int_0^{T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{2C_2}{\eta}\int_0^{T\wedge\tau_N}\|k(s)\|^2_{\mathbb{L}^2}\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\frac{3\eta T}{2}\Big\}.
\end{aligned}$$

Finally, an application of Gronwall's inequality yields

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N}\|u^{\varepsilon}_{k^{\varepsilon}}(t)\|^2_{\mathbb{L}^2}\Big\}+\eta\mathbb{E}\int_0^{T\wedge\tau_N}\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad\le C\Big\{\|u_0\|^2_{\mathbb{L}^2}+\frac{3\eta T}{2}\Big\}\exp\Big(\tilde{C}T+\frac{2C_2}{\eta}\int_0^T\|k(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big).
\end{aligned}$$

From this estimate, we observe that, as $N\to\infty$, $T\wedge\tau_N$ increases to T and, hence, we finally obtain the required estimate (14) with K as in (15). □

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Similar to the estimate attained in Lemma 4, it can be established that the following energy estimate holds for the solution u_k of (12):

$$\sup_{0\le t\le T} \|u_k(t)\|_{\mathbb{L}^2}^2 + \eta \int_0^T \|\nabla u_k(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le \tilde{K} \tag{17}$$

for some positive constant $\tilde{K} > 0$. The compactness argument (assumption (ii) in Theorem 1) is established by the following proposition.

Proposition 1. (Compactness) Let M be any finite fixed positive number. Let $K_M := \{\mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s) \colon k \in S_M\}$, where $\mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s)$ denotes the unique solution u_k in $\mathcal{Z}$ of the controlled equation (12) with $u_k(0) = u_0 \in \mathbb{L}^2(\mathcal{O})$. Then K_M is compact in $\mathcal{Z}$.

Proof. Let $\{u_{k_n}\} \in K_M$ denote the solution of (12) with the control k replaced by $k_n \in S_M$, $n \in \mathbb{N}$. Since S_M is weakly compact, there exists a subsequence of $\{k_n\}$ (still denoted by $\{k_n\}$) which converges to a limit k weakly in $\mathbb{L}^2((0,T);\mathcal{H}_0)$. Take $w_n = u_{k_n} - u_k$. Then

$$\mathrm{d}w_n + Aw_n\,\mathrm{d}t = [f(u_{k_n}) - f(u_k)]\,\mathrm{d}t + [\sigma(t,u_{k_n})k_n - \sigma(t,u_k)k]\,\mathrm{d}t. \tag{18}$$

Integrating and taking the inner product of (18) with w_n, we obtain

$$\begin{aligned}
&\|w_n(t)\|_{\mathbb{L}^2}^2 + 2\int_0^t (Aw_n(s), w_n(s))\,\mathrm{d}s \\
&\quad = 2\int_0^t (f(u_{k_n}(s)) - f(u_k(s)), w_n(s))\,\mathrm{d}s \\
&\qquad + 2\int_0^t (\sigma(s,u_{k_n}(s))k_n(s) - \sigma(s,u_k(s))k(s), w_n(s))\,\mathrm{d}s.
\end{aligned} \tag{19}$$

Here, the integrand

$$\begin{aligned}
&2(\sigma(s,u_{k_n})k_n - \sigma(s,u_k)k, w_n) \\
&\quad = 2((\sigma(s,u_{k_n}) - \sigma(s,u_k))k_n, w_n) + 2(\sigma(s,u_k)(k_n - k), w_n) \\
&\quad \le 2\|\sigma(s,u_{k_n}) - \sigma(s,u_k)\|_{L_Q}\|k_n\|_0\|w_n\|_{\mathbb{L}^2} + 2\|\sigma(s,u_k)(k_n-k)\|_{\mathbb{L}^2}\|w_n\|_{\mathbb{L}^2} \\
&\quad \le \frac{\eta}{2}\|\nabla w_n\|_{\mathbb{L}^2}^2 + \frac{2C_1}{\eta}\|k_n\|_0^2\|w_n\|_{\mathbb{L}^2}^2 + \|\sigma(s,u_k)(k_n-k)\|_{\mathbb{L}^2}^2 + \|w_n\|_{\mathbb{L}^2}^2.
\end{aligned}$$

Using this inequality in (19) along with the coercivity of the linear operator A given by (9) and the Lipschitz continuity of the nonlinear functional response $f(\cdot)$ given by (11), we have, after simplification,

$$\begin{aligned}
&\|w_n(t)\|_{\mathbb{L}^2}^2 + \frac{\eta}{2}\int_0^t \|\nabla w_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\quad \le K_1\int_0^t \|w_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + K_2\int_0^t \|w_n(s)\|_{\mathbb{L}^2}^2(\|u_{k_n}(s)\|_{\mathbb{L}^2}^2 + \|u_k(s)\|_{\mathbb{L}^2}^2)\,\mathrm{d}s \\
&\qquad + \frac{2C_1}{\eta}\int_0^t \|k_n(s)\|_0^2\|w_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \int_0^t \|\sigma(s,u_k(s))(k_n(s)-k(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s,
\end{aligned}$$

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where $K_1 = \max\{2\alpha, 8\gamma^2C^2/\eta\} + 2\beta^2C^2/\eta + 1$ and $K_2 = 4(1+\beta^2)/\eta + \gamma^2/\eta$. Finally, an application of Gronwall's inequality results in

$$\begin{aligned}
&\|w_n(t)\|_{\mathbb{L}^2}^2 + \frac{\eta}{2}\int_0^t \|\nabla w_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\quad \le C\int_0^t \|\sigma(s, u_k(s))(k_n(s) - k(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\qquad \times \exp\biggl(K_1T + K_2\int_0^T (\|u_{k_n}(s)\|_{\mathbb{L}^2}^2 + \|u_k(s)\|_{\mathbb{L}^2}^2)\,\mathrm{d}s + \frac{2C_1}{\eta}\int_0^T \|k_n(s)\|_0^2\,\mathrm{d}s\biggr),
\end{aligned}$$

where C is an arbitrary positive constant. The exponential term is bounded by virtue of (17) as k and k_n are controls from S_M. Also, since $k_n \to k$ weakly in $\mathbb{L}^2((0,T); \mathcal{H}_0)$ as $n \to \infty$ and σ is a Hilbert–Schmidt operator and, hence, compact, we have $\sigma k_n \to \sigma k$ strongly in $\mathbb{L}^2((0,T); \mathbb{L}^2(\mathcal{O}))$ and so $w_n = u_{k_n} - u_k \to 0$ in $\mathcal{Z}$. □

Proposition 2. (Weak convergence.) *Let $\{k^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$ converge in distribution to k with respect to the weak topology on $\mathbb{L}^2((0,T); \mathcal{H}_0)$. Then $\mathcal{G}^\varepsilon(W(\cdot) + (1/\sqrt{\varepsilon})\int_0^{\cdot} k^\varepsilon(s)\,\mathrm{d}s)$ converges in distribution to $\mathcal{G}^0(\int_0^{\cdot} k(s)\,\mathrm{d}s)$ in $\mathcal{Z}$ as $\varepsilon \to 0$.*

Proof. Let k^ε converge to k in distribution as random elements taking values in S_M, where S_M is equipped with the weak topology. Let $w^\varepsilon = u^\varepsilon_{k^\varepsilon} - u_k$, where u_k and $u^\varepsilon_{k^\varepsilon}$ are solutions of (12) and (13), respectively. Then w^ε corresponds to the solution of the SDE

$$\mathrm{d}w^\varepsilon + Aw^\varepsilon\,\mathrm{d}t = [f(u^\varepsilon_{k^\varepsilon}) - f(u_k)]\,\mathrm{d}t + [\sigma(t, u^\varepsilon_{k^\varepsilon})k^\varepsilon - \sigma(t, u_k)k]\,\mathrm{d}t + \sqrt{\varepsilon}\sigma(t, u^\varepsilon_{k^\varepsilon})\,\mathrm{d}W(t).$$

An application of Itô's formula [9] yields

$$\begin{aligned}
&\|w^\varepsilon(t)\|_{\mathbb{L}^2}^2 + 2\int_0^t (Aw^\varepsilon(s), w^\varepsilon(s))\,\mathrm{d}s \\
&\quad = 2\int_0^t (f(u^\varepsilon_{k^\varepsilon}(s)) - f(u_k(s)), w^\varepsilon(s))\,\mathrm{d}s \\
&\qquad + 2\int_0^t (\sigma(s, u^\varepsilon_{k^\varepsilon}(s))k^\varepsilon(s) - \sigma(s, k(s))k(s), w^\varepsilon(s))\,\mathrm{d}s \\
&\qquad + \varepsilon\int_0^t \operatorname{tr}(\sigma(s, u^\varepsilon_{k^\varepsilon}(s))Q\sigma^*(s, u^\varepsilon_{k^\varepsilon}(s)))\,\mathrm{d}s + 2\sqrt{\varepsilon}\int_0^t (w^\varepsilon(s), \sigma(s, u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}W(s)).
\end{aligned}$$

Using the coercivity of the diffusion operator A and the Lipschitz continuity of f given by (9) and (11), the above equation simplifies to

$$\begin{aligned}
&\|w^\varepsilon(t)\|_{\mathbb{L}^2}^2 + \eta\int_0^t \|\nabla w^\varepsilon(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\quad \le 2\alpha\int_0^t \|w_1^\varepsilon(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\qquad + \frac{4(1+\beta^2)+\gamma^2}{\eta}\int_0^t \|w^\varepsilon(s)\|_{\mathbb{L}^2}^2(\|u^\varepsilon_{k^\varepsilon}(s)\|_{\mathbb{L}^2}^2 + \|u_k(s)\|_{\mathbb{L}^2}^2)\,\mathrm{d}s \\
&\qquad + \frac{2\beta^2C^2}{\eta}\int_0^t \|w^\varepsilon(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{8\gamma^2C^2}{\eta}\int_0^t \|w_2^\varepsilon(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s
\end{aligned}$$

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$$
\begin{aligned}
&+2\int_0^t ((\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))-\sigma(s,u_k(s)))k^{\varepsilon}(s),w^{\varepsilon}(s))\,\mathrm{d}s\\
&+2\int_0^t (\sigma(s,u_k(s))(k^{\varepsilon}(s)-k(s)),w^{\varepsilon}(s))\,\mathrm{d}s\\
&+\varepsilon C_2\int_0^t (1+\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s+2\sqrt{\varepsilon}\bigg|\int_0^t (w^{\varepsilon}(s),\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))\,\mathrm{d}W(s))\bigg|.
\end{aligned}
$$

As in the proof of compactness, applying the Cauchy–Schwarz and Young inequalities yields

$$
\begin{aligned}
&\|w^{\varepsilon}(t)\|^2_{\mathbb{L}^2}+\eta\int_0^t \|\nabla w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad\le 2\alpha\int_0^t \|w^{\varepsilon}_1(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\frac{2\beta^2C^2}{\eta}\int_0^t \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{4(1+\beta^2)+\gamma^2}{\eta}\int_0^t \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}(\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}+\|u_k(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s\\
&\qquad+\frac{8\gamma^2C^2}{\eta}\int_0^t \|w^{\varepsilon}_2(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{\eta}{2}\int_0^t \|\nabla w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\frac{2C_1^2}{\eta}\int_0^t \|k^{\varepsilon}(s)\|^2_0\|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\int_0^t \|\sigma(s,u_k(s))(k^{\varepsilon}(s)-k(s))\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\int_0^t \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\varepsilon C_2\int_0^t (1+\|\nabla u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s+2\sqrt{\varepsilon}\bigg|\int_0^t (w^{\varepsilon}(s),\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))\,\mathrm{d}W(s))\bigg|. \qquad (20)
\end{aligned}
$$

Then taking the supremum in (20) over the interval 0 to T, taking the expectation, and using Lemma 4 given by (14), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\|w^{\varepsilon}(t)\|^2_{\mathbb{L}^2}\Big]+\frac{\eta}{2}\mathbb{E}\int_0^T \|\nabla w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad\le 2\alpha\mathbb{E}\int_0^T \|w^{\varepsilon}_1(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{4(1+\beta^2)+\gamma^2}{\eta}\mathbb{E}\int_0^T \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}(\|u^{\varepsilon}_{k^{\varepsilon}}(s)\|^2_{\mathbb{L}^2}+\|u_k(s)\|^2_{\mathbb{L}^2})\,\mathrm{d}s\\
&\qquad+\frac{2\beta^2C^2}{\eta}\mathbb{E}\int_0^T \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\frac{8\gamma^2C^2}{\eta}\mathbb{E}\int_0^T \|w^{\varepsilon}_2(s)^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\frac{2C_1^2}{\eta}\mathbb{E}\int_0^T \|k^{\varepsilon}(s)\|^2_0\|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\mathbb{E}\int_0^T \|w^{\varepsilon}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\\
&\qquad+\mathbb{E}\int_0^T \|\sigma(s,u_k(s))(k^{\varepsilon}(s)-k(s))\|^2_{\mathbb{L}^2}\,\mathrm{d}s+\varepsilon C_2(T+K)\\
&\qquad+2\sqrt{\varepsilon}\mathbb{E}\bigg\{\sup_{0\le t\le T}\bigg|\int_0^t (w^{\varepsilon}(s),\sigma(s,u^{\varepsilon}_{k^{\varepsilon}}(s))\,\mathrm{d}W(s))\bigg|\bigg\}. \qquad (21)
\end{aligned}
$$

Making use of Burkholder–Davis–Gundy inequality for the stochastic integral term on the right-hand side, we obtain

$$2\sqrt{\varepsilon}\mathbb{E}\bigg\{\sup_{0\le t\le T}\bigg|\int_0^t (w^\varepsilon(s), \sigma(s, u^\varepsilon_{k^\varepsilon}(s))\,\mathrm{d}W(s))\bigg|\bigg\} \le \frac{1}{2}\mathbb{E}\bigg\{\sup_{0\le t\le T}\|w^\varepsilon(t)\|^2_{\mathbb{L}^2}\bigg\} + 4\varepsilon C_2^2(T+K).$$

Using the above estimate in (21) and finally applying Gronwall's inequality, we end up with

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}\|w^\varepsilon(t)\|^2_{\mathbb{L}^2}\Big] &+ \frac{\eta}{2}\mathbb{E}\int_0^T \|\nabla w^\varepsilon(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s \\ &\le C\bigg\{\mathbb{E}\int_0^T \|\sigma(s,u_k(s))(k^\varepsilon(s)-k(s))\|^2_{\mathbb{L}^2}\,\mathrm{d}s + \varepsilon C_2(T+K) + 4\varepsilon C_2^2(T+K)\bigg\} \\ &\quad \times \exp\bigg(K_1 T + 2K_2 K + \frac{2C_1^2}{\eta}M\bigg), \end{aligned} \tag{22}$$

where K_1, K_2, and C are appropriate positive constants. Also, note that, as $\varepsilon \to 0$, the right-hand side of (22) converges to 0 resulting in

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|w^\varepsilon(t)\|^2_{\mathbb{L}^2}\Big] + \frac{\eta}{2}\mathbb{E}\int_0^T \|\nabla w^\varepsilon(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s \to 0.$$

Since convergence in expectation always implies convergence in probability, we have $w^\varepsilon \to 0$ in probability in the space $\mathbb{L}^2(\Omega; \mathbb{C}([0,T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2((0,T); \mathbb{H}^1(\mathcal{O})))$. □

Hence, the large deviation result is established with Proposition 1 and Proposition 2.

Appendix A. Proof of Lemma 2

Let us fix the time t since we deal here only with spatial inner products and norms. Consider

$$(f(u), u) = -(u_1^2, u_1) - \bigg(\frac{\beta u_1^2 u_2}{1+u_1^2}, u_1\bigg) + \bigg(\frac{\gamma u_1^2 u_2}{1+u_1^2}, u_2\bigg).$$

Since (u_1, u_2) is a positive solution to the system (6), we have $-(u_1^2, u_1) \le 0$. Using the Holder and Young inequalities, the second and third inner products can be estimated as

$$\begin{aligned}\bigg(\frac{\beta u_1^2 u_2}{1+u_1^2}, u_1\bigg) &\le \beta\int_{\mathcal{O}} \frac{u_1^2(x,t)}{1+u_1^2(x,t)}|u_1(x,t)||u_2(x,t)|\,\mathrm{d}x \\ &\le \beta\|u_1\|_{\mathbb{L}^2}\|u_2\|_{\mathbb{L}^2} \\ &\le \frac{\beta}{2}\|u_1\|^2_{\mathbb{L}^2} + \frac{\beta}{2}\|u_2\|^2_{\mathbb{L}^2}\end{aligned}$$

and, similarly,

$$\bigg(\frac{\gamma u_1^2 u_2}{1+u_1^2}, u_2\bigg) \le \gamma\|u_2\|^2_{\mathbb{L}^2}.$$

Combining the above three estimates, we obtain (10). In order to prove the Lipschitz continuity of $f(\cdot)$, let us first consider

$$\begin{aligned}
&2(f(u)-f(v),z)\\
&\quad=-2(u_1^2-v_1^2,z_1)-2\beta\left(\frac{u_1^2u_2}{1+u_1^2}-\frac{v_1^2v_2}{1+v_1^2},z_1\right)+2\gamma\left(\frac{u_1^2u_2}{1+u_1^2}-\frac{v_1^2v_2}{1+v_1^2},z_2\right)\\
&\quad=\mathrm{IP}_1+\mathrm{IP}_2+\mathrm{IP}_3.
\end{aligned}\tag{23}$$

The first inner product (IP_1) on the right-hand side can be estimated by means of applying the Holder inequality followed by the Ladyzhenskaya and Young inequalities as

$$\begin{aligned}
\mathrm{IP}_1&\le 2\|u_1-v_1\|_{\mathbb{L}^4}\|u_1+v_1\|_{\mathbb{L}^2}\|z_1\|_{\mathbb{L}^2}\\
&=2\|z_1\|_{\mathbb{L}^4}^2\|u_1+v_1\|_{\mathbb{L}^2}\\
&\le\sqrt{2}\|z_1\|_{\mathbb{L}^2}\|\nabla z_1\|_{\mathbb{L}^2}\|u_1+v_1\|_{\mathbb{L}^2}\\
&\le\frac{\eta}{4}\|\nabla z_1\|_{\mathbb{L}^2}^2+\frac{4}{\eta}\|z_1\|_{\mathbb{L}^2}^2(\|u_1\|_{\mathbb{L}^2}^2+\|v_1\|_{\mathbb{L}^2}^2).
\end{aligned}\tag{24}$$

To evaluate the second and third inner products in (23), first consider

$$\left|\frac{u_1^2u_2}{1+u_1^2}-\frac{v_1^2v_2}{1+v_1^2}\right|=\left|\frac{u_1^2u_2-v_1^2v_2+u_1^2v_1^2(u_2-v_2)}{(1+u_1^2)(1+v_1^2)}\right|\le|F_1|+|F_2|.$$

For the term F_1, utilizing the algebraic identity $a^2b-c^2d=(a-c)(ab+cd)+ac(b-d)$, the boundedness could be assured as

$$|F_1|=\left|\frac{(u_1-v_1)(u_1u_2+v_1v_2)+u_1v_1(u_2-v_2)}{(1+u_1^2)(1+v_1^2)}\right|\le|u_1-v_1|(|u_2|+|v_2|)+|u_2-v_2|.$$

Also, for the term F_2, we obtain

$$|F_2|\le|u_2-v_2|.$$

Thus, IP_2 can be estimated similar to IP_1, in addition using the imbedding result on general $\mathbb{L}^p$ spaces for bounded domains as

$$\begin{aligned}
\mathrm{IP}_2&\le 2\beta\||z_1|(|u_2|+|v_2|)+2|z_2|\|_{\mathbb{L}^{4/3}}\|z_1\|_{\mathbb{L}^4}\\
&\le 2\beta[\|z_1\|_{\mathbb{L}^4}(\|u_2\|_{\mathbb{L}^2}+\|v_2\|_{\mathbb{L}^2})+2(\mathrm{area}(\mathcal{O}))^{1/4}\|z_2\|_{\mathbb{L}^2}]\|z_1\|_{\mathbb{L}^4}\\
&\le\frac{\eta}{4}\|\nabla z_1\|_{\mathbb{L}^2}^2+\frac{4\beta^2}{\eta}\|z_1\|_{\mathbb{L}^2}^2(\|u_2\|_{\mathbb{L}^2}^2+\|v_2\|_{\mathbb{L}^2}^2)+\frac{\eta}{4}\|\nabla z\|_{\mathbb{L}^2}^2+\frac{2\beta^2C_a}{\eta}\|z\|_{\mathbb{L}^2}^2,
\end{aligned}\tag{25}$$

where the constant $C_a=(\mathrm{area}(\mathcal{O}))^{1/2}$. Likewise, the third inner product (IP_3) can be evaluated to be

$$\mathrm{IP}_3\le\frac{\eta}{4}\|\nabla z\|_{\mathbb{L}^2}^2+\frac{\gamma^2}{\eta}\|z\|_{\mathbb{L}^2}^2(\|u_2\|_{\mathbb{L}^2}^2+\|v_2\|_{\mathbb{L}^2}^2)+\frac{\eta}{4}\|\nabla z_2\|_{\mathbb{L}^2}^2+\frac{8\gamma^2C_a}{\eta}\|z_2\|_{\mathbb{L}^2}^2.\tag{26}$$

The required estimate (11) follows from the estimates (24)–(26).

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