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Conditions for equality between Lyapunov and Morse decompositions

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Abstract. Let $Q \to X$ be a continuous principal bundle whose group G is reductive. A flow ϕ of automorphisms of Q endowed with an ergodic probability measure on the compact base space X induces two decompositions of the flag bundles associated to Q: a continuous one given by the finest Morse decomposition and a measurable one furnished by the multiplicative ergodic theorem. The second is contained in the first. In this paper we find necessary and sufficient conditions so that they coincide. The equality between the two decompositions implies continuity of the Lyapunov spectra under perturbations leaving unchanged the flow on the base space.

1. *Introduction*

The purpose of this paper is to give necessary and sufficient conditions for the equality of Morse and Oseledets decompositions of a continuous flow on a flag bundle.

We consider a continuous principal bundle $Q \to X$ with group G, which is assumed to be semi-simple or reductive. A continuous automorphism $\phi \in \mathrm{Aut}(Q)$ of Q defines a discrete-time flow ϕ^n, $n \in \mathbb{Z}$, on Q. For instance, $Q \to X$ could be the bundle of frames of a d-dimensional vector bundle $\mathcal{V} \to X$ over X, in which case G is the reductive group $\mathrm{Gl}(d, \mathbb{R})$. Since a linear flow on a vector bundle lifts to the bundle of frames, our setup includes this classical case.

The flow of automorphisms on Q induces a flow on the base space X, also denoted by ϕ. It also induces flows on bundles having as typical fiber a space F where G acts. Such bundle is built via the associated bundle construction and is denoted by $Q \times_G F$. If there is no risk of confusion, the flows on the associated bundles are denoted by ϕ as well.

When G is a reductive group we are especially interested in its flag manifolds $\mathbb{F}_\Theta$, distinguished by the subindex Θ, which are compact homogeneous spaces of G. We write $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$ for the corresponding flag bundle. For the flow ϕ induced on $\mathbb{E}_\Theta$, it was proved in [**3**, **15**] that it has a finest Morse decomposition (under the mild assumption that the flow on the base space X is chain transitive). Each Morse component of this finest decomposition meets a fiber of $\mathbb{E}_\Theta \to X$ in an algebraic submanifold of $\mathbb{F}_\Theta$. This submanifold is defined as a set of fixed points for some $g \in G$ acting on $\mathbb{F}_\Theta$. For instance, in a projective bundle the fibers of a Morse component are subspaces, which can be seen as sets of fixed points on the projective space of diagonalizable matrices (see also Selgrade [**21**] for the Morse decomposition on a projective bundle). The Morse decomposition is thus described by a continuous section χ_{Mo} of an associated bundle $Q \times_G (\mathrm{Ad}(G)H_{\mathrm{Mo}})$, whose typical fiber is an adjoint orbit $\mathrm{Ad}(G)H_{\mathrm{Mo}}$ of G. Here H_{Mo} belongs to the Lie algebra $\mathfrak{g}$ of G and its adjoint $\mathrm{ad}(H_{\mathrm{Mo}})$ has real eigenvalues. The Morse components are then built from the section χ_{Mo} and the fixed point sets of $\exp H_{\mathrm{Mo}}$ on the flag manifolds. (See [**15**, Theorem 7.5].)

On the other hand, we also have the Oseledets decomposition, coming from the multiplicative ergodic theorem (as proved in [**1**]). To consider this decomposition, there is required a ϕ-invariant measure ν on the base space. If ν is ergodic and $\mathrm{supp}\,\nu = X$ (which provides chain transitivity on X), then the multiplicative ergodic theorem yields an analogous decomposition to the Morse decomposition that describes the level sets of the $\mathfrak{a}$-Lyapunov exponents (see [**1**, **19**]). Again there are an adjoint orbit $\mathrm{Ad}(G)H_{\mathrm{Ly}}$ and a section χ_{Ly} of the associated bundle $Q \times_G (\mathrm{Ad}(G)H_{\mathrm{Ly}})$ such that the Oseledets decomposition is built from χ_{Ly} and the fixed point sets of $\exp H_{\mathrm{Ly}}$. The section χ_{Ly} is now only measurable and defined up to a set of ν-measure 0.

It turns out that any component of the Oseledets decomposition is contained in a component of the Morse decomposition (see §6 below). This means that the eigenspaces of $\mathrm{ad}(H_{\mathrm{Mo}})$ are contained in the eigenspaces of $\mathrm{ad}(H_{\mathrm{Ly}})$, that is, the multiplicities of the eigenvalues of $\mathrm{ad}(H_{\mathrm{Mo}})$ are larger than those of $\mathrm{ad}(H_{\mathrm{Ly}})$.

In this paper we write down three conditions that together are necessary and sufficient for both decompositions to coincide (see §9). In this case the Morse decomposition is a continuous extension of the Oseledets decomposition.

The first of these conditions requires boundedness of the measurable section χ_{Ly}, which means that different components of the Oseledets decomposition do not approach each other. The other two conditions are about the Oseledets decomposition for the other ergodic measures on $\mathrm{supp}\,\nu = X$. They can be summarized by saying that if ρ is an ergodic measure, then its Oseledets decomposition is finer than the decomposition for ν.

It is easy to prove that each of the three conditions is necessary. Our main result is to prove that together they imply equality of the decompositions.

Now we describe the contents of the paper and say some words about other results that have independent interest.

Sections 2–4 are preliminary. Section 2 contains notation and general facts about flag manifolds, while in §3 we recall the results of [**1**, **3**, **15**, **19**] about Morse decomposition, Morse and Lyapunov spectra and the multiplicative ergodic theorem on flag bundles. In §4 we discuss briefly flows over periodic orbits.

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Section 5 is devoted to the analysis of ergodic measures on the flag bundles. We exploit the Krylov–Bogolyubov technique of occupation measures to see that any Lyapunov exponent coming from the multiplicative ergodic theorem is an integral over an ergodic measure and conversely. Combining this with the fact that an ergodic measure charges just one Oseledets component allows us to introduce what we call attractor and repeller measures. Later their supports will provide attractor–repeller pairs on the flag bundles, thus relating them to the finest Morse decomposition.

In §6 we use the attractor and repeller measures to check that the components of the Oseledets decomposition are indeed contained in the finest Morse decomposition.

Another tool is developed in §7, namely the Lyapunov exponents of the derivative flow on the tangent space of the fibers of the flag bundles. The knowledge of these exponents allows us to find ω-limits in the bundles themselves.

In §8 we prove our main technical lemma that furnishes attractor–repeller pairs on the flag bundles.

In §9 we state our conditions and prove that they are necessary. Their sufficiency is proved in §10.

In the next two sections (§§11 and 12) we discuss two cases that go in opposite directions. Namely, flows where the base space is uniquely ergodic (§11) and products of independent identically distributed (i.i.d.) sequences. For a uniquely ergodic base space the second and third conditions are vacuous and it follows by previous results that the Morse spectrum is a polyhedron that degenerates to a point if the first condition is satisfied. As to the i.i.d. case, there are plenty of invariant measures enabling us to find examples that violate our second condition. We do that with the aid of a result by Guivarch' and Raugi [**11**].

Section 13 is independent of the rest of the paper. It contains a result that motivates the study of the equality between Oseledets and Morse decompositions. We prove that if both decompositions coincide for ϕ, then the Lyapunov spectrum is continuous under perturbations $\sigma\phi$ of ϕ with σ varying in the gauge group $\mathcal{G}$ of Q. This continuity is a consequence of the differentiability result of [**9**]. By that result, there exists a subset Φ_{Mo} of linear functionals defined from the finest Morse decomposition such that the map $\sigma \mapsto \alpha(H_{\mathrm{Ly}}(\sigma\phi))$ is differentiable with respect to σ (at the identity) if $\alpha \in \Phi_{\mathrm{Mo}}$, where $H_{\mathrm{Ly}}(\sigma\phi)$ is the vector Lyapunov spectrum of $\sigma\phi$. Having equality of the decompositions, we can exploit upper semi-continuity of the spectrum to prove continuity of $\beta(H_{\mathrm{Ly}}(\sigma\phi))$ with β in a basis that contains Φ_{Mo}.

Finally, we mention that for a linear flow ϕ on a vector bundle $\mathcal{V} \to X$, the topological property given by the finest Morse decomposition of the flow induced on the projective bundle $\mathbb{P}\mathcal{V} \to X$ can be given an analytic characterization via exponential separation of vector subbundles of $\mathcal{V}$ (see Colonius and Kliemann [**5**, Ch. 5] and Bonatti *et al* [**2**, Appendix B]). In fact, by a theorem of Bronstein and Chernii [**4**] (quoted from [**5**]), the finest Morse decomposition on $\mathbb{P}\mathcal{V}$ corresponds to the finest decomposition of $\mathcal{V}$ into exponentially separated subbundles (see [**5**, Theorem 5.2.10]). Hence, our main result gives, in particular, necessary and sufficient conditions ensuring that the Oseledets decomposition of a vector bundle is exponentially separated. Our conditions are given in terms of the mutual distances of the Oseledets spaces as well as on invariant measures on the flag bundles defined by the Oseledets filtrations for the forward and backward flows.

Furthermore, the result of §13 shows that if the Oseledets decomposition is exponentially separated, then the Lyapunov spectrum changes continuously when ϕ is perturbed in such a way that the flow on the base X is kept fixed.

2. *Flag manifolds*

We explain here our notation about semi-simple (or reductive) Lie groups and their flag manifolds. We refer to Knapp [**13**], Duistermat *et al* [**7**] and Warner [**22**].

Let $\mathfrak{g}$ be a semi-simple non-compact Lie algebra. In order to make the paper understandable to readers without acquaintance with Lie theory, we adopted the strategy of defining the notation by writing explicitly their meanings for the special linear group $\mathrm{Sl}(d, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(d, \mathbb{R})$ (or $\mathrm{Gl}(d, \mathbb{R})$ and $\mathfrak{gl}(d, \mathbb{R})$ in the reductive case). We hope that the reader with expertise in semi-simple theory will recognize the notation for the general objects (e.g. $\mathfrak{k}$ is a maximal compact embedded subalgebra, etc).

At the Lie algebra level the Cartan decomposition reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, where $\mathfrak{k} = \mathfrak{so}(d)$ is the subalgebra of skew-symmetric matrices and $\mathfrak{s}$ is the space of symmetric matrices with zero trace. The Iwasawa decomposition of the Lie algebra is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is the subalgebra of diagonal matrices and $\mathfrak{n}$ is the subalgebra of upper triangular matrices with zeros on the diagonal.

The set of roots is denoted by Π. These are linear maps $\alpha_{ij} \in \mathfrak{a}^*$, $i \neq j$, defined by $\alpha_{ij}(\mathrm{diag}\{a_1, \ldots, a_d\}) = a_i - a_j$. The set of positive roots is $\Pi^+ = \{\alpha_{ij} : i < j\}$ and the set of simple roots is $\Sigma = \{\alpha_{ij} : j = i + 1\}$. The root space is $\mathfrak{g}_\alpha$ ($\mathfrak{g}_{\alpha_{ij}}$ is spanned by the basic matrix E_{ij}) and

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha,$$

where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{k}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ ($\mathfrak{m} = 0$ in $\mathfrak{sl}(d, \mathbb{R})$). The basic (positive) Weyl chamber is denoted by

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0, \alpha \in \Sigma\}$$

(cone of diagonal matrices $\mathrm{diag}\{a_1, \ldots, a_d\}$ satisfying $a_1 > \cdots > a_d$). Its closure $\mathrm{cl}\mathfrak{a}^+$ is formed by diagonal matrices with decreasing eigenvalues.

At the Lie group level the Cartan decomposition reads $G = KS$, $K = \exp \mathfrak{k}$ and $S = \exp \mathfrak{s}$ (K is the group $\mathrm{SO}(d)$ and S the space of positive-definite symmetric matrices in $\mathrm{Sl}(d, \mathbb{R})$). The Iwasawa decomposition is $G = KAN$, $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. The Cartan decomposition splits further into the polar decomposition $G = K(\mathrm{cl}A^+)K$, $A^+ = \exp \mathfrak{a}^+$.

The group $M = \mathrm{Cent}_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K (diagonal matrices with entries ± 1), $M^* = \mathrm{Norm}_K(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K (signed permutation matrices) and $\mathcal{W} = M^*/M$ is the *Weyl group* (for $\mathrm{Sl}(d, \mathbb{R})$, it is the group of permutations in d letters, which acts in $\mathfrak{a}$ by permuting the entries of a diagonal matrix).

The (standard) minimal parabolic subalgebra is $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (= upper triangular matrices), and a general standard parabolic subalgebra $\mathfrak{p}_\Theta$ is defined by a subset $\Theta \subset \Sigma$ as

$$\mathfrak{p}_\Theta = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_{-\alpha},$$

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where $\langle\Theta\rangle$ is the set of roots spanned (over $\mathbb{Z}$) by Θ and $\langle\Theta\rangle^+ = \langle\Theta\rangle \cap \Pi^+$. That is, $\mathfrak{p}_\Theta = \mathfrak{p} \oplus \mathfrak{n}^-(\Theta)$, where $\mathfrak{n}^\pm(\Theta) = \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_{\pm\alpha}$.

Alternatively, given Θ, take $H_\Theta \in \mathrm{cl}\mathfrak{a}^+$ such that $\alpha(H_\Theta) = 0$, $\alpha \in \Sigma$, if and only if $\alpha \in \Theta$. Such H_Θ exists and we call it a characteristic element of Θ. Then $\mathfrak{p}_\Theta$ is the sum of eigenspaces of $\mathrm{ad}(H_\Theta)$ having eigenvalues ≥ 0. In $\mathfrak{sl}(d, \mathbb{R})$, $H_\Theta = \mathrm{diag}(a_1, \ldots, a_d)$ with $a_1 \geq \cdots \geq a_d$, where the multiplicities of the eigenvalues are prescribed by $a_i = a_{i+1}$ if $\alpha_{i,i+1} \in \Theta$, that is, $\mathfrak{p}_\Theta$ is the subalgebra of matrices that are upper triangular in blocks, whose sizes are the multiplicities of the eigenvalues of H_Θ. When Θ is empty, $\mathfrak{p}_\emptyset$ reduces to the minimal parabolic subalgebra $\mathfrak{p}$.

Conversely, if $H \in \mathrm{cl}\mathfrak{a}^+$, then $\Theta_H = \{\alpha \in \Sigma : \alpha(H) = 0\}$ defines a flag manifold $\mathbb{F}_{\Theta_H}$ (e.g. the Grassmannian $\mathrm{Gr}_k(d)$ is a flag manifold of $\mathrm{Sl}(d, \mathbb{R})$ defined by $H = \mathrm{diag}\{a, \ldots, a, b, \ldots, b\}$ with $(n-k)a + kb = 0$). For $H_1, H_2 \in \mathrm{cl}\mathfrak{a}^+$, we say that H_1 refines H_2 in case $\Theta_{H_1} \subset \Theta_{H_2}$. In $\mathfrak{sl}(d, \mathbb{R})$, this means that the blocks determined by the multiplicities of the eigenvalues of H_1 are contained in the blocks of H_2.

The *parabolic subgroup* P_Θ, associated to Θ, is defined as the normalizer of $\mathfrak{p}_\Theta$ in G (as a group of matrices, it has the same block structure as $\mathfrak{p}_\Theta$). It decomposes as $P_\Theta = K_\Theta A N$, where $K_\Theta = \mathrm{Cent}_K(H_\Theta)$ is the centralizer of H_Θ in K. We usually omit the subscript when $\Theta = \emptyset$ and $P = P_\emptyset$ is the minimal parabolic subgroup.

The *flag manifold* associated to Θ is the homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$ (just $\mathbb{F}$ when $\Theta = \emptyset$). If $\Theta_1 \subset \Theta_2$, then the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $\pi^{\Theta_1}_{\Theta_2} : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ given by $gP_{\Theta_1} \mapsto gP_{\Theta_2}$ (just π_{Θ_2} if $\Theta_1 = \emptyset$). For the matrix group the flag manifold $\mathbb{F}_\Theta$ identifies with the manifold of flags of subspaces $V_1 \subset \cdots \subset V_k$, where the differences $\dim V_{i+1} - \dim V_i$ are the sizes of the blocks defined by Θ (or rather the diagonal matrix H_Θ). The projection $\pi^{\Theta_1}_{\Theta_2} : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ is defined by 'forgetting subspaces'.

The flag manifold dual of $\mathbb{F}_\Theta$ is defined as follows: let w_0 be the principal involution of $\mathcal{W}$, that is, the only element of $\mathcal{W}$ such that $w_0\mathfrak{a}^+ = -\mathfrak{a}^+$, and put $\iota = -w_0$. Then $\iota(\Sigma) = \Sigma$ and for $\Theta \subset \Sigma$ write $\Theta^* = \iota(\Theta)$. Then $\mathbb{F}_{\Theta^*}$ is called the flag manifold dual of $\mathbb{F}_\Theta$. For the matrix group the vector subspaces of the flags in $\mathbb{F}_{\Theta^*}$ have complementary dimensions to those in $\mathbb{F}_\Theta$ (for instance, the dual of a Grassmannian $\mathrm{Gr}_k(d)$ is the Grassmannian $\mathrm{Gr}_{d-k}(d)$).

We say that two elements $b_1 \in \mathbb{F}_\Theta$ and $b_2 \in \mathbb{F}_{\Theta^*}$ are transversal if (b_1, b_2) belongs to the unique open G-orbit in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$, by the action $g(b_1, b_2) = (gb_1, gb_2)$. For instance, $b_1 \in \mathrm{Gr}_k(d)$ and $b_2 \in \mathrm{Gr}_{d-k}(d)$ are transversal if and only if they are transversal as subspaces of $\mathbb{R}^d$. In general, transversality can be expressed in terms of transversality of subalgebras of $\mathfrak{g}$ (see e.g. [**18**]). By the very definition, transversality is an open condition and if $g \in G$ then gb_1 is transversal to gb_2 if and only if b_1 is transversal to b_2. The following lemma about transversality will be used afterwards.

LEMMA 2.1. *Let b^*_n be a sequence in $\mathbb{F}_{\Theta^*}$ with $\lim b^*_n = b^*$. Suppose that $b \in \mathbb{F}_\Theta$ is not transversal to b^*. Then there exists a sequence $b_n \in \mathbb{F}_\Theta$ with b_n not transversal to b^*_n such that $\lim b_n = b$.*

Proof. There exists a sequence $k_n \in K$ with $b^*_n = k_n b^*$ and $k_n \to 1$. Since b is not transversal to b^*, it follows that $k_n b$ is not transversal to b^*_n. Hence, $b_n = k_n b$ is the required sequence. □

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We consider now the fixed point set of the action of $h = \exp H$, $H \in \mathrm{cl}\mathfrak{a}^+$, on a flag manifold $\mathbb{F}_\Theta$. Look first at the example of the projective space $\mathbb{R}P^{d-1}$. The fixed point set is the union of the eigenspaces of h. The eigenspace associated to the biggest eigenvalue is the only attractor (for the iterations h^n) that has an open and dense stable manifold. In the same way, the eigenspace of the smallest eigenvalue is the unique repeller with open and dense unstable manifold.

In general, the flow defined by $\exp tH$ is gradient in any flag manifold $\mathbb{F}_\Theta$ (see [**7**]). Its fixed point set is given by the union of the orbits

$$Z_H \cdot wb_\Theta = K_H \cdot wb_\Theta, \quad w \in \mathcal{W},$$

where $b_\Theta = 1 \cdot P_\Theta$ is the origin of $\mathbb{F}_\Theta = G/P_\Theta$, $Z_H = \{g \in G : \mathrm{Ad}(g)H = H\}$, $K_H = Z_H \cap K$ and w runs through the Weyl group $\mathcal{W}$. We write $\mathrm{fix}_\Theta(H, w) = Z_H \cdot wb_\Theta$ and refer to it as the set of H-fixed points of type w. In addition, $\mathrm{fix}_\Theta(H, 1)$ is the only attractor while $\mathrm{fix}_\Theta(H, w_0)$ is the unique repeller, where w_0 is the principal involution of $\mathcal{W}$.

As to the stable and unstable sets of $\mathrm{fix}_\Theta(H, w)$, let $\Theta_H = \{\alpha \in \Sigma : \alpha(H) = 0\}$. The subspaces

$$\mathfrak{n}_H^+ = \sum_{\alpha \in \Pi^+ \setminus \langle \Theta_H \rangle} \mathfrak{g}_\alpha, \quad \mathfrak{n}_H^- = \sum_{\alpha \in \Pi^+ \setminus \langle \Theta_H \rangle} \mathfrak{g}_{-\alpha}$$

are nilpotent subalgebras having connected subgroups $N_H^\pm = \exp \mathfrak{n}_H^\pm$. Put

$$\mathrm{st}_\Theta(H, w) = N_H^- K_H \cdot wb_\Theta, \quad \mathrm{un}_\Theta(H, w) = N_H^+ K_H \cdot wb_\Theta.$$

Then $\mathrm{st}_\Theta(H, w)$ and $\mathrm{un}_\Theta(H, w)$ are the stable and unstable sets of $\mathrm{fix}_\Theta(H, w)$, respectively.

More generally, if $D = \mathrm{Ad}(g)H$, $g \in G$ and $H \in \mathrm{cl}\mathfrak{a}^+$, then the dynamics of $\exp tD$ is conjugate under g to the dynamics of $\exp tH$. Hence, $\mathrm{fix}_\Theta(D, w) = g \cdot \mathrm{fix}_\Theta(H, w)$, $\mathrm{st}_\Theta(D, w) = g \cdot \mathrm{st}_\Theta(H, w)$ and $\mathrm{un}_\Theta(D, w) = g \cdot \mathrm{un}_\Theta(H, w)$. It follows that

$$\mathrm{st}_\Theta(D, w) = P_D^- \cdot gwb_\Theta, \quad \mathrm{un}_\Theta(H, w) = P_D^+ \cdot gwb_\Theta,$$

where $P_D^\pm = gN_H^\pm K_H g^{-1} = N_D^\pm K_D$, $N_D^\pm = gN_H^\pm g^{-1}$ and $K_D = gK_H g^{-1}$.

For any $D = \mathrm{Ad}(g)H$, there is just one fixed point component (namely, $\mathrm{fix}_\Theta(D, 1)$) whose stable manifold is open and dense. We denote this component by $\mathrm{att}_\Theta(D)$. Analogously, there is a unique component ($\mathrm{fix}_\Theta(D, w_0)$, where w_0 is the principal involution) whose unstable manifold is open and dense. We denote this component by $\mathrm{rp}_\Theta(D)$. We use the same notation for $d = \exp D$.

If H_1 refines H_2, then the centralizers satisfy $Z_{H_1} \subset Z_{H_2}$ and hence the fixed point set of $\exp tH_1$ in a flag manifold $\mathbb{F}_\Theta$ is contained in the fixed point set of $\exp tH_2$.

The following lemma shows that we can control the inclusion of fixed point sets for different elements by looking at the attractor and repeller fixed point sets in the right flag manifolds.

LEMMA 2.2. *Suppose that H_1 refines H_2, take $S \in \mathrm{Ad}(G)H_1$ and $T \in \mathrm{Ad}(G)H_2$, and put $s = \exp S$, $t = \exp T$. Suppose that $\mathrm{att}_{\Theta(H_1)}(s) \subset \mathrm{att}_{\Theta(H_1)}(t)$ and $\mathrm{rp}_{\Theta(H_1)^*}(s) \subset \mathrm{rp}_{\Theta(H_1)^*}(t)$. Then the fixed point set of s in any flag manifold is contained in the fixed point set of t.*

Moreover, the fixed points are the same in case these attractor and repeller fixed points coincide.

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Proof. If we identify $\mathrm{Ad}(G)H_1$ with the open orbit in $\mathbb{F}_{\Theta(H_1)} \times \mathbb{F}_{\Theta(H_1)^*}$, then S is identified with the pair $(\mathrm{att}_{\Theta(H_1)}(s), \mathrm{rp}_{\Theta(H_1)}(s))$. In the same way, T is identified with the pair $(\mathrm{att}_{\Theta(H_2)}(t), \mathrm{rp}_{\Theta(H_2)}(t)) \in \mathbb{F}_{\Theta(H_2)} \times \mathbb{F}_{\Theta(H_2)^*}$. Now, since H_1 refines H_2, there are fibrations $p : \mathrm{Ad}(G)H_1 \to \mathrm{Ad}(G)H_2$, $\pi_1 : \mathbb{F}_{\Theta(H_1)} \to \mathbb{F}_{\Theta(H_2)}$ and $\pi_2 : \mathbb{F}_{\Theta(H_1)^*} \to \mathbb{F}_{\Theta(H_2)^*}$ with the equalities $\pi_1(\mathrm{att}_{\Theta(H_1)}(t)) = \mathrm{att}_{\Theta(H_2)}(t)$ and $\pi_2(\mathrm{att}_{\Theta(H_1)}(t)) = \mathrm{att}_{\Theta(H_2)}(t)$. Hence, we have $p(S) = T$. This means that there exists $g \in G$ such that $S = \mathrm{Ad}(g)H_1$ and $T = \mathrm{Ad}(g)H_2$. Hence, the fixed point set of s (respectively t) in a flag manifold $\mathbb{F}_\Theta$ is the image under g of the fixed point set of $\exp H_1$ (respectively $\exp H_2$), which implies the lemma. □

3. *Lyapunov and Morse spectra and decompositions*

From now on we consider a discrete-time continuous flow ϕ_n on a continuous principal bundle (Q, X, G), where the base space X is a compact metric space endowed with an ergodic invariant measure ν with $\mathrm{supp}\,\nu = X$. The structural group G is assumed to be semi-simple and non-compact or, slightly more generally, G is reductive with non-compact semi-simple component. We fix once and for all a maximal compact subgroup $K \subset G$ and a K-subbundle $R \subset Q$. (For a bundle of frames of a vector bundle $\mathcal{V} \to X$, this amounts to the choice of a Riemannian metric on $\mathcal{V}$. In case of a trivial bundle $Q = X \times G$, the reduction is $R = X \times K$.)

The Iwasawa $(G = KAN)$ and Cartan $(G = KS)$ decompositions of G yield decompositions of $Q = R \times AN$ and $Q = R \times S$ by writing $q \in Q$ as

$$q = r \cdot hn \quad \text{and} \quad q = r \cdot s,$$

$r \in R$, $hn \in AN$ and $s \in S$. In what follows we write for $q \in Q$,

$$\mathsf{a}(q) = \log \mathsf{A}(q) \in \mathfrak{a},$$

where $\mathsf{A}(q)$ is the projection onto A against the Iwasawa decomposition. Also, we write $\mathsf{S} : Q \to S$ as the projection onto S of $Q = R \times S$. By the polar decomposition $G = K(\mathrm{cl}A^+)K$, we get a map $\mathsf{A}^+ : Q \to \mathrm{cl}A^+$ by $\mathsf{S}(q) = k\mathsf{A}^+(q)k^{-1}$, $k \in K$. We write

$$\mathsf{a}^+(q) = \log \mathsf{A}^+(q) \in \mathrm{cl}\mathfrak{a}^+.$$

Now the flow ϕ_n on Q induces a flow ϕ_n^R on R by declaring $\phi_n^R(r)$ to be the projection of ϕ_n onto R against the decomposition $Q = R \times AN$ (ϕ_n^R is indeed a flow because AN is a group). The projections a and a^+ define maps (denoted by the same letters) $\mathsf{a} : \mathbb{Z} \times R \to \mathfrak{a}$ and $\mathsf{a}^+ : \mathbb{Z} \times R \to \mathrm{cl}\mathfrak{a}^+$ by

$$\mathsf{a}(n, r) = \mathsf{a}(\phi_n(r)) \quad \text{and} \quad \mathsf{a}^+(n, r) = \mathsf{a}^+(\phi_n(r)).$$

It turns out that $\mathsf{a}(n, r)$ is an additive cocycle over ϕ_n^R. This cocycle factors to a cocycle (also denoted by a) over the flow induced on $\mathbb{E} = Q \times_G \mathbb{F}$, an associated bundle of Q with typical fiber the maximal flag manifold $\mathbb{F}$. The $\mathfrak{a}$-Lyapunov exponent of ϕ_n in the direction of $\xi \in \mathbb{E}$ is defined by

$$\lambda(\xi) = \lim_{k \to +\infty} \frac{1}{k} \mathsf{a}(k, \xi) \in \mathfrak{a}, \qquad \xi \in \mathbb{E}.$$

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The polar exponent is defined by

$$H_\phi(r) = \lim_{k\to+\infty} \frac{1}{k}\mathsf{a}^+(k, r) \in \mathrm{cl}\mathfrak{a}^+, \quad r \in R.$$

It turns out that $H_\phi(r)$ is constant along the fibers of R (when it exists), so is written $H_\phi(x)$, $x \in X$. The existence of these limits is ensured by the multiplicative ergodic theorem, as follows.

MULTIPLICATIVE ERGODIC THEOREM [**1**]. *The polar exponent $H_\phi(x)$ exists for x in a set of total measure Ω. Assume that ν is ergodic. Then there is $H_{\mathrm{Ly}} = H_{\mathrm{Ly}}(\nu) \in \mathrm{cl}\mathfrak{a}^+$ such that $H_\phi(\cdot)$ is almost surely equal to H_{Ly}. Put $\mathbb{E}_\Omega = \pi^{-1}(\Omega)$, where $\pi : \mathbb{E} \to X$ is the projection. Then:*

(1) *$\lambda(\xi)$ exists for every $\xi \in \mathbb{E}_\Omega$ and the map $\lambda : \mathbb{E}_\Omega \to \mathfrak{a}$ assumes values in the finite set $\{wH_{\mathrm{Ly}} : w \in \mathcal{W}\}$;*

(2) *there exists a measurable section χ_{Ly} of the bundle $Q \times_G \mathrm{Ad}(G)(H_{\mathrm{Ly}})$, defined on Ω, such that $\lambda(\xi) = w^{-1}H_{\mathrm{Ly}}$ if $\xi \in \mathrm{st}(\chi_{\mathrm{Ly}}(x), w)$, $x = \pi(\xi)$.*

(To be rigorous, the stable set $\mathrm{st}(\chi_{\mathrm{Ly}}(x), w)$, simply denoted by $\mathrm{st}(x, w)$, must be defined using the formalism of fiber bundles. If $Q = X \times G$ is trivial, then $\chi_{\mathrm{Ly}} : X \to \mathrm{Ad}(G)(H_{\mathrm{Ly}})$ and $\mathrm{st}(\chi_{\mathrm{Ly}}(x), w)$ is the stable set discussed in the last section.)

We write $\mathrm{st}(w)$ for the union of the sets $\mathrm{st}(\chi_{\mathrm{Ly}}(x), w)$ with x running through Ω. In the same way, we let $\mathrm{fix}(w)$ be the union of the fixed point sets $\mathrm{fix}(\chi_{\mathrm{Ly}}(x), w)$.

By analogy with the multiplicative ergodic theorem on vector bundles, the union of the sets $\mathrm{fix}(w)$, $w \in \mathcal{W}$, is called the Oseledets decomposition of $\mathbb{E}$. These sets project to a partial flag bundle $\mathbb{E}_\Theta$ to fixed point sets $\mathrm{fix}_\Theta(w)$ that form the Oseledets decomposition of $\mathbb{E}_\Theta$.

To the exponent $H_{\mathrm{Ly}}(\nu) \in \mathrm{cl}\mathfrak{a}^+$ we associate the subset of the simple system of roots

$$\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Ly}}(\nu) = \{\alpha \in \Sigma : \alpha(H_{\mathrm{Ly}}(\nu)) = 0\}.$$

The corresponding flag manifold $\mathbb{F}_{\Theta_{\mathrm{Ly}}}$ and flag bundle $\mathbb{E}_{\Theta_{\mathrm{Ly}}} = Q \times_G \mathbb{F}_{\Theta_{\mathrm{Ly}}}$ play a prominent role in the proofs. (For a linear flow on a vector bundle $\mathbb{F}_{\Theta_{\mathrm{Ly}}}$, it is the manifold of flags $(V_1 \subset \cdots \subset V_k)$ of subspaces of $\mathbb{R}^d$ having the same dimensions as the subspaces of the Oseledets splitting when the Lyapunov spectrum is ordered decreasingly.) We refer to $\mathbb{F}_{\Theta_{\mathrm{Ly}}}$ as the flag type of ϕ with respect to ν.

Remark. We mention that the section χ_{Ly} yields (actually is built from) two sections ξ and ξ^* of the flag bundles $\mathbb{F}_{\Theta_{\mathrm{Ly}}}$ and $\mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$, respectively. Their images are defined from level sets of Lyapunov exponents and hence are measurable (see [**1**, §7.1]).

On the other hand, there are continuous decompositions of the flag bundles (defined in the same way as sets of fixed points) obtained by working out the concept of Morse decomposition of the flows on the bundles (see Conley [**6**] and Colonius and Kliemann [**5**]). It was proved in [**3**] and [**15**] that if the flow on the base space is chain transitive then the flow on any flag bundle $\mathbb{E}_\Theta$ admits a finest Morse decomposition with Morse sets $\mathcal{M}(w)$, also parametrized by $w \in \mathcal{W}$. Analogous to the Oseledets

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decomposition, the Morse sets are built as fixed point sets defined by a continuous section of an adjoint bundle $\chi_{\mathrm{Mo}} : X \to Q \times_G \mathrm{Ad}(G)H_{\mathrm{Mo}}$, where $H_{\mathrm{Mo}} \in \mathrm{cl}\mathfrak{a}^+$ as well. There is just one attractor Morse component, which is given by $\mathcal{M}^+ = \mathcal{M}(1)$. There is a unique repeller component as well, which is $\mathcal{M}^- = \mathcal{M}(w_0)$, where w_0 is the principal involution.

The assumption that the invariant measure ν is ergodic with support $\mathrm{supp}\,\nu = X$ implies chain transitivity on X.

We write

$$\Theta_{\mathrm{Mo}} = \Theta_{\mathrm{Mo}}(\phi) = \{\alpha \in \Sigma : \alpha(H_{\mathrm{Mo}}) = 0\}$$

and refer to $\mathbb{F}_{\Theta_{\mathrm{Mo}}}$ as the flag type of ϕ (with respect to the Morse decomposition).

The spectral counterpart of the Morse decomposition is the Morse spectrum associated to the cocycle $\mathsf{a}(n, \xi)$. This spectrum was originally defined by Colonius and Kliemann [**5**] for a flow on a vector bundle and extended to flag bundles (and vector-valued cocycles) in [**19**]. By the results of [**19**], each Morse set $\mathcal{M}(w)$ has a Morse spectrum $\Lambda_{\mathrm{Mo}}(w)$ which is a compact convex subset of $\mathfrak{a}$ and contains any $\mathfrak{a}$-Lyapunov exponent $\lambda(\xi)$, $\xi \in \mathcal{M}(w)$. The attractor Morse component is given by the identity $1 \in \mathcal{W}$ and we write $\Lambda_{\mathrm{Mo}} = \Lambda_{\mathrm{Mo}}(1)$, which is the only Morse spectrum meeting $\mathrm{cl}\mathfrak{a}^+$. The Morse spectrum Λ_{Mo} satisfies the following properties:

(1) Λ_{Mo} is invariant under the group $\mathcal{W}_{\Theta_{\mathrm{Mo}}}$ generated by reflections with respect to the roots $\alpha \in \Theta_{\mathrm{Mo}}$ (see [**19**, Theorem 8.3]);

(2) $\alpha(H) > 0$ if $H \in \Lambda_{\mathrm{Mo}}$ and α is a positive root that does not belong to the set $\langle\Theta_{\mathrm{Mo}}\rangle^+$ spanned by Θ_{Mo} (see [**19**, Corollary 7.4]).

By the last statement, $\alpha(H_{\mathrm{Ly}}) > 0$ if α is a simple root outside Θ_{Mo} because $H_{\mathrm{Ly}} \in \Lambda_{\mathrm{Mo}}$. Hence, $\alpha \notin \Theta_{\mathrm{Ly}}$ by definition of Θ_{Ly}. It follows that $\Theta_{\mathrm{Ly}} \subset \Theta_{\mathrm{Mo}}$. Below in §6 we improve this statement by proving, with the aid of invariant measures on flag bundles, that the Oseledets decomposition is contained in the Morse decomposition.

Our objective is to find necessary and sufficient conditions ensuring that $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ and hence that the Oseledets decomposition coincides with the Morse decomposition.

4. *Flows over periodic orbits*

Before proceeding, let us recall the case where the base space is a single periodic orbit $X = \{x_0, \ldots, x_{\omega-1}\}$ of period ω, which will be used later to reduce some arguments to non-periodic orbits.

In the periodic case we have $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ since, as is well known, the asymptotics depends ultimately on iterations of a fixed element in the group G. Here the principal bundle is $Q = X \times G$ and the flow is given by

$$\phi(x_i, h) = (x_{i+1 (\mathrm{mod}\, \omega)}, A(x_i)h)$$

for a map $A : X \to G$, so that

$$\phi^n(x_i, h) = (x_{i+n (\mathrm{mod}\, \omega)}, g_{n,i}h),$$

where $g_{n,i} = A(x_{i+n-1 (\mathrm{mod}\, \omega)}) \cdots A(x_{i+1})A(x_i)$. We have $g_{n+m,i} = g_{n,i+m (\mathrm{mod}\, \omega)} g_{m,i}$, so that $g_{k\omega,i} = g_{\omega,i}^k$. Hence, the asymptotics of an orbit starting at a point above x_i is dictated by the iterations of the action of $g_{\omega,i}$. The iterations for the action of a fixed $g \in G$ on the flag manifolds, as well as the continuous-time version of periodicity, were

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studied by Ferraiol *et al* [**8**]. Let $g_{n,i} = u_{n,i} h_{n,i} x_{n,i}$ be the Jordan decomposition of $g_{n,i}$ with $u_{n,i}$, $h_{n,i}$ and $x_{n,i}$ elliptic, hyperbolic and unipotent, respectively. There is a choice of an Iwasawa decomposition $G = KAN$ such that $u_{n,i} \in K$, $h_{n,i} \in A$ and $x_{n,i} \in N$. It follows that the Lyapunov spectrum is given by $\log h_{\omega,i}$, which is the same for any $i = 0, \ldots, \omega - 1$ (because $g_{\omega,i+1} = A(x_i) g_{\omega,i} A(x_i)^{-1}$). Also, as proved in [**8**], the Morse decomposition is given by the fixed point sets of $h_{\omega,i}$. Hence, $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$.

5. *Invariant measures on the bundles and* $\mathfrak{a}$*-Lyapunov exponents*

Let μ be an invariant measure for the flow on the maximal flag bundle $\pi : \mathbb{E} \to X$. Then the integral

$$\int q \, d\mu, \quad q(\xi) = \mathsf{a}(1, \xi)$$

is an $\mathfrak{a}$-Lyapunov exponent for the cocycle $\mathsf{a}(n, \xi)$ (see [**19**]). On the other hand, by applying the multiplicative ergodic theorem to an invariant measure ν on the base space, we obtain $\mathfrak{a}$-Lyapunov exponents, which we call regular Lyapunov exponents with respect to ν (because they are obtained as limits of sequences in $\mathfrak{a}$, which in turn come from regular sequences in G, see [**1**]).

In this section we show that these Lyapunov exponents coincide. Namely, if ν is an ergodic measure on X, then any of its $\mathfrak{a}$-Lyapunov exponents is an integral over an ergodic measure μ that projects onto ν, i.e. $\pi_* \mu = \nu$, and conversely any such integral is a regular Lyapunov exponent.

Fix an ergodic invariant measure ν on the base space and let $\Omega \subset X$ be the set of ν-total measure given by the multiplicative ergodic theorem (as proved in [**1**]). Recall that

$$\pi^{-1}(\Omega) = \dot{\bigcup_{w \in \mathcal{W}_{\Theta_{\mathrm{Ly}}} \backslash \mathcal{W}}} \mathrm{st}(w)$$

and $\lambda(\xi) = w^{-1} H_{\mathrm{Ly}}$ if $\xi \in \mathrm{st}(w)$, where H_{Ly} is the polar exponent with respect to ν.

PROPOSITION 5.1. *Let* μ *be an ergodic measure on* $\mathbb{E}$ *that projects onto* ν. *Then there exists* $w \in \mathcal{W}$ *such that* $\mu(\mathrm{st}(w)) = 1$ *and* $\mu(\mathrm{st}(w')) = 0$ *if* $\mathcal{W}_{\Theta_{\mathrm{Ly}}} w \neq \mathcal{W}_{\Theta_{\mathrm{Ly}}} w'$. *In this case,*

$$\int q \, d\mu = w^{-1} H_{\mathrm{Ly}}.$$

Proof. By ergodicity of μ, the ergodic theorem applied to μ and $q(\xi) = \mathsf{a}(1, \xi)$ implies that there exists a measurable set $\mathcal{I} \subset \mathbb{E}$ with $\mu(\mathcal{I}) = 1$ and

$$\lambda(\xi) = \lim_{k \to \infty} \frac{1}{k} \mathsf{a}(k, \xi) = \int q \, d\mu, \quad \xi \in \mathcal{I}.$$

Now $\mu(\pi^{-1}(\Omega) \cap \mathcal{I}) = 1$ and $\pi^{-1}(\Omega) \cap \mathcal{I}$ is the disjoint union of the sets $\mathrm{st}(w) \cap \mathcal{I}$. In each $\mathrm{st}(w) \cap \mathcal{I}$, $w \in \mathcal{W}$, λ is defined and is a constant equal to $w^{-1} H_{\mathrm{Ly}}$. Since λ is constant on $\mathcal{I}$, it follows that $\pi^{-1}(\Omega) \cap \mathcal{I} \subset \mathrm{st}(w)$ for some $w \in \mathcal{W}$. Then, for any $\xi \in \mathcal{I}$,

$$\int q \, d\mu = \lambda(\xi) = w^{-1} H_{\mathrm{Ly}}.$$

Finally, $\mu(\mathrm{st}(w)) \geq \mu(\pi^{-1}(\Omega) \cap \mathcal{I}) = 1$, which implies that $\mu(\mathrm{st}(w')) = 0$ if $\mathrm{st}(w') \neq \mathrm{st}(w)$, that is, if $W_{\Theta_{\mathrm{Ly}}} w \neq W_{\Theta_{\mathrm{Ly}}} w'$. □

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COROLLARY 5.2. *Let $\Lambda_{\mathrm{Mo}}(w) \subset \mathfrak{a}$ be the Morse spectrum of the Morse set $\mathcal{M}(w)$. Then the extremal points of the compact convex set $\Lambda_{\mathrm{Mo}}(w)$ are regular Lyapunov exponents for ergodic measures on the base space.*

Proof. In fact, it was proved in [**19**, (see Theorem 3.2(6))] that any extremal point of $\Lambda_{\mathrm{Mo}}(w)$ is an integral $\int q\, d\mu$ with respect to an ergodic measure μ on $\mathbb{E}$. (See also [**5**, Lemma 5.4.10].) □

The converse to the above proposition says that any regular Lyapunov exponent is the integral of q with respect to some ergodic measure projecting onto ν. In order to prove the converse, we recall the Krylov–Bogolyubov procedure of constructing invariant measures as occupation measures (see e.g. [**5**]). Let ψ_n, $n \in \mathbb{Z}$, be a flow on a compact metric space Y. Then this means that

$$(L_{n,x} f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(\psi_k x), \qquad x \in Y,$$

define linear maps on the space $C_0(Y)$ of continuous functions and hence Borel probability measures ρ_n. An accumulation point $\rho_x = \lim_k \rho_{n_k}$ is called an (invariant) occupation measure. When the limit $\widetilde{f}(x) = \lim_n 1/n \sum_{k=0}^{n-1} f(\psi_k x)$ exists, it is an integral $\widetilde{f}(x) = \int f(y) \mu_x(dy)$ with respect to an occupation measure. The following properties will be used below.

(1) Let ρ be an ergodic probability measure on Y. Then, for ρ-almost every $y \in Y$, any occupation measure $\rho_y = \rho$ (this is an easy consequence of the Birkhoff ergodic theorem).
(2) There exists a set $\mathcal{J} \subset Y$ of total probability (that is, $\rho(\mathcal{J}) = 1$ for every invariant measure ρ) such that for all $y \in \mathcal{J}$ there exists an ergodic occupation measure ρ_y (see Mañé [**14**, Ch. II, §6]).

PROPOSITION 5.3. *Given $w \in \mathcal{W}$, there exists an invariant ergodic measure μ^w on $\mathbb{E}$ with $\pi_* \mu^w = \nu$ such that*

$$\int q\, d\mu^w = w^{-1} H_{\mathrm{Ly}}$$

and $\mu^w(\mathrm{st}(w)) = 1$.

Proof. If $\xi \in \mathrm{st}(w)$, then

$$\lambda(\xi) = \lim_{k \to +\infty} \frac{1}{k} \mathsf{a}(k, \xi) = w^{-1} H_{\mathrm{Ly}}$$

and, since $\mathsf{a}(k, \xi)$ is a cocycle, it follows that there exists an occupation measure μ_ξ such that

$$w^{-1} H_{\mathrm{Ly}} = \int q\, d\mu_\xi.$$

Note that $\pi_*(\mu_\xi)$ is an occupation measure ρ_x with $x = \pi(\xi)$. Since ν is ergodic for ν-almost all x, $\rho_x = \nu$ and hence we can choose ξ with $\pi_*(\mu_\xi) = \nu$.

It is not clear in advance that μ_ξ is ergodic. Nevertheless, we can decompose μ_ξ into ergodic components θ_η with η ranging through a set $\mathcal{A}$ of μ_ξ total probability, that is,

$$\mu_\xi(\cdot) = \int \theta_\eta(\cdot)\, d\mu_\xi(\eta).$$

Since $\pi_* \mu_\xi = \nu$, it follows that $\pi_* \theta_\eta = \nu$ for μ_ξ-almost all η.

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We claim that

$$w^{-1}H_{\mathrm{Ly}} = \int q \, d\theta_\eta$$

for almost all $\eta \in \mathcal{A}$. In fact,

$$w^{-1}H_{\mathrm{Ly}} = \int_{\mathbb{E}} \left(\int q \, d\theta_\eta \right) d\mu_\xi(\eta).$$

Hence, $w^{-1}H_{\mathrm{Ly}}$ belongs to the convex closure of the set $\{\int q \, d\theta_\eta \in \mathfrak{a}; \eta \in \mathcal{A}\}$. However, by Proposition 5.1, for any ergodic θ_η there exists $u \in \mathcal{W}$ such that $\int q \, d\theta_\eta = u^{-1}H_{\mathrm{Ly}}$, so that $w^{-1}H_{\mathrm{Ly}}$ is a convex combination of points of the orbit $\mathcal{W} \cdot H_{\mathrm{Ly}}$. But this is possible only if $\int q \, d\theta_\eta = w^{-1}H_{\mathrm{Ly}}$ for almost all η because the convex closure of the orbit $\mathcal{W} \cdot H_{\mathrm{Ly}}$ is a polyhedron whose vertices (extremal points) are the points of the orbit. Hence, there exists μ^w yielding the Lyapunov exponent $w^{-1}H_{\mathrm{Ly}}$. Finally, the equality $\mu^w(\mathrm{st}(w)) = 1$ follows by the previous proposition. □

Now we select two special kinds of ergodic measures on the flag bundles.

Definition 5.4. An ergodic measure μ on the maximal flag bundle $\mathbb{E}$, which projects to ν, is said to be an attractor measure for the flow if $\int q \, d\mu \in \mathrm{cl}\, \mathfrak{a}^+$. A measure μ_Θ in $\mathbb{E}_\Theta$ is an attractor measure if $\mu_\Theta = \pi_{\Theta *}\mu$ with μ attractor in $\mathbb{E}$, where $\pi_\Theta : \mathbb{E} \to \mathbb{E}_\Theta$ is the canonical projection.

Similarly, a measure μ in $\mathbb{E}$, which projects to ν, is a repeller measure if $\int q \, d\mu^w \in -\mathrm{cl}\, \mathfrak{a}^+$, and μ_Θ in $\mathbb{E}_\Theta$ is a repeller measure if $\mu_\Theta = \pi_{\Theta *}\mu$ with μ repeller in $\mathbb{E}$.

Proposition 5.3 ensures the existence of both attractor and repeller measures.

PROPOSITION 5.5. *A repeller measure is an attractor measure for the backward flow.*

Proof. Let μ be a repeller measure on $\mathbb{E}$ and write $q^-(\cdot) = \mathsf{a}(-1, \cdot)$. Then, by the cocycle property, $q^-(\xi) = -\mathsf{a}(1, \phi_{-1}(\xi)) = -q(\phi_{-1}(\xi))$, so that

$$\int q^- \, d\mu = -\int q \, d\mu \in \mathrm{cl}\, \mathfrak{a}^+$$

because $\int q \, d\mu \in -\mathrm{cl}\, \mathfrak{a}^+$. Thus, μ is an attractor measure for the backward flow. This proves the statement on the maximal flag bundle $\mathbb{E}$. On the other bundles the result follows by definition. □

Now we relate the supports of the attractor and repeller measures with the decomposition given by the multiplicative ergodic theorem on the flag bundle $\mathbb{E}_{\Theta_{\mathrm{Ly}}(\nu)}$ and in its dual $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}(\nu)}$. This decomposition is given by sections ξ and ξ^* of $\mathbb{E}_{\Theta_{\mathrm{Ly}}(\nu)}$ and $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}(\nu)}$, respectively.

We write simply $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Ly}}(\nu)$ and distinguish the several projections as $\pi : \mathbb{E} \to X$, $\pi_{\Theta_{\mathrm{Ly}}} : \mathbb{E} \to \mathbb{E}_{\Theta_{\mathrm{Ly}}}$, $\pi_{\Theta^*_{\mathrm{Ly}}} : \mathbb{E} \to \mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$ and p for either $\mathbb{E}_{\Theta_{\mathrm{Ly}}} \to X$ or $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}} \to X$.

Let μ be a repeller measure on $\mathbb{E}$ and put $\mu_{\Theta^*_{\mathrm{Ly}}} = \pi_{\Theta^*_{\mathrm{Ly}}*}(\mu)$ for the corresponding repeller measure on $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$. We have $p_*(\mu_{\Theta^*_{\mathrm{Ly}}}) = \nu$ because $p \circ \pi_{\Theta^*_{\mathrm{Ly}}} = \pi$ and $\pi_*\mu = \nu$. Hence, we can disintegrate $\mu_{\Theta^*_{\mathrm{Ly}}}$ with respect to ν to get

$$\mu_{\Theta^*_{\mathrm{Ly}}}(\cdot) = \int_X \rho_x(\cdot) \, d\nu(x),$$

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where $x \in X \mapsto \rho_x \in \mathbb{M}^+(\mathbb{E}_{\Theta^*_{\mathrm{Ly}}})$ is a measurable map into the space of probability measures on $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$.

LEMMA 5.6. *For ν-almost all $x \in X$, the component ρ_x in the above disintegration is a Dirac measure at $\xi^*(x)$, that is, $\rho_x = \delta_{\xi^*(x)}$.*

Proof. Let Z be the Borel set

$$Z = \{\mathrm{im}\,\xi^*\}^c = \mathbb{E}_{\Theta^*_{\mathrm{Ly}}} \backslash \{\mathrm{im}\,\xi^*\}$$

(see the remark in §3). Then

$$\mu_{\Theta^*_{\mathrm{Ly}}}(Z) = \mu(\pi^{-1}_{\Theta^*_{\mathrm{Ly}}}(Z)) = \mu(\mathbb{E}\backslash \mathrm{st}(w_0)) = 0$$

because μ is a repeller measure. However,

$$0 = \mu_{\Theta^*_{\mathrm{Ly}}}(Z) = \int_X \rho_x(Z)\, d\nu(x)$$

and, since ρ_x is supported at $\pi^{-1}_{\Theta^*_{\mathrm{Ly}}}(x)$, it follows that $\rho_x(\mathbb{E}_{\Theta^*_{\mathrm{Ly}}} \backslash \{\xi^*(x)\}) = 0$ for ν-almost all $x \in X$. □

This lemma shows also that a repeller measure on the dual flag manifold $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$ is unique. Now we can apply the same argument for the reverse flow ϕ_{-t}, and get a similar result for an attracting measure on $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ with ξ^* replaced by ξ.

For later reference, we summarize these facts in the following proposition.

PROPOSITION 5.7. *There exists a unique attractor measure $\mu^+_{\Theta_{\mathrm{Ly}}}$ for ϕ_t in its flag type $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$, which is a Dirac measure on $\xi(x)$, that is, it disintegrates as*

$$\mu^+_{\Theta_{\mathrm{Ly}}}(\cdot) = \int \delta_{\xi(x)}(\cdot)\, d\nu(x)$$

with respect to ν. There also exists a unique repeller measure $\mu^-_{\Theta_{\mathrm{Ly}}}$ on $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$, which is Dirac at ξ^*.

COROLLARY 5.8. *There exists a unique attractor (respectively repeller) measure in $\mathbb{E}_\Theta$ if $\Theta_{\mathrm{Ly}} \subset \Theta$ (respectively $\Theta^*_{\mathrm{Ly}} \subset \Theta$).*

Proof. This is true because the projection $\mathbb{E} \to \mathbb{E}_\Theta$ factors through $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ if $\Theta_{\mathrm{Ly}} \subset \Theta$: $\mathbb{E} \to \mathbb{E}_{\Theta_{\mathrm{Ly}}} \to \mathbb{E}_\Theta$. Hence, a measure in $\mathbb{E}_\Theta$ is an attractor if and only if it is the projection of the attractor measure in $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$. □

6. *Morse decomposition × Oseledets decomposition*

In this section we use the concepts of attractor and repeller measures developed above to relate the Oseledets decomposition and the Morse decomposition on a flag bundle $\mathbb{E}_\Theta$, as well as the Lyapunov spectrum H_{Ly} and the Morse spectrum Λ_{Mo}.

First we have the following consequence of Proposition 5.3.

PROPOSITION 6.1. *Suppose that $\alpha(\Lambda_{\mathrm{Mo}}) = 0$ for all $\alpha \in \Theta_{\mathrm{Mo}}$. Then $\Theta_{\mathrm{Ly}}(\rho) = \Theta_{\mathrm{Mo}}$ for every ergodic measure ρ on the base space.*

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Proof. As checked in §3, we have $\Theta_{\mathrm{Ly}}(\rho) \subset \Theta_{\mathrm{Mo}}$. On the other hand, by Proposition 5.3, any regular Lyapunov exponent is a Morse exponent, that is, $H_{\mathrm{Ly}}(\rho) \subset \Lambda_{\mathrm{Mo}}$. Hence, $\alpha(H_{\mathrm{Ly}}(\rho)) = 0$ if $\alpha \in \Theta_{\mathrm{Mo}}$, showing that $\Theta_{\mathrm{Mo}} \subset \Theta_{\mathrm{Ly}}(\rho)$. □

Now we look at the decompositions of the flag bundles.

PROPOSITION 6.2. *Let μ be an attractor measure on $\mathbb{E}$. Then its support* supp μ *is contained in the unique attractor component $\mathcal{M}^+$ of the finest Morse decomposition.*

Proof. Each point in suppμ is recurrent and hence belongs to the set of chain recurrent points which is the union of the Morse components.

Now, since μ is an attractor measure we have

$$\lambda_\mu = \int q \, d\mu \in \mathrm{cl}\mathfrak{a}^+.$$

This integral is the $\mathfrak{a}$-Lyapunov exponent of μ-almost all $z \in \mathrm{supp}\mu$ and hence is contained in the Morse spectrum. Actually, $\lambda_\mu \in \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$, the Morse spectrum of $\mathcal{M}^+$, because this is the only Morse component whose spectrum meets $\mathrm{cl}\mathfrak{a}^+$. Therefore, for μ-almost all $z \in \mathrm{supp}\mu$, $z \in \mathcal{M}^+$. Since $\mathcal{M}^+$ is compact, it follows that $\mathrm{supp}\mu \subset \mathcal{M}^+$. □

By taking the backward flow, we get a similar result for the repeller measures.

PROPOSITION 6.3. *Let μ be a repeller measure on $\mathbb{E}$. Then its support* suppμ *is contained in the unique repeller component $\mathcal{M}^-$ of the finest Morse decomposition.*

PROPOSITION 6.4. *Let $\mathcal{O}$ be a component of the Oseledets decomposition in a flag bundle $\mathbb{E}_\Theta$. Then there exists a component $\mathcal{M}$ of the Morse decomposition of $\mathbb{E}_\Theta$ such that $\mathcal{O} \subset \mathcal{M}$.*

Proof. Let $\mu_{\Theta_{\mathrm{Ly}}}$ be the only attractor measure in $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ and $\mu_{\Theta^*_{\mathrm{Ly}}}$ the repeller measure in $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$. These are projections of attractor and repeller measures on $\mathbb{E}$. Hence, the above lemmas imply that $\mathrm{supp}\mu_{\Theta_{\mathrm{Ly}}} \subset \mathcal{M}^+_{\Theta_{\mathrm{Ly}}}$ and $\mathrm{supp}\mu_{\Theta^*_{\mathrm{Ly}}} \subset \mathcal{M}^+_{\Theta^*_{\mathrm{Ly}}}$. However, we checked in §3 that $\Theta_{\mathrm{Ly}} \subset \Theta_{\mathrm{Mo}}$. Hence, by Lemma 2.2, we conclude that the fixed point set—in any flag bundle—of the Oseledets section χ_{Ly} is contained in the fixed point set of the Morse section χ_{Mo}. This means that the Oseletets components are contained in the Morse components. □

Remark. It is proved in [**5**, Corollary 5.5.17] that the Oseledets decomposition is contained in the Morse decomposition for a linear flow on a vector bundle.

7. *Lyapunov exponents in the tangent bundle $T^f\mathbb{E}_{\Theta_{\mathrm{Ly}}}$*

A fiber of a flag bundle $\mathbb{E}_\Theta$ is a differentiable manifold and hence has a tangent bundle. Gluing together the tangent bundles to the fibers of $\mathbb{E}_\Theta$, we get a vector bundle $T^f\mathbb{E}_\Theta \to \mathbb{E}_\Theta$ over $\mathbb{E}_\Theta$. The flow ϕ_t on $\mathbb{E}_\Theta$ is differentiable along the fibers with differential map ψ_t, a linear map of the vector bundle $T^f\mathbb{E}_\Theta$. (See [**16**] for a construction of this vector bundle as an associated bundle $Q \times_G V$.)

We look here at the Lyapunov exponents for the linear flow ψ_t on $T^f\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ with respect to an attractor measure of ϕ_t. These Lyapunov exponents will be used in the proof of the main technical lemma to describe the ω-limit sets (with respect to ϕ_t) in the flag bundles.

Equip the bundle $T^f\mathbb{E}_\Theta$ with a Riemannian metric $\langle\cdot\,,\cdot\rangle$, which can be built from a K-reduction R of the principal bundle Q. (Roughly, the metric $\langle\cdot\,,\cdot\rangle$ is constructed by piecing together K-invariant metrics on the fibers. See [**16**] for details.)

PROPOSITION 7.1. *Let ν be an ergodic measure on X and denote by μ its attractor measure on the bundle $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$. Let $H(\nu)\in \mathrm{cl}\mathfrak{a}^+$ be the polar exponent of ν. Then the Lyapunov spectrum of ψ with respect to μ is $\mathrm{ad}(H(\mu))_{|\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}}}$, which is a diagonal linear map of $\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}}$ (that is, an element of a Weyl chamber $\mathrm{cl}\mathfrak{a}^+$ of $\mathfrak{gl}(\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}})$).*

Proof. Denote by $\mathcal{O}(\nu)$ the $Z_{H_{\mathrm{Ly}}}$-measurable reduction of Q, corresponding to the Oseledets section of ν. This reduction is a principal bundle with structural group $Z_{H_{\mathrm{Ly}}}$ over a set $\Omega\subset X$ with $\nu(\Omega)=1$ (see [**1**]).

The section $\xi_{\mathrm{Ly}}:\Omega\to\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ gives a disintegration of μ with respect to ν by Dirac measures. Let $\Omega^\#$ be the image of this section. Then the restriction of $T^f\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ to $\Omega^\#$ is a vector bundle $T^f\Omega^\#\to\Omega^\#$, which is invariant by the differential flow ψ_t.

We can build the vector bundle $T^f\Omega^\#$ as an associated bundle $\mathcal{O}(\nu)$ through the adjoint representation θ of $Z_{H_{\mathrm{Ly}}}$ in $\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}}$.

Then, if we take compatible Cartan decompositions of $\theta(Z_{H_{\mathrm{Ly}}})$ and $\mathrm{Gl}(\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}})$, it follows that the polar exponent of ψ_t is precisely $\theta(H(\mu))$. By the constructions of [**1**, §8], it follows that the Lyapunov exponents of ψ_t are the eigenvalues of $\theta(H(\mu))$ as linear maps of $\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}}$. □

COROLLARY 7.2. *Suppose that $\Theta_{\mathrm{Ly}}\subset\Theta$. Then the Lyapunov exponents of ψ_t in $T^f\mathbb{E}_\Theta$ with respect to the attractor measure μ are strictly negative.*

Later on, we will combine this corollary with the following general fact about Lyapunov exponents on vector bundles. Let $p:V\to X$ be a continuous vector bundle endowed with a norm $\|\cdot\|$. Let Φ_n be a continuous linear flow on V. If ν is a Φ-invariant ergodic measure on the base X, then Φ has a Lyapunov spectrum $H_{\mathrm{Ly}}(\nu)=\{\lambda_1\geq\cdots\geq\lambda_n\}$ with respect to ν, as ensured by the multiplicative ergodic theorem. The following lemma may be well known. For the sake of completeness, we prove it here using the Morse spectrum of the linear flow.

LEMMA 7.3. *Suppose that for every Φ-invariant ergodic measure ν on X the spectrum with respect to ν is strictly negative. Then, for every $v\in V$,*

$$\lim_{n\to+\infty}\|\Phi_n v\|=0.$$

Proof. Let $p:\mathbb{P}V\to X$ be the projective bundle of V. The cocycle $\rho(n,v)=\|\Phi_n v\|/\|v\|$ on V induces the additive cocycle $a(n,\eta)=\log\rho(n,\eta)$, $\eta\in\mathbb{P}V$, whose asymptotics gives the Lyapunov spectrum of Φ. Write $q(\cdot)=a(1,\cdot)$. Then, by general results on the Morse spectrum of an additive cocycle (see [**19**, §3] and references therein), the Morse spectrum of a is a union of intervals whose extreme points are integrals $\int q(x)\mu\,(dx)$ with respect

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to ergodic invariant measures μ for the flow on $\mathbb{P}V$. By the Birkhoff ergodic theorem, it follows that for μ-almost all $\eta \in \mathbb{P}V$,

$$\lim_{n\to\infty} \frac{1}{n} a(n, \eta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} q(\Phi_k \eta) = \int q(z)\mu(dz).$$

On the other hand, the projection $p_*\mu = \nu$ is ergodic on the base X. Hence, by assumption, the spectrum with respect to ν, given by the multiplicative ergodic theorem, is strictly negative. This means that for ν-almost all $x \in X$, $\lim_{n\to\infty} (1/n)a(n, \eta)$ exists for every $\eta \in p^{-1}\{x\}$ and is strictly negative. Combining these two facts, we conclude that

$$\int q(z)\mu(dz) < 0$$

and therefore the Morse spectrum is contained in $(-\infty, 0)$.

Now, for every $\eta \in \mathbb{P}V$, $\limsup_{n\to+\infty} (1/n)a(n, \eta)$ belongs to the Morse spectrum (see [**5**, Theorem 5.3.6]). Hence, for every $0 \neq v \in V$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi_n v\| < 0.$$

This implies that for large n, $\|\Phi_n v\| < e^{cn}$, $c < 0$, proving the lemma. □

Applying the lemma for the backward flow, we have the following result.

COROLLARY 7.4. *With the same assumptions of the lemma, we have* $\lim_{n\to-\infty} \|\Phi_n v\| = \infty$ *if* $v \neq 0$.

8. *Main technical lemma*

LEMMA 8.1. *Let $\mathbb{E}_\Theta$ be a flag manifold with dual $\mathbb{E}_{\Theta^*}$. Suppose that there are three compact ϕ-invariant subsets A, $B \subset \mathbb{E}_\Theta$ and $C \subset \mathbb{E}_{\Theta^*}$ that project onto X and such that:*

(1) *$A \cap B = \emptyset$;*

(2) *B^c is the set of elements transversal to C (that is, an element $v \in \mathbb{E}_\Theta$ belongs to B if and only if it is not transversal to some $w \in C$ in the same fiber as v);*

(3) *for any ergodic measure ρ for the flow on the base space X, we have $\Theta_{\mathrm{Ly}}(\rho) \subset \Theta$. By Corollary 5.8, this implies that there are a unique attractor measure $\mu^+_\Theta(\rho)$ for ρ on $\mathbb{E}_\Theta$ and a unique repeller measure $\mu^-_{\Theta^*}(\rho)$ on $\mathbb{E}_{\Theta^*}$;*

(4) *for any ergodic measure ρ on X, $\mathrm{supp}\mu^+_\Theta(\rho) \subset A$ and $\mathrm{supp}\mu^-_{\Theta^*}(\rho) \subset C$.*

Then (A, B) is an attractor–repeller pair on $\mathbb{E}_\Theta$. That is, the ω-limit set $\omega(v) \subset A$ if $v \notin B$ while the α-limit set $\omega^*(v) \subset B$ if $v \notin A$.

The proof of this lemma will be done in several steps. Before starting, we define a fourth set $D \subset Q \times_G (\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*})$ by

$$D = \pi_1^{-1}(A) \cap \pi_2^{-1}(C),$$

where $\pi_1 : Q \times_G (\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}) \to \mathbb{E}_\Theta$ and $\pi_2 : Q \times_G (\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}) \to \mathbb{E}_{\Theta^*}$ are the projections. This set is compact and invariant and, by the transversality given by the first and second conditions in the lemma, we can view D as a compact subset of the bundle

$$\mathcal{A}_\Theta = Q \times_G \mathrm{Ad}(G)H_\Theta,$$

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where $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. In fact, $\mathrm{Ad}(G)H_\Theta$ is in bijection with the set of transversal pairs in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ (see e.g. [**20**]).

Now, to start the proof, fix $x \in X$ that has a periodic orbit $\mathcal{O}(x)$. Then the Oseledets decomposition coincides with the Morse decomposition above $\mathcal{O}(x)$, which, by §4, is built from the dynamics of the action of a $g_x \in G$. Clearly, the homogeneous measure θ on the periodic orbit is an ergodic invariant measure on X. By the third condition of the lemma, g_x has one attractor fixed point at $\mathbb{F}_\Theta$, say b^+, and a repeller fixed point $b^- \in \mathbb{F}_{\Theta^*}$. The Morse decomposition of g_x is the union of $\{b^+\}$ with subsets whose elements are not transversal to b^-. It follows that the attractor $\mu^+_\Theta(\theta)$ and the repeller $\mu^-_{\Theta^*}(\theta)$ measures are homogeneous measures on periodic orbits. By the fourth condition of the lemma, $\mathrm{supp}\mu^+_\Theta(\theta) \subset A$ and $\mathrm{supp}\mu^-_{\Theta^*}(\theta) \subset C$, so that by the second condition the Morse decomposition above $\mathcal{O}(x)$ is the union of $\mathrm{supp}\mu^+_\Theta(\theta)$ with subsets contained in B. Hence, if v is in the fiber above x, then $\omega(v) \subset A$ if $v \notin B$, while $\omega^*(v) \subset B$ if $v \notin A$.

Therefore, from now on we look at ω-limits $\omega(v)$ and $\omega^*(v)$ assuming that the orbit $\mathcal{O}(x)$ of $x = \pi(v)$ is not periodic, that is, the map $n \in \mathbb{Z} \mapsto x_n = \phi_n(x) \in X$ is injective and $\mathcal{O}(x)$ is in bijection with $\mathbb{Z}$.

To prove that the ω-limits are contained in A, we will use the following consequence of Lemma 7.3.

LEMMA 8.2. *Given $v \in A$, let $w \in T^f_v \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ be a tangent vector at v. Then $\lim_{t\to+\infty} \|\psi_t w\| = 0$.*

Proof. Is an immediate consequence of Lemma 7.3, combined with the third and fourth conditions of the lemma. □

Now, above the non-periodic orbit $\mathcal{O}(x)$ we reduce the flow to just a sequence g_n of elements of the subgroup Z_{H_Θ}. The construction is the following: start with an element $\eta_0 \in D$ in the fiber over x. The orbit $\mathcal{O}(\eta_0)$ is the sequence $\eta_n = \phi_n(\eta_0)$, $n \in \mathbb{Z}$, that can be viewed as a section over $\mathcal{O}(x)$ by $x_n \mapsto \eta_n$. The elements of the associated bundle $\mathcal{A}_\Theta$ are written as $p \cdot H_\Theta$, $p \in R$, where as before R is the K-reduction of $Q \to X$. Hence, there exists a sequence $p_n \in R$ such that $\eta_n = p_n \cdot H_\Theta$.

Since p_{n+m} and $\phi_n(p_m)$ are in the same fiber, we have $\phi_n(p_m) = p_{n+m} \cdot g_{n,m}$ with $g_{n,m} \in G$, $n, m \in \mathbb{Z}$. Actually, $g_{n,m} \in Z_{H_\Theta}$ because $\phi_n(\eta_m) = \eta_{n+m}$, so that

$$p_{n+m} \cdot \mathrm{Ad}(g_{n,m})H_\Theta = \phi_n(\eta_m) = \eta_{n+m} = p_{n+m} \cdot H_\Theta.$$

We write ξ_n and ξ^*_n for the projections of η_n into $\mathbb{E}_\Theta$ and $\mathbb{E}_{\Theta^*}$, respectively. By definition of D, we have $\xi_n \in A$ and $\xi^*_n \in C$. Hence, by the second assumption of the lemma, the elements in $\mathbb{E}_\Theta$ that are not transversal to ξ^*_n are contained in B. In other words, we have the following result.

LEMMA 8.3. *Take $v \notin B$ in the fiber of x. Then v is transversal to ξ^*_0.*

Now we use Lyapunov exponents of the lifting ψ_n of ϕ_n to $T^f\mathbb{E}_\Theta$ to show that $\omega(v) \subset A$ if $v \notin B$ is in the fiber of x.

To do that, we first note that if the starting element $\eta_0 \in \mathcal{A}_\Theta$ is written as $\eta_0 = p \cdot H_\Theta$, $p \in R$, then the set of points that are transversal to ξ^*_0 is given algebraically by

$$T = p \cdot (N^-_\Theta \cdot b_0) = \{p \cdot nb_0 : n \in N^-_\Theta\},$$

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where b_0 is the origin of the flag $\mathbb{F}_\Theta$ and N_Θ^- is the nilpotent subgroup with Lie algebra $\mathfrak{n}_\Theta^- = \sum_{\alpha\notin\langle\Theta\rangle,\alpha<0} \mathfrak{g}_\alpha$ (lower triangular matrices).

Since $\exp : \mathfrak{n}_\Theta^- \to N_\Theta^-$ is a diffeomorphism, we have also $T = \{p \cdot (\exp Y \cdot b_0) : Y \in \mathfrak{n}_\Theta^-\}$.

The action of ϕ_n on an element $p \cdot (\exp Y \cdot b_0) \in T$ is given as follows: put $g_n = g_{n,0} \in Z_{H_\Theta}$. Then, as remarked above, $\phi_n(p) = p_n \cdot g_n$, so that

$$\phi_n(p \cdot (\exp Y \cdot b_0)) = p_n \cdot (g_n \exp Y g_n^{-1} \cdot g_n b_0).$$

But $g_n b_0 = b_0$ because $g_n \in Z_H$, and $g_n \exp Y g_n^{-1} = \exp(\mathrm{Ad}(g_n)Y)$. Hence,

$$\phi_n(p \cdot (\exp Y \cdot b_0)) = p_n \cdot \exp(\mathrm{Ad}(g_n)Y)b_0. \tag{1}$$

The next lemma relates this action with the lifting ψ_n of ϕ_n to the tangent space $T^f\mathbb{E}_\Theta$.

LEMMA 8.4. *Given $Y \in \mathfrak{g}$, denote by $p \cdot Y$ the vertical tangent vector $(d/dt)(p \cdot (\exp tY \cdot b_0))_{t=0} \in T^f_{p\cdot b_0}\mathbb{E}_\Theta$. Then $p \cdot Y$, $Y \in \mathfrak{n}_\Theta^-$, fulfill the vertical tangent space $T^f_{p\cdot b_0}\mathbb{E}_\Theta$, and the derivative ψ_n of ϕ_n at $p \cdot b_0$ satisfies*

$$\psi_n(p \cdot Y) = p_n \cdot \mathrm{Ad}(g_n)Y.$$

Proof. The fact that any tangent vector in $T^f_{p\cdot b_0}\mathbb{E}_\Theta$ is given by $p \cdot Y$ for some $Y \in \mathfrak{n}_\Theta^-$ is due to the fact that $N_\Theta^- \cdot b_0$ is an open submanifold of $\mathbb{F}_\Theta$. For the last statement, we have

$$\begin{aligned}\psi_n(p \cdot Y) &= \frac{d}{dt}\phi_n(p \cdot (\exp tY \cdot b_0))_{t=0} \\ &= p_n \cdot \frac{d}{dt}(\exp(t\mathrm{Ad}(g_n)Y)b_0)_{t=0} = p_n \cdot \mathrm{Ad}(g_n)Y.\end{aligned}$$ □

We are now prepared to prove that $\omega(v) \subset A$ if $v \notin B$ is in the fiber of x. We have $v = p \cdot (\exp Y \cdot b_0)$ for some $Y \in \mathfrak{n}_\Theta^-$, so that by (1) $\phi_n(v) = p_n \cdot \exp(\mathrm{Ad}(g_n)Y)b_0$.

Now, by Corollary 7.2 and Lemma 7.3, we have $\lim_{n\to+\infty} \|\psi_n w\| = 0$ if $w \in T^f_{p\cdot b_0}\mathbb{E}_\Theta$. Taking $w = p \cdot Y$, we have $\|p \cdot Y\| = \|Y\|$ because $p \in R$ and hence it is an isometry between the flag manifold $\mathbb{F}_\Theta$ and the corresponding fiber of $\mathbb{E}_\Theta$ (see [**16**] for the construction of the norm in $T^f\mathbb{E}_\Theta$). Since the same remark holds for $p_n \in R$, we have $\|\psi_n w\| = \|\mathrm{Ad}(g_n)Y\|$, so that

$$\mathrm{Ad}(g_n)Y \to 0 \quad \text{and} \quad \exp \mathrm{Ad}(g_n)Y \to 1.$$

This implies that if d is the metric on $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$, then $d(\phi_n(v), p_n \cdot b_0) \to 0$. But $p_n \cdot b_0 = \xi_n \in A$ as well as its limit points, by invariance and compactness of A. Therefore, we conclude that $\omega(v) \subset A$ if $v \notin B$.

We turn now to the proof that $\omega^*(v) \subset B$ if $v \notin A$, again with v above a non-periodic orbit.

Take a sequence $n_k \to -\infty$ such that $\phi_{n_k} v$ converges in $\mathbb{E}_\Theta$. Taking subsequences, we assume the convergences $p_{n_k} \to p \in R$, $\eta_{n_k} \to \eta$, $\xi_{n_k} \to \xi$ and $\xi^*_{n_k} \to \xi^*$.

By invariance, it is enough to take $v \notin B$, so that we can write $v = p_0 \cdot (\exp Y)b_0$ with $Y \in \mathfrak{n}_\Theta^-$ and $Y \neq 0$ (because $v \notin A$).

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Taking subsequences again, we assume that $g_{n_k}(\exp Y)b_0$ converges to $b_1 \in \mathbb{F}_\Theta$. Since $g_n \in Z_{H_\Theta}$, we have $g_n b_0 = b_0$ and hence

$$(\exp \operatorname{Ad}(g_{n_k})Y)b_0 = g_{n_k}(\exp Y)b_0 \to b_1.$$

Now $\operatorname{Ad}(g_n)Y \to \infty$ in $\mathfrak{n}^-_{\Theta_{\mathrm{Ly}}}$ because the Lyapunov exponents for the backward flow are greater than zero. This implies that b_1 is not transversal to the origin b_0^* of $\mathbb{F}_{\Theta^*}$. Hence, $p_{n_k} \cdot b_1$ is not transversal to $p_{n_k} \cdot b_0^*$, so that $p_{n_k} \cdot b_1 \in B$. But

$$\phi_{n_k} v = p_{n_k} g_{n_k} \cdot (\exp Y)b_0 = p_{n_k} \cdot (\exp \operatorname{Ad}(g_{n_k})Y)b_0,$$

so that $\lim \phi_{n_k} v = p \cdot b_1$, showing that $\omega^*(v) \subset B$.

In conclusion, we have compact invariant sets A and B that satisfy $\omega(v) \subset A$ and $\omega^*(v) \subset B$ if $v \notin A \cup B$. Hence, A and B define a Morse decomposition of $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ with A the attractor component.

9. *Three conditions*

In this section we state three conditions that together are necessary and sufficient to have equality between the Lyapunov and Morse decompositions over an ergodic invariant measure.

Thus, as in §3, let ϕ_n be a continuous flow on a continuous principal bundle $\pi : Q \to X$ whose structural group G is reductive and non-compact. We fix once and for all an ergodic invariant measure ν on the base space having support $\operatorname{supp} \nu = X$. Then the $\mathfrak{a}$-Lyapunov exponents of ϕ_n select a flag type, which is expressed by a subset Θ_{Ly} of simple roots. The flow on X is chain transitive, so it also has a flag type Θ_{Mo} coming from the Morse decomposition and $\mathfrak{a}$-Morse spectrum.

We start by writing down the three conditions and check that they are necessary. In the next section we prove that together they are also sufficient to have $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$.

9.1. *Bounded section.*

The Oseledets section is a measurable section $\chi_{\mathrm{Ly}} : \Omega \to Q \times_G \mathcal{O}_{\mathrm{Ly}}$ of the associated bundle $Q \times_G \mathcal{O}_{\mathrm{Ly}} \to X$ above the set of full ν-measure Ω. The fiber of this bundle is the adjoint orbit $\mathcal{O}_{\mathrm{Ly}} = \operatorname{Ad}(G)H_{\mathrm{Ly}}$. Associated to this section there is the equivariant map $f_{\mathrm{Ly}} : Q_\Omega \to \mathcal{O}_{\mathrm{Ly}}$ defined above Ω, where $Q_\Omega = \pi^{-1}(\Omega)$ and $\pi : Q \to X$ is the projection. The map f_{Ly} is defined by $f_{\mathrm{Ly}}(q) = q^{-1} \cdot \chi_{\mathrm{Ly}}(x)$ with $x = \pi(q)$, where q is viewed as a map from $\mathcal{O}_{\mathrm{Ly}}$ to a fiber of $Q \times_G \mathcal{O}_{\mathrm{Ly}}$.

Let $R \subset Q \to X$ be a (continuous) K-reduction of Q and write $R_\Omega = \pi^{-1}(\Omega)$. Then we say that the Oseledets section is *bounded* if:

- f_{Ly} is bounded in R_Ω.

This definition does not depend on the specific K-reduction because the base space X is assumed to be compact.

If $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$, then we can take $H_{\mathrm{Mo}} = H_{\mathrm{Ly}}$ and $\chi_{\mathrm{Mo}} = \chi_{\mathrm{Ly}}$, so that f_{Ly} is continuous and hence bounded by compactness.

Hence, boundedness is a necessary condition.

Example. Let ϕ be a linear flow on a d-dimensional trivial vector bundle $X \times V$ with two Lyapunov exponents $\lambda_1 > \lambda_2$ whose Oseledets subspaces have dimensions k and $d - k$.

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Then H_{Ly} is chosen to be a diagonal matrix $\mathrm{diag}\{\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2\}$ with λ_1 having multiplicity k. Since the bundle is trivial, the Oseledets section is ultimately given by a map $f_{\mathrm{Ly}} : \Omega \to \mathrm{Ad}(G)H_{\mathrm{Ly}}$, $f_{\mathrm{Ly}}(x) = f_{\mathrm{Ly}}(x, 1)$, with obvious abuse of notation. Then the Oseledets subspaces at $x \in X$ are the eigenspaces $V_{\lambda_1}(x)$ and $V_{\lambda_2}(x)$ of $f_{\mathrm{Ly}}(x) \in \mathrm{Ad}(G)H_{\mathrm{Ly}}$. To say that f_{Ly} is bounded means that the subspaces $V_{\lambda_1}(x)$ and $V_{\lambda_2}(x)$ have a positive distance.

9.2. *Refinement of the Lyapunov spectrum of other invariant measures.* Denote by $\mathcal{P}_X(\phi)$ the set of ϕ-invariant probability measures on X. For each ergodic measure $\rho \in \mathcal{P}_X(\phi)$, we have its Lyapunov spectrum $H_{\mathrm{Ly}}(\rho)$ and the corresponding flag type $\Theta_{\mathrm{Ly}}(\rho) = \{\alpha \in \Sigma : \alpha(H_{\mathrm{Ly}}(\rho)) = 0\}$. Our second condition is:

- $\Theta_{\mathrm{Ly}}(\rho) \subset \Theta_{\mathrm{Ly}}(\nu)$ for every ergodic $\rho \in \mathcal{P}_X(\phi)$.

This is a necessary condition for $\Theta_{\mathrm{Ly}}(\nu) = \Theta_{\mathrm{Mo}}$. To see this, let $\rho \in \mathcal{P}_X(\phi)$ be ergodic and denote by $Y \subset X$ its support. Let $\Theta_{\mathrm{Mo}}(Y)$ be the flag type of the Morse decomposition of the flow restricted to Y (that is, to the fibers above Y). We have $\Theta_{\mathrm{Mo}}(Y) \subset \Theta_{\mathrm{Mo}}$ because the Morse components of the flow restricted to Y are contained in the components over X. However, $\Theta_{\mathrm{Ly}}(\rho) \subset \Theta_{\mathrm{Mo}}(Y)$, so that the equality $\Theta_{\mathrm{Mo}} = \Theta_{\mathrm{Ly}}(\nu)$ implies that

$$\Theta_{\mathrm{Ly}}(\rho) \subset \Theta_{\mathrm{Mo}}(Y) \subset \Theta_{\mathrm{Mo}} = \Theta_{\mathrm{Ly}}(\nu).$$

Example. Let ϕ be a linear flow on a d-dimensional vector bundle $X \times V$ with Lyapunov spectrum $\lambda_1 > \cdots > \lambda_s$ with multiplicities $k_1, \ldots, k_s$ with respect to ν. Then this condition means that the Lyapunov spectrum $\mu_1 > \cdots > \mu_t$ with respect to another ergodic measure ρ has multiplicities $r_1, \ldots, r_t$ that satisfy $r_1 + \cdots + r_{i_1} = k_1$, $r_{i_1} + \cdots + r_{i_2} = k_2$, etc.

9.3. *Attractor and repeller measures.* Recall that we defined an attractor measure on a partial flag manifold $\mathbb{E}_\Theta$ to be the projection of an ergodic invariant measure μ on $\mathbb{E}$ such that

$$H_{\mathrm{Ly}}(\mu) = \int q \, d\mu \in \mathrm{cl}\mathfrak{a}^+.$$

In the specific flag bundle $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$, the attractor measure $\mu^+_{\Theta_{\mathrm{Ly}}}$ is unique and has a disintegration over ν by Dirac measures on the fibers above a set Ω of total measure ν. We denote by $\mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ the support of $\mu^+_{\Theta_{\mathrm{Ly}}}$.

Analogously, in the dual flag bundle $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$ there is a unique repeller measure $\mu^-_{\Theta_{\mathrm{Ly}}}$. We denote by $\mathrm{rep}_{\Theta^*_{\mathrm{Ly}}}(\nu)$ the support of $\mu^-_{\Theta_{\mathrm{Ly}}}$.

Let $\rho \in \mathcal{P}_X(\phi)$ be an ergodic measure with support the compact set $Y \subset X$, which is an invariant subset. The subset $\pi^{-1}_{\Theta_{\mathrm{Ly}}}(Y) \cap \mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ is invariant as well. We denote by $\mathcal{E}^+_{\Theta_{\mathrm{Ly}}}(\rho)$ the set of ergodic probability measures with support contained in $\pi^{-1}_{\Theta_{\mathrm{Ly}}}(Y) \cap \mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ that project down to ρ. Also, we put $\mathcal{E}^-_{\Theta^*_{\mathrm{Ly}}}(\rho)$ for the set of ergodic probability measures with support in $\pi^{-1}_{\Theta_{\mathrm{Ly}}}(Y) \cap \mathrm{rep}_{\Theta^*_{\mathrm{Ly}}}(\nu)$ that project down to ρ. Both sets $\mathcal{E}^+_{\Theta_{\mathrm{Ly}}}(\rho)$ and $\mathcal{E}^-_{\Theta^*_{\mathrm{Ly}}}(\rho)$ are not empty, since in the compact invariant sets $\pi^{-1}_{\Theta_{\mathrm{Ly}}}(Y) \cap \mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ and

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$\pi_{\Theta_{\mathrm{Ly}}}^{-1}(Y) \cap \mathrm{rep}_{\Theta_{\mathrm{Ly}}^{*}}(\nu)$ there are occupation measures that project down to ρ (cf. the proof of Proposition 5.3).

Now we can state our third condition.

- Any $\theta \in \mathcal{E}_{\Theta_{\mathrm{Ly}}}^{+}(\rho)$ is an attractor measure and any $\theta \in \mathcal{E}_{\Theta_{\mathrm{Ly}}^{*}}^{-}(\rho)$ is a repeller measure for ϕ.

If $\Theta_{\mathrm{Mo}} = \Theta_{\mathrm{Ly}}(\nu)$, then the attractor Morse component $\mathcal{M}_{\Theta_{\mathrm{Ly}}} = \mathcal{M}_{\Theta_{\mathrm{Ly}}}(1)$ in $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ is the image of a section $\xi : X \to \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ and contains the support $\mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ of the attractor measure. This implies that $\mathcal{M}_{\Theta_{\mathrm{Ly}}} = \mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$, so that $\mathcal{M} = \pi_{\Theta_{\mathrm{Ly}}}^{-1}(\mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu))$ is the attractor Morse component $\mathcal{M}$ in the maximal flag bundle. Now the Morse spectrum $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ of $\mathcal{M}$ is contained in the cone

$$\mathfrak{a}_{\Theta_{\mathrm{Mo}}}^{+} = \{H \in \mathfrak{a} : \forall \alpha \notin \langle \Theta_{\mathrm{Mo}} \rangle, \alpha(H) > 0\}$$

(see [**19**]). Hence, any Lyapunov exponent of $\mathcal{M}$ belongs to $\mathfrak{a}_{\Theta_{\mathrm{Mo}}}^{+}$. By projecting down to $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ the measures with support in $\mathcal{M}$, we see that any θ with support in $\pi_{\Theta_{\mathrm{Ly}}}^{-1}(Y) \cap \mathrm{att}_{\Theta_{\mathrm{Ly}}}(\nu)$ is an attractor measure.

The same proof with the backward flow shows that $\theta \in \mathcal{E}_{\Theta_{\mathrm{Ly}}^{*}}^{-}(\rho)$ is a repeller measure.

9.4. *Oseledets decompositions for other measures.* The second and third conditions above refer to ergodic measures ρ on X different from the initial measure ν. These two conditions can be summarized in just one condition on the Oseledets section for the ergodic measures $\rho \in \mathcal{P}_X(\phi)$.

Given $\rho \in \mathcal{P}_X(\phi)$, write χ^{ρ} for its Oseledets section and ξ^{ρ} and $\xi^{\rho *}$ for the corresponding sections on $\mathbb{E}_{\Theta_{\mathrm{Ly}}(\rho)}$ and $\mathbb{E}_{\Theta_{\mathrm{Ly}}^{*}(\rho)}$, respectively.

Definition 9.1. We say that χ^{ρ} is contained in the Oseledets section of ν in case the following two conditions are satisfied.

(1) $\Theta_{\mathrm{Ly}}(\rho) \subset \Theta_{\mathrm{Ly}}(\nu)$. In this case there is the fibration $p : Q \times_G \mathrm{Ad}(G)(H_{\mathrm{Ly}}(\rho)) \to Q \times_G \mathrm{Ad}(G)(H_{\mathrm{Ly}}(\nu))$.

(2) $p(\mathrm{im}\chi^{\rho}) \subset \mathrm{cl}(\mathrm{im}\chi)$, where χ is the Oseledets section of ν.

The second condition implies that the images of the sections ξ^{ρ} and $\xi^{\rho *}$ project onto $\mathrm{cl}(\mathrm{im}\xi)$ and $\mathrm{cl}(\mathrm{im}\xi^{*})$, by the fibrations $\mathbb{E}_{\Theta_{\mathrm{Ly}}(\rho)} \to \mathbb{E}_{\Theta_{\mathrm{Ly}}(\nu)}$ and $\mathbb{E}_{\Theta_{\mathrm{Ly}}^{*}(\rho)} \to \mathbb{E}_{\Theta_{\mathrm{Ly}}^{*}(\nu)}$, respectively.

Since the attractor and repeller measures for ρ disintegrate according to the sections ξ^{ρ} and $\xi^{\rho *}$, respectively, it follows that the second and third conditions above are equivalent to having χ^{ρ} contained in χ.

10. *Sufficiency of the conditions*

We apply here the main lemma (Lemma 8.1) to get sufficiency of the conditions of the last section and thus prove the following characterization for the equality of Morse and Oseledets decompositions.

THEOREM 10.1. *Suppose that the invariant measure on the base space is ergodic. Then the three conditions together—bounded section (9.1), refinement of Lyapunov*

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spectrum (9.2) and attractor–repeller measures (9.3)—are necessary and sufficient to have $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ *and* $\chi_{\mathrm{Ly}} = \chi_{\mathrm{Mo}}$.

As before, we have the sections $\xi : \Omega \to \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ and $\xi^* : \Omega \to \mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$, respectively, that are combined to give the Oseledets section $\chi_{\mathrm{Ly}} : \Omega \subset X \to \mathcal{A}_{\Theta_{\mathrm{Ly}}}$.

We apply Lemma 8.1 with:

(1) $A = \mathrm{cl}(\mathrm{im}\xi)$, which is the support of the unique attractor measure $\mu^+_{\Theta_{\mathrm{Ly}}}$ in $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$;

(2) $C = \mathrm{cl}(\mathrm{im}\xi^*)$, which is the support of the unique repeller measure $\mu^-_{\Theta_{\mathrm{Ly}}}$ in $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$; and

(3) $B = \mathrm{cl} \bigcup_{w \neq 1} \mathrm{st}_{\Theta_{\mathrm{Ly}}}(x, w)$, $x \in \Omega$. That is, B is the closure of the set of elements that are *not* transversal to $\xi^*(x)$, $x \in \Omega$.

Alternatively, we have the following characterization of B in terms of the closure of the dual section ξ^*.

PROPOSITION 10.2. *An element* $v \in \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ *belongs to* B *if and only if it is not transversal to some* $w \in \mathrm{cl}(\mathrm{im}\xi^*)$ *in the same fiber as* v.

Proof. Take local trivializations so that locally $\mathbb{E}_{\Theta_{\mathrm{Ly}}} \simeq U \times \mathbb{F}_{\Theta_{\mathrm{Ly}}}$, $\mathbb{E}_{\Theta^*_{\mathrm{Ly}}} \simeq U \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$ ($U \subset X$ open), $\xi : U \to \mathbb{F}_{\Theta_{\mathrm{Ly}}}$ and $\xi^* : U \to \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$. If $v = (x, b) \in B$, then there exists a sequence $(x_n, b_n) \to v$ with b_n not transversal to $\xi^*(x_n)$. By taking a subsequence, we can assume that $\xi^*(x_n)$ converges to $b^* \in \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$. Then the pair $(b_n, \xi^*(x_n))$ converges to $(b, b^*) \in \mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$. Now the set of non-transversal pairs in $\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$ is closed. Hence, b and b^* are not transversal, showing that $v = (x, b)$ is not transversal to $w = (x, b^*) \in \mathrm{cl}(\mathrm{im}\xi^*)$.

Conversely, suppose that $v = (x, b) \in \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ is not transversal to $w = (x, b^*) \in \mathrm{cl}(\mathrm{im}\xi^*)$. Then $b^* = \lim \xi^*(x_n)$ with $\lim x_n = x$. By Lemma 2.1, there exists a sequence $b_n \in \mathbb{F}_{\Theta_{\mathrm{Ly}}}$ such that b_n is not transversal to $\xi^*(x_n)$ and $\lim b_n = b$. Hence, $(x_n, b_n) \in B$ and $\lim(x_n, b_n) = (x, b) = v$, showing that $v \in B$. □

Clearly, A, B and C are compact sets. Also, A and C are invariant because the sections ξ and ξ^* are invariant and, since transversality is preserved by the flow, it follows that B is invariant as well.

Now we verify that the assumptions of Lemma 8.1 hold in the presence of the three conditions of Theorem 10.1. Statements (3) and (4) of Lemma 8.1 are the same as the refinement of Lyapunov spectrum and attractor-repeller measure conditions, respectively. Item (2) of Lemma 8.1 is the above proposition. So, it remains to prove that A and B are disjoint. This is the only place where the boundedness condition is used.

PROPOSITION 10.3. *We have* $A \cap B = \emptyset$. *Precisely, if* $v \in A$ *and* $w \in \mathrm{cl}(\mathrm{im}\xi^*)$, *then* v *and* w *are transversal and, if* $v \in B$, *then there exists* $w \in \mathrm{cl}(\mathrm{im}\xi^*)$ *in the same fiber which is not transversal to* v.

Proof. Since the restriction of f_{Ly} to R_Ω is bounded, its image in $\mathcal{O}_{\mathrm{Ly}} = \mathrm{Ad}(G)H_{\mathrm{Ly}}$ has compact closure. By the equality $\chi_{\mathrm{Ly}}(x) = p \cdot f_{\mathrm{Ly}}(p)$ ($p \in Q$ with $\pi(p) = x$), it follows that $\mathrm{cl}(\mathrm{im}\chi_{\mathrm{Ly}})$ is a compact subset of the bundle $Q \times_G \mathcal{O}_{\mathrm{Ly}}$. After identifying $\mathcal{O}_{\mathrm{Ly}}$ with an open subset of $\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}$, we get a section of $Q \times_G (\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}})$ over Ω also denoted by χ_{Ly}. The image of this section is contained in the open subset of those pairs in $Q \times_G (\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}})$ that are transversal to each other. The closure of the image of χ_{Ly}

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is a compact subset contained in the open subset of $Q \times_G (\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}})$ identified with $Q \times_G \mathcal{O}_{\mathrm{Ly}}$. Hence, the closure of the image of χ_{Ly} contains only transversal pairs.

Now let $p : Q \times_G (\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}) \to \mathbb{E}_{\Theta_{\mathrm{Ly}}}$ and $p^* : Q \times_G (\mathbb{F}_{\Theta_{\mathrm{Ly}}} \times \mathbb{F}_{\Theta^*_{\mathrm{Ly}}}) \to \mathbb{E}_{\Theta^*_{\mathrm{Ly}}}$ be the canonical projections. Then

$$\xi = p \circ \chi_{\mathrm{Ly}} \quad \text{and} \quad \xi^* = p^* \circ \chi_{\mathrm{Ly}}.$$

Hence, by compactness, $p(\mathrm{cl}(\mathrm{im}\chi_{\mathrm{Ly}})) = \mathrm{cl}(\mathrm{im}\xi)$ and $p^*(\mathrm{cl}(\mathrm{im}\chi_{\mathrm{Ly}})) = \mathrm{cl}(\mathrm{im}\xi^*)$. It follows that two elements $v \in A = \mathrm{cl}(\mathrm{im}\xi)$ and $w \in \mathrm{cl}(\mathrm{im}\xi^*)$ are transversal to each other if they are in the same fiber.

On the other hand, if $v \in B$, then, by Proposition 10.2, there exists $w \in \mathrm{cl}(\mathrm{im}\xi^*)$ such that v and w are in the same fiber and are not transversal. Hence, $v \notin A$, concluding that A and B are disjoint. □

End of proof of Theorem 10.1. By Lemma 8.1, A and B define a Morse decomposition of $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$ with A the attractor component. Hence, $A = \mathrm{cl}(\mathrm{im}\xi)$ contains the unique attractor component $\mathcal{M}^+_{\Theta_{\mathrm{Ly}}}$ of the finest Morse decomposition of $\mathbb{E}_{\Theta_{\mathrm{Ly}}}$. On the other hand, by Proposition 6.4, the Oseledets component $\mathrm{im}\xi \subset \mathcal{M}^+_{\Theta_{\mathrm{Ly}}}$. Therefore, $A = \mathrm{cl}(\mathrm{im}\xi) \subset \mathcal{M}^+_{\Theta_{\mathrm{Ly}}}$, so that they are equal. □

11. *Uniquely ergodic base spaces*

When the flow on the base space has unique invariant (and hence ergodic) probability measure ν, the second and third conditions of §9 are meaningless. Hence, in this case, a necessary and sufficient condition to have equality of Oseledets and Morse decompositions is that the Oseledets section for ν is bounded (first condition of §9).

From another point of view, the Morse spectrum Λ_{Mo} of the attractor component $\mathcal{M}^+$ is a compact convex set whose extremal points are Lyapunov exponents given by integrals with respect to invariant measures on the maximal flag bundle. By the results of §5, any such integral Lyapunov exponent is a regular Lyapunov exponent of an invariant probability in the base space. Just one of these Lyapunov exponents belongs to $\mathrm{cl}\mathfrak{a}^+$, which is the polar exponent H_{Ly} associated to the measure.

Hence, Λ_{Mo} has a unique extremal point in $\mathrm{cl}\mathfrak{a}^+$ if the flow on the base space is uniquely ergodic.

PROPOSITION 11.1. *Suppose that the flow on the base space X has a unique invariant probability measure ν with* $\mathrm{supp}\,\nu = X$. *Let $H_{\mathrm{Ly}} = H_{\mathrm{Ly}}(\nu)$ be its polar exponent. Then Λ_{Mo} is the polyhedron whose vertices are wH_{Ly}, $w \in \mathcal{W}_{\Theta_{\mathrm{Mo}}}$.*

Proof. The convex set Λ_{Mo} is invariant by $\mathcal{W}_{\Theta_{\mathrm{Mo}}}$ (see §3 above and [**19**, Theorem 8.3]). Since $H_{\mathrm{Ly}} \in \Lambda_{\mathrm{Mo}}$, the polyhedron with vertices in $\mathcal{W}_{\Theta_{\mathrm{Mo}}}(H_{\mathrm{Ly}})$ is contained in Λ_{Mo}. Conversely, suppose that H is an extremal point of Λ_{Mo}. Then there exists $w \in \mathcal{W}$ such that $wH \in \mathrm{cl}\mathfrak{a}^+$. We claim that $w \in \mathcal{W}_{\Theta_{\mathrm{Mo}}}$. In fact, by Weyl group invariance of the Morse spectrum, there exists a Morse component $\mathcal{M}$ such that $wH \in \Lambda_{\mathrm{Mo}}(\mathcal{M})$ (see [**19**]). But the attractor component $\mathcal{M}^+$ is the only one whose Morse spectrum meets $\mathrm{cl}\mathfrak{a}^+$, so that $\mathcal{M} = \mathcal{M}^+$ and $wH \in \Lambda_{\mathrm{Mo}} = \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ and, since the spectra of distinct Morse components are disjoint, we have $w\Lambda_{\mathrm{Mo}} = \Lambda_{\mathrm{Mo}}$, implying that $w \in \mathcal{W}_{\Theta_{\mathrm{Mo}}}$.

Now $wH \in \mathrm{cl}\mathfrak{a}^+$ is an extremal point of $\Lambda_{\mathrm{Mo}} = w\Lambda_{\mathrm{Mo}}$ and hence $wH = H_{\mathrm{Ly}}$. Therefore, $\mathcal{W}_{\Theta_{\mathrm{Mo}}} H_{\mathrm{Ly}}$ is the set of extremal points of Λ_{Mo}, concluding the proof. □

THEOREM 11.2. *Suppose that the flow on the base space X has a unique invariant probability measure ν with* supp $\nu = X$. *Then the following conditions are equivalent.*

(1) $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$.

(2) *The Oseledets section for ν is bounded.*

(3) $\alpha(\Lambda_{\mathrm{Mo}}) = \{0\}$ *for all* $\alpha \in \Theta_{\mathrm{Mo}}$.

If these conditions hold, then $\Lambda_{\mathrm{Mo}} = \{H_{\mathrm{Ly}}\}$.

Proof. As mentioned above, the equivalence between the first two conditions is a consequence of the main theorem (Theorem 10.1) and the fact that ν is the only ergodic measure on X. Now $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ means that $\alpha(H_{\mathrm{Ly}}) = 0$ for all $\alpha \in \Theta_{\mathrm{Mo}}$. If this happens, then any $\alpha \in \Theta_{\mathrm{Mo}}$ annihilates on the polyhedron with vertices wH_{Ly}, $w \in \mathcal{W}_{\Theta_{\mathrm{Mo}}}$. Hence, $\alpha \in \Theta_{\mathrm{Mo}}$ is zero on Λ_{Mo} by the above proposition. Conversely, if (3) holds, then $\alpha(H_{\mathrm{Ly}}) = 0$ for all $\alpha \in \Theta_{\mathrm{Mo}}$ because $H_{\mathrm{Ly}} \in \Lambda_{\mathrm{Mo}}$.

Finally, if $\alpha(H_{\mathrm{Ly}}) = 0$ for all $\alpha \in \Theta_{\mathrm{Mo}}$, then $wH_{\mathrm{Ly}} = H_{\mathrm{Ly}}$ for every $w \in \mathcal{W}_{\Theta_{\mathrm{Mo}}}$, so that $\Lambda_{\mathrm{Mo}} = \{H_{\mathrm{Ly}}\}$ by the previous proposition. □

By piecing together known results in the literature, we can have examples of flows over uniquely ergodic systems for which $\Theta_{\mathrm{Ly}} \neq \Theta_{\mathrm{Mo}}$. Indeed, as proved by Furman [**10**], the Lyapunov spectrum is discontinuous at a non-uniform cocycle with values in $\mathrm{Gl}(d, \mathbb{R})$, that is, at a flow on the trivial bundle $X \times \mathrm{Gl}(d, \mathbb{R})$. The result of [**10**, Theorem 5] makes the assumption that the flow on the base space is equicontinuous, which is satisfied, e.g., by the translations on compact groups, like an irrational rotation on the circle S^1.

In Herman [**12**], there is an example of a non-uniform cocycle with values in $\mathrm{Sl}(2, \mathbb{R})$ over the irrational rotation. Thus, that example is a discontinuity point of the Lyapunov spectrum. Finally, in §13 below, we prove continuity of the whole Lyapunov spectrum if $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$. Hence, we get $\Theta_{\mathrm{Ly}} \neq \Theta_{\mathrm{Mo}}$ for the example in [**12**].

12. *Products of i.i.d. sequences*

The product of i.i.d. random elements in G yields flows on product spaces $\mathcal{X} \times G$ with plenty of invariant measures on $\mathcal{X}$. In this section we provide an example of such flow that violates the second condition of §9 and hence has distinct Morse and Oseledets decompositions.

Let $C \subset G$ be a compact subset and form the product $\mathcal{X} = C^{\mathbb{Z}}$ endowed with the compact product topology. The shift $\tau((x_n)) = (x_{n+1})_{n\in\mathbb{Z}}$ is a homeomorphism and hence defines a continuous flow on $\mathcal{X}$.

Now let μ be a probability measure with $\mathrm{supp}\mu = C$ and take the product measure $\mu^{\times\mathbb{Z}}$ on $\mathcal{X}$. Then $\mu^{\times\mathbb{Z}}$ is ergodic with respect to the shift τ and $\mathrm{supp}\mu^{\times\mathbb{Z}} = \mathcal{X}$.

These data define the continuous flow ϕ_n^μ on $\mathcal{X} \times G$ by $\phi_n^\mu(\mathbf{x}, g) = (\tau^n(\mathbf{x}), \rho^\mu(n, \mathbf{x})g)$, where $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in C^{\mathbb{Z}} \subset G^{\mathbb{Z}}$ and

$$\rho^\mu(n, \mathbf{x}) = \begin{cases} x_{n-1} \cdots x_0 & \text{if } n \geq 0, \\ x_1^{-1} \cdots x_n^{-1} & \text{if } n < 0. \end{cases}$$

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The $\mathfrak{a}$-Lyapunov spectrum of ϕ^{μ} was founded by Guivarch' and Raugi [**11**]. To state their result, we recall the following concepts.

(1) A subgroup $H \subset G$ is *totally irreducible* if it does not leave invariant a subset which is a finite union of complements of Bruhat cells in their respective closures (Schubert cells).

(2) A sequence $g_n \in G$ is said to be *contracting* with respect to the maximal flag manifold if its polar decomposition $g_n = u_n h_n v_n \in K(\mathrm{cl}A^+)K$ is such that

$$\lim_{n\to\infty} \alpha(\log h_n) = \infty$$

for every positive root α.

Now denote by G_μ and S_μ the subgroup and semigroup generated by $\mathrm{supp}\mu = C$, respectively. Then we have the following result of [**11**, Theorem 2.6].

THEOREM 12.1. *Let μ be a probability measure on G, and suppose that:*

(1) *the subgroup G_μ is totally irreducible;*

(2) *the semigroup S_μ has a contracting sequence with respect to $\mathbb{F}$.*

Then the polar exponent of ϕ^{μ} is regular, that is, it belongs to $\mathfrak{a}^+$. This means that $\Theta_{\mathrm{Ly}}(\mu^{\times\mathbb{Z}}) = \emptyset$.

Both conditions of this theorem are satisfied if S_μ has non-empty interior in G:

(1) if $\mathrm{int}S_\mu \neq \emptyset$, then $G_\mu = G$ because G is assumed to be connected;

(2) if $\mathrm{int}S_\mu \neq \emptyset$, then there exists a regular $h \in \mathrm{int}S_\mu$ (see [**17**, Lemma 3.2]). Then $h^n \in S_\mu$ is a contracting sequence with respect to $\mathbb{F}$ because $h^n \in \mathrm{cl}A^+$ (hence its polar decomposition has $u_n = v_n = 1$) and $\alpha(\log h^n) = n\alpha(\log h)$.

Hence, we get the following consequence of Guivarch' and Raugi's [**11**] result.

COROLLARY 12.2. *If $\mathrm{int}S_\mu \neq \emptyset$, then $\Theta_{\mathrm{Ly}}(\phi^{\mu}) = \emptyset$.*

Now it is easy to give an example that does not satisfy the second condition of §9 and hence $\Theta_{\mathrm{Ly}}(\phi^{\mu}) \neq \Theta_{\mathrm{Mo}}(\phi^{\mu})$. In fact, take a non-regular element $h = \exp H \in \mathrm{cl}A^+$, that is, $\Theta(H) = \{\alpha \in \Sigma : \alpha(H) = 0\} \neq \emptyset$, and a probability μ whose support C contains h in its interior. For instance,

$$\mu = \frac{1}{I} f \cdot \eta,$$

where η is Haar measure and $I = \int_G f(g)\eta(dg) < \infty$ with $f : G \to \mathbb{R}$ a non-negative function with $\mathrm{supp} f = C$.

Let $\rho = \delta_{\mathbf{x}_h}$ be the Dirac measure at the constant sequence $\mathbf{x}_h = (x_n)_{n\in\mathbb{Z}}$, $x_n = h$. Clearly, ρ is τ-invariant and ergodic. Since

$$\phi_n^{\mu}(\mathbf{x}_h, 1) = (\tau^n(\mathbf{x}_h), \rho(n, \mathbf{x}_h)) = (\mathbf{x}_h, h^n),$$

the polar exponent of $\delta_{\mathbf{x}_h}$ is

$$\lim_{n\to+\infty} \frac{1}{n} \log h^n = \log h = H,$$

which is not regular. Hence, $\Theta_{\mathrm{Ly}}(\delta_{\mathbf{x}_h}) = \Theta(H) \neq \emptyset$ is not contained in $\Theta_{\mathrm{Ly}}(\mu^{\times\mathbb{Z}}) = \emptyset$. Therefore, the second condition of §9 is violated and the flag types $\Theta_{\mathrm{Ly}}(\mu^{\times\mu})$ and $\Theta_{\mathrm{Mo}}(\phi^{\mu})$ are different.

13. *Continuity of the Lyapunov spectrum*

In this section we apply the differentiability result of [**9**] to show that the equality $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ implies continuity of the Lyapunov spectrum by perturbations of the original ϕ that do not change the flow on the base space.

Let $\mathcal{G} = \mathcal{G}(Q)$ be the gauge group of $Q \to X$, that is, the group of automorphisms of Q that project to the identity map of X. It is well known that $\mathcal{G}$ is a Banach Lie group.

If $\sigma \in \mathcal{G}$, then ϕ and $\sigma\phi$ induce the same map on X and hence have the same ergodic measure ν. Denote by $H_{\mathrm{Ly}}^{\sigma\phi}$ the polar spectrum of $\sigma\phi$ with respect to ν. Assume as before that ν has full support. Then we have the following continuity result.

THEOREM 13.1. *If $\Theta_{\mathrm{Ly}}(\phi) = \Theta_{\mathrm{Mo}}$, then the map $\sigma \in \mathcal{G} \mapsto H_{\mathrm{Ly}}^{\sigma\phi} \in \mathrm{cl}\mathfrak{a}^+$ is continuous at $\sigma = \mathrm{id}$.*

We work out separately the proof for $\mathrm{Sl}(n, \mathbb{R})$ in order to explain it in concrete terms. For this group $\mathfrak{a}$ is the algebra of zero-trace diagonal matrices and $\mathfrak{a}^+$ are those with strictly decreasing eigenvalues. The simple set of roots is $\Sigma = \{\alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_i = \alpha_{i,i+1} = \lambda_i - \lambda_{i+1}$ and $\lambda_i \in \mathfrak{a}^*$ maps the diagonal $H \in \mathfrak{a}$ to its ith diagonal entry. We denote by $\Delta = \{\delta_1, \ldots, \delta_{n-1}\}$ the set of fundamental weights, which is defined by

$$\frac{2\langle \alpha_i, \delta_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}$$

and is given by $\delta_j = \lambda_1 + \cdots + \lambda_j$. The Morse decomposition of ϕ on the flag bundles is determined by the subset $\Theta_{\mathrm{Mo}} \subset \Sigma$. Alternatively, we can look at the partition

$$\{1, \ldots, n\} = \{1, \ldots, r_1\} \cup \{r_1 + 1, \ldots, r_2\} \cup \cdots \cup \{r_k + 1, \ldots, n\},$$

where $\Sigma \setminus \Theta_{\mathrm{Mo}} = \{\alpha_{r_1}, \ldots, \alpha_{r_k}\}$. From the partition, we recover Θ_{Mo} as the set of $\alpha_{j,j+1}$ such that if $[r, s]$ is the interval of the partition containing j, then $r \leq j < s$.

For $H \in \mathrm{cl}\mathfrak{a}^+$ with $\alpha(H) = 0$ for all $\alpha \in \Theta_{\mathrm{Mo}}$, its eigenvalues a_i are such that $a_i = a_j$ if the indices i, j belong to the same set of the partition. If furthermore H is such that $\Theta_{\mathrm{Mo}} = \{\alpha \in \Sigma : \alpha(H) = 0\}$, then the multiplicities of the eigenvalues of H are the sizes of the sets of the partition.

Hence, $\Theta_{\mathrm{Ly}} = \Theta_{\mathrm{Mo}}$ means that the multiplicities of the Lyapunov exponents are given by the partition associated to Θ_{Mo}.

Now it was proved in [**9**] that the map

$$\sigma \in \mathcal{G} \mapsto \delta_j(H_{\mathrm{Ly}}^{\sigma\phi}) \in \mathbb{R}_+$$

is differentiable at $\sigma = \mathrm{id}$ for any index j such that $\alpha_{j,j+1} \notin \Theta_{\mathrm{Mo}}$. Since Δ is a basis of $\mathfrak{a}^*$, we get continuity of $H_{\mathrm{Ly}}^{\sigma\phi}$ if we prove that $\delta_j(H_{\mathrm{Ly}}^{\sigma\phi})$ is continuous when $\alpha_{j,j+1} \in \Theta_{\mathrm{Mo}}$.

For this purpose, we recall from [**1**] that $\delta_j(H_{\mathrm{Ly}}^{\sigma\phi})$ is obtained as a limit furnished by the subadditive ergodic theorem. Namely,

$$\delta_j(H_{\mathrm{Ly}}^{\sigma\phi}) = \lim \frac{1}{k}\delta_j(\mathsf{a}_\sigma^+(k, x)) = \inf_{k \geq 1} \frac{1}{k} \int \delta_j(\mathsf{a}_\sigma^+(k, x))\nu\,(dx), \tag{2}$$

where $\mathsf{a}_\sigma^+(k, x)$, $x \in X$, is the polar component of the flow defined by $\sigma\phi$. (See [**1**, §3.2]. Since δ_j is a fundamental weight $\delta_j(\mathsf{a}_\sigma^+(k, x))$, it is a subadditive cocycle on the

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base space. As shown in [**1**], this cocycle can be written as a norm in the space of a representation of G, which in this case is the j-fold exterior product of $\mathbb{R}^n$.)

By (2), we have that $\sigma \mapsto \delta_j(H_{\mathrm{Ly}}^{\sigma\phi})$ is upper semi-continuous.

To prove continuity, take j with $\alpha_{j,j+1} \in \Theta_{\mathrm{Mo}}$ and let $[r, s]$, $r \le j < s$, be the interval of the partition that contains j. Assume by contradiction that there exist $c > 0$ and a sequence $\sigma_k \in \mathcal{G}$ converging to id such that

$$\delta_j(H_{\mathrm{Ly}}^{\sigma_k\phi}) < \delta_j(H_{\mathrm{Ly}}^{\phi}) - c.$$

Then we have two cases, as follows.

(1) $s < n$. Then $\alpha_{s,s+1} \notin \Theta_{\mathrm{Mo}}$, so that $\sigma \mapsto \delta_s(H_{\mathrm{Ly}}^{\sigma\phi})$ is continuous. In the same way, $\delta_{r-1}(H_{\mathrm{Ly}}^{\sigma\phi})$ is continuous (where $\delta_{r-1} = 0$ if $r = 1$). Then, for large k, we have

$$\delta_{r-1}(H_{\mathrm{Ly}}^{\sigma_k\phi}) > \delta_{r-1}(H_{\mathrm{Ly}}^{\phi}) - c/2.$$

Since $\lambda_{i_1} \ge \lambda_{i_2}$ on $\mathrm{cl}\mathfrak{a}^+$ if $i_1 \le i_2$ and the polar exponents $H_{\mathrm{Ly}}^{\sigma_k\phi} \in \mathrm{cl}\mathfrak{a}^+$, we get

$$\delta_j(H_{\mathrm{Ly}}^{\phi}) - c > \delta_{r-1}(H_{\mathrm{Ly}}^{\sigma_k\phi}) + (j - r + 1)\lambda_j(H_{\mathrm{Ly}}^{\sigma_k\phi}).$$

Hence, for large k, we have

$$\delta_j(H_{\mathrm{Ly}}^{\phi}) - c > \delta_{r-1}(H_{\mathrm{Ly}}^{\phi}) - c/2 + (j - r + 1)\lambda_j(H_{\mathrm{Ly}}^{\sigma_k\phi}),$$

that is,

$$\lambda_j(H_{\mathrm{Ly}}^{\sigma_k\phi}) < \frac{1}{j - r + 1}(\delta_j(H_{\mathrm{Ly}}^{\phi}) - \delta_{r-1}(H_{\mathrm{Ly}}^{\phi}) - c/2).$$

By the inequality $\delta_s = \delta_j + \lambda_{j+1} + \cdots + \lambda_s \le \delta_j + (s - j)\lambda_j$ that holds on $\mathrm{cl}\mathfrak{a}^+$, we get

$$\delta_s(H_{\mathrm{Ly}}^{\sigma_k\phi}) \le \delta_j(H_{\mathrm{Ly}}^{\phi}) - c + \frac{s - j}{j - r + 1}(\delta_j(H_{\mathrm{Ly}}^{\phi}) - \delta_{r-1}(H_{\mathrm{Ly}}^{\phi}) - c/2). \tag{3}$$

Now we use the assumption that $\Theta_{\mathrm{Ly}}(\phi) = \Theta_{\mathrm{Mo}}$, which implies that

$$\delta_j(H_{\mathrm{Ly}}^{\phi}) = \delta_{r-1}(H_{\mathrm{Ly}}^{\phi}) + (j - r + 1)\lambda_j(H_{\mathrm{Ly}}^{\phi})$$

and

$$\delta_s(H_{\mathrm{Ly}}^{\phi}) = \delta_j(H_{\mathrm{Ly}}^{\phi}) + (s - j)\lambda_j(H_{\mathrm{Ly}}^{\phi}).$$

Hence, the last term in (3) becomes

$$\frac{s - j}{j - r + 1}((j - r + 1)\lambda_j(H_{\mathrm{Ly}}^{\phi}) - c/2) = (s - j)\lambda_j(H_{\mathrm{Ly}}^{\phi}) - \frac{s - j}{j - r + 1}\frac{c}{2},$$

so that for large k we have

$$\delta_s(H_{\mathrm{Ly}}^{\sigma_k\phi}) \le \delta_s(H_{\mathrm{Ly}}^{\phi}) - c - \frac{s - j}{j - r + 1}\frac{c}{2}, \tag{4}$$

which contradicts the continuity of $\delta_s(H_{\mathrm{Ly}}^{\sigma\phi})$.

(2) $s = n$. If $r = 1$, then $\Theta_{\mathrm{Ly}}(\phi) = \Theta_{\mathrm{Mo}} = \Sigma$, so that $H_{\mathrm{Ly}}^{\phi} = 0$ and continuity follows by upper semi-continuity. When $r \ne 1$, we get continuity of $\delta_{r-1}(H_{\mathrm{Ly}}^{\sigma\phi})$. By arguing

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as in the first case, we get the same estimate (4) for $0 = \delta_n = \lambda_1 + \cdots + \lambda_n$, which is a contradiction.

This proves Theorem 13.1 for the group $\mathrm{Sl}(d, \mathbb{R})$.

Now we consider a general semi-simple group G. As before, let $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ and $\Delta = \{\delta_1, \ldots, \delta_l\}$ be the simple system of roots and fundamental weights, respectively. Given $\Theta \subset \Sigma$, let $\Delta_\Theta \subset \Delta$ be the set of fundamental weights δ_j such that the root with the same index $\alpha_j \in \Theta$. We put

$$\mathfrak{a}(\Theta) = \mathrm{span}(\Theta), \qquad \mathfrak{a}_\Theta = \mathrm{span}(\Delta \backslash \Delta_\Theta).$$

These subspaces are orthogonal to each other and, since $\Theta \cup (\Delta \setminus \Delta_\Theta)$ is a basis of $\mathfrak{a}^*$, we have $\mathfrak{a} = \mathfrak{a}(\Theta) \oplus \mathfrak{a}_\Theta$.

The proof of continuity will be an easy consequence of the following algebraic lemma.

LEMMA 13.2. *If $\delta \in \Delta_\Theta$, then its coordinates with respect to $\Theta \cup (\Delta \setminus \Delta_\Theta)$ are non-negative.*

Proof. Let γ_1 and γ_2 be the orthogonal projections of δ on $\mathfrak{a}(\Theta)$ and $\mathfrak{a}_\Theta$, respectively. First we check that the coefficients of γ_1 with respect to Θ are non-negative. By definition, there exists just one root $\alpha \in \Theta$ such that $2\langle \alpha, \delta\rangle / \langle \alpha, \alpha\rangle = 1$ and $2\langle \beta, \delta\rangle / \langle \beta, \beta\rangle = 0$ if $\beta \neq \alpha$. But, if $\beta \in \Theta$, then $2\langle \beta, \delta\rangle / \langle \beta, \beta\rangle = 2\langle \beta, \gamma_1\rangle / \langle \beta, \beta\rangle$. Hence, γ_1 is a fundamental weight for the root system defined by Θ. Its coefficients with respect to Θ are the entries of the inverse of the Cartan matrix, which are non-negative.

As to γ_2, it is given by the mean

$$\gamma_2 = \frac{1}{|\mathcal{W}_\Theta|} \sum_{w \in \mathcal{W}_\Theta} w\delta$$

because γ_2 is orthogonal to Θ and hence $w\gamma_2 = \gamma_2$ for every $w \in \mathcal{W}_\Theta$. Moreover, $\langle \Theta \rangle$ is a root system in $\mathfrak{a}(\Theta)$ with Weyl group $\mathcal{W}_\Theta$ which has no fixed points in $\mathfrak{a}(\Theta)$ besides 0. Hence, the mean applied to γ_1 is 0, since it is a fixed point.

Now, if $\mathfrak{a}^+ = \{\beta \in \mathfrak{a}^* : \forall \alpha \in \Sigma,\ \langle \alpha, \beta\rangle > 0\}$ is the Weyl chamber in $\mathfrak{a}^*$, then the fundamental weight $\delta \in \mathrm{cl}\mathfrak{a}^+$. Hence,

$$\gamma_2 \in \mathcal{W}_\Theta(\mathrm{cl}\mathfrak{a}^+) = \bigcup_{w \in \mathcal{W}_\Theta} w(\mathrm{cl}\mathfrak{a}^+)$$

because $\mathcal{W}_\Theta(\mathrm{cl}\mathfrak{a}^+)$ is a cone. On the other hand, if $\beta \notin \Theta$ and $\gamma \in \mathcal{W}_\Theta(\mathrm{cl}\mathfrak{a}^+)$, then $\langle \beta, \gamma\rangle \geq 0$ (see e.g. [**19**, Lemma 7.5]). But

$$\gamma_2 = \sum_{\beta \notin \Theta} \frac{2\langle \gamma_2, \beta\rangle}{\langle \beta, \beta\rangle} \delta_\beta,$$

where δ_β is the fundamental weight corresponding to β. Hence, the coefficients of γ_2 are non-negative, concluding the proof. □

Proof of Theorem 13.1 for G semi-simple. Let $\Theta = \Theta_{\mathrm{Ly}}(\phi) = \Theta_{\mathrm{Mo}}$ and take $\delta \in \Delta_\Theta$. Then $\sigma \mapsto \delta(H_{\mathrm{Ly}}^{\sigma\phi})$ is upper semi-continuous. Write

$$\delta = \sum_{\alpha \in \Theta} a_\alpha \alpha + \sum_{\delta \in \Delta \backslash \Delta_\Theta} b_\delta \delta$$

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with $a_\alpha \geq 0$ by the lemma. We have

$$\delta(H_{\mathrm{Ly}}^{\sigma\phi}) = \sum_{\alpha\in\Theta} a_\alpha \alpha(H_{\mathrm{Ly}}^{\sigma\phi}) + \sum_{\lambda\in\Delta\backslash\Delta_\Theta} b_\lambda \lambda(H_{\mathrm{Ly}}^{\sigma\phi}),$$

where the last sum is continuous by the differentiability result of [**9**]. Hence, the first sum is upper semi-continuous as well. The assumption that $\Theta_{\mathrm{Ly}}(\phi) = \Theta_{\mathrm{Mo}}$ implies that the first sum is zero at $\sigma = \mathrm{id}$. Since it is non-negative because $\alpha(H_{\mathrm{Ly}}^{\sigma\phi}) \geq 0$ and $a_\alpha \geq 0$, we conclude that δ is continuous at id, proving Theorem 13.1. □

It remains to consider the reductive groups, which amounts to checking continuity of the central component defined in [**1**, Section 3.3]. The continuity of this component holds without any further assumption. This is because this central component is given by an integral

$$\int \mathfrak{a}_\sigma^+(1, x)\nu\,(dx)$$

on the base space whose integrand is the time 1 of a cocycle $\mathfrak{a}_\sigma^+(n, x)$ that depends continuously of $\sigma \in \mathcal{G}$ (see [**1**] for the details).

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References

[**1**] L. A. Alves and L. A. B. San Martin. Multiplicative ergodic theorem on flag bundles of semi-simple Lie groups. *Discrete Contin. Dyn. Syst.* **33** (2013), 1247–1273.

[**2**] C. Bonatti, L. J. Diaz and M. Viana. *Dynamics Beyond Uniform Hyperbolicity (Encyclopedia of Mathematical Sciences, 102)*. Springer, Berlin, 2005.

[**3**] C. J. Braga Barros and L. A. B. San Martin. Chain transitive sets for flows on flag bundles. *Forum Math.* **19** (2007), 19–60.

[**4**] I. U. Bronstein and V. F. Chernii. Linear extensions satisfying Perron's condition I. *Differ. Equ.* **14** (1978), 1234–1243.

[**5**] F. Colonius and W. Kliemann. *The Dynamics of Control*. Birkhäuser, Boston, MA, 2000.

[**6**] C. Conley. *Isolated Invariant Sets and the Morse Index (CBMS Regional Conference Series in Mathematics, 38)*. American Mathematical Society, Providence, RI, 1978.

[**7**] J. J. Duistermat, J. A. C. Kolk and V. S. Varadarajan. Functions, flows and oscillatory integral on flag manifolds. *Compos. Math.* **49** (1983), 309–398.

[**8**] T. Ferraiol, M. Patrão and L. Seco. Jordan decomposition and dynamics on flag manifolds. *Discrete Contin. Dyn. Syst.* **26**(3) (2010), 923–947.

[**9**] T. Ferraiol and L. A. B. San Martin. Differentiability of Lyapunov exponents and Morse decompositions. *Preprint*, 2014, to appear, arXiv:1405.1262.

[**10**] A. Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** (1997), 797–815.

[**11**] Y. Guivarch' and A. Raugi. Frontière de Furstenberg, proprietés de contraction et théorèmes de convergence. *Probab. Theory Relat. Top.* **69** (1985), 187–242.

[**12**] M. R. Herman. Construction d'un diffeomorphisme minimal d'entropie topologique non nulle. *Ergod. Th. & Dynam. Sys.* **1** (1981), 65–76.

[**13**] A. W. Knapp. *Lie Groups: Beyond an Introduction (Progress in Mathematics, 140)*, 2nd edn. Birkhäuser, Basel, 2004.

[**14**] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, Berlin, 1989.

[**15**] M. Patrão and L. A. B. San Martin. Morse decomposition of semiflows on fiber bundles. *Discrete Contin. Dyn. Syst.* **17** (2007), 561–587.
[**16**] M. Patrão, L. A. B. San Martin and L. Seco. Conley indexes and stable sets for flows on flag bundles. *Dyn. Syst.* **24** (2009), 249–276.
[**17**] L. A. B. San Martin. Invariant control sets on flag manifolds. *Math. Control Signals Systems* **6** (1993), 41–61.
[**18**] L. A. B. San Martin. Maximal semigroups in semi-simple Lie groups. *Trans. Amer. Math. Soc.* **353** (2001), 5165–5184.
[**19**] L. A. B. San Martin and L. Seco. Morse and Lyapunov spectra and dynamics on flag bundles. *Ergod. Th. & Dynam. Sys.* **30** (2010), 893–922.
[**20**] L. A. B. San Martin, O. G. Rocio and M. A. Verdi. Semigroup actions on adjoint orbits. *J. Lie Theory* **22** (2012), 931–948.
[**21**] J. Selgrade. Isolated invariant sets for flows on vector bundles. *Trans. Amer. Math. Soc.* **203** (1975), 259–390.
[**22**] G. Warner. *Harmonic Analysis on Semi-simple Lie Groups I*. Springer, Berlin, 1972.

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