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EXCURSION SETS OF INFINITELY DIVISIBLE RANDOM FIELDS WITH CONVOLUTION EQUIVALENT LÉVY MEASURE

ANDERS RØNN-NIELSEN,* *Copenhagen Business School*
EVA B. VEDEL JENSEN,** *Aarhus University*

Abstract

We consider a continuous, infinitely divisible random field in $\mathbb{R}^d$, $d = 1, 2, 3$, given as an integral of a kernel function with respect to a Lévy basis with convolution equivalent Lévy measure. For a large class of such random fields, we compute the asymptotic probability that the excursion set at level x contains some rotation of an object with fixed radius as $x \to \infty$. Our main result is that the asymptotic probability is equivalent to the right tail of the underlying Lévy measure.

Keywords: Convolution equivalence; excursion set; infinite divisibility; Lévy-based modelling

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Secondary 60E07; 60D05

1. Introduction

In this paper we investigate the extremal behaviour of excursion sets for a field $(X_t)_{t\in B}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t-s|)M(\mathrm{d}s), \tag{1.1}$$

where M is an infinitely divisible, independently scattered random measure on $\mathbb{R}^d$, f is some kernel function, and B is a compact index set. We assume that the Lévy measure of the random measure M has a convolution equivalent right tail; see [5], [6], [10]. In [13] it was shown under some regularity conditions that the distribution of $\sup_{t\in B} X_t$ has a similar convolution equivalent tail. In this paper we are interested in the excursion set

$$A_x = \{t : X_t > x\}.$$

Under the additional assumption (2.10) below, we show that the asymptotic probability of the excursion set at level x containing some rotation of an object with a fixed radius r has a tail that is equivalent to the tail of the underlying Lévy measure. A more precise definition of the event that is studied asymptotically is found in Section 2 below. Measures with a convolution equivalent tail cover the important cases of an inverse Gaussian and a normal inverse Gaussian (NIG) basis, respectively, see [13].

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* Postal address: Department of Finance, Copenhagen Business School, Solbjerg Plads 3, DK-2000 Frederiksberg, Denmark. Email address: aro.fi@cbs.dk
** Postal address: Department of Mathematics and Centre for Stochastic Geometry and Advanced Bioimaging, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark. Email address: eva@imf.au.dk

Lévy models as defined in (1.1) provide a flexible and tractable modelling framework that recently has been used for a variety of modelling purposes, including modelling of turbulent flows [4], growth processes [8], Cox point processes [7], and brain imaging data [9]. In [9], a model (1.1) with M following a NIG distribution was suitable for modelling the neuroscience data under consideration. For such data it is typically of interest to detect for which $t \in B$ a given field attains values that are significantly large. The results in this paper make it possible to discuss whether a cluster of $t \in B$ with large observations jointly form an extreme observation.

For Gaussian random fields it is known that the distribution of the supremum of the field can be approximated by the expected Euler characteristic of an excursion set (see [1] and the references therein). The supremum and excursion sets of a non-Gaussian field given by integrals with respect to an infinitely divisible random measure have already been studied in the case that the random measure has regularly varying tails. Results for the asymptotic distribution of the supremum can be found in [11], and these results were refined in [2] and [3], where results were obtained on the asymptotic joint distribution of the number of critical points of the excursion sets. The arguments used there, as also in the present paper, are based on finding the Lévy measure of a dense countable subset of the field. However, otherwise the proofs of [2], [3], and [11] rely heavily on the assumption of regularly varying tails and, therefore, cannot be translated into the convolution equivalent framework.

Note that convolution equivalent distributions have heavier tails than Gaussian distributions and lighter tails than those of regularly varying distributions. The latter statement follows from the fact that convolution equivalent distributions have exponential tails while regularly varying distributions have power function tails.

The present paper is organised as follows. In Section 2 we define the random field (1.1) and introduce the necessary assumptions. In Section 3 we prove three technical lemmas concerning the asymptotic behaviour of deterministic fields. These results are used in Section 4 where we give the main result of the paper. The proof is comprised of several steps, exploiting the fact that X can be decomposed as $X^1 + X^2$, where X^1 is a compound Poisson sum and X^2 has a lighter tail than X^1. The proofs in Section 4 apply techniques similar to those of [13].

2. Preliminaries

We make the same general assumptions as in [13] except for the additional assumption (2.10) below. For completeness, we present all assumptions in the following.

Consider an independently scattered random measure M on $\mathbb{R}^d$, $d = 1, 2, 3$. Then for a sequence of disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ in $\mathcal{B}(\mathbb{R}^d)$, the random variables $(M(A_n))_{n\in\mathbb{N}}$ are independent and satisfy $M(\cup A_n) = \sum M(A_n)$. Further, assume that $M(A)$ is infinitely divisible for all $A \in \mathcal{B}(\mathbb{R}^d)$. Then M is called a Lévy basis (see [4] and the references therein).

For a random variable X let $\kappa_X(\lambda)$ denote its cumulant function $\log \mathbb{E}[\mathrm{e}^{i\lambda X}]$. We shall assume that the Lévy basis is stationary and isotropic such that for $A \in \mathcal{B}(\mathbb{R}^d)$ the variable $M(A)$ has a Lévy–Khintchine representation given by

$$\kappa_{M(A)}(\lambda) = i\lambda a m_d(A) - \frac{1}{2}\lambda^2\theta m_d(A) + \int_{A\times\mathbb{R}} (\mathrm{e}^{i\lambda z} - 1 - i\lambda z 1_{[-1,1]}(z)) F(\mathrm{d}s, \mathrm{d}z), \quad (2.1)$$

where m_d denotes the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $a \in \mathbb{R}$, $\theta \geq 0$, and F is a measure on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ of the form

$$F(A \times B) = m_d(A)\rho(B). \quad (2.2)$$

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We assume that ρ in (2.2) has an exponential tail with index $\beta > 0$, i.e. for all $y \in \mathbb{R}$,

$$\frac{\rho((x-y,\infty))}{\rho((x,\infty))} \to \mathrm{e}^{\beta y} \quad \text{as } x \to \infty. \tag{2.3}$$

Here the assumption $\beta > 0$ excludes the subexponential case. Let ρ_1 be a normalization of the restriction of ρ to $(1,\infty)$, and note that ρ_1 also has an exponential tail with index $\beta > 0$. Assume also that

$$\frac{(\rho_1 * \rho_1)((x,\infty))}{\rho_1((x,\infty))} \to 2m \quad \text{as } x \to \infty, \tag{2.4}$$

where $m < \infty$. This makes ρ_1 a convolution equivalent distribution. (Formally, a distribution is said to be *convolution equivalent*, if it has an exponential tail and satisfies (2.4).) Here $\rho_1 * \rho_1$ denotes the convolution. In fact, $m = \int_{\mathbb{R}_+} \mathrm{e}^{\beta z}\rho_1(\mathrm{d}z)$; cf. [10, Corollary 2.1(ii)]. Writing $\rho((x,\infty)) = L(x)\mathrm{e}^{-\beta x}$, it is seen from (2.3) that, for all $y \in \mathbb{R}$,

$$\frac{L(x-y)}{L(x)} \to 1 \quad \text{as } x \to \infty. \tag{2.5}$$

For each $a, b \in \mathbb{R}$, the limit (2.5) holds uniformly in $y \in [a,b]$; cf. [10, p. 408]. Further, we assume that

$$\int_{\mathbb{R}} z^2 \rho(\mathrm{d}z) < \infty. \tag{2.6}$$

Now assume that $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly decreasing kernel function satisfying

$$\int_{\mathbb{R}^d} f(|s|)\mathrm{d}s < \infty, \tag{2.7}$$

and

$$f(x) \le \frac{K_1}{(x+1)^d} \quad \text{for all } x \ge 0 \tag{2.8}$$

for some finite, positive constant K_1. Further, assume that f is differentiable with f' satisfying

$$|f'(x)| \le \frac{K_2}{(x+1)^d} \quad \text{for all } x \ge 0 \tag{2.9}$$

for some finite, positive constant K_2. Finally, let $r > 0$ be fixed and assume that there exists a g such that both $f(x) \le g(x)$ for all $x \ge 0$ and

$$g(x) = f'(r)(x-r) + f(r) \quad \text{for all } x \in [0, 2r]. \tag{2.10}$$

Note that such a g exists in the particular case that f is concave on $[0, 2r]$. Further, we can and do choose g on $[2r, \infty)$ such that it satisfies (2.7)–(2.9).

Let B be a compact, convex subset of $\mathbb{R}^d$ with $m_d(B) > 0$ and define the set $B \oplus C_r = \{x + y\colon x \in B, y \in C_r(0)\}$, where $C_r(0)$ is the ball with radius r and centre at 0. We consider the family of random variables $(X_t)_{t \in B \oplus C_r}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t-s|)M(\mathrm{d}s)$$

(see [13] for the existence of these integrals).

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Example 2.1. (*Gaussian kernel function.*) Suppose that $f(x) = \mathrm{e}^{-\sigma x^2}$, $\sigma > 0$. Then the assumptions (2.7)–(2.9) are satisfied, and f is concave on the interval $[0, 1/\sqrt{2\sigma}]$. In particular, assumption (2.10) is satisfied for $r \leq \frac{1}{2}/\sqrt{2\sigma}$.

Example 2.2. (*Matérn kernel function.*) Suppose that

$$f(x) = \frac{|\lambda x|^{\eta} K_{\eta}(\lambda|x|)}{2^{\eta-1}\Gamma(\eta)},$$

where K_η is the modified Bessel function of the second kind, index $\eta \geq \frac{1}{2}$, and $\lambda > 0$. It can be shown that the Matérn kernel satisfies assumptions (2.7)–(2.9) (see [12, Example 2.5] and the references therein for details). Further, [12, Example 2.5] provides identities for the derivatives of f from which it can be shown that when $\eta > \frac{1}{2}$, f is concave in an interval $(0, \delta)$ close to 0. In particular, assumption (2.10) is satisfied.

For $s \in B$, let $C_r(s)$ be the ball in $\mathbb{R}^d$ with radius r and centre s, and let $\mathbb{S}^{d-1} = \{\alpha \in \mathbb{R}^d : |\alpha| = 1\}$ denote the unit sphere. Let $D \subseteq C_r(0)$ be a set with radius r in the sense that there exists $\beta \in \mathbb{S}^{d-1}$ such that $\{-r\beta, r\beta\} \subseteq D$. Further, let $\mathrm{SO}(d)$ denote the special orthogonal group, i.e. the set of all orthogonal matrices with determinant 1. Hence, each $R \in \mathrm{SO}(d)$ represents a rotation in $\mathbb{R}^d$. For $R \in \mathrm{SO}(d)$ and $s \in \mathbb{R}^d$, define $D^R(s) = RD + s$. Recalling the definition of the excursion set, $A_x = \{t \in B \oplus C_r : X_t > x\}$, we are interested in the event

$$\{\text{there exist } t \in B \text{ and } R \in \mathrm{SO}(d) \colon D^R(t) \subseteq A_x\}.$$

This event can be alternatively expressed as

$$\Big\{\sup_{t_0 \in B} \sup_{R \in \mathrm{SO}(d)} \inf_{t \in D^R(t_0)} X_t > x\Big\}.$$

Example 2.3. One possible choice of D is $C_r(0)$; then the rotations of D are unnecessary. Another choice is $D = \{r\alpha_0, -r\alpha_0\}$ for some fixed $\alpha_0 \in \mathbb{S}^{d-1}$. A third possibility is the line segment connecting the points $r\alpha$ and $-r\alpha$. For convenience, we let $\alpha_0 = 1$, $\alpha_0 = (1, 0)$, $\alpha_0 = (1, 0, 0)$ for $d = 1, 2, 3$ respectively.

For our study of the extremal behaviour of $(X_t)_{t \in B \oplus C_r}$, it is crucial that the field $(X_t)_{t \in T}$ should itself be infinitely divisible, where $T = (B \oplus C_r) \cap \mathbb{Q}^d$ and $\mathbb{Q}^d$ denotes the rational numbers in $\mathbb{R}^d$ (see [13] and the references therein for details). The Lévy measure of $(X_t)_{t \in T}$ is the measure ν on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ defined by $\nu = F \circ V^{-1}$, where $V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^T$ is given by

$$V(s, z) = (zf(|t - s|))_{t \in T}.$$

Because $(X_t)_{t \in T}$ is infinitely divisible, we have the following decomposition (see, for example, [11]):

$$X_t = X_t^1 + X_t^2,$$

where the fields $(X_t^1)_{t \in T}$ and $(X_t^2)_{t \in T}$ are independent. The first field, $(X_t^1)_{t \in T}$, is a compound Poisson sum

$$X_t^1 = \sum_{n=0}^{N} U_t^n,$$

where N is Poisson distributed with parameter $\nu(A) < \infty$ and $A = \{x \in \mathbb{R}^T : \sup_{t \in T} x_t > 1\}$.

The fields $(U_t^n)_{t\in T}$ are independent and identically distributed (i.i.d.) with common distribution $\nu_1 = \nu_A/\nu(A)$, where ν_A is the measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ obtained by restricting ν to A. Further, $(X_t^2)_{t\in T}$ is infinitely divisible with a Lévy measure ν_{A^c}, the restriction of ν to A^c.

As argued in [13], all the fields U^n, X^1 and X^2 have continuous extensions to $B \oplus C_r$. Note also that each of the fields $(U_t^n)_{t\in B\oplus C_r}$ can be represented by $(Zf(|t-S|))_{t\in B\oplus C_r}$, where $(S, Z) \in \mathbb{R}^d \times \mathbb{R}_+$ has distribution F_1, the restriction of the measure F to the set

$$V^{-1}(A) = \Big\{(s,z) \in \mathbb{R}^d \times \mathbb{R}\colon \sup_{t\in T} zf(|t-s|) > 1\Big\}.$$

3. Asymptotic results for deterministic fields

An important property for the arguments in [13] is that for a continuous field $(y_t)_{t\in B\oplus C_r}$, the limit relation

$$\inf_{t\in B} \frac{x-y_t}{f(|t-s|)} - (x-y_s) \to 0 \quad (x\to\infty) \tag{3.1}$$

holds for all $s \in B$. In this paper we need a similar but more involved result concerning the asymptotic behaviour of

$$\inf_{t_0\in B}\ \inf_{R\in \mathrm{SO}(d)}\ \sup_{t\in D^R(t_0)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)}, \tag{3.2}$$

where $\mathrm{SO}(d)$ and $D^R(t)$ are as defined in Section 2.

Lemma 3.1. *Let $(y_t)_{t\in B\oplus C_r}$ be a continuous field. Then there exists a function $\lambda_s((y_t)_{t\in B\oplus C_r})$ such that, for each $s\in B$,*

$$\inf_{t_0\in B}\ \inf_{R\in \mathrm{SO}(d)}\ \sup_{t\in D^R(t_0)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)} + \lambda_s((y_t)_{t\in B\oplus C_r}) \to 0 \quad (x\to\infty). \tag{3.3}$$

Further,

(i) *if $(y_t)_{t\in B\oplus C_r}$ is constant-valued and equal to y then $\lambda_s((y_t)_{t\in B\oplus C_r}) = y/f(r)$ for all s;*

(ii) *for constant y, $\lambda_s((y+y_t)_{t\in B\oplus C_r}) = y/f(r) + \lambda_s((y_t)_{t\in B\oplus C_r})$;*

(iii) *for any $\varepsilon > 0$, $\lambda_s((y_t)_{t\in B\oplus C_r})$ depends only on $(y_t)_{t\in C_{r+\varepsilon}(s)}$.*

Proof. Let $y^* = \sup_{t\in B\oplus C_r}$ and $y_* = \inf_{t\in B\oplus C_r}$. Then the expression in (3.2) is bounded above by

$$\frac{x-y_*}{\sup_{t_0,R}\inf_{t\in D^R(t_0)} f(|t-s|)} - \frac{x}{f(r)} = \frac{x-y_*}{f(r)} - \frac{x}{f(r)} = \frac{-y_*}{f(r)}.$$

Similarly, the expression is bounded below by $-y^*/f(r)$. Assertion (i) for a constant field (y_t) follows from these two bounds. Similarly, assertion (ii), when a constant y is added to (y_t), follows once the existence of the limit $\lambda_s((y_t)_{t\in B\oplus C_r})$ is established. For each $x > 0$ we can choose $t_x \in B$ and $R_x \in \mathbb{S}^{d-1}$ such that

$$\inf_{t_0\in B}\ \inf_{R\in \mathrm{SO}(d)}\ \sup_{t\in D^R(t_0)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)} = \sup_{t\in D^{R_x}(t_x)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)}. \tag{3.4}$$

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First, we show that $t_x \to s$ as $x \to \infty$. We find

$$\frac{x - \inf_{t\in C_r(s)} y_t}{f(r)} = \sup_{t\in C_r(s)} \frac{x - y_t}{f(r)} \geq \sup_{t\in D^{R_x}(t_x)} \frac{x - y_t}{f(|t-s|)} \geq \frac{x - y^*}{\inf_{t\in D^{R_x}(t_x)} f(|t-s|)}.$$

On using $\inf_{t\in D^{R_x}(t_x)} f(|t-s|) \leq f(r)$ this yields

$$\frac{x - y^*}{x - \inf_{t\in C_r(s)} y_t} \leq \frac{\inf_{t\in D^{R_x}(t_x)} f(|t-s|)}{f(r)} \leq 1,$$

so $\inf_{t\in D^{R_x}(t_x)} f(|t-s|) \to f(r)$ as $x \to \infty$. Since also $\inf_{t\in D^R(t_0)} f(|t-s|) < f(r)$ for all $t_0 \neq s$ and $R \in \mathrm{SO}(d)$, we can conclude that $t_x \to s$. From this we can conclude that $\lambda_s((y_t)_{t\in B\oplus C_r})$ depends only on y_t for t close to $C_r(s)$, i.e. assertion (iii).

In fact we need a stronger version of (iii). The differentiability of f in r implies that, for $u \geq 0$,

$$\frac{1}{f(u)} - \frac{1}{f(r)} = b(u-r) + (u-r)\phi(u-r) \tag{3.5}$$

for $b > 0$ and some continuous function ϕ with $\phi(0) = 0$. Since f is decreasing, for each $K > 0$,

$$x\left(\frac{1}{f(u)} - \frac{1}{f(r)}\right) \begin{cases} \leq -bK + \phi\left(-\dfrac{K}{x}\right) & \text{for } 0 < u < r - K/x, \\ \geq bK + \phi\left(\dfrac{K}{x}\right) & \text{for } u > r + K/x. \end{cases}$$

In particular, we can choose K and x_0 such that, for all $x > x_0$,

$$\frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)} \begin{cases} < -\dfrac{y^*}{f(r)} & \text{for } |t-s| < r - K/x, \\ > -\dfrac{y_*}{f(r)} & \text{for } |t-s| > r + K/x. \end{cases}$$

With this choice of K, we have, for $x > x_0$,

$$\inf_{t_0\in B} \inf_{R\in\mathrm{SO}(d)} \sup_{t\in D^R(t_0)} \frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)} = \inf_{t_0\in B} \inf_{R\in\mathrm{SO}(d)} \sup_{t\in D^R(t_0)\cap H_x} \frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)}, \tag{3.6}$$

where $H_x = \{t \in \mathbb{R}^d : r - K/x \leq |t-s| \leq r + K/x\}$. Define

$$h(\ell) = \sup\{|\phi(u-r)| : r - \ell \leq u \leq r + \ell\},$$

and note that $h(\ell) \to 0$ as $\ell \to 0$.

We now show the convergence result (3.3) by contradiction. To this end, assume that there is a sequence $x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 < \cdots$ and constants a and $\varepsilon > 0$ such that

$$\sup_{t\in D^{R_n}(t_n)} \frac{x_n - y_t}{f(|t-s|)} - \frac{x_n}{f(r)} \leq a \qquad \sup_{t\in D^{\tilde{R}_n}(\tilde{t}_n)} \frac{\tilde{x}_n - y_t}{f(|t-s|)} - \frac{\tilde{x}_n}{f(r)} \geq a + \varepsilon \tag{3.7}$$

for all n, where $R_n = R_{x_n}$, $\tilde{R}_n = R_{\tilde{x}_n}$ are the corresponding rotation matrices, and $t_n = t_{x_n}$, $\tilde{t}_n = t_{\tilde{x}_n}$ correspond to the relevant displacements, chosen according to (3.4). By using subsequences we can assume that $|t_n - s|$ is decreasing and that (R_n) is convergent. Let ℓ be chosen such that $h(\ell) < 1/m$, where $m \in \mathbb{N}$ will be determined later. Let S_n be the rotation that is needed to rotate $D^{R_{n+1}}(0)$ into $D^{R_n}(0)$: $S_n D^{R_{n+1}}(0) = D^{R_n}(0)$. Choose $\delta > 0$ according to the uniform continuity of $(z_t)_{t\in B\oplus C_r} = (y_t/f(|t-s|))_{t\in B\oplus C_r}$ such that $|z_{s_2} - z_{s_1}| < \frac{1}{4}\varepsilon$ if $|s_2 - s_1| < \delta$. Further, δ should be chosen so small that $\delta < \frac{1}{2}\ell$. Choose $\tilde{x} > x_0$ such that $\delta + K/\tilde{x} < \ell$. Now choose n such that $|t_n - t_{n+1}| < \frac{1}{2}\delta$, such that $|S_n u - u| < \frac{1}{2}\delta$ for all $u \in B \oplus C_r$, and such that $K/x_n + |t_n - t_{n+1}| < K/\tilde{x}$.

Recall that $D^{R_n}(t_n)$ can be parameterised by $\{R_n t + t_n : t \in D\}$ and, similarly, $D^{R_{n+1}}(t_{n+1})$ is parameterised by $\{R_{n+1}t + t_{n+1} : t \in D\}$. Choose $D_{\tilde{x}} \subseteq D$ such that $D^{R_n}(t_n) \cap H_{\tilde{x}} = \{R_n t + t_n : t \in D_{\tilde{x}}\}$. From the definition of t_n, it follows that

$$\sup_{t\in D^{R_n}\cap H_{\tilde{x}}} \frac{x_n - y_t}{f(|t-s|)} - \frac{x_n}{f(r)} \le a. \tag{3.8}$$

Further, let $\tilde{D}_{\tilde{x}}^{R_{n+1}}$ be the rotation by S_n of $D^{R_n}(t_n) \cap H_{\tilde{x}}$ centred in s: $\tilde{D}_{\tilde{x}}^{R_{n+1}} = S_n(D^{R_n}(t_n) \cap H_{\tilde{x}} - s) + s$. Now $\tilde{D}_{\tilde{x}}^{R_{n+1}}$ has the form $\{R_{n+1}t + \tilde{t} : t \in D_{\tilde{x}}\}$ for some $\tilde{t}$, where in fact $\tilde{t} = S_n(t_n - s) + s$ but this is not important in the following. Since for $t \in D_{\tilde{x}}$ each $R_{n+1}t + \tilde{t} \in \tilde{D}_{\tilde{x}}^{R_{n+1}}$ is the rotation around s of $R_n t + t_n \in D^{R_n}(t_n) \cap H_{\tilde{x}}$, the distance to s is unchanged. Since also $|R_{n+1}t + \tilde{t} - (R_n t + t_n)| < \delta$ for $t \in D_{\tilde{x}}$ because of the choice of S_n, the inequality (3.8) now leads to

$$\sup_{t\in \tilde{D}_{\tilde{x}}^{R_{n+1}}} \frac{x_n - y_t}{f(|t-s|)} - \frac{x_n}{f(r)} \le a + \frac{1}{4}\varepsilon,$$

which can be reparameterised as

$$\sup_{t\in D_{\tilde{x}}} x_n\left(\frac{1}{f(|R_{n+1}t + \tilde{t} - s|)} - \frac{1}{f(r)}\right) - z_{R_{n+1}t+\tilde{t}} \le a + \frac{1}{4}\varepsilon. \tag{3.9}$$

Define in the same way $D_{\tilde{x}}^{R_{n+1}}(t_{n+1}) = \{R_{n+1}t + t_{n+1} : t \in D_{\tilde{x}}\}$ as a reduced version of $D^{R_{n+1}}(t_{n+1})$. From the definition of t_{n+1}, we have, similarly,

$$\sup_{t\in D_{\tilde{x}}} x_n\left(\frac{1}{f(|R_{n+1}t + t_{n+1} - s|)} - \frac{1}{f(r)}\right) - z_{R_{n+1}t+t_{n+1}} \le a,$$

and by the uniform continuity of (z_t) and the small distance between t_{n+1} and $\tilde{t}$, we have

$$\sup_{t\in D_{\tilde{x}}} x_n\left(\frac{1}{f(|R_{n+1}t + t_{n+1} - s|)} - \frac{1}{f(r)}\right) - z_{R_{n+1}t+\tilde{t}} \le a + \frac{1}{4}\varepsilon. \tag{3.10}$$

Note that $D_{\tilde{x}}^{R_{n+1}}(t_{n+1})$ is a translation of $\tilde{D}_{\tilde{x}}^{R_{n+1}}$. We parameterise all the intermediate translations by

$$D_{u,\tilde{x}} = \{R_{n+1}t + \gamma(u) : t \in D_{\tilde{x}}\} \quad \text{for } u \in [0, 1].$$

Here $\gamma(u) = \tilde{t} + u(t_{n+1} - \tilde{t})$ is a linear parameterisation of the line segment from $\tilde{t}$ to t_{n+1}. Note that $D_{0,\tilde{x}} = \tilde{D}_{\tilde{x}}^{R_{n+1}}$ and $D_{1,\tilde{x}} = D_{\tilde{x}}^{R_{n+1}}(t_{n+1})$. Now define $x(u) = K/(1 - u + C)$ for $u \in [0, 1]$, where $C, K > 0$ are chosen such that $x(0) = x_n$ and $x(1) = x_{n+1}$ (see Lemma A.1). Suppose that we can show, for all $u \in [0, 1]$,

$$\sup_{t\in D_{\tilde{x}}} x(u)\left(\frac{1}{f(|R_{n+1}t + \gamma(u)) - s|)} - \frac{1}{f(r)}\right) - z_{R_{n+1}t+\tilde{t}} \le a + \frac{1}{2}\varepsilon. \tag{3.11}$$

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Then choosing u such that $x(u) = \tilde{x}_n$ and defining $\tilde{t}_n = \gamma(u)$ yields

$$\sup_{t \in D_{\tilde{x}}} \tilde{x}_n \left(\frac{1}{f(|R_{n+1}t + \tilde{t}_n) - s|)} - \frac{1}{f(r)} \right) - z_{R_{n+1}t+\tilde{t}} \leq a + \frac{1}{2}\varepsilon.$$

Using the uniform continuity of (z_t) again together with a reparameterisation gives

$$\sup_{t \in D_{u,\tilde{x}}} \frac{\tilde{x}_n - y_t}{f(|t-s|)} - \frac{\tilde{x}_n}{f(r)} \leq a + \frac{3}{4}\varepsilon.$$

Note that from the choice of $\tilde{x}$ and the fact that $x_n < \tilde{x}_n$, $D^{R_{n+1}}(\tilde{t}_n) \cap H_{\tilde{x}_n} \subseteq D_{u,\tilde{x}}$. In combination with (3.6) this yields the desired contradiction to (3.7).

Thus, the proof will be complete, if we can show (3.11). First, we observe that the cases $u = 0$ and $u = 1$ follow from (3.9) and (3.10). The result for a general $u \in (0, 1)$ then follows if, for any given $t \in D_{\tilde{x}}$, we can show that

$$x(u)F(u) \leq a + \tilde{z} + \tfrac{1}{2}\varepsilon \quad \text{for all } u \in [0, 1], \tag{3.12}$$

where

$$F(u) = \frac{1}{f(|\tilde{\gamma}(u) - s|)} - \frac{1}{f(r)}, \qquad \tilde{z} = z_{R_{n+1}t+\tilde{t}}, \qquad \tilde{\gamma}(u) = R_{n+1}t + \gamma(u).$$

For ease of notation, t is suppressed. To obtain (3.12), we use the facts that, for all t such that $r \leq |t-s| \leq r + \ell$,

$$\left(b - \frac{1}{m}\right)(|t-s| - r) \leq \left(\frac{1}{f(|t-s|)} - \frac{1}{f(r)}\right) \leq \left(b + \frac{1}{m}\right)(|t-s| - r), \tag{3.13}$$

and, for $r - \ell \leq |t-s| \leq r$,

$$\left(b + \frac{1}{m}\right)(|t-s| - r) \leq \left(\frac{1}{f(|t-s|)} - \frac{1}{f(r)}\right) \leq \left(b - \frac{1}{m}\right)(|t-s| - r), \tag{3.14}$$

where we have applied (3.5) and the fact that $h(\ell) < 1/m$. Note that the assumptions above imply that $||\tilde{\gamma}(u) - s| - r| < \ell$ for all $u \in [0, 1]$. Note also that $F(u) > 0$ if and only if $|\tilde{\gamma}(u) - s| - r > 0$. Consider the four cases (1°) $F(0)$, $F(1) > 0$; (2°) $F(0)$, $F(1) < 0$; (3°) $F(0) < 0 < F(1)$; (4°) $F(0) > 0 > F(1)$.

For case (1°), using (3.13), we find, for $u = 0, 1$,

$$\left(b - \frac{1}{m}\right)x(u)(|\tilde{\gamma}(u) - s| - r) \leq a + \tilde{z} + \frac{1}{4}\varepsilon. \tag{3.15}$$

Now let $G(u)$ be the linear interpolant such that $G(0) = (|\tilde{\gamma}(0) - s| - r)$ and $G(1) = (|\tilde{\gamma}(1) - s| - r)$. Then since, for $u = 0, 1$,

$$\left(b - \frac{1}{m}\right)x(u)G(u) \leq a + \tilde{z} + \frac{1}{2}\varepsilon, \tag{3.16}$$

and since $u \mapsto x(u)G(u)$ is monotone, Lemma A.1 implies that inequality (3.16) is satisfied for all $u \in [0, 1]$. Also, since $u \mapsto |\tilde{\gamma}(u) - s|$ is convex, (3.15) is satisfied for all $u \in [0, 1]$; thus, for all u in $[0, 1]$,

$$\left(b + \frac{1}{m}\right)x(u)(|\tilde{\gamma}(u) - s| - r) \leq \left(a + \tilde{z} + \frac{1}{4}\varepsilon\right)\frac{b + 1/m}{b - 1/m}.$$

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Another appeal to (3.13) then shows that

$$x(u)F(u) \le \left(a + \tilde{z} + \frac{1}{4}\varepsilon\right)\frac{b+1/m}{b-1/m}. \tag{3.17}$$

For the (2°) case, since $F(u) < 0$ if both $F(0) < 0$ and $F(1) < 0$, inequality (3.12) is trivially satisfied if $a + \tilde{z} + \frac{1}{2}\varepsilon \ge 0$. So assume that $a + \tilde{z} + \frac{1}{2}\varepsilon < 0$. Then, similarly, using (3.14),

$$x(u)F(u) \le \left(a + \tilde{z} + \frac{1}{4}\varepsilon\right)\frac{b-1/m}{b+1/m}. \tag{3.18}$$

Case (3°) is trivially satisfied because $u \mapsto F(u)$ is increasing.

For case (4°), we only need to show that $x(u)F(u) \le a + \tilde{z} + \frac{1}{4}\varepsilon$ for all $u \in [0, u_0]$, where $F(u_0) = 0$. To do so, we can repeat the technique from (1°), since now $x(u)F(u) \le a + \tilde{z} + \frac{1}{4}\varepsilon$ for $u = 0, u_0$.

The desired inequality (3.12) follows from (3.17) and (3.18) by letting $m \to \infty$. Note that this can be carried out uniformly in t because the field (z_t) is bounded. □

The next lemma describes λ_s for a particularly simple set D.

Lemma 3.2. *If $D = \{-\alpha_0 r, \alpha_0 r\}$ with α_0 as defined in Example 2.3 then*

$$\lambda_s((y_t)_{t\in B\oplus C_r}) = \sup_{\alpha\in\mathbb{S}^{d-1}} \frac{1}{2f(r)}(y_{s+\alpha r} + y_{s-\alpha r}).$$

Proof. First, we introduce the notation $D^\alpha(s) = \{s - \alpha r, s + \alpha r\}$ for $\alpha \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}^d$. Then, for such s fixed, $\{D^\alpha(s) \colon \alpha \in \mathbb{S}^{d-1}\} = \{D^R(s) \colon R \in \mathrm{SO}(d)\}$, so for D chosen as in the lemma, we can use unit vectors to parameterise all rotations. Now define $u_{s,\alpha} = s + r\alpha$ for $\alpha \in \mathbb{S}^{d-1}$ and $u_{s,t,\gamma,\alpha} = s + t\gamma + r\alpha$ for $t \ge 0$ and $\gamma \in \mathbb{S}^{d-1}$. The latter parameterises points on the boundary of a ball with radius r and centre in $s + t\gamma$. Note that $u_{s,0,\gamma,\alpha} = u_{s,\alpha}$ and that $\lim_{t\to 0} u_{s,t,\gamma,\alpha} = u_{s,\gamma,\alpha}$. Further,

$$|u_{t\gamma,\alpha} - s| = |t\gamma + r\alpha| = \sqrt{t^2 + r^2 + 2tr\cos\angle(\alpha,\gamma)},$$

where $\angle(\alpha, \gamma)$ denotes the angle between α and γ (for example, in the one-dimensional case, we have $\angle(1, -1) = \pi$). Because f is differentiable in r, we can write

$$\begin{aligned}
&\left|\frac{1}{t}\left[\frac{1}{f(|u_{s,t,\gamma,\alpha} - s|)} - \frac{1}{f(r)}\right] - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right| \\
&\quad = \left|\frac{1}{t}\left[\frac{-f'(r)}{f(r)^2}(|t\gamma + r\alpha| - r) + \phi(|t\gamma + r\alpha| - r)(|t\gamma + r\alpha| - r)\right]\right. \\
&\qquad \left. - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right|,
\end{aligned}$$

where ϕ is continuous with $\phi(0) = 0$. Using a second-order Taylor approximation around 0 of $t \mapsto \sqrt{t^2 + r^2 + 2tr\cos\angle(\alpha,\gamma)}$, we see that $\big(|t\gamma + r\alpha| - r\big)/t$ converges to $\cos\angle(\alpha,\gamma)$ uniformly in α, γ as $t \to 0$. Thus, for all $s \in B$,

$$\sup_{\gamma,\alpha}\left|\frac{1}{t}\left(\frac{1}{f(|u_{s,t,\gamma,\alpha} - s|)} - \frac{1}{f(r)}\right) - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right| \to 0 \quad (t \to 0).$$

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Since $y_{u_{s,t,\gamma,\alpha}} \to y_{u_{s,\alpha}}$ uniformly in $\alpha, \gamma \in \mathbb{S}^{d-1}$ due to uniform continuity of the (y_t)-field, we find that if (t_x) is a sequence decreasing to 0 such that $xt_x \to C$ as $x \to \infty$, then

$$\sup_{\gamma,\alpha}\left|\frac{x - y_{u_{s,t_x,\gamma,\alpha}}}{f(|u_{s,t_x,\gamma,\alpha} - s|)} - \frac{x}{f(r)} - \left(C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{u_{s,\alpha}}}{f(r)}\right)\right| \to 0 \quad (x \to \infty).$$

From this we find, still for $x \to \infty$,

$$\sup_{\gamma,\alpha}\left|\max_{t\in D^\alpha(s+\gamma t)}\left(\frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)}\right) - \max_{t\in D^\alpha(s)}\left(C\frac{-f'(r)}{f(r)^2}\cos\angle(t-s,\gamma) - \frac{y_t}{f(r)}\right)\right| \to 0.$$

Next, we claim that, for all $\alpha, \gamma \in \mathbb{S}^{d-1}$ and $C \geq 0$,

$$\begin{aligned}\max_{t\in D^\alpha(s)}&\left(C\frac{-f'(r)}{f(r)^2}\cos\angle(t-s,\gamma) - \frac{y_t}{f(r)}\right)\\ &= \max\left\{C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{s+r\alpha}}{f(r)}, -C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{s-r\alpha}}{f(r)}\right\}\\ &\geq \sup_{\alpha\in\mathbb{S}^{d-1}}\frac{1}{2f(r)}(y_{s+r\alpha} + y_{s-r\alpha}),\end{aligned}$$

with equality if

$$\alpha_0 = \arg\max_{\alpha\in\mathbb{S}^{d-1}}\left\{\frac{y_{s+r\alpha} + y_{s-r\alpha}}{2f(r)} : y_{s+r\alpha} \geq y_{s-r\alpha}\right\},$$

and, further, $\gamma_0 = \alpha_0$ and $C_0 = f(r)/(-2f'(r))(y_{s+r\alpha} - y_{s-r\alpha})$. For the proposed choice of α_0, γ_0, and C_0, it is easily seen that

$$C_0\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha_0,\gamma_0) - \frac{y_{s+r\alpha_0}}{f(r)} = -C_0\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha_0,\gamma_0) - \frac{y_{s-r\alpha_0}}{f(r)},$$

and that the common value equals the desired lower bound. It is also seen that any other choice of α, γ and C can only increase one of the two terms above.

Now let (α_n) and (γ_n) be sequences in $\mathbb{S}^{d-1}$, (t_n) a sequence of positive numbers, and (x_n) a sequence increasing to ∞. Then the results above show that

$$\liminf_{n\to\infty}\max_{t\in D_n^\alpha(s+\gamma_n t_n)}\left(\frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)}\right) \geq \sup_{\alpha\in\mathbb{S}^{d-1}}\frac{y_{s+r\alpha} + y_{s-r\alpha}}{2f(r)},$$

and that there is equality if $\alpha_n = \alpha_0$, $\gamma_n = \gamma_0$, and $x_n t_n \to C_0$ with α_0, γ_0, and C_0 as proposed above. Combined with Lemma 3.1 this gives the desired result. □

Lemma 3.3. *Let $n \in \mathbb{N}$ and assume that, for each $i = 1, \ldots, n$, $(y_t^i)_{t\in B\oplus C_r}$ has the form*

$$y_t^i = z^i f(|t - s^i|) \quad \text{for all } t \in B \oplus C_r,$$

where all $z^i \geq 0$ and $s^i \in \mathbb{R}^d$. Let g be as defined in (2.10). Define, for $s \in \mathbb{R}^d$,

$$\varphi(s) = f(r)1_{B\oplus C_r}(s) + 1_{(B\oplus C_r)^c}(s)\sup_{t\in B} g(|t-s|). \tag{3.19}$$

Then

$$\sup_{s\in B}\lambda_s\left(\left(\sum_{i=1}^n y_t^i\right)_{t\in B\oplus C_r}\right) \leq \frac{1}{f(r)}\sum_{i=1}^n z^i\varphi(s^i), \qquad \sup_{t_0\in B}\sup_{\alpha\in\mathbb{S}^{d-1}}\inf_{t\in D^\alpha(t_0)}\sum_{i=1}^n y_t^i \leq \sum_{i=1}^n z^i\varphi(s^i).$$

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Proof. Assume that $s^i \in B \oplus C_r$. For each $\alpha \in \mathbb{S}^{d-1}$ and $s \in B$, if $\min\{|s + r\alpha - s^i|, |s - r\alpha - s^i|\} = r - \delta$ for some $\delta > 0$, then $\max\{|s + r\alpha - s^i|, |s - r\alpha - s^i|\} \geq r + \delta$. Using assumption (2.10) then yields

$$\tfrac{1}{2}(y^i_{s+r\alpha} + y^i_{s-r\alpha}) \leq \tfrac{1}{2} z^i (g(r-\delta) + g(r+\delta)) = z^i f(r) = z^i \varphi(s^i).$$

Clearly, this inequality is also satisfied if both $|s + r\alpha - s^i| \geq r$ and $|s - r\alpha - s^i| \geq r$. If $s^i \in (B \oplus C_r)^c$ then for all choices of $s \in B$ and $\alpha \in \mathbb{S}^{d-1}$, we must have

$$\tfrac{1}{2}(y^i_{s+r\alpha} + y^i_{s-r\alpha}) \leq \tfrac{1}{2} z^i (g(|s + r\alpha - s^i|) + g(|s - r\alpha - s^i|)) \leq z^i \varphi(s^i).$$

Recall that for a given rotation matrix $R \in \mathrm{SO}(d)$, there exists $\alpha \in \mathbb{S}^{d-1}$ such that $\{s - r\alpha, s + r\alpha\} \subseteq D^R(s)$; then combined with Lemma 3.2, we now see that, for each $s \in B$,

$$\begin{aligned}\lambda_s\bigg(\bigg(\sum_{i=1}^n y^i_t\bigg)_{t\in B\oplus C_r}\bigg) &\leq \sup_{\alpha\in\mathbb{S}^{d-1}} \frac{1}{2f(r)}\bigg(\sum_{i=1}^n y^i_{s+\alpha r} + \sum_{i=1}^n y^i_{s-\alpha r}\bigg) \\ &\leq \sum_{i=1}^n \frac{1}{2f(r)} \sup_{\alpha\in\mathbb{S}^{d-1}} (y^i_{s+\alpha r} + y^i_{s-\alpha r}) \\ &\leq \frac{1}{f(r)} \sum_{i=1}^n z^i \varphi(s^i).\end{aligned}$$

Taking the supremum over $s \in B$ yields the first statement of the lemma.

For the second statement, similarly, we have, for each $t_0 \in B$ and $R \in \mathrm{SO}(d)$,

$$\inf_{t\in D^R(t_0)} \sum_{i=1}^n y^i_t \leq \min\bigg\{\sum_{i=1}^n y^i_{t_0+r\alpha}, \sum_{i=1}^n y^i_{t_0-r\alpha}\bigg\} \leq \frac{1}{2}\bigg(\sum_{i=1}^n y^i_{t_0+\alpha r} + \sum_{i=1}^n y^i_{t_0-\alpha r}\bigg) \leq \sum_{i=1}^n z^i \varphi(s^i),$$

where, again, $\alpha \in \mathbb{S}^{d-1}$ is chosen such that $\{s - r\alpha, s + r\alpha\} \subseteq D^R(s)$. The result follows by taking the supremum over $t_0 \in B$ and $R \in \mathrm{SO}(d)$. □

4. The main theorem

In this section we derive the main result, namely Theorem 4.4 below. For $x > 0$, define the set

$$\Lambda(x) = \Big\{(y_t)_{t\in B\oplus C_r} : \sup_{t_0\in B} \sup_{R\in\mathrm{SO}(d)} \inf_{t\in D^R(t_0)} y_t > x\Big\}.$$

Note that for a random field $(Y_t)_{t\in B\oplus C_r}$ with excursion set $A_x = \{t \in B \oplus C_r : Y_t > x\}$,

$$\mathbb{P}((Y_t)_{t\in B\oplus C_r} \in \Lambda(x)) = \mathbb{P}(\text{there exist } t \in B \text{ and } R \in \mathrm{SO}(d) \colon D^R(t) \subseteq A_x).$$

Our first step is to determine the asymptotic behaviour of excursion sets for a field U with distribution ν_1. Recall the definition of $L(x)$ from (2.5).

Theorem 4.1. *Assume that $(U_t)_{t\in B\oplus C_r}$ has distribution ν_1 and let $(y_t)_{t\in B\oplus C_r}$ be continuous. Then, as $x \to \infty$,*

$$\frac{\mathbb{P}((U_t + y_t)_{t\in B\oplus C_r} \in \Lambda(x))}{L(x/f(r)) \exp(-\beta x/f(r))} \to \frac{1}{\nu(A)} \int_B \exp(\beta\lambda_s((y_t)_{t\in B\oplus C_r}))\mathrm{d}s, \tag{4.1}$$

$$\frac{\mathbb{P}((U_t)_{t\in B\oplus C_r} \in \Lambda(x))}{L(x/f(r)) \exp(-\beta x/f(r))} \to \frac{1}{\nu(A)} m_d(B), \tag{4.2}$$

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and

$$\frac{\mathbb{P}((U_t+y_t)_{t\in B\oplus C_r}\in\Lambda(x))}{\mathbb{P}((U_t)_{t\in B\oplus C_r}\in\Lambda(x))}\to\frac{\int_B\exp(\beta\lambda_s((y_t)_{t\in B\oplus C_r}))\,\mathrm{d}s}{m_d(B)}. \tag{4.3}$$

Proof. The limit results (4.2) and (4.3) follow directly from (4.1), so we only need to prove (4.1). We can assume that $(y_t)_{t\in B\oplus C_r}$ is nonnegative: simply write $x=x'-x_0$ for a suitable x_0 such that $(x_0+y_t)_{t\in B\oplus C_r}$ is nonnegative; we shall find

$$\lim_{x'\to\infty}\frac{\mathbb{P}((U_t+x_0+y_t)_{t\in B\oplus C_r}\in\Lambda(x'))}{L(x'/f(r))\exp(-\beta x'/f(r))}.$$

Consider

$$\begin{aligned}
&\mathbb{P}((U_t+y_t)_{t\in B\oplus C_r}\in\Lambda(x))\\
&\quad=\frac{1}{\nu(A)}F\bigg(\bigg\{(s,z)\in\mathbb{R}^d\times\mathbb{R}\colon\sup_{t_0\in B}\sup_{R\in\mathrm{SO}(d)}\inf_{t\in D^R(t_0)}zf(|t-s|)+y_t>x\bigg\}\bigg)\\
&\quad=\frac{1}{\nu(A)}F\bigg(\bigg\{(s,z)\in\mathbb{R}^d\times\mathbb{R}\colon z>\inf_{t_0,R}\sup_{t\in D^R(t_0)}\frac{x-y_t}{f(|t-s|)}\bigg\}\bigg)\\
&\quad=\frac{1}{\nu(A)}\int_B L\bigg(\inf_{t_0,\alpha}\sup_{t\in D^R(t_0)}\frac{x-y_t}{f(|t-s|)}\bigg)\exp\bigg(-\beta\inf_{t_0,R}\sup_{t\in D^R(t_0)}\frac{x-y_t}{f(|t-s|)}\bigg)\mathrm{d}s\\
&\qquad+\frac{1}{\nu(A)}\int_{\mathbb{R}^d\setminus B}L\bigg(\inf_{t_0,R}\sup_{t\in D^R(t_0)}\frac{x-y_t}{f(|t-s|)}\bigg)\exp\bigg(-\beta\inf_{t_0,\alpha}\sup_{t\in D^R(t_0)}\frac{x-y_t}{f(|t-s|)}\bigg)\mathrm{d}s.
\end{aligned} \tag{4.4}$$

Start by showing that the second term in (4.4) is $o(L(x/f(r))\exp(-\beta x/f(r)))$. Let $y^*=\sup_{s\in B\oplus C_r}y_s$. Observe that $L(x)\mathrm{e}^{-\beta x}$ decreases in x so if $x>y^*$ then the second term is bounded above by

$$\frac{1}{\nu(A)}\int_{\mathbb{R}^d\setminus B}L\bigg(\frac{x-y^*}{f_0(s)}\bigg)\exp\bigg(-\beta\frac{x-y^*}{f_0(s)}\bigg)\mathrm{d}s, \tag{4.5}$$

where $f_0(s):=\sup_{t_0,R}\inf_{t\in D^R(t_0)}f(|t-s|)$. From arguments similar to those used to prove [13, Theorem 3.1] it can be seen that for all $\gamma>0$ there exists $x_0>0$ and $C>0$ such that

$$\frac{L(ax)}{L(x)}\le C\mathrm{e}^{(a-1)\gamma x}\quad\text{for all } x\ge x_0 \text{ and } a\ge 1. \tag{4.6}$$

Note that the convexity of B implies that $f_0(s)<f(r)$ for all $s\in\mathbb{R}^d\setminus B$. Combining this with (2.5), relation (4.6), and the fact that $L(x)\mathrm{e}^{-\gamma x}\to 0$ for all $\gamma>0$ shows that the integrand in (4.5) is $o(L(x/f(r))\exp(-\beta x/f(r)))$.

Denote the integrand of (4.5) by $h(s;x)$. If we can find an integrable function g such that

$$\frac{h(s;x)}{L(x/f(r))\exp(-\beta x/f(r))}\le g(s),\qquad s\in\mathbb{R}^d,$$

then the dominated convergence theorem implies that (4.5) is $o(L(x/f(r))\exp(-\beta x/f(r)))$. Let $0<\gamma<\beta$. Using (4.6) and the boundedness of $L((x-y^*)/f(r))/L(x/f(r))$, we can find a constant $\tilde{C}$ and $x_0 > y^*$ such that, for $x \geq x_0$,

$$\frac{h(s;x)}{L(x/f(r))\exp(-\beta x/f(r))} \leq \tilde{C}\exp\bigg(\frac{\beta y^*}{f(r)}\bigg)\exp\bigg(-(\beta-\gamma)\bigg[\frac{1}{f_0(s)}-\frac{1}{f(r)}\bigg](x_0-y^*)\bigg). \tag{4.7}$$

Now choose $D > 0$ such that $B \oplus C_r \subseteq C_D(0)$ and $\sup_{t\in B} f(|t-s|) < f(r)$ for all $s \notin C_D(0)$. Then using (2.8), we find for $s \notin C_D(0)$ that

$$f_0(s) \leq \sup_{t\in B} f(|t-s|) \leq \sup_{t\in C_D(0)} f(|t-s|) \leq \sup_{t\in C_D(0)} \frac{1}{(|t-s|+1)^d} = \frac{1}{(|s|-D+1)^d}.$$

It follows that the function (4.7) is integrable.

The theorem now follows by applying dominated convergence to the first term of (4.4). From Lemma 3.1, we have, for $s \in B$,

$$\frac{L(\inf_{t_0,R}\sup_{t\in D^R(t_0)}(x-y_t)/f(|t-s|))(-\beta\inf_{t_0,R}\sup_{t\in D^R(t_0)}(x-y_t)/f(|t-s|))}{L(x/f(r))\exp(-\beta x/f(r))} \\ \to \mathrm{e}^{\beta\lambda_s((y_t)_t)}.$$

Again using the fact that $L(x)\mathrm{e}^{-\beta x}$ is decreasing, it follows that, for large enough x,

$$\begin{aligned}&\bigg|\frac{L(\inf_{t_0,R}\sup_{t\in D^R(t_0)}(x-y_t)/f(|t-s|))\exp(-\beta\inf_{t_0,R}\sup_{t\in D^R(t_0)}(x-y_t)/f(|t-s|))}{L(x/f(r))\exp(-\beta x/f(r))} \\ &\quad - \mathrm{e}^{\beta\lambda_s((y_t)_t)}\bigg| \\ &\qquad \leq \frac{L((x-y^*)/f(r))\exp(-\beta(x-y^*)/f(r))}{L(x/f(r))\exp(-\beta x/f(r))} + \mathrm{e}^{\beta\lambda_s((y_t)_t)} \\ &\qquad \leq (C+1)\mathrm{e}^{\beta y^*},\end{aligned}$$

where C is chosen such that

$$\frac{L(x-y^*)/f(r)}{L(x/f(r))} \leq C.$$

The result is integrable over B. □

The next step is to extend the result of Theorem 4.1 to the case $\mathbb{P}((U^1+\cdots+U^n+y_t)_t \in \Lambda(x))$, where U^i, $i=1,\dots,n$, are independent with common distribution ν_1. Recall that each $(U_t^i)_{t\in B\oplus C_r}$ can be represented by $\big(Z^i f(|t-S^i|)\big)_{t\in B\oplus C_r}$, where (S^i, Z^i) has distribution F_1. For this purpose we need the following lemma and corollary.

Lemma 4.1. *Let (S,Z) be distributed according to F_1. Then, when $x\to\infty$,*

$$\frac{\mathbb{P}(Z\varphi(S) > x)}{L(x/f(r))\exp(-\beta x/f(r))} \to \frac{m_d(B\oplus C_r)}{\nu(A)}.$$

In particular,

$$\mathbb{E}\bigg[\exp\bigg(\frac{\beta}{f(r)Z\phi(S)}\bigg)\bigg] < \infty.$$

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Proof. Similar to the proof of Theorem 4.1, we can write

$$\begin{aligned}\mathbb{P}(Z\phi(S) > x) &= \frac{1}{\nu(A)} F(\{(s,z) \in \mathbb{R}^d \times \mathbb{R} \colon z\varphi(s) > x\}) \\ &= \frac{1}{\nu(A)} \int_{B\oplus C_r} L\bigg(\frac{x}{f(r)}\bigg) \exp\bigg(-\frac{\beta x}{f(r)}\bigg) \mathrm{d}s \\ &\quad + \frac{1}{\nu(A)} \int_{B\oplus C_r} L\bigg(\frac{x}{\sup_{t\in B} g(|t-s|)}\bigg) \exp\bigg(-\frac{\beta x}{\sup_{t\in B} g(|t-s|)}\bigg) \mathrm{d}s,\end{aligned}$$

where the first term equals $L(x/f(r)) \exp(-\beta x/f(r))$ times the desired limit while a dominated convergence argument (using $\sup_{t\in B} g(|t-s|) < f(r)$ for all $s \in (B \oplus C_r)^c$) shows that the second term is $o(L(x/f(r)) \exp(-\beta x/f(r))$.

The second particular result follows from [10, Corollary 2.1(ii)]. □

Corollary 4.1. *For $U^1, U^2, \dots$ i.i.d. with distribution ν_1,*

$$\mathbb{E}\Big[\exp\Big(\beta \sup_{s\in B} \lambda_s((U_t^1 + \cdots + U_t^n)_{t\in B\oplus C_r})\Big)\Big] < \infty \quad (\textit{all } n \in \mathbb{N}).$$

Proof. Because each U^i has the form $(Z^i f(|t - S^i|))_{t\in B\oplus C_r}$, the result follows from Lemmas 3.3 and 4.1. □

Theorem 4.2. *Let $U^1, U^2, \dots$ be i.i.d. with distribution ν_1 and assume that $(y_t)_{t\in B\oplus C_r}$ is continuous. For all $n \in \mathbb{N}$ and $x \to \infty$,*

$$\begin{aligned}&\frac{\mathbb{P}((U_t^1 + \cdots + U_t^n + y_t)_t \in \Lambda(x))}{\mathbb{P}((U_t^1)_t \in \Lambda(x))} \\ &\quad \to \frac{n}{m_d(B)} \int_B \mathbb{E}[\exp(\beta\lambda_s((U_t^1 + \cdots + U_t^{n-1} + y_t)_t))]\mathrm{d}s.\end{aligned}$$

Proof. As in the proof of Theorem 4.1, we can assume that $(y_t)_{t\in B\oplus C_r}$ is nonnegative. The result is shown by induction over n. For $n = 1$, the result is shown in Theorem 4.1. Assume now that the theorem is correct for some $n \in \mathbb{N}$. For convenience, write $V = U^1 + \cdots + U^n$ and recall the representation $U_t^i = Z^i f(|t - S^i|)$. Then

$$\begin{aligned}&\mathbb{P}((V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)) \\ &\quad = \mathbb{P}\bigg(\sum_{i=1}^n Z^i\varphi(S^i) > \frac{1}{2}x,\ Z^{n+1}\varphi(S^{n+1}) > \frac{1}{2}x,\ (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\bigg) \\ &\qquad + \mathbb{P}\bigg(\sum_{i=1}^n Z^i\varphi(S^i) \le \frac{1}{2}x,\ (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\bigg) \\ &\qquad + \mathbb{P}\big(Z^{n+1}\varphi(S^{n+1}) \le \tfrac{1}{2}x,\ (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\big).\end{aligned} \tag{4.8}$$

The first term on the right-hand side is bounded above by the product

$$\mathbb{P}\bigg(\sum_{i=1}^n Z^i\varphi(S^i) > \frac{1}{2}x\bigg)\mathbb{P}\bigg(Z^{n+1}\varphi(S^{n+1}) > \frac{1}{2}x\bigg).$$

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In Lemma 4.1 we showed that the distribution of each $Z^i\varphi(S^i)$ is convolution equivalent. Thus, both terms of the product are asymptotically equivalent with $\rho_1((x/(2f(r)),\infty))$, and then it follows from the proof of [5, Lemma 2] that the product is $o((\rho_1 * \rho_1)((x/f(r),\infty)))$. In particular, the product above is $o(\rho_1((x/f(r),\infty)))$ due to convolution equivalence (see around (2.5) above concerning these ideas for ρ and ρ_1).

The two remaining terms in (4.8) divided by $\mathbb{P}((U_t^1)_t \in \Lambda(x))$ can be rewritten as below:

$$\int_{C_x} \frac{\mathbb{P}((U_t^{n+1} + \sum_{i=1}^n z^i f(|t-s^i|) + y_t)_t \in \Lambda(x))}{\mathbb{P}((U_t^1)_t \in \Lambda(x))} F_1^{*\otimes n}(\mathrm{d}(s^1, z^1; \dots; s^n, z^n)) \\ + \int_{\tilde{C}_x} \frac{\mathbb{P}((V_t + zf(|t-s|) + y_t)_t \in \Lambda(x))}{\mathbb{P}((U_t^1)_t \in \Lambda(x))} F_1(\mathrm{d}(s,z)). \tag{4.9}$$

Here $F_1^{*\otimes n}$ is the n-fold product measure of F_1, and we have used the fact that $(V_t)_t$ can be represented by $(\sum_{i=1}^n Z^i f(|t-S^i|))_t$. The sets C_x and $\tilde{C}_x$ in (4.9) are defined by

$$C_x = \left\{(s^1, z^1; \dots; s^n, z^n) \colon \sum_{i=1}^n z^i\varphi(s^i) \le \frac{1}{2}x\right\} \quad \text{and} \quad \tilde{C}_x = \left\{(s,z) \colon z\varphi(s) \le \frac{1}{2}x\right\}.$$

Using Theorem 4.1 and the induction assumption, the two integrands of (4.9) multiplied by 1_{C_x} and $1_{\tilde{C}_x}$ respectively, converge as $x \to \infty$ to

$$f_1(s^1, z^1; \dots; s^n, z^n) = \frac{1}{m_d(B)} \int_B \exp\biggl(\beta\lambda_s\biggl(\biggl(y_t + \sum_{i=1}^n z^i f(|t-s^i|)\biggr)_{t\in B\oplus C_r}\biggr)\biggr)\mathrm{d}s$$

and

$$f_2(s,z) = \frac{n}{m_d(B)} \int_B \mathbb{E}[\exp(\beta\lambda_s((U_t^1 + \dots + U_t^{n-1} + zf(|t-s|) + y_t)_{t\in B\oplus C_r}))]\mathrm{d}s,$$

respectively. We want to show that (4.9) converges to

$$\int f_1(s^1, z^1; \dots; s^n, z^n) F_1^{*\otimes n}(\mathrm{d}(s^1, z^1; \dots; s^n, z^n)) + \int f_2(s,z) F_1(\mathrm{d}(s,z)) \\ = \frac{n+1}{m_d(B)} \int_B \mathbb{E}[\exp(\beta\lambda_s((U_t^1 + \dots + U_t^n + y_t)_t))]\mathrm{d}s.$$

To this end, it is enough to find integrable functions $g_1(s^1, z^1; \dots; s^n, z^n; x)$ and $g_2(s,z;x)$ that are upper bounds of the two integrands of (4.9) such that the limits $g_1(s^1, z^1; \dots; s^n, z^n) = \lim_{x\to\infty} g_1(s^1, z^1; \dots; s^n, z^n; x)$ and $g_2(s,z) = \lim_{x\to\infty} g_2(s,z;x)$ exist with

$$\int_{C_x} g_1(s^1, z^1; \dots; s^n, z^n; x) F_1^{*\otimes n}(\mathrm{d}(s^1, z^1; \dots; s^n, z^n)) + \int_{\tilde{C}_x} g_2(s,z;x) F_1(\mathrm{d}(s,z)) \tag{4.10}$$

converging to the similar integrals with $g_1(s^1, z^1; \dots; s^n, z^n)$ and $g_2(s,z)$, and use Fatou's lemma. Using Lemma 3.3 we find that as functions $g_1(s^1, z^1; \dots; s^n, z^n; x)$ and $g_2(s,z;x)$ we can use

$$g_1(s^1, z^1; \dots; s^n, z^n; x) = \frac{\mathbb{P}(Z^1\varphi(S^1) > x - y^* - \sum_{i=1}^n z^i\varphi(s^i))}{\mathbb{P}((U_t)_t \in \Gamma(x))},$$

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where, as previously, $y^* = \sup_{t\in B\oplus C_r} y_t$, and

$$g_2(s, z; x) = \frac{\mathbb{P}(\sum_{i=1}^n Z^i\varphi(S^i) > x - y^* - z\varphi(s))}{\mathbb{P}((U_t)_t \in \Gamma(x))}.$$

From Theorem 4.1 and Lemma 4.1, we see that, for large x,

$$\mathbb{P}((U_t)_t \in \Gamma(x)) \sim \frac{m_d(B)}{m_d(B\oplus C_r)}\mathbb{P}(Z^1\varphi(S^1) > x),$$

and, hence,

$$\begin{aligned} g_1(s^1, z^1; \dots; s^n, z^n; x) &\to g_1(s^1, z^1; \dots; s^n, s^n) \\ &= \frac{m_d(B\oplus C_r)}{m_d(B)}\exp\biggl(\frac{\beta}{f(r)(y^* + \sum_{k=1}^n z^i\varphi(s^i))}\biggr). \end{aligned}$$

Since the distribution of $\sum_{i=1}^n Z^i\varphi(S^i)$ is convolution equivalent, [6, Corollary 2.11] yields

$$g_2(s, z; x) \to g_2(s, z) = \frac{m_d(B\oplus C_r)}{m_d(B)} n\cdot \mathrm{e}^{\beta/f(r)(y^*+z\varphi(s))}\bigl(\mathbb{E}[\mathrm{e}^{\beta/f(r)Z^1\varphi(S^1)}]\bigr)^{n-1}.$$

We observe that

$$\begin{aligned} &\int g_1(s^1, z^1; \dots; s^n, z^n)F_1^{*\otimes n}(\mathrm{d}(s^1, z^1; \dots; s^n, z^n)) + \int g_2(s, z)F_1(\mathrm{d}(s, z)) \\ &\quad = \frac{m_d(B\oplus C_r)}{m_d(B)}(n+1)\cdot \mathrm{e}^{\beta/f(r)y^*}\bigl(\mathbb{E}[\mathrm{e}^{\beta/f(r)Z^1\varphi(S^1)}]\bigr)^{n}. \end{aligned} \tag{4.11}$$

Since the tails of $\sum_{i=1}^n Z^i\varphi(S^i)$ and $Z^1\varphi(S^1)$, in particular, are exponential with index $\beta/f(r)$, appealing to [5, Lemma 2] shows that (4.10) is asymptotically equal to

$$\mathrm{e}^{\beta/f(r)y^*}\frac{\mathbb{P}(\sum_{i=1}^{n+1} Z^i\varphi(S^i) > x)}{\mathbb{P}(Z^1\varphi(S^1) > x)}$$

which, by another reference to [6, Corollary 2.11], is seen to converge to (4.11). □

For a dominated convergence argument, we need the lemma below.

Lemma 4.2. *Let $U^1, U^2, \dots$ be i.i.d. with distribution ν_1, and assume that (S, Z) has distribution F_1. There exists a constant K such that*

$$\mathbb{P}((U_t^1 + \dots + U_t^n)_t \in \Lambda(x)) \le K^n\mathbb{P}(Z\varphi(S) > x) \quad (\textit{all } n\in\mathbb{N} \textit{ and all } x \ge 0).$$

Proof. Since $Z\varphi(S)$ has a convolution equivalent tail according to Corollary 4.1, it follows from [6, Lemma 2.8] that there exists K such that

$$\mathbb{P}\biggl(\sum_{i=1}^n Z^i\varphi(S^i) > x\biggr) \le K^n\mathbb{P}(Z\varphi(S) > x).$$

The result now follows directly from Lemma 3.3. □

Recall that we can write the field $(X_t)_{t\in T}$ as $X_t = X_t^1 + X_t^2$, where the field X^1 is obtained from the fields $U^1, U^2, \dots$ and an independent Poisson distributed variable N with parameter $\nu(A)$ by $X_t^1 = \sum_{n=1}^N U_t^n$.

Theorem 4.3. (a) *For each* $s \in B$, $\mathbb{E}[\exp(\beta\lambda_s((X_t^1)_{t\in B\oplus C_r}))] < \infty$.

(b) *For a continuous field* $(y_t)_{t\in B\oplus C_r}$,

$$\lim_{x\to\infty} \frac{\mathbb{P}((X_t^1 + y_t)_t \in \Lambda(x))}{L(x/f(r))\exp(-\beta x/f(r))} = \int_B \mathbb{E}[(\exp(\beta\lambda_s((X_t^1 + y_t)_{t\in B\oplus C_r})))]\mathrm{d}s.$$

Proof. To prove part (a), recall that $\lambda_s((X_t^1)_{t\in B\oplus C_r}) \le (1/f(r))\sum_{n=0}^N Z^i\varphi(S^i)$ and that $\mathbb{E}[\exp(\beta/f(r)Z^1\varphi(S^1))]$ is finite.

To prove (b), we use the relation

$$\mathbb{P}((X_t^1 + y_t)_t \in \Lambda(x)) = \mathrm{e}^{-\nu(A)} \sum_{n=1}^{\infty} \frac{\nu(A)^n}{n!} \mathbb{P}((U_t^1 + \cdots + U_t^n + y_t)_t \in \Lambda(x)).$$

For the sum here, we use Lemma 4.2 and the notation $y^* = \sup_{t\in B\oplus C_r} y_t$ as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\nu(A)^n}{n!} \frac{\mathbb{P}((U_t^1 + \cdots + U_t^n + y_t)_t \in \Lambda(x))}{\mathbb{P}(Z\varphi(S) > x - y^*)} \\
&\quad\le \sum_{n=1}^{\infty} \frac{\nu(A)^n}{n!} \frac{\mathbb{P}((U_t^1 + \cdots + U_t^n)_t \in \Lambda(x - y^*))}{\mathbb{P}(Z\varphi(S) > x - y^*)} \\
&\quad\le \sum_{n=1}^{\infty} \frac{K^n\nu(A)^n}{n!} \frac{\mathbb{P}(Z\varphi(S) > x - y^*)}{\mathbb{P}(Z\varphi(S) > x - y^*)} \\
&\quad= \sum_{n=1}^{\infty} \frac{K^n\nu(A)^n}{n!} \\
&\quad< \infty.
\end{aligned}$$

Now we obtain from Lemma 4.1 and Theorem 4.2 that

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\mathbb{P}((U_t^1 + \cdots + U_t^n + y_t)_t \in \Lambda(x))}{\mathbb{P}(Z\varphi(S) > x - y^*)} \\
&\qquad= \frac{n}{\mathrm{e}^{\beta/f(r)y^*} m_d(B\oplus C_r)} \int_B \mathbb{E}[\mathrm{e}^{\beta\lambda_s((U_t^1+\cdots+U_t^{n-1}+y_t)_t)}]\mathrm{d}s,
\end{aligned}$$

with the convention that $U_t^1 + \cdots + U_t^{n-1} = 0$ if $n = 1$. Then dominated convergence yieldss

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\mathbb{P}((X_t^1 + y_t)_t \in \Lambda(x))}{\mathbb{P}(Z\varphi(S) > x - y^*)} \\
&\quad= \frac{\mathrm{e}^{-\nu(A)}}{\mathrm{e}^{\beta/f(r)y^*} m_d(B\oplus C_r)} \sum_{n=1}^{\infty} \frac{\nu(A)^n}{n!} n \int_B \mathbb{E}[\mathrm{e}^{\beta\lambda_s((U_t^1+\cdots+U_t^{n-1}+y_t)_t)}]\mathrm{d}s \\
&\quad= \frac{\nu(A)}{\mathrm{e}^{\beta/f(r)y^*} m_d(B\oplus C_r)} \sum_{n=0}^{\infty} \mathrm{e}^{-\nu(A)} \frac{\nu(A)^n}{n!} \int_B \mathbb{E}[\mathrm{e}^{\beta\lambda_s((U_t^1+\cdots+U_t^{n}+y_t)_t)}]\mathrm{d}s \\
&\quad= \frac{\nu(A)}{\mathrm{e}^{\beta/f(r)y^*} m_d(B\oplus C_r)} \int_B \mathbb{E}[\mathrm{e}^{\beta\lambda_s((U_t^1+\ldots+U_t^{N}+y_t)_t)}]\mathrm{d}s \\
&\quad= \frac{\nu(A)}{\mathrm{e}^{\beta/f(r)y^*} m_d(B\oplus C_r)} \int_B \mathbb{E}[\mathrm{e}^{\beta\lambda_s((X_t^1+y_t)_t)}]\mathrm{d}s,
\end{aligned}$$

which with a final reference to Theorem 4.1 and Lemma 4.1 concludes the proof. □

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The theorem below is the main result of our paper. In the formulation of the theorem, we explicitly state the assumptions under which the limit holds.

Theorem 4.4. *When M satisfies assumptions (2.1)–(2.6) and f satisfies (2.7)–(2.10), we have* $\mathbb{E}[\exp(\beta\lambda_{t_0}((X_t)_{t\in B\oplus C_r})] < \infty$ *and*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t_0\in B}\sup_{R\in \mathrm{SO}(d)}\inf_{t\in D^R(t_0)} X_t > x)}{L(x/f(r))\exp(-\beta x/f(r))} = \mathbb{E}[\exp(\beta\lambda_{t_0}((X_t)_{t\in B\oplus C_r})m_d(B)],$$

where $t_0 \in B$ is chosen arbitrarily and λ_{t_0} is as defined in Lemma 3.1.

Proof. First we note that $\mathbb{E}[\exp(\gamma \sup_{t\in B\oplus C_r} X_t^2)] < \infty$ for all $\gamma > 0$ according to [13, Lemma 4.1]. Since, by Lemma 3.1, we also have

$$\lambda_{t_0}((X_t)_t) \le \lambda_{t_0}\Big(\Big(X_t^1 + \sup_t X_t^2\Big)_t\Big) = \lambda_{t_0}((X_t^1)_t) + \frac{\sup_t X_t^2}{f(r)},$$

the first statement follows from part (a) of Theorem 4.3. Let π be the distribution of $(X_t^2)_{t\in B\oplus C_r}$. We find that

$$\frac{\mathbb{P}((X_t)_t\in\Lambda(x))}{\mathbb{P}((X_t^1)_t\in\Lambda(x))} = \int \frac{\mathbb{P}((X_t^1+y_t)_t\in\Lambda(x))}{\mathbb{P}((X_t^1)_t\in\Lambda(x))}\pi(\mathrm{d}y) = \int f(y;x)\pi(\mathrm{d}y),$$

where

$$f(y;x) = \frac{\mathbb{P}((X_t^1+y_t)_t\in\Lambda(x))}{\mathbb{P}((X_t^1)_t\in\Lambda(x))}.$$

From Theorem 4.3 it is seen that as $x\to\infty$,

$$f(y;x)\to f(y) := \frac{\int_B \mathbb{E}[(\exp(\beta\lambda_s((X_t^1+y_t)_{t\in B\oplus C_r})))]\mathrm{d}s}{\int_B \mathbb{E}[(\exp(\beta\lambda_s((X_t^1)_{t\in B\oplus C_r})))]\mathrm{d}s}.$$

If we can show that

$$\int f(y;x)\pi(\mathrm{d}y) \to \int f(y)\pi(\mathrm{d}y) \quad \text{as } x\to\infty, \tag{4.12}$$

then the theorem follows with another reference to Theorem 4.3 and by recalling that $(X_t)_{t\in B\oplus C_r}$ is stationary. According to Fatou's lemma, (4.12) follows if we can find integrable nonnegative functions $g(y;x)$ and $g(y)$ such that

$$f(y;x) \le g(y;x), \tag{4.13}$$

$$g(y;x)\to g(y), \tag{4.14}$$

$$\int g(y;x)\pi(\mathrm{d}y) \to \int g(y)\pi(\mathrm{d}y). \tag{4.15}$$

For this purpose, let

$$g(y;x) = \frac{\mathbb{P}((X_t^1+\sup_t y_t)_t\in\Lambda(x))}{\mathbb{P}((X_t^1)_t\in\Lambda(x))}.$$

Then (4.13) is satisfied. Further, using Theorem 4.3 and Lemma 3.1, we find that (4.14) is satisfied with $g(y) = \exp(\beta/f(r)\sup_t y_t)$. To prove (4.15), we have

$$\int g(y;x)\pi_2(\mathrm{d}y) = \frac{\mathbb{P}(\sup_{t_0,R}\inf_{t\in D^R(t_0)} X_t^1 + \sup_t X_t^2 > x)}{\mathbb{P}(\sup_{t_0,R}\inf_{t\in D^R(t_0)} X_t^1 > x)}.$$

Note that $\sup_{t_0,R} \inf_{t \in D^R(t_0)} X_t^1$ has a convolution equivalent tail according to Theorem 4.3 and [10, Lemma 2.4 (i)]. Since $\mathbb{E}[\exp(\gamma \sup_t X_t^2)] < \infty$ for all $\gamma > 0$, it follows from [10, Lemma 2.1] and [10, Lemma 2.4 (ii)] that

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t_0,R} \inf_{t\in D^R(t_0)} X_t^1 + \sup_t X_t^2 > x)}{\mathbb{P}(\sup_{t_0,R} \inf_{t\in D^R(t_0)} X_t^1 > x)} = \mathbb{E}\bigg[\exp\bigg(\frac{\beta}{f(r)\sup_t X_t^2}\bigg)\bigg] = \int g(y)\pi(\mathrm{d}y).$$

Thus, (4.15) is satisfied. □

Appendix A.

The following simple lemma is used in Lemma 3.1.

Lemma A.1. *Let* $0 < x_n < x_{n+1}$ *be given. Then there exist constants* $C, D > 0$ *such that* $x\colon [0, 1] \to [0, \infty)$ *defined by*

$$x(u) = \frac{C}{1 - u + D} \tag{A.1}$$

is strictly increasing with $x(0) = x_n$ *and* $x(1) = x_{n+1}$*. Further, if* $g(u) = au + b$ *then* $u \mapsto x(u)g(u)$ *is monotone on* $[0, 1]$.

Proof. Any function of the form (A.1) is clearly strictly increasing on $[0, 1]$. The constants C, D that yield $x(0)$ and $x(1)$ as stated are found by straightforward manipulation. The last result is proved by differentiating $u \mapsto x(u)g(u)$. □

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