Proceedings of the Royal Society of Edinburgh, **152**, 626–648, *2022*
DOI:10.1017/prm.2021.23

The homotopy classification of four-dimensional toric orbifolds

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(Received 28 January 2020; accepted 24 April 2021)

Let X be a 4-dimensional toric orbifold. If $H^3(X)$ has a non-trivial odd primary torsion, then we show that X is homotopy equivalent to the wedge of a Moore space and a CW-complex. As a corollary, given two 4-dimensional toric orbifolds having no 2-torsion in the cohomology, we prove that they have the same homotopy type if and only their integral cohomology rings are isomorphic.

Keywords: Cohomological rigidity; toric orbifold

2020 *Mathematics Subject Classification:* Primary: 57R18; 55P15
Secondary: 55P60

1. Introduction

One of the central problems in topology is the rigidity question, namely when a weaker equivalence between two spaces implies a stronger equivalence between them. Freedman's work [**11**] on the classification of closed oriented simply connected topological 4-manifolds via the intersection form is a good example of this type of question. In toric topology, a similar type of question was posed in [**15**], which is now called the *cohomological rigidity problem*, which asks if homeomorphism/diffeomorphism classes of quasitoric manifolds can be classified by their integral cohomology rings.

Although the problem looks overambitious, it is a sensible question to ask on the following basis. No counter-example has been found since it was formulated. On the contrary, there is a piece of evidence supporting the cohomological rigidity of quasitoric manifolds. Indeed, the classification result in [**16**] together with the description of the cohomology ring of a quasitoric manifold [**9**, theorem 4.14] implies the cohomological rigidity of 4-dimensional quasitoric manifolds. Besides, many affirmative answers have been proved, for instance certain Bott manifolds [5], generalized Bott manifolds [6] and 6-dimensional quasitoric manifolds associated to 3-dimensional Pogorelov polytopes [1].

Being a generalized notion of quasitoric manifold, a toric orbifold [9] is a $2n$-dimensional compact orbifold equipped with a locally standard T^n-action whose orbit space is a simple polytope. It is known that the cohomology rings fail to classify toric orbifolds up to homeomorphism. For instance, there are weighted projective spaces with isomorphic cohomology rings that are not homeomorphic. Therefore, toric orbifolds do not satisfy cohomological rigidity. However, in the above counter-examples, two weighted projective spaces with isomorphic cohomology rings are homotopy equivalent [2]. Hence, we take a step back and ask a homotopical version of the cohomological rigidity:

QUESTION 1.1. *Are two toric orbifolds homotopy equivalent if their integral cohomology rings are isomorphic as graded rings?*

This paper aims to answer this question for certain 4-dimensional toric orbifolds. We first study certain CW-complexes which model 4-dimensional toric orbifolds and investigate their homotopy theory. In what follows, $H^*(X)$ denotes the cohomology ring with integral coefficients unless otherwise stated, and $P^3(k)$ denotes the 3-dimensional mod-k Moore space for $k > 1$. It is known that $H^3(X)$ is a finite cyclic group for all 4-dimensional toric orbifolds X. We refer to [10, 13]. Let $H^3(X) \cong \mathbb{Z}_m$ with $m = 2^s q$ for q odd and $s \geqslant 0$. When $q > 1$, we show that X decomposes into a wedge of $P^3(q)$ and a recognizable space.

THEOREM 1.2. *Let X be a 4-dimensional toric orbifold such that $H^3(X) \cong \mathbb{Z}_m$. If $m = 2^s q$ for an odd integer $q > 1$ and $s \geqslant 0$, then X is homotopy equivalent to $\hat{X} \vee P^3(q)$, where $\hat{X}$ is a simply connected 4-dimensional CW-complex with $H^3(\hat{X}) = \mathbb{Z}_{2^s}$ and $H^i(\hat{X}) \cong H^i(X)$ for $i \neq 3$.*

If m is odd or equivalently $s = 0$, then theorem 1.2 implies $X \simeq \hat{X} \vee P^3(m)$ where $H^3(\hat{X}) = 0$. As an application, we can answer question 1.1 for certain 4-dimensional toric orbifolds in the following theorem.

THEOREM 1.3. *Let X and X' be 4-dimensional toric orbifolds such that $H^3(X)$ and $H^3(X')$ have no 2-torsion. Then X is homotopy equivalent to X' if and only if there is a ring isomorphism $H^*(X) \cong H^*(X')$.*

This paper is organized as follows. In § 2, we review the constructive definition of a 4-dimensional toric orbifold X. In particular, it is important to see that X is the mapping cone of a map from a lens space to a wedge of 2-spheres. This phenomenon is motivated by the study of [4] and can also be understood in terms of a **q**-CW complex studied in [3]. In § 3, we define a category $\mathscr{C}_{n,m}$ of certain CW-complexes which model 4-dimensional toric orbifolds and study the homotopy theory of $\mathscr{C}_{n,m}$. Section 4 aims to give a necessary and sufficient condition for $X \in \mathscr{C}_{n,m}$ to decompose into a wedge of $P^3(q)$ and a space in $\mathscr{C}_{n,2^s}$. In § 5, we study the p-local version of the discussion of § 4 for some odd prime p and apply this to

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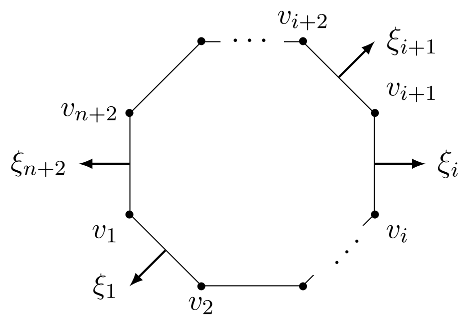

Figure 1. $(n+2)$-gon with primitive vectors on facets.

4-dimensional toric orbifolds. Combining the equivalent condition (Proposition 4.8) and the p-local decomposition (Proposition 5.3), we finally complete the proofs of theorems 1.2 and 1.3 in § 6.

2. Toric orbifolds of dimension 4

We begin with a summary of the constructive definition of a toric orbifold. For our purpose, we focus on the 4-dimensional case. For more details on toric orbifolds see [**9**, § 7], [**18**, § 2] and [**7**, chapters 3, 10].

Let P be an $(n+2)$-gon on vertices $v_1, \ldots, v_{n+2}$ for some $n \geqslant 0$. We denote by E_i the edge connecting v_i and v_{i+1} for $i = 1, \ldots, n+2$, where we take indices modulo $n+2$. To each edge E_i, assign a primitive vector $\xi_i = (a_i, b_i) \in \mathbb{Z}^2$ such that two adjacent vectors ξ_i and ξ_{i+1} are linearly independent. We often describe this combinatorial data as in Figure 1.

Identify $\mathbb{Z}^2$ with $\mathrm{Hom}(S^1, T^2)$. Each ξ_i defines a one-parameter subgroup of T^2

$$S^1_{\xi_i} = \{(t^{a_i}, t^{b_i}) \in T^2 \mid t \in S^1\}.$$

Now, define the following identification space

$$X = P \times T^2/_{\sim} \tag{2.1}$$

where we identify (p, g) and (p, h) for $gh^{-1} \in S^1_{\xi_i}$ if p is in the relative interior of E_i, and for all $g, h \in T^2$ if p is a vertex of P. Note that there is no identification between (p, g) and (q, h) unless $p = q$. Here, the torus T^2 acts on X by the multiplication on the second factor, which yields the orbit map $\pi \colon X \to P$ by the projection onto the first factor.

We roughly describe the orbifold structure on X following the identification (2.1). First, there is a standard presentation of $\mathbb{C}^2$ given by a homeomorphism $\mathbb{R}^2_{\geqslant} \times T^2/_{\sim_{std}} \cong \mathbb{C}^2$ that maps $[(x, y), (t, s)]$ in $\mathbb{R}^2_{\geqslant} \times T^2/_{\sim_{std}}$ to (xt, ys) in $\mathbb{C}^2$. Here, the standard identification $\sim_{std}$ is given by $((x, y), g) \sim_{std} ((x, y), h)$

(1) for $gh^{-1} \in 1 \times S^1$ if $x = 0$ and $y \neq 0$;

(2) for $gh^{-1} \in S^1 \times 1$ if $x \neq 0$ and $y = 0$;

(3) for all $g, h \in T^2$ if $x = y = 0$.

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Note that there is no identification between $((x_1, y_1), g)$ and $((x_2, y_2), h)$ in $\mathbb{R}^2_{\geqslant} \times T^2$ unless $(x_1, y_1) = (x_2, y_2)$.

Let U_i be a neighbourhood of v_i in P, which is homeomorphic to $\mathbb{R}^2_{\geqslant}$ as a manifold with corners. Let ψ_i be a homeomorphism $\mathbb{R}^2_{\geqslant} \cong U_i$ and let $\rho_i\colon T^2 \twoheadrightarrow T^2$ be an endomorphism of T^2 given by

$$\rho_i\colon T^2 \to T^2, \quad (t_1, t_2) \xmapsto{\rho_i} (t_1^{a_i} t_2^{a_{i+1}}, t_1^{b_i} t_2^{b_{i+1}}). \tag{2.2}$$

Since $\xi_i = (a_i, b_i)$ and $\xi_{i+1} = (a_{i+1}, b_{i+1})$ are linearly independent, the kernel $K_i = \ker \rho_i$ is a cyclic subgroup of T^2. Then the map $\psi_i \times \rho_i$ induces a surjection

$$\mathbb{C}^2 \cong \mathbb{R}^2_{\geqslant} \times T^2/_{\sim_{std}} \xrightarrow{\psi_i \times \rho_i} U_i \times T^2/_{\sim}. \tag{2.3}$$

This shows that $U_i \times T^2/_{\sim}$ is homeomorphic to the quotient $\mathbb{C}^2/K_i$, where K_i acts on $\mathbb{C}^2$ as a subgroup of T^2. Hence, the map (2.3) forms an orbifold chart around the point $[v_i, g] \in X$. The gluing maps among these orbifold charts are determined by the underlying polygon.

A certain cofibration construction of X is studied in [**4**] based on the orbifold structure on X. Pick a vertex v_i of P and U_i is its neighbourhood as above. Consider a line segment ℓ_i in P connecting two points lying in the relative interior of E_i and E_{i+1}, respectively. The restriction of identification (2.1) to ℓ_i gives rise to a subspace of X

$$L_i = \ell_i \times T^2/_{\sim}.$$

By assuming that the homeomorphism $\psi_i\colon \mathbb{R}^2 \to U_i$ sends the arc $S^1_{\geqslant} = S^1 \cap \mathbb{R}^2_{\geqslant}$ to ℓ_i, the restriction of (2.3) to $S^1_{\geqslant} \times T^2/_{\sim_{std}}$ induces a homeomorphism $(S^1_{\geqslant} \times T^2/_{\sim_{std}})/K_i \cong L_i$. Here, we notice that K_i is isomorphic to $\mathbb{Z}_{m_{i,i+1}}$, where

$$m_{i,j} = |\det \begin{bmatrix} \xi_i^t & \xi_j^t \end{bmatrix}|. \tag{2.4}$$

As $S^1_{\geqslant} \times T^2/_{\sim_{std}}$ is homeomorphic to S^3, we conclude that L_i is homeomorphic to $S^3/\mathbb{Z}_{m_{i,i+1}}$ which is S^3 if $m_{i,i+1} = 1$ and is a lens space otherwise. This description can be found in [**19**, Proposition 2.3] including higher dimensional cases.

Moreover, the subspace $U_i \times T^2/_{\sim}$ is homeomorphic to a tubular neighbourhood of the cone on L_i. Let B be the union of all edges E_j where $j \neq i, i+1$. The subspace $B \times T^2/_{\sim}$ is homotopic to a wedge of n copies of 2-spheres and the subspace $(P - \{v_i\}) \times T^2/_{\sim}$ retracts to $B \times T^2/_{\sim}$. As X is a union of $(P - \{v_i\}) \times T^2/_{\sim}$ and $U_i \times T^2/_{\sim}$, it implies a homotopy cofibration

$$L_i \xrightarrow{f_i} \bigvee_{j=1}^{n} S^2 \to X \tag{2.5}$$

where the map f_i is induced by the composition of the inclusion ι and the retraction r

$$\ell_i \times T^2/_{\sim} \xhookrightarrow{\iota} (P - \{v_i\}) \times T^2/_{\sim} \xrightarrow{r} B \times T^2/_{\sim}.$$

See Figure 2 for a pictorial illustration of (2.5).

Applying the cohomology functor to the cofibre sequence (2.5) and referring to [**3**, theorem 1.1], we can compute the free part of $H^*(X)$. The cohomology of X has

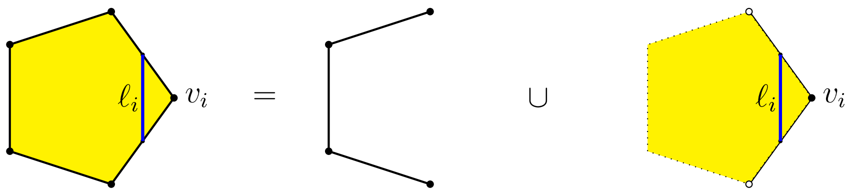

Figure 2. $X = \bigvee_{i=1}^{3} S^2 \cup_{f_i} CL_i$.

been discussed using various tools in the studies [**13**, theorem 2.5.5], [**10**, theorem 2.3] and [**14**, corollary 5.1] which can be summarized as follows.

PROPOSITION 2.1. *Let X be a toric orbifold of dimension 4. Then we have*

$$\begin{array}{c|cccccc} i & 0 & 1 & 2 & 3 & 4 & \geqslant 5 \\ \hline H^i(X) & \mathbb{Z} & 0 & \mathbb{Z}^n & \mathbb{Z}_m & \mathbb{Z} & 0 \end{array} \tag{2.6}$$

where m is the greatest common divisor of $\{m_{i,j} \mid 1 \leqslant i < j \leqslant n+2\}$ for $m_{i,j}$'s defined in (2.4). We set $\mathbb{Z}_m = 0$ if $m = 1$.

REMARK 2.2. The way of realizing X as a cofibre in (2.5) can be understood in a more general framework of a **q**-*CW complex*. A **q**-CW complex is defined inductively starting from a discrete set X_0 of points. Then, X_i is defined by the pushout

$$\begin{array}{ccc} \coprod_\alpha S^{i-1}/K_\alpha & \hookrightarrow & \coprod_\alpha e^i/K_\alpha \\ \downarrow{\scriptstyle \{\phi_\alpha\}} & & \downarrow \\ X_{i-1} & \longrightarrow & X_i, \end{array}$$

where e^i and S^{i-1} are i-dimensional cell and its boundary, respectively, and K_α is a finite group acting linearly on e_i. Every toric orbifold is a **q**-CW complex. We refer to [**3**] for more details.

3. Cohomology of 4-dimensional CW-complexes

3.1. A category of 4-dimensional CW-complexes

Suppose that X is a simply connected CW-complex satisfying (2.6). By [**12**, Proposition 4H.3] it is homotopy equivalent to a CW-complex

$$\left(\bigvee_{i=1}^{n} S^2 \vee P^3(m)\right) \cup_f e^4 \tag{3.1}$$

where $f\colon S^3 \to \bigvee_{i=1}^{n} S^2 \vee P^3(m)$ is the attaching map of the 4-cell. In this section, we study the homotopy theory of CW-complexes in this form.

Define $\mathscr{C}_{n,m}$ to be the full subcategory of Top_* consisting of mapping cones as in (3.1). Here, the orientation of the 4-cell e^4 is the induced orientation of the upper

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hemisphere in S^5. We label the ith copy of 2-spheres in $\bigvee_{i=1}^n S^2$ by S_i^2 for $1 \leqslant i \leqslant n$ and write

$$Y = \bigvee_{i=1}^{n} S_i^2 \vee P^3(m)$$

for short. Let $\mu_i, \nu \in H_2(Y)$ be homology classes representing S_i^2 and the 2-cell of $P^3(m)$ respectively. Then, we have

$$H_2(Y) \cong \mathbb{Z}\langle \mu_1, \dots, \mu_n\rangle \oplus \mathbb{Z}_m\langle \nu\rangle. \tag{3.2}$$

Let $g\colon Y \to Y$ be a map. Then the induced homology map $g_*\colon H_2(Y) \to H_2(Y)$ is given by $g_*(\mu_i) = \sum_{j=1}^n x_{ij}\mu_j + y_i\nu$ and $g_*(\nu) = z\nu$ for some integers x_{ij} and mod-m congruence classes y_i and z. Conversely, we have the following lemma.

Lemma 3.1. *Given a vector $(y_1, \dots, y_n, z) \in (\mathbb{Z}_m)^{n+1}$ and an $(n \times n)$-integral matrix*

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in Mat_n(\mathbb{Z}),$$

there exists a map $g\colon Y \to Y$ such that $g_*(\mu_i) = \sum_{j=1}^n x_{ij}\mu_j + y_i\nu$ and $g_*(\nu) = z\nu$.

Proof. First, consider the string of isomorphisms

$$\begin{aligned} \left[\bigvee_{i=1}^n S_i^2, \bigvee_{j=1}^n S_j^2 \vee P^3(m)\right] &\cong \bigoplus_{i=1}^n \left[S_i^2, \bigvee_{j=1}^n S_j^2 \vee P^3(m)\right] \\ &\cong \bigoplus_{i=1}^n \pi_2\left(\bigvee_{j=1}^n S_j^2 \vee P^3(m)\right) \\ &\cong \bigoplus_{i=1}^n H_2\left(\bigvee_{j=1}^n S_j^2 \vee P^3(m)\right) \\ &\cong \bigoplus_{i=1}^n \left(\bigoplus_{j=1}^n \mathbb{Z} \oplus \mathbb{Z}_m\right) \end{aligned}$$

where the third isomorphism is due to the Hurewicz theorem. Under these isomorphisms, take $g'\colon \bigvee_{i=1}^n S_i^2 \to Y$ to be the map corresponding to

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \oplus (y_1, \dots, y_n) \in \left(\bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathbb{Z}\right) \oplus \left(\bigoplus_{i=1}^n \mathbb{Z}_m\right).$$

Then $g'_*(\mu_i) = \sum_{j=1}^n x_{ij}\mu_j + y_i\nu$.

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Next, for $z \in \mathbb{Z}_m$ let $g'' \colon P^3(m) \to Y$ be the composition

$$g'' \colon P^3(m) \xrightarrow{\underline{z}} P^3(m) \hookrightarrow Y,$$

where $\underline{z} \colon P^3(m) \to P^3(m)$ is the degree-z map. Let $g : Y \to Y$ be the wedge sum $g = g' \vee g''$. Then $g_*(\mu_i) = \sum_{j=1}^{n} x_{ij}\mu_j + y_i\nu$ and $g_*(\nu) = z\nu$. □

3.2. Cellular cup product representation

Let $C_f \in \mathscr{C}_{n,m}$ be the mapping cone of a map $f \colon S^3 \to Y$. As the inclusion $Y \hookrightarrow C_f$ induces an isomorphism $H_2(Y) \to H_2(C_f)$, we do not distinguish $\mu_i, \nu \in H_2(Y)$ and their images in $H_2(C_f)$. Let $u_i \in H^2(C_f)$ and $e \in H^4(C_f)$ be cohomology classes dual to μ_i and the homology class represented by the 4-cell in C_f respectively. Let $v \in H^3(C_f)$ be the Ext image of ν. Then

$$H^2(C_f) \cong \mathbb{Z}\langle u_1, \ldots, u_n\rangle, \quad H^3(C_f) \cong \mathbb{Z}_m\langle v\rangle, \quad H^4(C_f) \cong \mathbb{Z}\langle e\rangle. \tag{3.3}$$

We call the set $\{u_1, \ldots, u_n, v, e\}$ the *cellular basis of* $H^*(C_f)$.

With coefficient $\mathbb{Z}_m$, let $\bar{u}_i \in H^2(C_f;\mathbb{Z}_m)$ and $\bar{e} \in H^4(C_f;\mathbb{Z}_m)$ be the mod-m images of u_i and e, and let $\bar{v} \in H^2(C_f;\mathbb{Z}_m)$ be the cohomology class dual to ν. Then

$$H^2(C_f;\mathbb{Z}_m) \cong \mathbb{Z}_m\langle \bar{u}_1, \ldots, \bar{u}_n, \bar{v}\rangle, \quad H^3(C_f;\mathbb{Z}_m) \cong \mathbb{Z}_m\langle \beta(\bar{v})\rangle,$$
$$H^4(C_f;\mathbb{Z}_m) \cong \mathbb{Z}_m\langle \bar{e}\rangle,$$

where β is the Bockstein homomorphism. We call the set $\{\bar{u}_1, \ldots, \bar{u}_n, \bar{v}; \bar{e}\}$ the *mod-m cellular basis* of $H^*(C_f;\mathbb{Z}_m)$.

Definition 3.2. Let C_f be a mapping cone in $\mathscr{C}_{n,m}$. Then the *cellular cup product representation* $M_{cup}(C_f)$ of C_f is $A \in \mathrm{Mat}_n(\mathbb{Z})$ if $m = 1$, and is a triple $(A, \mathbf{b}, c) \in \mathrm{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m$ if $m > 1$, where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ and } \mathbf{b} = (b_1, \ldots, b_n)$$

are given by $u_i \cup u_j = a_{ij}e$, $\bar{u}_i \cup \bar{v} = b_i\bar{e}$ and $\bar{v} \cup \bar{v} = c\bar{e}$.

Here, A is a symmetric matrix since it is the matrix representation of the bilinear form

$$(-\cup-)_{\mathbb{Z}} \colon H^2(C_f;\mathbb{Z}) \otimes H^2(C_f;\mathbb{Z}) \to H^4(C_f;\mathbb{Z})$$

with respect to the cellular basis $\{u_1, \ldots, u_n; e\}$. Furthermore, universal coefficient theorem implies $\bar{u}_i \cup \bar{u}_j = a_{ij}\bar{e} \pmod m$. So, $(-\cup-)_{\mathbb{Z}_m} \colon H^2(C_f;\mathbb{Z}_m) \otimes H^2(C_f;\mathbb{Z}_m) \to H^4(C_f;\mathbb{Z}_m)$ can be recovered from $M_{cup}(C_f)$ as well.

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REMARK 3.3. When X is an oriented compact smooth 4-manifold, the *intersection form* $I(X)$ is the bilinear form given by cup products of degree 2 cohomology classes modulo torsion

$$I(X)\colon H^2(X)/\mathrm{Tor} \otimes H^2(X)/\mathrm{Tor} \to \mathbb{Z}, \quad x \otimes y \mapsto \langle x \cup y, [X] \rangle,$$

where $[X] \in H_4(X)$ is the fundamental class. Although defined in a similar fashion, $I(X)$ and $M_{cup}(X)$ are different. First, $I(X)$ only concerns cup products of free elements in $H^2(X)$ and its matrix representation is a symmetric matrix, while $M_{cup}(X)$ concerns cup products of cohomology with integral and $\mathbb{Z}_m$-coefficients and is a triple consisting of a matrix, a mod-m vector and a mod-m congruence class that record all data. Second, a matrix representation of $I(X)$ depends on the choice of generators of $H^2(X)$, whereas we define $M_{cup}(X)$ using a fixed CW-complex structure of X. In the following section, we will discuss the transformation between cellular map representations of two CW-complex structures of the same X. It is similar to matrix congruence but is slightly more complicated, as cup products of cohomology with $\mathbb{Z}_m$ coefficient are involved.

Let $g\colon S^3 \to Y$ be another map and let $C_g \in \mathscr{C}_{n,m}$ be its mapping cone. Recall that $f+g$ is the composition

$$f+g\colon S^3 \xrightarrow{\text{comult}} S^3 \vee S^3 \xrightarrow{f\vee g} Y \vee Y \xrightarrow{\text{fold}} Y.$$

Denote its mapping cone by C_{f+g}.

LEMMA 3.4. *Let Y be (1) $S^2_1 \vee S^2_2$ or (2) $S^2_1 \vee P^3(m)$ and let $f, g : S^3 \to Y$ be two maps. Then $M_{cup}(C_{f+g}) = M_{cup}(C_f) + M_{cup}(C_g)$.*

Proof. In the following, we only prove case (2). The proof also works for case (1) but is simpler. Let

- $\{u, v; e\}$, $\{u_1, v_1; e_1\}$ and $\{u_2, v_2; e_2\}$ be the cellular bases of $H^*(C_{f+g}), H^*(C_f)$ and $H^*(C_g)$, respectively;
- $\{\bar{u}, \bar{v}; \bar{e}\}$, $\{\bar{u}_1, \bar{v}_1; \bar{e}_1\}$ and $\{\bar{u}_2, \bar{v}_2; \bar{e}_2\}$ be the mod-m cellular bases of $H^*(C_{f+g}; \mathbb{Z}_m)$, $H^*(C_f; \mathbb{Z}_m)$ and $H^*(C_g; \mathbb{Z}_m)$, respectively;
- the cellular cup product representations of C_{f+g}, C_f and C_g be $M_{cup}(C_{f+g}) = (A, \mathbf{b}, c)$, $M_{cup}(C_f) = (A_1, \mathbf{b}_1, c_1)$ and $M_{cup}(C_g) = (A_2, \mathbf{b}_2, c_2)$, respectively.

Here, A, A_1, A_2 are integers and $\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2$ are mod-m congruence classes. We claim that

$$A = A_1 + A_2,\ \mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 \text{ and } c = c_1 + c_2.$$

Consider the mapping cone $C' = Y \cup_{f\vee g} (e^4_1 \vee e^4_2)$ of $g \vee h\colon S^3 \vee S^3 \to Y$. Let

- $u' \in H^2(C')$, $e'_1, e'_2 \in H^4(C')$ be cohomology classes dual to S^2, e^4_1 and e^4_2;
- $\bar{u}' \in H^2(C'; \mathbb{Z}_m)$, $\bar{e}'_1, \bar{e}'_2 \in H^4(C'; \mathbb{Z}_m)$ be the mod-m images of u, e'_1 and e'_2;
- $\bar{v}' \in H^2(C'; \mathbb{Z}_m)$ be the cohomology class dual to the 2-cell of $P^3(m)$.

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Observe that C_f and C_g are subcomplexes of C'. Let $\imath_1\colon C_f \to C'$ and $\imath_2\colon C_g \to C'$ be natural inclusions and let $q\colon C_{f+g} \to C'$ be the map collapsing the equatorial disk of the 4-cell in C_{f+g} to a point. Then

$$q^*(u') = u, \quad q^*(\bar{v}') = \bar{v}, \quad q^*(e_1') = q^*(e_2') = e,$$
$$\imath_j^*(u') = u_j, \quad \imath_j^*(\bar{v}') = \bar{v}_j, \quad \imath_j^*(e_k') = \delta_{jk} e_j$$

for $j, k \in \{1, 2\}$, where δ_{jk} is the Kronecker symbol. On the one hand, $u' \cup u' = \alpha_1 e_1' + \alpha_2 e_2'$ for some integers α_1 and α_2. Now the naturality of cup products implies

$$\begin{aligned} \imath_j^*(u' \cup u') &= \imath_j^*(\alpha_1 e_1' + \alpha_2 e_2') \\ \imath_j^*(u') \cup \imath_j^*(u') &= \alpha_1 \imath_j^*(e_1') + \alpha_2 \imath_j^*(e_2') \\ u_j \cup u_j &= \alpha_j e_j \end{aligned}$$

for $j \in \{1, 2\}$. So, $\alpha_j = A_j$. On the other hand,

$$\begin{aligned} u \cup u &= q^*(u') \cup q^*(u') \\ &= q^*(u' \cup u') \\ &= q^*(A_1 e_1' + A_2 e_2') \\ &= A_1 q^*(e_1') + A_2 q^*(e_2') \\ &= (A_1 + A_2) e. \end{aligned}$$

So, $A = A_1 + A_2$. Similarly we can show $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ and $c = c_1 + c_2$. Therefore, we have

$$M_{cup}(C_{f+g}) = M_{cup}(C_f) + M_{cup}(C_g). \qquad \square$$

3.3. Cellular map representations

Let $f, f'\colon S^3 \to Y$ be two maps and $C_f, C_{f'} \in \mathscr{C}_{n,m}$ be their mapping cones. Let

- $\{u_1, \dots, u_n, v, e\}$ and $\{u_1', \dots, u_n', v', e'\}$ be the cellular bases of $H^*(C_f)$ and $H^*(C_{f'})$,
- $\{\bar{u}_1, \dots, \bar{u}_n, \bar{v}, \bar{e}\}$ and $\{\bar{u}_1', \dots, \bar{u}_n', \bar{v}', \bar{e}'\}$ be the mod-m cellular bases of $H^*(C_f; \mathbb{Z}_m)$ and $H^*(C_{f'}; \mathbb{Z}_m)$.

Given a map $\psi\colon C_{f'} \to C_f$ and a coefficient ring R, let

$$\psi_R^*\colon H^2(C_f; R) \to H^2(C_{f'}; R)$$

be the induced morphism on the second cohomology with coefficient R.

Definition 3.5. Let $\psi\colon C_{f'} \to C_f$ be a map. Then the *cellular map representation* $M(\psi)$ of ψ is $W \in \mathrm{Mat}_n(\mathbb{Z})$ if $m = 1$, and is the triple $(W, \mathbf{y}, z) \in \mathrm{Mat}_n(\mathbb{Z}) \oplus$

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$(\mathbb{Z}_m)^n \oplus \mathbb{Z}_m$ if $m > 1$, where

$$W = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \text{ and } \mathbf{y} = (y_1, \ldots, y_n) \tag{3.4}$$

are given by $\psi^*_{\mathbb{Z}}(u_j) = \sum_{i=1}^n x_{ij} u'_i$ and $\psi^*_{\mathbb{Z}_m}(\bar{v}) = \sum_{i=1}^n y_i \bar{u}'_i + z\bar{v}'$.

Lemma 3.6. *For $R = \mathbb{Z}$ or $\mathbb{Z}_m$, consider ψ, ψ^*_R and $M(\psi)$ as above. For $1 \leqslant j \leqslant n$, we have $\psi^*_{\mathbb{Z}_m}(\bar{u}_j) = \sum_{i=1}^n x_{ij} \bar{u}'_i$. Furthermore, if ψ is a homotopy equivalence, then W is an invertible matrix and z is a unit in $\mathbb{Z}_m$.*

Proof. Since C_f and $C_{f'}$ are simply connected, universal coefficient theorem implies that

$$H^2(C_f; R) \cong \mathrm{Hom}(H_2(C_f), R), H^2(C_{f'}; R) \cong \mathrm{Hom}(H_2(C_{f'}), R) \text{ and}$$
$$\psi^*_R = \mathrm{Hom}(\psi_*, R)$$

is dual to $\psi_* \colon H_2(C_{f'}) \to H_2(C_f)$. So, $\psi^*_{\mathbb{Z}_m}(\bar{u}_j)$ is the mod-m image of $\psi^*_{\mathbb{Z}}(u_j)$ and the first part of the lemma follows.

If ψ is a homotopy equivalence, then $W \in \mathrm{Mat}_n(\mathbb{Z})$ and

$$\begin{pmatrix} \bar{x}_{11} & \cdots & \bar{x}_{1n} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ \bar{x}_{n1} & \cdots & \bar{x}_{nn} & y_n \\ 0 & \cdots & 0 & z \end{pmatrix} \in \mathrm{Mat}_{n+1}(\mathbb{Z}_m), \quad \text{where } \bar{x}_{ij} \equiv x_{ij} \pmod{m}$$

are invertible matrices. So, the second part follows. □

The cellular map representation records the data of $\psi^*_{\mathbb{Z}}$ and $\psi^*_{\mathbb{Z}_m}$. The square matrix W in (3.4) is the map representation of $\psi^*_{\mathbb{Z}}$ with respect to bases $\{u_1, \ldots, u_n\}$ and $\{u'_1, \ldots, u'_n\}$. Lemma 3.6 implies that

$$\begin{pmatrix} \overline{W} & \mathbf{y}^t \\ \mathbf{0} & z \end{pmatrix} \in \mathrm{Mat}_{n+1}(\mathbb{Z}_m),$$

is the matrix representation of $\psi^*_{\mathbb{Z}_m}$ with respect to bases $\{\bar{u}_1, \ldots, \bar{u}_n, \bar{v}\}$ and $\{\bar{u}'_1, \ldots, \bar{u}'_n, \bar{v}'\}$, where $\overline{W}$ is the mod-m image of W and $\mathbf{0} = (0, \ldots, 0)$.

Recall that in linear algebra, matrix representations of a bilinear form $V \otimes V \to \mathbb{Z}$ with respect to different bases of V are congruent to each other. So, we have the following lemma.

Lemma 3.7. *For $C_f, C_{f'} \in \mathscr{C}_{n,m}$, let $M_{cup}(C_f) = (A, \boldsymbol{b}, c)$ and $M_{cup}(C_{f'}) = (A', \boldsymbol{b}', c')$ be their cellular cup product representations. If there is a homotopy equivalence $\psi \colon C_{f'} \to C_f$ with $M(\psi) = (W, \boldsymbol{y}, z) \in GL_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}^*_m$, then*

$$A' = W^t A W, \quad \boldsymbol{b}' = \boldsymbol{y} A W + z \boldsymbol{b} W, \quad c' = \boldsymbol{y} A \boldsymbol{y}^t + 2z\boldsymbol{y}\boldsymbol{b}^t + z^2 c.$$

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In particular, if two maps f and $f'\colon S^3 \to Y$ are homotopic, then the matrix cup product representations of their mapping cones are the same.

LEMMA 3.8. *If f is homotopic to f', then $M_{cup}(C_f) = M_{cup}(C_{f'})$.*

Proof. Take a homotopy $\phi\colon S^3 \times I \to Y$ between f and f'. It induces a homotopy equivalence $\Phi\colon C_f \to C_{f'}$ such that its restriction to Y is the identity map. So, $M(\Phi) = (I_n, \mathbf{0}, 1)$. Then the lemma follows from lemma 3.7. □

LEMMA 3.9. *Let $C_f \in \mathscr{C}_{n,m}$ and let $(W, \boldsymbol{y}, z)$ be a triple in $GL_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m^*$. Then there exist a CW-complex $C_{f'} \in \mathscr{C}_{n,m}$ and a homotopy equivalence $\psi\colon C_f \to C_{f'}$ such that the cellular map representation $M(\psi)$ is $(W, \boldsymbol{y}, z)$.*

Proof. Let

$$W = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \text{ and } \mathbf{y} = (y_1, \ldots, y_n).$$

By lemma 3.1, there exists a map $\tilde{\psi}\colon Y \to Y$ such that $\tilde{\psi}_*(\mu_i) = \sum_{j=1}^n x_{ij}\mu_j + y_i\nu$ and $\tilde{\psi}_*(\nu) = z\nu$, where $\mu_1, \ldots, \mu_n$ and ν are elements in $H_2(Y)$ as in (3.2). Thus, we have $\tilde{\psi}^*_{\mathbb{Z}}(u_i) = \sum_{j=1}^n x_{ji}u_j$ and $\tilde{\psi}^*_{\mathbb{Z}_m}(\bar{v}) = \sum_{i=1}^n y_i\bar{u}_i + z\bar{v}$.

Let $f' = \tilde{\psi} \circ f$ and let $C_{f'}$ be its mapping cone. Then there is a diagram of cofibration sequences

$$\begin{array}{ccccc} S^3 & \xrightarrow{f} & Y & \longrightarrow & C_f \\ \| & & \downarrow{\scriptstyle\tilde{\psi}} & & \downarrow{\scriptstyle\psi} \\ S^3 & \xrightarrow{f'} & Y & \longrightarrow & C_{f'}, \end{array}$$

where ψ is an induced map. Since $W \in GL_n(\mathbb{Z})$ and $z \in \mathbb{Z}_m^*$, the middle vertical arrow $\tilde{\psi}$ induces an isomorphism in homology. By five lemma, ψ_* is an isomorphism, which implies that ψ is a homotopy equivalence. Finally, we have $M(\psi) = (W, \mathbf{y}, z)$ by the construction. □

4. The homotopy theory of complexes in $\mathscr{C}_{n,m}$

4.1. The $\mathscr{C}_{n,1}$ case

When $m = 1$, the mapping cone $C_f \in \mathscr{C}_{n,1}$ is in the form $\bigvee_{i=1}^n S_i^2 \cup_f e^4$ where $f\colon S^3 \to \bigvee_{i=1}^n S_i^2$ is the attaching map of the 4-cell. The Hilton–Milnor theorem (see for instance [**20**, theorem 7.9.4]) implies that f is homotopic to a wedge sum

$$\sum_{i=1}^n a_i\eta_i + \sum_{1\leqslant j<k\leqslant n} a_{jk}\omega_{jk},$$

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for some integers a_i's and a_{jk}'s. Here η_i's and ω_{jk}'s are compositions

$$\eta_i\colon S^3 \xrightarrow{\eta} S_i^2 \hookrightarrow \bigvee_{\ell=1}^{n} S_\ell^2$$
$$\omega_{jk}\colon S^3 \xrightarrow{[\imath_1,\imath_2]} S_j^2 \vee S_k^2 \hookrightarrow \bigvee_{l=1}^{n} S_\ell^2$$

of Hopf map η, Whitehead product $[\imath_1, \imath_2]$ and canonical inclusions of S_i^2 and $S_j^2 \vee S_k^2$ into $\bigvee_{\ell=1}^{n} S_\ell^2$. The lemma below shows that the coefficients a_i and a_{jk} are determined by $M_{cup}(C_f)$.

LEMMA 4.1. *Let $C_f \in \mathscr{C}_{n,1}$ be the mapping cone of $f \simeq \sum_{i=1}^{n} a_i\eta_i + \sum_{1\leqslant j<k\leqslant n} a_{jk}\omega_{jk}$. If*

$$M_{cup}(C_f) = \begin{pmatrix} a'_1 & a'_{12} & \cdots & a'_{1n} \\ a'_{12} & a'_2 & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{1n} & a'_{2n} & \cdots & a'_n \end{pmatrix},$$

then $a_i = a'_i$ and $a_{jk} = a'_{jk}$ for all i, j and k.

Proof. By lemma 3.8, we may assume $f = \sum_{i=1}^{n} a_i\eta_i + \sum_{1\leqslant j<k\leqslant n} a_{jk}\omega_{jk}$. For $n = 2$, let C_1, C_2 and C_{12} be the mapping cones of $a_1\eta_1, a_2\eta_2$ and $a_{12}\omega_{12}$. Then their cellular cup product representations are

$$M_{cup}(C_1) = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{cup}(C_2) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix},$$
$$M_{cup}(C_{12}) = \begin{pmatrix} 0 & a_{12} \\ a_{12} & 0 \end{pmatrix}.$$

By lemma 3.4, we have

$$M_{cup}(C_f) = \begin{pmatrix} a_1 & a_{12} \\ a_{12} & a_2 \end{pmatrix}.$$

So, the lemma holds.

For $n \geqslant 3$, let $\{u_1, \ldots, u_n, e\}$ be the cellular basis of $H^*(C_f)$. We claim that

$$u_i \cup u_i = a_i e \quad \text{and} \quad u_j \cup u_k = a_{jk} e,$$

for each $1 \leqslant i \leqslant n$ and $1 \leqslant j < k \leqslant n$. The composition

$$f_{jk}\colon S^3 \xrightarrow{f} \bigvee_{l=1}^{n} S_l^2 \xrightarrow{\text{pinch}} S_j^2 \vee S_k^2$$

is homotopic to $a_j\eta'_1 + a_k\eta'_2 + a_{jk}\omega'_{12}$, where $\eta'_1\colon S^3 \xrightarrow{\eta} S_j^2 \hookrightarrow S_j^2 \vee S_k^2$ and $\eta'_2\colon S^3 \xrightarrow{\eta} S_k^2 \hookrightarrow S_j^2 \vee S_k^2$ are compositions of Hopf map η and canonical inclusions and

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$\omega'_{12}\colon S^3 \to S^2_j \vee S^2_k$ is the Whitehead product. Let C_{jk} be the mapping cone of f_{jk}. By lemma 3.8 and the above argument, we have

$$M_{cup}(C_{jk}) = \begin{pmatrix} a_j & a_{jk} \\ a_{jk} & a_k \end{pmatrix}.$$

Let $\{u'_j, u'_k; e'\}$ be the cellular basis of $H^*(C_{jk})$ and let $\alpha\colon C_f \to C_{jk}$ be the map which pinches all 2-spheres in C_f to the basepoint except for S^2_j and S^2_k. Then

$$\alpha^*(u'_j) = u_j,\ \alpha^*(u'_k) = u_k \text{ and } \alpha^*(e') = e.$$

By the naturality of cup products, we have

$$\begin{aligned} \alpha^*(u'_j) \cup \alpha^*(u'_k) &= \alpha^*(u'_j \cup u'_k) \\ u_j \cup u_k &= \alpha^*(a_{jk} e') \\ &= a_{jk} e. \end{aligned}$$

So, $a'_{jk} = a_{jk}$. Similarly we can show $a'_i = a_i$. Hence, the lemma follows. □

Now we classify the homotopy types of CW-complexes in $\mathscr{C}_{n,1}$ by their integral cohomology rings in the next statement.

PROPOSITION 4.2. *Let $f, f'\colon S^3 \to \bigvee_{i=1}^n S^2_i$ be two maps and let $C_f, C_{f'} \in \mathscr{C}_{n,1}$ be their mapping cones. Then $C_f \simeq C_{f'}$ if and only if there is a ring isomorphism $H^*(C_f) \cong H^*(C_{f'})$.*

Proof. The 'only if' part is trivial. Assume that $H^*(C_f) \cong H^*(C_{f'})$. Then there is an invertible matrix $W \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$W^t \cdot M_{cup}(C_f) \cdot W = \varepsilon M_{cup}(C_{f'}),$$

where ε is either 1 or -1. Suppose first $\varepsilon = 1$. By lemma 3.9, there is a CW-complex $\tilde{C} \in \mathscr{C}_{n,1}$ together with a homotopy equivalence $\psi\colon \tilde{C} \to C_f$ such that $M(\psi) = W$. We claim that $\tilde{C} \simeq C_{f'}$. By lemma 3.7, we have

$$M_{cup}(\tilde{C}) = W^t \cdot M_{cup}(C_f) \cdot W = M_{cup}(C_{f'}).$$

Let $\tilde{f}$ be the attaching map of the 4-cell in $\tilde{C}$ and let

$$M_{cup}(\tilde{C}) = M_{cup}(C_{f'}) = \begin{pmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ a_{12} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_n \end{pmatrix}.$$

Then lemma 4.1 implies that f' and $\tilde{f}$ are homotopic to the wedge sum

$$\sum_{i=1}^n a_i \eta_i + \sum_{1 \leqslant i < j \leqslant n} a_{ij} \omega_{ij},$$

which means $C_{f'} \simeq \tilde{C}$. Therefore, $C_f \simeq C_{f'}$.

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Suppose $\varepsilon = -1$. Let $C_{-f'}$ be the mapping cone of $-f' : S^3 \xrightarrow{-1} S^3 \xrightarrow{f'} \bigvee_{i=1}^n S_i^2$. Then

$$W^t \cdot M_{cup}(C_f) \cdot W = M_{cup}(C_{-f'})$$

and $C_{-f'} \simeq C_{f'}$. The above argument implies $C_f \simeq C_{-f'}$, so $C_f \simeq C_{f'}$. Hence, we have established the claim. □

COROLLARY 4.3. *Two 4-dimensional toric orbifolds without torsion in (co)homology are homotopy equivalent if and only if their integral cohomology rings are isomorphic.*

As 4-dimensional quasitoric manifolds always have torsion-free (co)homology, corollary 4.3 implies that the homotopy types of 4-dimensional quasitoric manifolds are classified by their cohomology rings. As we mentioned in Introduction, the homeomorphism types of 4-dimensional toric manifolds are cohomologically rigid. One can deduce the conclusion from the topological classification of 4-dimensional smooth manifolds with T^2-action studied in [**16**] together with the cohomology formula [**9**, theorem 4.14].

We note that the method in Proposition 4.2 applies to CW-complexes in $\mathscr{C}_{n,1}$ which are not necessarily manifolds. We also refer to [**8**, § 5] for the computation of the cohomology ring of toric orbifolds considered in corollary 4.3.

4.2. The $\mathscr{C}_{n,m}$ case

From now on, we assume $m = 2^s q$, where $q > 1$ is odd and $s \geqslant 0$. Recall from (3.3) that $H^3(C_f) \cong \mathbb{Z}_m$ for $C_f \in \mathscr{C}_{n,m}$. In this subsection, we discuss the homotopy type of C_f. To be more precise, we study a necessary and sufficient condition for a wedge decomposition

$$C_f \simeq \hat{C} \vee P^3(q) \tag{4.1}$$

where $\hat{C}$ is a complex in $\mathscr{C}_{n,2^s}$ so that $H^i(C_f) \cong H^i(\hat{C})$ for $i \neq 3$ and $H^3(\hat{C}) \cong \mathbb{Z}_{2^s}$.

LEMMA 4.4. *Let q be odd and greater than 1. Consider*

(i) *a map $g_1 \colon S^3 \to P^3(q)$ and its mapping cone C_1,*

(ii) *a map $g_2 \colon S^3 \to P^4(q)$ and the mapping cone C_2 of the composition*

$$S^3 \xrightarrow{g_2} P^4(q) \xrightarrow{[\kappa_1,\kappa_2]} S^2 \vee P^3(q),$$

where $[\kappa_1, \kappa_2]$ is the Whitehead product of inclusions

$$\kappa_1 \colon S^2 \to S^2 \vee P^3(q) \text{ and } \kappa_2 \colon P^3(q) \to S^2 \vee P^3(q).$$

For $i = 1$ or 2, if $H^*(C_i; \mathbb{Z}_m)$ has trivial cup products, then g_i is null homotopic.

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Proof. Let $q = p_1^{r_1} \dots p_\ell^{r_\ell}$ be a primary factorization of q such that p_j's are different odd primes and all r_j's are at least 1. By the Hurewicz theorem, $\pi_3(P^4(q)) \cong \mathbb{Z}_q$. By [**17**, theorem 4] and [**21**, lemma 2.1],

$$\pi_3(P^3(q)) \cong \bigoplus_{j=1}^{\ell} \pi_3(P^3(p_j^{r_j})) \cong \bigoplus_{j=1}^{\ell} \mathbb{Z}_{p_j^{r_j}} \cong \mathbb{Z}_q.$$

It suffices to prove the two cases after localization at p_j.

For the $i = 1$ case, the lemma is a special case of [**21**, Proposition 4.4]. For the $i = 2$ case, it can be proved by the argument of [**21**, Proposition 3.2] and replacing $P^3(p^t)$ by S^2 and the index t by ∞, respectively. □

LEMMA 4.5. *Let $m = 2^s q$, where q is odd and greater than 1. Let $f\colon S^3 \to \bigvee_{i=1}^n S_i^2 \vee P^3(m)$ be the attaching map of the 4-cell in C_f and let $M_{cup}(C_f) = (A, \boldsymbol{b}, c)$. If $\boldsymbol{b} \equiv (0, \dots, 0) \pmod q$ and $c \equiv 0 \pmod q$, then there is a CW-complex $\hat{C} \in \mathscr{C}_{n,2^s}$ such that $C_f \simeq \hat{C} \vee P^3(q)$.*

Proof. Since 2^s and q are coprime, we have $P^3(m) \simeq P^3(2^s) \vee P^3(q)$. By the Hilton–Milnor theorem, f is homotopic to a wedge sum

$$f \simeq \sum_{i=1}^n a_i \eta_i + \sum_{1 \leqslant j < k \leqslant n} a_{jk} \omega_{jk} + \eta' + \sum_{i=1}^n \omega_i' + \eta_q + \sum_{i=1}^n \omega_{iq}$$

for some integers a_i's and a_{jk}'s. Here, η', ω_i', η_q and ω_{iq} are compositions

$$\begin{aligned}
&\eta'\colon S^3 \xrightarrow{b'} P^3(2^s) \hookrightarrow \bigvee_{j=1}^n S_j^2 \vee P^3(2^s) \vee P^3(q)\\
&\omega_i'\colon S^3 \xrightarrow{b_i'} P^4(2^s) \xrightarrow{[\kappa_1', \kappa_2']} S_i^2 \vee P^3(2^s) \hookrightarrow \bigvee_{j=1}^n S_j^2 \vee P^3(2^s) \vee P^3(q)\\
&\eta_q\colon S^3 \xrightarrow{b_q} P^3(q) \hookrightarrow \bigvee_{j=1}^n S_j^2 \vee P^3(2^s) \vee P^3(q)\\
&\omega_{iq}\colon S^3 \xrightarrow{b_{iq}} P^4(q) \xrightarrow{[\kappa_1, \kappa_2]} S_i^2 \vee P^3(q) \hookrightarrow \bigvee_{j=1}^n S_j^2 \vee P^3(2^s) \vee P^3(q)
\end{aligned}$$

for some maps b', b_i' and b_q, b_{iq}. Here

$$[\kappa_1', \kappa_2'] : P^4(2^s) \to S_i^2 \vee P^3(2^s)$$

is the Whitehead product of inclusions $\kappa_1' : S_i^2 \to S_i^2 \vee P^3(2^s)$ and $\kappa_2' : P^3(2^s) \to S_i^2 \vee P^3(2^s)$, and

$$[\kappa_1, \kappa_2] : P^4(q) \to S_i^2 \vee P^3(q)$$

is the Whitehead product of inclusions $\kappa_1 : S_i^2 \to S_i^2 \vee P^3(q)$ and $\kappa_2 : P^3(q) \to S_i^2 \vee P^3(q)$. If η_q and ω_{iq}'s are null homotopic, then f factors through a map $\hat{f}\colon S^3 \to \bigvee_{i=1}^n S_i^2 \vee P^3(2^s)$. Let $\hat{C}$ be the mapping cone of $\hat{f}$. Then $C_f \simeq \hat{C} \vee P^3(q)$.

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Hence, it suffices to show that η_q and $\omega_{\ell q}$ are null homotopic for any ℓ with $1 \leqslant \ell \leqslant n$. After localization away from 2, $P^3(2^s)$ becomes contractible and the composition

$$f_{\ell q} \colon S^3 \xrightarrow{f} \bigvee_{j=1}^{n} S_j^2 \vee P^3(q) \xrightarrow{\text{pinch}} S_\ell^2 \vee P^3(q)$$

is homotopic to the wedge sum $a_\ell \tilde{\eta}_\ell + \jmath \circ b_q + [\kappa_1, \kappa_2] \circ b_{\ell q}$, where $\tilde{\eta}$ is the composition of Hopf map η and inclusion $S_\ell^2 \to S_\ell^2 \vee P^3(q)$, and $\jmath \colon P^3(q) \to S_\ell^2 \vee P^3(q)$ is the inclusion.

Consider the diagram of homotopy cofibration sequences

$$\begin{array}{ccccc} S^3 & \xrightarrow{f} & Y & \longrightarrow & C_f \\ \| & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi} \\ S^3 & \xrightarrow{f'} & Y & \longrightarrow & C_{f'}, \end{array}$$

where $C_{\ell q}$ is the mapping cone of $f_{\ell q}$ and π is an induced map. Let $\{\bar{u}_1, \ldots, \bar{u}_n, \bar{v}; \bar{e}\}$ and $\{\bar{u}', \bar{v}'; \bar{e}'\}$ be the mod-q cellular bases of $H^*(C_f; \mathbb{Z}_q)$ and $H^*(C_{\ell q}; \mathbb{Z}_q)$. Then

$$\pi^*(\bar{u}') = \bar{u}_\ell, \quad \pi^*(\bar{v}') = \bar{v}, \quad \pi^*(\bar{e}') = \bar{e}.$$

By the hypothesis, $\pi^*(\bar{u}' \cup \bar{v}') = \bar{u}_\ell \cup \bar{v} = 0$. Since $\pi^* \colon H^4(C_{\ell q}; \mathbb{Z}_q) \to H^4(C_f; \mathbb{Z}_q)$ is isomorphic, we have $\bar{u}' \cup \bar{v}' = 0$. Similarly, we have $\bar{v}' \cup \bar{v}' = 0$ and $\bar{u}' \cup \bar{u}' = a_\ell \bar{e}$ so that $M_{cup}(C_{\ell q}) = (a_\ell, 0, 0)$. Let C_1 and C_2 be the mapping cones of $[\kappa_1, \kappa_2] \circ b_{\ell q}$ and $\jmath \circ b_q$. By lemma 3.4,

$$M_{cup}(C_1) = M_{cup}(C_2) = (0, 0, 0)$$

so $H^*(C_1; \mathbb{Z}_q)$ and $H^*(C_2; \mathbb{Z}_q)$ have trivial cup products. By lemma 4.4, $b_{\ell q}$ is null homotopic and so is $\omega_{\ell q}$. Also, notice that $C_2 \simeq S_\ell^2 \vee C'$ where C' is the mapping cone of b_q. So $H^*(C'; \mathbb{Z}_q)$ has trivial cup products. By lemma 4.4, b_q is null homotopic and so is η_q. □

REMARK 4.6. In general, $\hat{C}$ cannot be further decomposed into a wedge of non-contractible spaces, for example $\hat{C} = \Sigma \mathbb{RP}^3$.

Notice that $\hat{C} \vee P^3(q)$ is not contained in $\mathscr{C}_{n,m}$, but it is homotopic to a mapping cone in $\mathscr{C}_{n,m}$ as follows. Since 2^s and q are coprime to each other, there exist integers α and β such that $2^s\alpha + q\beta = 1$, where the mod-q congruence class of α and the mod-2^s congruence class of β are unique. Identify $\mathbb{Z}_{2^s} \oplus \mathbb{Z}_q$ with $\mathbb{Z}_m$ via the isomorphism

$$\rho \colon \mathbb{Z}_{2^s} \oplus \mathbb{Z}_q \to \mathbb{Z}_m, \quad (x, y) \mapsto q\beta x + 2^s \alpha y. \tag{4.2}$$

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It induces a homotopy equivalence $\rho\colon P^3(2^s)\vee P^3(q)\to P^3(m)$. Consider the diagram of homotopy cofibrations

$$\begin{array}{ccccc}
* & \longrightarrow & \bigvee_{i\neq \ell}^{n} S_i^2 & = & \bigvee_{i\neq \ell}^{n} S_i^2 \\
\downarrow & & \downarrow & & \downarrow \\
S^3 & \xrightarrow{f} & \bigvee_{i=1}^{n} S_i^2 \vee P^3(q) & \longrightarrow & C_f \\
\| & & \downarrow{\scriptstyle \text{pinch}} & & \downarrow{\scriptstyle \pi} \\
S^3 & \xrightarrow{f_{\ell q}} & S_\ell^2 \vee P^3(q) & \longrightarrow & C_{\ell q}
\end{array} \tag{4.3}$$

where C' is the mapping cone of $(id\vee\rho)\circ(\hat{f}\vee *)$ and $\tilde{\rho}$ is an induced homotopy equivalence. Then $C'\simeq \hat{C}\vee P^3(q)$ via $\tilde{\rho}$.

Lemma 4.7. *Let $M_{cup}(C')=(A,\boldsymbol{b},c)$. Then $\boldsymbol{b}\equiv(0,\ldots,0)$ and $c\equiv 0 \pmod q$.*

Proof. We prove $b_i\equiv 0 \pmod q$. Let

- $\bar{u}_i\in H^2(\hat{C};\mathbb{Z}_m)$ and $\bar{e}\in H^4(\hat{C};\mathbb{Z}_m)$ be the mod-m cohomology classes dual to homology classes representing S_i^2 and the 4-cell in $\hat{C}$, respectively;

- μ_i, ω_{2^s}, $\omega_q\in H_2(\hat{C}\vee P^3(q);\mathbb{Z})$ be the homology classes representing S_i^2, the bottom cells of $P^3(2^s)$ and $P^3(q)$;

- $\bar{w}_{2^s}$, $\bar{w}_q\in H^2(\hat{C}\vee P^3(q);\mathbb{Z}_m)$ be the cohomology classes such that

$$\begin{array}{lll}\bar{w}_{2^s}(\omega_{2^s})\equiv q\beta & \bar{w}_{2^s}(\omega_q)\equiv 0 & \bar{w}_{2^s}(\mu_i)\equiv 0\\ \bar{w}_q(\omega_{2^s})\equiv 0 & \bar{w}_q(\omega_q)\equiv 2^s\alpha & \bar{w}_q(\mu_i)\equiv 0\end{array} \pmod m.$$

Denote $\bar{v}=\bar{w}_{2^s}+\bar{w}_q$. Then $\bar{u}_1,\ldots,\bar{u}_n$ and $\bar{v}$ form a basis of $H^2(\hat{C}\vee P^3(q);\mathbb{Z}_m)$. The right square of (4.3) implies

$$\tilde{\rho}^*(\bar{u}_i')=\bar{u}_i,\quad \tilde{\rho}^*(\bar{v}')=\bar{v}=\bar{w}_{2^s}+\bar{w}_q.$$

By the naturality of cup products, we have

$$\begin{aligned}\tilde{\rho}^*(\bar{u}_i'\cup\bar{v}')&=\tilde{\rho}^*(b_i\bar{e}')\\ \bar{u}_i\cup(\bar{w}_{2^s}+\bar{w}_q)&=b_i\bar{e}\\ \bar{u}\cup\bar{w}_{2^s}&=b_i\bar{e}.\end{aligned}$$

Since $\bar{w}_{2^s}$ is a multiple of q and $\bar{e}$ is a generator, $b_i\equiv 0 \pmod q$. Similarly we can show that $c\equiv 0 \pmod q$. □

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PROPOSITION 4.8. *Let $m = 2^s q$ as before. For $C_f \in \mathscr{C}_{n,m}$, let $M_{cup}(C_f) = (A, \boldsymbol{b}, c)$ where*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ and } \mathbf{b} = (b_1, \dots, b_n).$$

Then $C_f \simeq \hat{C} \vee P^3(q)$ for some $\hat{C} \in \mathscr{C}_{n,2^s}$ if and only if the system of mod-q linear equations

$$\begin{cases} a_{11}y_1 + \cdots + a_{1n}y_n & \equiv & -b_1 \\ \vdots & & \\ a_{n1}y_1 + \cdots + a_{nn}y_n & \equiv & -b_n \\ b_1y_1 + \cdots + b_ny_n & \equiv & -c \end{cases} \pmod{q} \tag{4.4}$$

has a solution $(y_1, \dots, y_n) \in (\mathbb{Z}_q)^n$.

Proof. Suppose $g\colon C' \simeq \hat{C} \vee P^3(q) \to C_f$ is a homotopy equivalence. Let $M(g) = (W, \mathbf{y}, z)$ and let $M_{cup}(C') = (A', \mathbf{b}', c')$. By lemma 3.7, we have

$$A' = W^tAW, \ \mathbf{b}' = \mathbf{y}AW + z\mathbf{b}W \ \text{ and } \ c' = \mathbf{y}A\mathbf{y}^t + 2z\mathbf{y}\mathbf{b}^t + z^2c. \tag{4.5}$$

Lemma 4.7 implies $\mathbf{b}' \equiv (0, \dots, 0)$ and $c' \equiv 0$ modulo q. Since W and z are invertible in $\mathbb{Z}_q$, we can rewrite equations in (4.5) as

$$z^{-1}\mathbf{y}A \equiv -\mathbf{b} \ \text{ and } \ z^{-1}\mathbf{y}\mathbf{b}^t \equiv -c \pmod{q}.$$

Therefore, $z^{-1}\mathbf{y}$ is a solution of (4.4).

Conversely, suppose there is a solution $\mathbf{y} = (y_1, \dots, y_n) \in (\mathbb{Z}_q)^n$ of (4.4). By lemma 3.9, there exist $C'' \in \mathscr{C}_{n,m}$ and a homotopy equivalence $g\colon C'' \to C_f$ such that $M(g) = (I, \mathbf{y}', 1)$ and

$$M_{cup}(C'') = \Big(A,\ \mathbf{y}'\bar{A} + \mathbf{b},\ \mathbf{y}'\bar{A}(\mathbf{y}')^t + 2\mathbf{b}(\mathbf{y}')^t + c\Big),$$

where $\mathbf{y}' = (\rho(y_1, 0), \dots, \rho(y_n, 0)) \in (\mathbb{Z}_m)^n$ for ρ defined in (4.2) and $\bar{A}$ is the mod-m image of A. Then

$$\mathbf{y}'\bar{A} + \mathbf{b} \equiv 0 \ \text{ and } \ \mathbf{y}'\bar{A}(\mathbf{y}')^t + 2\mathbf{b}(\mathbf{y}')^t + c \equiv 0 \pmod{q}.$$

Note that lemma 4.5 implies $C'' \simeq \hat{C} \vee P^3(q)$ for some $\hat{C} \in \mathscr{C}_{n,2^s}$. Consequently, we have $C_f \simeq \hat{C} \vee P^3(q)$. □

5. Odd primary local decomposition of toric orbifolds

Let $X = P \times T^2/_\sim$ be a 4-dimensional toric orbifold associated with the combinatorial data described in § 2. Since X is simply connected and $H^*(X)$ satisfies (2.6), [**12**, Proposition 4H.3] implies that X is in $\mathscr{C}_{n,m}$ up to homotopy. Let $m = 2^s q$,

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where q is odd and $s \geqslant 0$. In this section, we show that for any odd prime p, there is a p-local equivalence

$$X \simeq_{(p)} \hat{X} \vee P^3(q) \tag{5.1}$$

for a CW-complex $\hat{X}$ in $\mathscr{C}_{n,2^s}$ and $P^3(q)$ denotes a point if $q = 1$.

The **q**-CW complex structure of X with respect to a vertex v_i (see remark 2.2) implies that X is homotopy equivalent to the mapping cone of a map

$$f \colon L_i \to \bigvee\nolimits_{j=1}^{n} S^2$$

where L_i is the quotient $S^3/\mathbb{Z}_{m_{i,i+1}}$ and $m_{i,i+1} = |\det\left[\xi_i^t, \xi_{i+1}^t\right]|$. Recall that $\mathbb{Z}_{m_{i,i+1}}$ is isomorphic to a subgroup $\ker\rho_i$ of T^2, where ρ_i is defined in (2.2). The $\mathbb{Z}_{m_{i,i+1}}$-action on S^3 is given by the inclusion $\ker\rho_i \hookrightarrow T^2$ and the standard T^2-action on S^3. If $m_{i,i+1} = 1$, then $L_i \cong S^3$ and X is in $\mathscr{C}_{n,1}$. So, the equivalence (5.1) holds. If $m_{i,i+1} > 1$, then L_i is a lens space $L(m_{i,i+1}; k_i)$ for some k_i coprime to $m_{i,i+1}$.

In the following, the *p-component* $\nu_p(t)$ of a number t is defined to be the p-power p^r such that p^r divides t but p^{r+1} does not.

Lemma 5.1. *For p odd prime, let $\nu_p(m_{i,i+1}) = p^r$ and let $L_i = L(m_{i,i+1}; k_i)$ be a lens space. Then there is a map $\alpha_p \colon \Sigma L_i \to S^4 \vee P^3(p^r)$ that is a p-local equivalence.*

Proof. Let $m_{i,i+1} = p^r t$ where p and t are coprime. Then $P^3(m_{i,i+1}) \simeq P^3(p^r) \vee P^3(t)$. Consider the diagram of homotopy cofibration sequences

$$\begin{array}{ccccc}
* & \longrightarrow & P^3(t) & = & P^3(t) \\
\downarrow & & \downarrow & & \downarrow \\
S^3 & \xrightarrow{\phi} & P^3(m_{i,i+1}) & \longrightarrow & \Sigma L_i \\
\| & & \downarrow \text{pinch} & & \downarrow \alpha_p \\
S^3 & \xrightarrow{\phi'} & P^3(p^r) & \longrightarrow & C
\end{array}$$

where ϕ is the attaching map of the 4-cell in ΣL_i, ϕ' is the composition of ϕ and the pinch map, C is the mapping cone of ϕ' and α_p is an induced map. The right column induces an exact sequence

$$\cdots \to \tilde{H}^{i-1}(P^3(t); \mathbb{Z}_{p^r}) \to \tilde{H}^i(C; \mathbb{Z}_{p^r}) \xrightarrow{\alpha_p^*} \tilde{H}^i(\Sigma L_i; \mathbb{Z}_{p^r}) \to \tilde{H}^i(P^3(t); \mathbb{Z}_{p^r}) \to \cdots$$

Since $\tilde{H}^*(P^3(t); \mathbb{Z}_{p^r}) = 0$, the map $\alpha_p^* \colon H^*(C; \mathbb{Z}_{p^r}) \to H^*(\Sigma L_i; \mathbb{Z}_{p^r})$ is an isomorphism. Moreover, $H^*(C; \mathbb{Z}_{p^r})$ has trivial cup products because ΣL_i is a suspension. Now, lemma 4.4 shows that ϕ' is null homotopic, which means $C \simeq S^4 \vee P^3(p^r)$. Therefore, we consider α_p as a map from ΣL_i to $S^4 \vee P^3(p^r)$. Since $P^3(t)$ is contractible after p-localization, the right column implies that α_p is a p-local equivalence. $\square$

LEMMA 5.2. *Let p be a prime and let $H^3(X) \cong \mathbb{Z}_m$. Then there exists $i \in \{1, \ldots, n+2\}$ such that $\nu_p(m_{i,i+1}) = \nu_p(m)$.*

Proof. By [**14**, corollary 5.1] and [**10**, lemma 3.1], $m = \gcd\{m_{i,j} | 1 \leqslant i < j \leqslant n+2\}$. If $n = 1$, the lemma is trivial. So, we prove the lemma for $n \geqslant 2$.

Without loss of generality, suppose $\nu_p(m_{1,j}) = \nu_p(m) = p^r$ for some $j \in \{3, \ldots, n+1\}$. Let $\xi_1 = (a, b)$. Since a and b are coprime, there exist u and v such that

$$\begin{pmatrix} u & -b \\ v & a \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Changing the basis of $\mathbb{Z}^2$ if necessary, we may assume $\xi_1 = (1, 0)$.

Let $\xi_2 = (x, y)$ and $\xi_j = (z, w)$. Then

$$m_{1,2} = \left|\det \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}\right| = |y|, \quad m_{1,j} = \left|\det \begin{pmatrix} 1 & z \\ 0 & w \end{pmatrix}\right| = |w|.$$

Write $w = cp^r$ and $y = c'p^s$, where c and c' are integers coprime to p and $s \geqslant r$. If $s = r$, then $v_p(m_{1,2}) = \nu_p(m)$ and consequently the lemma holds. If $s > r$, then

$$m_{2,j} = |xw - yz| = |cxp^r - c'yp^s| = p^r|cx - c'yp^{s-r}|.$$

Since x is coprime to y, x is coprime to p. So, $cx - c'yp^{s-r}$ is coprime to p and $\nu_p(m_{2,j}) = p^r$. If $j = 3$, then we are done. If not, iterate this argument to $m_{2,j}$, ξ_3 and ξ_j. Then we can conclude that $\nu_p(m_{j-1,j}) = p^r$. □

For any odd prime p, let $\nu_p(m) = p^r$. By lemma 5.2, there is an $i \in \{1, \ldots, n+2\}$ such that $\nu_p(m_{i,i+1}) = \nu_p(m) = p^r$. Pick the vertex v_i and construct the **q**-CW-complex structure with respect to v_i. Then there is a homotopy cofibration sequence

$$L_i \xrightarrow{f} \bigvee_{j=1}^{n} S^2 \to X$$

with a coaction $c : X \to X \vee \Sigma L_i$. Furthermore, the 3-skeleton of X is $\bigvee_{j=1}^{n} S^2 \vee P^3(m)$ for $m = 2^s q$. Let $\hat{X}$ be the quotient $X/P^3(q)$ and let ϕ_p be the composition

$$\phi_p : X \xrightarrow{c} X \vee \Sigma L \xrightarrow{\jmath \vee \alpha} \hat{X} \vee P^3(p^r) \vee S^4 \xrightarrow{\text{pinch}} \hat{X} \vee P^3(p^r) \tag{5.2}$$

where α is the map in lemma 5.1 and $\jmath : X \to \hat{X}$ is the quotient map.

PROPOSITION 5.3 (p-local version of main theorem). *Let p be an odd prime. If $\nu_p(m) = p^r$ for some $r \geqslant 1$, then $\phi_p : X \to \hat{X} \vee P^3(p^r)$ is a p-local equivalence.*

Proof. We claim that the map ϕ_p in (5.2) induces an isomorphism on $\mathbb{Z}_{(p)}$-cohomology

$$\phi_p^* : H^*(\hat{X} \vee P^3(p^r); \mathbb{Z}_{(p)}) \to H^*(X; \mathbb{Z}_{(p)}) \tag{5.3}$$

where $\mathbb{Z}_{(p)}$ is the ring of p-local integers.

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The cofibration sequence $P^3(q) \hookrightarrow X \xrightarrow{\jmath} \hat{X}$ induces an exact sequence

$$\cdots \to \tilde{H}^{i-1}(P^3(q)) \to \tilde{H}^i(\hat{X}) \xrightarrow{\jmath^*} \tilde{H}^i(X) \to \tilde{H}^i(P^3(q)) \to \tilde{H}^{i+1}(\hat{X}) \to \cdots$$

For $i \neq 3$, since $\tilde{H}^i(P^3(p^r)) = 0$ and $\jmath^*\colon H^i(\hat{X}) \to H^i(X)$ is an isomorphism, the map (5.3) is an isomorphism.

Next, consider the cofibration sequence $L_i \xrightarrow{f} \bigvee_{i=1}^n S^2 \xrightarrow{\imath} X \xrightarrow{\delta} \Sigma L_i$, where $\imath$ is the inclusion and δ is the coboundary map. It induces an exact sequence

$$\cdots \to \tilde{H}^i(\Sigma L_i; \mathbb{Z}_{(p)}) \xrightarrow{\delta^*} \tilde{H}^i(X; \mathbb{Z}_{(p)}) \to \tilde{H}^i\left(\bigvee_{i=1}^n S^2; \mathbb{Z}_{(p)}\right) \to \tilde{H}^{i+1}(\Sigma L; \mathbb{Z}_{(p)}) \to \cdots$$

Since $\imath^*\colon H^2(X; \mathbb{Z}_{(p)}) \to H^2(\bigvee_{i=1}^n S_i^2; \mathbb{Z}_{(p)})$ is an isomorphism, $\delta^*\colon H^3(\Sigma L; \mathbb{Z}_{(p)}) \to H^3(X; \mathbb{Z}_{(p)})$ is an isomorphism. Consider the following commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{c} & X \vee \Sigma L_i & \xrightarrow{\jmath \vee \alpha_p} & \hat{X} \vee P^3(p^r) \vee S^4 & \longrightarrow & \hat{X} \vee P^3(p^r) \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\delta} & \Sigma L_i & \xrightarrow{\alpha_p} & P^3(p^r) \vee S^4 & \longrightarrow & P^3(p^r)
\end{array}$$

where the composite in the upper row is ϕ_p in (5.2) and the unnamed arrows are pinch maps. The left square commutes due to the property of the coaction map. By lemma 5.1, the map $\alpha_p^*\colon H^3(P^3(p^r) \vee S^4; \mathbb{Z}_{(p)}) \to H^3(\Sigma L; \mathbb{Z}_{(p)})$ is isomorphic, so the composite in the lower row induces an isomorphism $H^3(P^3(p^r); \mathbb{Z}_{(p)}) \to H^3(X; \mathbb{Z}_{(p)})$. Since $H^3(\hat{X}; \mathbb{Z}_{(p)}) = 0$, the map (5.3) is an isomorphism for $i = 3$. Therefore, $\phi_p^*\colon H^*(\hat{X} \vee P^3(p^r); \mathbb{Z}_{(p)}) \to H^*(X; \mathbb{Z}_{(p)})$ is an isomorphism and ϕ_p is a p-local equivalence. □

LEMMA 5.4. *Let X be a 4-dimensional toric orbifold with $H^3(X) \cong \mathbb{Z}_m$, and let $\nu_p(m) = p^r$ for some odd prime p and $r \geqslant 1$. If $M_{cup}(X) = (A, \boldsymbol{b}, c)$ where*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ and } \mathbf{b} = (b_1, \ldots, b_n),$$

then the system of mod-p^r linear equations

$$\begin{cases} a_{11}y_1 + \ldots + a_{1n}y_n & \equiv & -b_1 \\ & \vdots & \\ a_{n1}y_1 + \ldots + a_{nn}y_n & \equiv & -b_n \\ b_1 y_1 + \ldots + b_n y_n & \equiv & -c \end{cases} \pmod{p^r}$$

has a solution in $(\mathbb{Z}_{p^r})^n$.

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Proof. By Proposition 5.3 there is a map $\phi_p\colon X \to \hat{X} \vee P^3(p^r)$ that becomes a homotopy equivalence after localized at p, where $\hat{X} \in \mathscr{C}_{n,2^s}$ is the quotient $X/P^3(q)$. Let

$$M(\phi_p) = (W, \mathbf{y}, z) \in \mathrm{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m$$

be the cellular map representation of ϕ_p. After p-localization, W is an invertible matrix and z is a unit. The lemma follows from Proposition 4.8. □

6. Proof of the main theorems

Proof of theorem 1.2. Let $q = p_1^{r_1} \dots p_k^{r_k}$ be the primary factorization where p_i's are different odd primes and $r_i \geqslant 1$. For each prime p_i, lemma 5.4 implies that the mod-$p_i^{r_i}$ version of (4.4) has a solution. By Chinese Remainder theorem, they give a mod-q solution for (4.4). By Proposition 4.8, X is homotopy equivalent to $\hat{X} \vee P^3(q)$. □

Proof of theorem 1.3. The 'only if' part is trivial. To prove the 'if' part, let X and X' be 4-dimensional toric orbifolds such that $H^3(X) \cong \mathbb{Z}_m$ and $H^3(X') \cong \mathbb{Z}_{m'}$ for m and m' odd. The hypothesis implies that $H^3(X) \cong H^3(X')$, hence we have $m = m'$. By theorem 1.2, we have $X \simeq \hat{X} \vee P^3(m)$ and $X' \simeq \hat{X}' \vee P^3(m)$ for some $\hat{X}, \hat{X}' \in \mathscr{C}_{n,1}$. Since $H^i(X) \cong H^i(\hat{X})$ and $H^i(X') \cong H^i(\hat{X}')$ for $i \neq 3$, we have $H^*(\hat{X}) \cong H^*(\hat{X}')$. Then Proposition 4.2 implies that $\hat{X} \simeq \hat{X}'$, which yields $X \simeq X'$. □

Acknowledgement
We started this project during our participation to the Thematic Program on Toric Topology and Polyhedral Products at Fields Institute. We gratefully acknowledge the support of Fields Institute and the organizers of the programme. Furthermore, we thank Mikiya Masuda, Taras Panov, Dong Youp Suh and Donald Stanley for discussing the topics, and thank Stephen Theriault for proofreading our draft and giving helpful comments.

Financial support
The first author was supported by Fields Institute and is supported by the National Research Foundation of Korea funded by the Korean Government (NRF-2019R1A2C2010989), the second author is supported by Pacific Institute for the Mathematical Sciences (PIMS) Postdoctoral Fellowship and the third author is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07048480) and a KIAS Individual Grant (MG076101) at Korea Institute for Advanced Study.

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