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Stabilizing the second-order nonholonomic systems with chained form by finite-time stabilizing controllers

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SUMMARY
An underactuated mechanical system is generally a good test bed for advanced nonlinear controllers and can be applied to design a novel mechanical system with better energy efficiency and good controllability. It has been shown that the dynamics of many underactuated mechanical systems could be transformed into the chained canonical form. To improve the performance of the controllers presented in the literature, a novel controller design method is proposed in this paper. It is shown that the set-point stabilization problem of the second-order chained form systems can be changed into a trajectory-tracking problem based on the nonsmooth Hölder continuous feedback. By designing the tracked trajectory, the presented controller permits the achievement of exponential stability. Some numerical simulations demonstrate the stability of the proposed controller for an underactuated Hovercraft system.
KEYWORDS: Underactuation; Nonholonomic constraints; Global stability; Finite-time; Control.

1. Introduction

1.1. The motivations for the work

The underactuated mechanical systems are commonly used in real-word applications, such as the test beds of advanced nonlinear controllers and a novel mechanism scheme for designing a mechanical system with better energy efficiency and good controllability. For instance, the former includes the underactuated manipulators,[1] Acrobot,[2] Pendubot,[3] planar underactuated rigid,[4] underactuated ships,[5] and underactuated Hovercrafts,[6] etc. The later involves the novel underactuated hopping or running robots.[7] It has been shown that the dynamics of an underactuated mechanical system can be transformed into some kinds of special canonical forms if the system under consideration holds some kind of special differentially geometric or algebraic properties, such as differential flatness[8] or nilpotency,[9] and then designing a controller for the underactuated system is effectively simplified. On the basis of the chained form transformation theorem presented by Murray, Li and Sastry,[10] it has been shown many underactuated systems can be changed into the special regular form, such as the mobile robots[11] or the angular momentum conservation systems,[12] etc.

It has also been found that the dynamics of many underactuated mechanical systems can be transformed into the so-called second-order chained canonical form.[13] The relevant systems include RRR[14] (R indicates the last joint is passive, R denotes the rotationally actuated joint, P denotes the linearly telescopically actuated joint) and PPR[15] underactuated manipulators, planar underactuated rigid,[4] and underactuated Hovercrafts,[6] etc. Similar to the related research on the controller design for the first-order nonholonomic systems, due to the Brockett's theorem[16] that confirms a nonholonomic

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system cannot be stabilized by any smooth time-invariant feedback, the presented controllers in literature for the underactuated systems with acceleration constraints primarily belong to two classes. One class of them is the smooth time-varying feedback,[13,17] and another class is the discontinuous feedback.[14,15,18,19] Although a controller based on the smooth time-varying feedback shows asymptotically stabilizing performance, among other things, the settling-time of this class of controllers are commonly too long for some applications.[4] A discontinuous feedback controller can generally stabilize a system in finite settling time, however, this class of controllers tends to cause abrupt changes on the states of the controlled systems, thus the control outputs exceed limits of the physical actuators.

To improve the performance of the controllers presented in the literature for the second-order chained form systems, a novel controller design method is presented in this paper on the basis of Hölder continuous feedback.[20] A function $V(x) : R^n \to R$ is said to be Hölder continuous at $x_1 \in R^n$, if there exist $k > 0$ and an open neighborhood U of x_1 such that the inequality $|V(x_1) - V(x_2)| \leq k\|x_1 - x_2\|^\alpha$ holds, where $x_2 \in U$ and $\alpha > 0$. For the well-known Lipschiz continuity, the exponent of the inequality must satisfy $\alpha \geq 1$. Therefore, Hölder continuity is a more general notion than Lipschiz continuity. For instance, consider the scalar function $f(x) = |x|^\alpha$, $x \in R$, if the exponent α satisfies $\alpha \in (0, 1)$, then the function $f(x)$ is said to be Hölder continuous but not Lipschiz continuous. In this paper, the Hölder continuous feedback just indicates continuous but nonsmooth feedback, and the finite-time stabilizing controllers proposed in this paper are also stable in Lyapunov sense. For more thoroughly understanding the finite-time stability, the readers are encouraged to see the book.[36]

As analyzed in the next subsection, nonsmooth feedback might be the unique feasible selection for exponentially stabilizing a nonholonomic constraints system without controllable linearization. For the nonholonomic systems with controllable linearization, the Hölder continuous feedback guarantees the controlled system be (at least locally) stabilized in finite settling time. Whereas for the nonholonomic systems without controllable linearization, the finite-time stability could not be achieved by any continuous feedback. Hence, the exponential stability of the controller presented in this paper is appealing in practice, at least in the sense of reducing both the settling time and the impact effects, even though the settling time of the presented controller is actually infinite.

1.2. A concise overview of relevant works

In this subsection, we do not intend to give a thorough survey about the techniques related to the presented work, but concisely review some benchmark results in literature according to the technical relationships.

Two different methodologies have been proposed in the literature to design finite-time stabilizing controllers. One is the higher-order sliding mode control[21] presented by Levant and Fridman *et al.* By designing a variable structure control (VSC) for the virtual inputs that are the time-derivatives of the actual inputs, then the actual inputs do not show the "chattering" phenomenon and robustly stabilize a system in finite-time. However, the higher-order sliding mode controllers are generally required to use faster actuators and the stability of the controllers cannot be easily proved due to the discontinuous differential equations. The other finite-time controllers use the fractional-power-based Hölder continuous feedback.[22–25] Due to the limited mathematical tools that can be utilized to deal with nonsmooth differential equations, the finite-time controllers based on the Hölder continuous feedback techniques cannot be easily applied to control the general nonlinear systems. So far the Hölder continuous feedback techniques were primarily presented for perturbed linear systems,[22] and homogeneous nonlinear systems with special strict-feedback forms.[23,24] The mentioned references[22–24] can be regarded as the relevant research of the presented work, whereas, this research is not directly related to the nonholonomic systems without controllable linearization.

More directly related to this work, in the reference, Hong *et al.*[25] presented a finite-time stabilization approach for the first-order nonholonomic systems with uncertain chained form. We note that the presented method is a switch-based discontinuous control method. M'Closkey and Morin[4] proposed two methods to design the time-varying homogeneous feedback for stabilizing some special nonholonomic systems. Related to another elegant paper,[26] the first approach presented in ref. [4] depends on the prior known smooth time-varying controllers. The second approach proposed in ref. [4] depends on the homogeneous approximation system associated with the structure of the control Lie algebra of the nonholonomic systems, and applies a time-averaging technique to analyze the stability of the closed-loop homogeneous approximation systems.

Closely related to the work of,[4] Oriolo and Vendittelli[9,27] more clearly showed the powerful techniques of high-order approximation in constructing the time-varying control laws for general nonholonomic systems. It is different from the approach proposed in ref. [4] and the works in refs. [9, 27] extend the methodologies of homogeneous approximation to nonhomogeneous nilpotent approximation on the basis of homogeneous nilpotent approximation algorithm.[28] Even though the approaches presented in refs. [4, 9, 26–28] lead to extremely involved control laws with just localized stability when high-order Lie brackets of control vector fields are required to span the state space, and yields highly oscillatory movement trajectories, the methodologies presented in these works completely open the door for investigating the control problems for general highly nonlinear dynamics systems, which are not limited to the well developed triangular forms nonlinear systems[23,34] and not necessarily limited to nonholonomic systems.[24,30]

In recent years, the researches following this direction are devoted to controlling law synthesis for uncertain nonholonomic systems with special normal forms.[29,31] In the ref. [29], inspired by the approach presented in ref. [30], an adaptive controller was presented for the uncertain chained form systems with the help of Lyapunov-based method and time-rescaling technique. The work of[31] involved the stabilization of stochastic nonholonomic chained form systems in finite settling time. Similar to the approach presented in ref. [25], in order to stabilize the chained form systems in finite settling time, both of the controllers proposed in refs. [29] and [31] followed the switched-based control strategies, which are actually discontinuous control approach.

As for the opinions of the authors, the time-varying smooth feedback tenderly constrain the state of a nonholonomic system moving on a special smooth manifold, so that the motion efficiency is considerably reduced, thus the settling time increases. Contrarily, switch-based discontinuous feedback crustily constrains a system moving on several sub-manifolds. Even though the motion efficiency is improved, the impact effects caused by switched inputs make it infeasible for many real-world applications.

1.3. The contributions of the work

Different from the time-varying smooth feedback[13,17] and the switch-based controller design methods,[14,15,18,19] the new controller presented in this paper shows more appealing performance in balancing the settling time and the peaking of control inputs. In principle, the performance of the controller presented in this work and that presented in refs. [4, 26] are accordant, since the time-varying Hölder continuous feedback is used in both of the works. Whereas it is different from the approaches presented in refs. [4, 26], in this work, the presented controller design method does not depend on any prior known controllers, and is not necessary to use the complex time-periodic averaging techniques to analysis the stability of the closed-loop systems. Even though a class of specific nonholonomic systems with the second-order chained form is considered in this paper, this work provides a rather simple but effective controller design approach for stabilizing a class of high-order nonholonomic systems, and provides new insights to the drifts for maneuvering the nonholonomic systems (see remark 5 in Section 3.1). In Section 3.2, the robustness of the presented controller in this paper is also analyzed under upper bounded disturbances. In Section 3.3, we also show that the presented controller design method can be generalized to a class of uncertain higher-order chained form systems.

2. Problem Formulations and Some Useful Lemmas

2.1. Problem formulation

In this paper, we primarily consider the control problems of the second-order nonholonomic nonlinear systems with chained form[13−15,18,19]

$$\ddot{y}_1 = u_1, \quad \ddot{y}_2 = u_2, \quad \ddot{y}_3 = y_2 u_1, \tag{1}$$

where $y = [\, y_1 \ y_2 \ y_3 \,]^{\mathrm{T}} \in R^3$ are the generalized coordinates of the system, $u_1 \in R$ and $u_2 \in R$ are two inputs. Define a coordinate transformation, which is given by

$$z_1 = y_1, \ z_2 = \dot{y}_1, \ z_3 = y_3, \ z_4 = \dot{y}_3, \ z_5 = y_2, \ z_6 = \dot{y}_2,$$

then the system (1) can be written in state space form

$$\begin{array}{ll} & \dot{z}_3 = z_4 \\ \dot{z}_1 = z_2 & \dot{z}_4 = u_1 z_5 \\ \dot{z}_2 = u_1 & \dot{z}_5 = z_6 \\ & \dot{z}_6 = u_2 \end{array} \tag{2}$$

Defining vector $\mathbf{z} = [z_1, \ldots, z_6]^{\mathrm{T}}$, the global stabilization problem for the chained form nonholonomic system (2) can be stated as: to design control inputs $u_1 = u_1(\mathbf{z}, t)$ and $u_2 = u_2(\mathbf{z}, t)$ such that the origin of the system (2) is stable from any given initial state $\mathbf{z}(t_0)$.

2.2. *Some useful lemmas*

As they will be used in the sequence for synthesizing the new controller of the paper, some lemmas are provided in this section.

Lemma 1:[24] For any real numbers a_i, $i = 1, 2, \ldots, n$ and $0 < \gamma \leq 1$, the following inequality holds

$$\left(\sum_{i=1}^{n} |a_i|\right)^{\gamma} \leq \sum_{i=1}^{n} |a_i|^{\gamma}. \tag{3}$$

For $x \in R$, $y \in R$, when $0 < \gamma = p/q \leq 1$, where $p > 0$ and $q > 0$ are odd integers, then

$$|x^{\gamma} - y^{\gamma}| \leq 2^{1-\gamma} |x - y|^{\gamma} < 2 |x - y|^{\gamma}. \tag{4}$$

When $\gamma > 1$ is a constant, then

$$|x - y|^{\gamma} < 2^{\gamma-1} |x^{\gamma} - y^{\gamma}|. \tag{5}$$

Lemma 2:[22,24] Let m, n be positive real numbers, and $\gamma(x, y) \geq 0$ be a real-valued function, then the following inequality holds

$$|x|^m |y|^n \leq \frac{m\gamma(x, y)}{m+n} |x|^{m+n} + \frac{n\gamma^{-m/n}(x, y)}{m+n} |y|^{m+n}. \tag{6}$$

Remark 1: Lemma 2 can be proved by the Young's inequality,[32, pp.27] namely $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $p > 0, q > 0$. In particular, if let $a = |x|^m \beta^{\frac{m}{m+n}}$, $b = |y|^n \beta^{\frac{-m}{m+n}}$, $p = \frac{m+n}{m}$, $q = \frac{m+n}{n}$, and β is a non-negative real-valued function of (x, y), then the inequality (6) follows.

Lemma 3: Given $0 < \gamma = p/q \leq 1$, where $p > 0$ and $q > 0$ are odd integers, and $\xi \neq \alpha$, then the following inequality holds

$$W = \int_{\alpha}^{\xi} (s^{1/\gamma} - \alpha^{1/\gamma})^{2-\gamma} ds > 0. \tag{7}$$

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Remark 2: Lemma 3 can be proved using inequality (5) and the identity $(x)^\gamma = \text{sign}(x)|x|^\gamma$ for all $0 < \gamma = p/q \le 1$ with $p > 0$ and $q > 0$ are odd integers. Actually, it is easy to show that

$$\begin{aligned}\int_\alpha^\xi (s^{1/\gamma} - \alpha^{1/\gamma})^{2-\gamma} ds &= \int_\alpha^\xi \text{sign}(s^{1/\gamma} - \alpha^{1/\gamma}) \left|s^{1/\gamma} - \alpha^{1/\gamma}\right|^{2-\gamma} ds \\ &\overset{(5)}{>} 2^{-\frac{\gamma(2-\gamma)}{1-\gamma}} \int_\alpha^\xi \text{sign}(s-\alpha)\,|s-\alpha|^{\frac{2-\gamma}{\gamma}} ds \\ &= 2^{-\frac{\gamma(2-\gamma)}{1-\gamma}} \int_\alpha^\xi |s-\alpha|^{\frac{2-\gamma}{\gamma}} d(s-\alpha) \\ &= 2^{-\frac{\gamma(2-\gamma)}{1-\gamma}} \times \frac{\gamma}{2}(\xi-\alpha)^{2/\gamma} \\ &> 0.\end{aligned}$$

Remark 3: In the next section, the power integrator (7) will be used to design a Lyapunov function candidate on the basis of the powerful "adding a power integrator" technique.[21,23,24] Since the power integrator (7) is a first differentiable function, i.e. $W \in C^1$, thus the time derivate of W at origin will be Hölder continuous but not Lipschiz continuous. The Hölder but non-Lipschiz continuous differential equations are also called the finite-time differential equations.[20,33] Thus, the following Lemma 4 is also called the finite-time stability theorem.

Lemma 4:[33] For the non-Lipschitz autonomous system $\dot{x} = f(x)$, suppose there exists a continuous function $V(x) : D \to R$ defined on a neighborhood $N \subseteq D$ of the origin, such that the following conditions hold:

(a) $V(x)$ is positive definite on $D \subset R^n$;
(b) There exist real numbers $c > 0$ and $\gamma \in (0, 1)$, such that $\dot{V}(x) + cV^\gamma(x) \le 0$, $x \in N\backslash\{0\}$.

Then the origin of system $\dot{x} = f(x)$ is locally finite-time stable. The settling time, depending on the initial state $x(0) = x_0$, satisfies $T_x(x_0) \le V(x_0)^{1-\gamma}/[c(1-\gamma)]$ for all x_0 in some open neighborhood of the origin. If $D = R^n$ and $V(x)$ is also unbounded, then the origin of system $\dot{x} = f(x)$ is globally finite-time stable.

Remark 4: Lemma 4 is generally used as the foundation for designing the finite-time stabilizing controllers.[33] It is worth mentioning that Bhat and Bernstein presented the detailed relationships between the geometric homogenity and the finite-time stability in the ref.[20]. Referring to relevant refs. [9] and [26–28], the relationships between the homogeneous approximation and nilpotent approximation can be well understood. As to design the finite-time controllers, refs. [22–25] are recommended. Whereas, as mentioned in Section 1.2, the relevant finite-time controllers were primarily presented for homogeneous system with special strict feedback form,[23,24] or even perturbed linear systems.[22] These methods cannot be directly used to design the controllers for chained form nonholonomic systems since the linearization of nonholonomic systems is not controllable at origin.

3. Global Stabilization of the Second-Order Chained Systems

3.1. Controller synthesis

To design a controller for the chained form system (2), let's partition the system to two subsystems

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= u_1\end{aligned} \tag{8a}$$

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$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = u_1 z_5$$
$$\dot{z}_5 = z_6$$
$$\dot{z}_6 = u_2. \tag{8b}$$

The primary obstacle of designing a controller for the second-order chained form system (8) is that u_1 or z_5 should not be zero when $(z_3, z_4) \neq (0, 0)$. Otherwise, refer to (8b), if $\dot{z}_3 = z_4$, $\dot{z}_4 = 0$, which means $\dot{z}_3 = z_4 = c$, where $c \neq 0$ is a constant, then it follows that $z_3 = z_3(0) + ct$, thus z_3 will be unstable under the case. To solve the problem, the approaches presented in the references[25,29,31] generally design a bounded controller $u_1(t) = u_{1a}(t) \neq 0$, so that the subsystem (8b) can be stabilized to its origin $(z_3, \ldots, z_6) = (0, \ldots, 0)$ in finite settling time T_1. When the time satisfies $t \geq T_1$, then switches the controller $u_1(t)$ to another state $u_1(t) \neq u_{1b}(t)$, so that the subsystem (8a) is stabilized to its origin $(z_1, z_2) = (0, , 0)$ in finite time T_2. Therefore, the overall system (8a)–(8b) is stabilized in finite settling time $T \leq T_1 + T_2$ by switch-based discontinuous controllers.

Motivated by the limitations of time-varying smooth controllers and the switch-based controllers presented for nonholonomic systems in the literature, in this paper, we change the set-point stabilization problem for chained form system (8) to a trajectory tracking control problem. The desired trajectories of the system (8) are given by a group of smooth and exponentially stable functions, for instance

$$z_1^d(t) = \beta_1(t), z_3^d(t) = \beta_2(t), z_5^d(t) = \beta_3(t), \tag{9}$$

where $\beta_i(t) = \beta_{i0} \exp(-\lambda_i t)$, $\lambda_i > 0$ and $\beta_{i0} \neq 0$, $i = 1, 2, 3$, are constants. The time derivative of (9) is given by

$$z_2^d(t) = \dot{z}_1^d(t) = -\lambda_1 \beta_1$$
$$z_4^d(t) = \dot{z}_3^d(t) = -\lambda_2 \beta_2$$
$$z_6^d(t) = \dot{z}_5^d(t) = -\lambda_3 \beta_3.$$

Notably, the given exponentially stable trajectory satisfies $(z_1^d(t), \ldots, z_6^d(t)) \neq 0$ for all $\lambda_i > 0$ and $\beta_{i0} \neq 0$. This point is important for designing the controller for chained form nonholonomic systems of which the origin is not controllable.

Now let's define the error variables $\xi_i = z_i - z_i^d$, $i = 1, \ldots, 6$, then the error systems of the chained form system (8) can be written as

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = u_1 - \lambda_1^2 \beta_1 \tag{10a}$$

$$\dot{\xi}_3 = \xi_4$$
$$\dot{\xi}_4 = u_1 (\xi_5 + \beta_3) - \lambda_2^2 \beta_2$$
$$\dot{\xi}_5 = \xi_6$$
$$\dot{\xi}_6 = u_2 - \lambda_3^2 \beta_3. \tag{10b}$$

It will be intuitional, if the error systems (10a)–(10b) could be stabilized to origin $(\xi_1, \ldots, \xi_6) = (0, \ldots, 0)$ in finite settling time T, then after the time $t \geq T$, the motions of the original systems (8a)–(8b) accurately satisfy the exponentially stable trajectories given by (9). Therefore, the origin of the original chained form system (8) is exponentially stable. To this end, a finite-time stabilizing controller for the error systems (10) is presented as following proposition.

Proposition 1: For the time-varying nonlinear system (10) endowed with a set of exponentially stable functions $\beta_i = \beta_{i0} \exp(-\lambda_i t)$, $i = 1, 2, 3$, where $\lambda_i > 0$ and β_{i0} are constants, if the functions

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satisfy

$$\lambda_2^2\beta_2(t) = \lambda_1^2\beta_1(t)\beta_3(t), \tag{11}$$

then there exists a set of constants $k_i > 0$, $i = 1, \ldots, 6$, such that the controllers

$$u_1 = \lambda_1^2\beta_1 - k_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}$$
$$u_2 = \lambda_3^2\beta_3 - k_6\left(\xi_6^3 - \alpha_4^3\right)^{1/9} \tag{12}$$

where

$$\alpha_4 = -k_5\left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^{1/3}, \quad \alpha_3 = -\frac{1}{u_1}k_4\left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{5/9},$$

$\alpha_2 = -k_3\xi_3^{7/9}$, and $\alpha_1 = -k_1\xi_1^{7/9}$, globally stabilize the time-varying nonlinear system (10) to origin $(\xi_1, \ldots, \xi_6) = 0$ in finite settling time.

Proof: see Appendix A. □

Remark 5: Synthesizing the controllers for a nonholonomic system with drifts is generally a difficult task.[4] However, in this work we deliberately utilize the nonzero but exponentially stable drifts to generate the virtual control input α_3 of the controller (12), such that the coupled vector filed u_1z_5 of (8b) is bounded and then guarantees the state variables (ξ_3, ξ_4) do not escape in any finite time. Due to the special cascade of chained form system (8), the structure characteristic is inherited by the error system (10). The proof of proposition 1 follows a "nearly" standard back-stepping procedure.[34] The slight difference is that the stability of the subsystem (10b) depends on the stability of the subsystem (10a). Thus we have to design a nonzero and bounded control input $u_1(t) \neq 0$, such that $u_1(t)$ and $u_2(t)$ simultaneously stabilize both of the time-varying error subsystems (10a) and (10b) in finite settling time.

Remark 6: In proposition 1, the selected functions $\beta_i(\beta_{i0}, \lambda_i, t)$, $i = 1, 2, 3$ should satisfy the constraint (11). Refer to the subsystem (8b), one can find that the motions of the system satisfy relationship $\dot{z}_4 = u_1z_5$. When the error system (10) is stabilized to its origin $(\xi_1, \ldots, \xi_6) = (0, \ldots, 0)$, then the system (8) is stabilized to the exponentially stable trajectories given by (9), and the relevant variables of differential equation $\dot{z}_4 = u_1z_5$ satisfy

$$u_1 = \ddot{z}_1^d = \lambda_1^2\beta_1(t),$$
$$\dot{z}_4 = \dot{z}_4^d = \lambda_2^2\beta_2(t),$$
$$z_5 = z_5^d = \beta_3(t).$$

Thus the design parameters of $\beta_i(\beta_{i0}, \lambda_i)$, $i = 1, 2, 3$ should satisfy the relationship (11). More specifically, the constraint Eq. (11) can be given by

$$\lambda_1 = \lambda_2 - \lambda_3,\ \beta_{10} = \beta_{20}\lambda_2^2/\beta_{30}(\lambda_2 - \lambda_3)^2. \tag{13}$$

Remark 7: The partial differentials of the powerintegrators given by (7) are repeatedly used in the proof of proposition 1. It can be proved that,[22,24]

$$\frac{\partial W}{\partial \xi_k} = (\xi_k^{1/\gamma} - \alpha^{1/\gamma})^{2-\gamma}, \tag{14}$$

$$\frac{\partial W}{\partial \xi_i} = -(2-\gamma)\left(\frac{\partial \alpha^{1/\gamma}}{\partial \xi_i}\right)\int_{\alpha}^{\xi_k}(s^{1/\gamma} - \alpha^{1/\gamma})^{1-\gamma}ds, \quad i = 1, \ldots, k-1, \tag{15}$$

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where

$$W = \int_{\alpha}^{\xi_k} (s^{1/\gamma} - \alpha^{1/\gamma})^{2-\gamma} ds, \quad \alpha = \alpha(\xi_i), i = 1, \ldots, k-1,$$

is considered. Using the relationships (14) and (15), and considering the Lemmas 1 and 2, then proposition 1 can be proved with step by step.

On the basis of proposition 1, we can give the following proposition that ensures the origin's exponential stability of the chained form nonholonomic system (8), when the system (8) is controlled by the presented controller (12).

Proposition 2: If the origin of the error system (10) is globally finite-time stable, then the origin of the original chained form system (8) is globally exponentially stable.

Proof: According to the condition of the proposition, the error system (10) is stabilized in finite-time $T(\boldsymbol{\xi}_0) \leq \frac{18V(\boldsymbol{\xi}_0)^{1/9}}{k}$, thus if $t > T(\boldsymbol{\xi}_0)$ is satisfied, then the motions of the system (8) accurately satisfy the following equations □

$$\begin{aligned} \dot{z}_1 &= -\beta_{10}\lambda_1 \exp(-\lambda_1 t) \\ \dot{z}_2 &= \beta_{10}\lambda_1^2 \exp(-\lambda_1 t) \end{aligned} \tag{16a}$$

$$\begin{aligned} \dot{z}_3 &= -\beta_{20}\lambda_2 \exp(-\lambda_2 t) \\ \dot{z}_4 &= \beta_{10}\beta_{30}\lambda_1^2 \exp\left[-(\lambda_1 + \lambda_3)t\right] \\ \dot{z}_5 &= -\beta_{30}\lambda_3 \exp(-\lambda_3 t) \\ \dot{z}_6 &= \beta_{30}\lambda_3^2 \exp(-\lambda_3 t), \end{aligned} \tag{16b}$$

where $\lambda_i > 0$, $i = 1, 2, 3$ are constants. Therefore, the origin of the chained form system (8) is exponentially stable. Since the claim of proposition 1 is globally stable, and there is not any additional condition in proposition 2, thus the origin's stability of the chained form system (8) is global. □

Remark 8: The controller design method presented by proposition 1 and 2 can be easily applied to design a controller for the first-order chained form nonholonomic system,[25] and this problem is further addressed in Section 3.3.

3.2. Robust controllers for disturbed second-order chained form systems

Nonsmoothness is a kind of highly nonlinear characteristic. The robustness of Hölder continuous feedback has been illustrated by the homogenous systems.[4,9,22,25–28] In this subsection, we will further clearly show the appealing robustness of the nonsmooth but Hölder continuous feedback by the disturbed second-order chained form systems.

Let's consider the disturbed version of the system (10), suppose the systems has the following form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \phi_1(\boldsymbol{\xi}) \\ \dot{\xi}_2 &= u_1 - \lambda_1^2\beta_1 + \phi_2(\boldsymbol{\xi}) \end{aligned} \tag{17a}$$

$$\begin{aligned} \dot{\xi}_3 &= \xi_4 + \phi_3(\boldsymbol{\xi}) \\ \dot{\xi}_4 &= u_1(\xi_5 + \beta_3) - \lambda_2^2\beta_2 + \phi_4(\boldsymbol{\xi}) \\ \dot{\xi}_5 &= \xi_6 + \phi_5(\boldsymbol{\xi}) \\ \dot{\xi}_6 &= u_2 - \lambda_3^2\beta_3 + \phi_6(\boldsymbol{\xi}) \end{aligned} \tag{17b}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_6)$, and $\phi_i(\boldsymbol{\xi})$, $i = 1, \ldots, 6$ are disturbances. In this paper, we assume the disturbance terms are bounded, and satisfy $|\phi_i(\boldsymbol{\xi})| \leq \rho_i(\boldsymbol{\xi}) \sum_{j=1}^{i} |\xi_j| < M_i$, where $0 < \rho_i(\boldsymbol{\xi}) < \rho_{i0}$, both ρ_{i0} and $M_i > 0$ are constants.

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Remark 9: For achieving global stability, the disturbance terms with lower triangular form are assumed in this paper in order to make the structure of the uncertain terms consistent with the nominal chained form systems. As to more general nonholonomic systems, only local stability can be obtained in the refs. [4, 9, 27, 28], since in those cases the homogeneous approximation technique has to be used to simplify the nonlinear model of the considered systems. In fact, for a general nonlinear system, even the linearization of it is controllable, global stabilization is still a very open problem.

In order to understand the robustness of the Hölder continuous controller (12), for the uncertain second-order chained form systems (17), the following proposition can be directly proved.

Proposition 3: The controller (12) globally stabilizes the disturbed second-order chained form system (17) to origin in finite settling time.

Proof: The structure of the two subsystems of (17a) and (17b) is identical due to the lower triangular form of the bounded disturbance terms. As shown in the proof of proposition 1, we just need to prove the stability of the subsystem (17a). If the fact is that, then the stability of the overall system (17a)–(17b) can be proved without any difficult. □

Let's consider the subsystem (17a), and given a Lyapunov function candidate

$$V_1 = 0.5\xi_1^2 + W_1, \quad W_1 = \int_{\alpha_1}^{\xi_2} \left(s_1^{9/7} - \alpha_1^{9/7}\right)^{11/9} ds_1, \tag{18}$$

then the time derivate of V is given by

$$\dot{V}_1 = \xi_1 (\xi_2 + \phi_1(\xi) - \alpha_1) + \xi_1\alpha_1 + \frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2 + \frac{\partial W_1}{\partial \xi_2}\left(u_1 - \lambda_1^2\beta_1 + \phi_2(\xi)\right). \tag{19}$$

Considering the controller $u_1 = \lambda_1^2\beta_1 - k_2(\xi_2^{9/7} - \alpha_1^{9/7})^{5/9}$ with $\alpha_1 = -k_1\xi_1^{7/9}$ of (12), then (19) can be written as

$$\dot{V}_1 = -k_1\xi_1^{16/9} + \xi_1 (\xi_2 - \alpha_1) + \xi_1\phi_1(\xi) + \frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2 + \frac{\partial W_1}{\partial \xi_2}\left(-k_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9} + \phi_2(\xi)\right). \tag{20}$$

Using the formula (14), it can be shown that

$$\frac{\partial W_1}{\partial \xi_2} = \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{11/9}. \tag{21}$$

Substitute (21) into (20), and then the Eq. (20) can be written as

$$\begin{aligned}\dot{V}_1 &= -k_1\xi_1^{16/9} - k_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{16/9} + \xi_1 (\xi_2 - \alpha_1) \\ &\quad + \frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2 + \xi_1\phi_1(\xi) + \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{11/9}\phi_2(\xi).\end{aligned} \tag{22}$$

The first uncertain terms in (22) can be estimated as follows

$$\left|\xi_1\phi_1(\xi)\right| \le \rho_{10} |\xi_1|^2 = \left(\rho_{10}\xi_1^{2/9}\right) |\xi_1|^{16/9}.$$

Considering $|\phi_1(\xi)| < M_1$, thus there exist a positive constant $\bar{\rho}_{10}$, such that

$$\left|\xi_1\phi_1(\xi)\right| \le \bar{\rho}_{10} |\xi_1|^{16/9}. \tag{23}$$

The second uncertain term in right-hand side of (22) can be estimated as

$$\left|\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{11/9}\phi_2(\xi)\right| \le \rho_{20}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{11/9}|\xi_1| + \rho_{20}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{11/9}|\xi_2|. \tag{24}$$

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By the inequality (6) of Lemma 2, it not difficult to show there necessarily exist two positive constants c_1 and c_2, such that (24) satisfies that

$$\left|\left(\xi_2^{9/7}-\alpha_1^{9/7}\right)^{11/9}\phi_2(\xi)\right| \leq c_1\left|\xi_1\right|^{16/9}+c_2\left|\xi_2^{9/7}-\alpha_1^{9/7}\right|^{16/9}. \tag{25}$$

Proceeding along similar lines, the bounds of the terms $\xi_1(\xi_2-\alpha_1)$ and $\frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2$ can be estimated, and then the Eq. (22) can be expressed as following form

$$\dot{V}_1 \leq -k\xi_1^{16/9}-k\left(\xi_2^{9/7}-\alpha_1^{9/7}\right)^{16/9}, \tag{26}$$

where k is a positive constant associated with a pair of sufficiently large control parameters (k_1,k_2) of the controller $u_1(t)$. On the other hand, from the definition (18), it is easy to show

$$V_1 \leq 2\xi_1^2+2\left(\xi_2^{9/7}-\alpha_1^{9/7}\right)^{16/9}. \tag{27}$$

By applying inequality (3), then from (26) and (27), we can conclude

$$\dot{V}_1 \leq -\frac{k}{2}V_1^{8/9}. \tag{28}$$

According to the Lemma 4, the subsystem (17a) is globally finite-time stable due to $\lim_{\|(\xi_1,\xi_2)\|\to\infty} V_1=\infty$.

By a similar procedure as given above (possibly, if necessary, refer to the Appendix A), the global finite-time stability of overall system (17) can be concluded. □

Remark 10: Even though only bounded and "matched" disturbances are considered in proposition 3, it is not difficult to show the controller (12) is still valid for bounded "unmatched" disturbances with higher order homogeneous degree, due to the Theorem 7.4 of.[20] It is worth pointing out that, for more general disturbances, the structured characteristics of the chained form systems might be changed by the uncertain terms. Under those more general cases, the homogeneous approximation techniques have to be used for getting locally stabilizing controllers.[4,9,27,28]

3.3. Robust stabilization of a class of uncertain high-order chained form systems

In this subsection, we further show the controller design method presented in Section 3.1 can be generalized to more generally disturbed chained form systems. Let's consider a class of disturbed high-order chained form systems[25,29]

$$\begin{aligned}
\dot{z}_0 &= u_0+\bar{\phi}_0(z)\\
\dot{z}_1 &= u_0 z_2+\bar{\phi}_1(z)\\
&\vdots\\
\dot{z}_{n-1} &= u_0 z_n+\bar{\phi}_{n-1}(z)\\
\dot{z}_n &= u_1+\bar{\phi}_n(z),
\end{aligned} \tag{29}$$

where $z=(z_0,z_1,\ldots,z_n)\in R^{n+1}$, u_0,u_1 are control inputs, $\bar{\phi}_i(z)$, $i=0,1,\ldots,n$ denote the disturbances terms. As that does for stabilizing the second-order chained form systems in the former subsections, we change the set-point stabilization problem for (29) to an exponential stable trajectory-tracking problem. Define the target trajectories

$$z_i^d=\beta_i=\beta_{i0}\exp(-\lambda_i t), \quad i=0,1,\ldots,n \tag{30}$$

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and the error variables

$$\xi_i = z_i - z_i^d, \quad i = 0, 1, \ldots, n, \tag{31}$$

then the error system of (29) is given by

$$\begin{aligned}
\dot{\xi}_0 &= u_0 + \phi_0(\xi) + \lambda_0\beta_0 \\
\dot{\xi}_1 &= u_0\,(\xi_2 + \beta_2) + \phi_1(\xi) + \lambda_1\beta_1 \\
&\vdots \\
\dot{\xi}_{n-1} &= u_0\,(\xi_n + \beta_n) + \phi_{n-1}(\xi) + \lambda_{n-1}\beta_{n-1} \\
\dot{\xi}_n &= u_1 + \phi_n(\xi) + \lambda_n\beta_n.
\end{aligned} \tag{32}$$

A globally finite-time stabilizing controller for the disturbed error system (32) can be presented as following proposition 4.

Proposition 4: Suppose the given exponential functions $\beta_i = \beta_{i0}\exp(-\lambda_i t)$, $\lambda_i > 0$, $\beta_{i0} \neq 0$, for $i = 0, 1, \ldots, n$, satisfy the relationships $\lambda_i\beta_i = \lambda_0\beta_0\beta_{i+1}$, for $i = 1, \ldots, n-1$, namely

$$\beta_i\lambda_i = \beta_0\lambda_0\beta_{i+1}, \quad \lambda_i = \lambda_0 + \lambda_{i+1}. \tag{33}$$

And the disturbances terms $\phi_i(\xi)$, $i = 0, 1, \ldots, n$, satisfy

$$|\phi_i(\xi)| \leq \rho_i(\xi)\sum_{j=1}^{i}|\xi_i| < M_i, \tag{34}$$

where $0 < \rho_i(\xi) < \rho_{i0}$, both ρ_{i0} and $M_i > 0$ are constants. Then there exists a set of positive constants k_i, $i = 0, 1, \ldots, n$, and the controller

$$\begin{aligned}
u_0 &= -\left[M_0 + \lambda_0\beta_0\right] - k_0\xi_0^{(2n-1)/(2n+1)} \\
u_1 &= -\left[M_n + \lambda_n\beta_n\right] - k_n\left(\xi_n^{(2n+1)/3} - \alpha_{n-1}^{(2n+1)/3}\right)^{\frac{1}{2n+1}},
\end{aligned} \tag{35}$$

where

$$\alpha_i = -\xi_i\left(u_0\beta_{i+1} + M_i + \lambda_i\beta_i\right) - \frac{k_i}{u_0}\left(\xi_i^{1/r_i} - \alpha_{i-1}^{1/r_i}\right)^{\frac{4n}{2n+1}-(2-r_i)}$$

with $r_i = \frac{2n-2i+3}{2n+1}$, for $i = 2, \ldots, n-1$, and $\alpha_1 = -\xi_1(u_0\beta_2 + M_1 + \lambda_1\beta_1) - \frac{k_1}{u_0}\xi_1^{(2n-1)/(2n+1)}$, globally stabilize the error system (32) to origin in finite settling time.

Proof: This claim is an extension of proposition 3. By defining the Lyapunov function candidate □

$$V = 0.5\xi_0^2 + 0.5\xi_1^2 + \sum_{i=1}^{n-1} W_i, \tag{36}$$

where $W_i = \int_{\alpha_i}^{\xi_{i+1}}(s^{1/r_i} - \alpha_i^{1/r_i})^{2-r_i}ds$, $i = 2, \ldots, n-1$, then by a standard back-stepping procedure,[34] and using the inequalities (4)–(6) to estimate the bounds of the terms in $\dot{V}$ of (36) for the error system (32), proposition 4 can be proved without any difficulties. □

Remark 11: The assumption (34) is presented due to the bounded property of the state variables of an actual physical system.

In refs. [25], [29] and [31], controller design for uncertain chained form systems closely related to system (32) had also been investigated. As pointed out in Sections 1.2 and 3.1, the relevant

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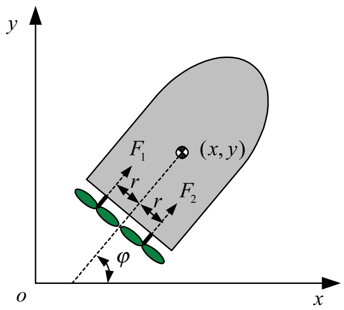

Fig. 1. An underactuated hovercraft model.

researches adopted the switch-based control strategy. Whereas, propositions 1 4 clearly show the considered disturbed chained form systems can be globally exponentially stabilized by non-switch-based controllers. The nonsmooth but Hölder continuous feedback controllers proposed in this work are also different from the methods presented in refs. [4] and [26]. We noted that some prior known sinusoid time-varying smooth feedback controllers are necessary in those works to synthesize the homogeneous feedback controllers for chained form systems. In refs. [4] and [9], homogeneous approximation techniques were also used to design Hölder continuous feedback controllers for more general nonholonomic systems without chained forms.

4. Application Example

4.1. The dynamics and the normal form of a simplified hovercraft model
A simple model of an underactuated hovercraft system is illustrated in Fig. 1. Let $o - xy$ denote the inertial coordinate system, $o' - uv$ is a coordinate system fixed to the mass center of the hovercraft. Suppose the hovercraft system has only two actuators at the rear of it, and the produced forces of the actuators are F_1 and F_2 respectively, the minimum distance between the mass center of the hovercraft and the lines along the driving forces is r, the direction of thrust force parallels to the symmetric axis of the hovercraft. We also suppose the mass of the system is m, the inertia of the system is I, the angle between the symmetric axis of the hovercraft and the coordinate axis x is φ, and the positive direction of φ is defined to be counter clockwise. Then the dynamics of the system can be expressed as

$$\begin{aligned} m\ddot{x} &= (F_1 + F_2)\cos\varphi \\ m\ddot{y} &= (F_1 + F_2)\sin\varphi \\ I\ddot{\varphi} &= (F_2 - F_1)r \end{aligned} \tag{37}$$

Obviously, the system (37) is an underactuated system since the three DOF systems are actuated by two inputs F_1, F_2. The generalized coordinates of the system can be given by (x, y, φ). To simplify the control design, a normal form transformation approach for the nonlinear system (37) is provided by the following proposition.

Proposition 5: The dynamics of the underactuated system (37) can be transformed into the canonical form (1) by the following coordinate transformation

$$y_1 = x, \quad y_2 = \tan\varphi, \quad y_3 = y, \tag{38}$$

and the input changes

$$\begin{aligned} u_1 &= \frac{1}{m}(F_1 + F_2)\cos\varphi \\ u_2 &= \frac{r}{I}(F_2 - F_1)\sec^2\varphi + 2\dot{\varphi}^2\tan\varphi\sec^2\varphi \end{aligned} \tag{39}$$

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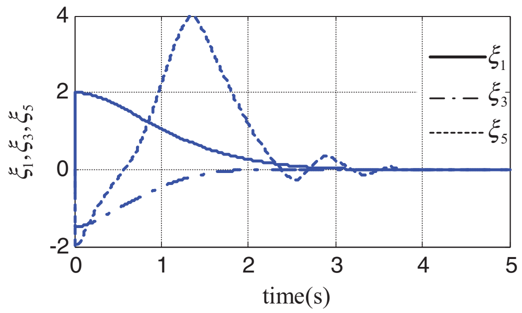

Fig. 2. Trajectories of the position error variables of the nominal system (10) in simulation 1.

Proof: Substitute (38) and (39) into (37), then the results follow. □

Remark 12: The inverse transformations of (38) and (39) are respectively given by

$$x = y_1, \quad \varphi = \text{atan}\,(y_2)\,, \quad y = y_3, \tag{40}$$

and

$$F_1 = \frac{1}{2}\left[(m\sec\varphi)\,u_1 - \frac{I\cos^2\varphi}{r}\left(u_2 - 2\dot{\varphi}^2\tan\varphi\sec^2\varphi\right)\right]$$
$$F_2 = \frac{1}{2}\left[(m\sec\varphi)\,u_1 + \frac{I\cos^2\varphi}{r}\left(u_2 - 2\dot{\varphi}^2\tan\varphi\sec^2\varphi\right)\right] \tag{41}$$

Then the actual configurations and the inputs of the underactuated hovercraft systems can be recovered from the transformed coordinates $y = (y_1, y_2, y_3)$ and the inputs (u_1, u_2).

It is worth noting that $\varphi = \pm k\pi + \frac{\pi}{2}$, $k \in \mathrm{N}$ are mathematically singular points for the system (38), but not physically singular points for the underactuated hovercraft system shown in Fig. 1. One can use any rotational coordinate transformations to avoid these mathematically singular points in practice. In this paper, these singular points do not occur due to the particular selections of the smooth and exponentially stable trajectories (9), which could not be zeros.

4.2. Numerical simulations

In this subsection, the physical parameters of the hovercraft system (37) are given by $m = 100\,\text{kg}$, $I = 100\ \text{kg}\cdot\text{m}^2$, and $r = 0.5$ m. Four kinds of numerical simulation results are presented as follows.

Simulation 1: Nominal system (10) with the first kind of conditions

Given an initial state $\xi(0) = [\,2\ 0\ {-1.5}\ 0\ {-2}\ 0\,]^{\mathrm{T}}$, design the desired trajectory $\beta_2(t) = -\exp(-1.25t)$, $\beta_3(t) = \exp(-t)$, and $\beta_1(t)$ is calculated to be $\beta_1(t) = -25\exp(-0.25t)$ by applying (11). If we select the controller parameters of the controller (12) to be $(k_1, \ldots, k_6) = (1, 2, 1, 2.5, 4, 30)$, and then the numerical simulation results are illustrated in Figs. 2–6. Figures 2 and 3 show the trajectories of the state variables of the system (10) controlled by (12). Figure 4 shows the trajectories of virtual inputs u_1 and u_2. Figure 5 shows the actual inputs F_1 and F_2, which are calculated from formulation (41). Figure 6 plots the movement path of the hovercraft with data from Fig. 2. It is well known that a nonholonomic constraint system cannot perform a motion violating the intrinsic nonholonomic constraints. For the first-order nonholonomic systems, such as a car, which cannot move along lateral direction. However, for the second-order nonholonomic constraints systems, such as the hovercraft system addressed in this paper, the nonholonomic constraints are accelerate level but not speed level, the motion of a hovercraft system is free in every movement directions due to air suspension, therefore the system can perform a motion in the lateral direction as that shown in Fig. 6.

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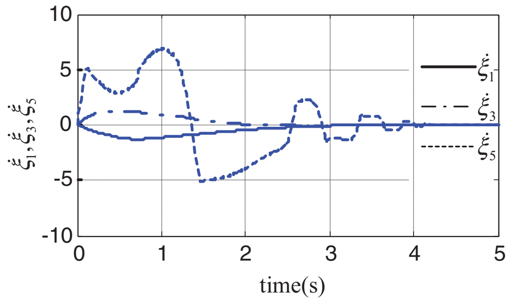

Fig. 3. Trajectories of the speed error variables of the nominal system (10) in simulation 1.

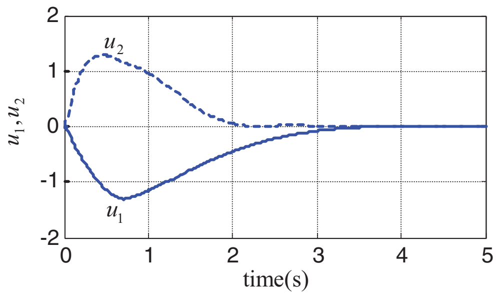

Fig. 4. Trajectories of the virtual inputs u_1 and u_2 of the nominal system (10) in simulation 1.

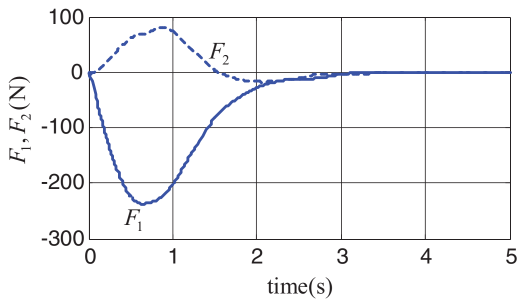

Fig. 5. Trajectories of the actual inputs F_1 and F_2 corresponding to the virtual inputs u_1 and u_2 shown in Fig. 4.

Simulation 2: Disturbed system (42) with same conditions as simulation 1

To test the robustness of the controller (12), the following disturbed system (42) is considered in this numerical simulation. All of the given parameters including the initial state, desired trajectories, and the controller parameters are same as that in simulation 1. The corresponding simulation results are plotted in Figs. 7 and 8.

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + 0.1\,(\xi_1 + \xi_2 + \xi_3)\sin(10t)\\ \dot{\xi}_2 &= u_1 - \lambda_1^2\beta_1 - 0.1(\xi_1 + \xi_2)\sin(5t)\end{aligned} \tag{42a}$$

$$\begin{aligned}\dot{\xi}_3 &= \xi_4 - 0.1\xi_3\sin(3t)\\ \dot{\xi}_4 &= u_1\,(\xi_5 + \beta_3) - \lambda_2^2\beta_2 - 0.2\,(\xi_3 + \xi_4)\sin(10t)\\ \dot{\xi}_5 &= \xi_6 - 0.5\,(\xi_3 + \xi_4 + \xi_5)\sin(5t)\\ \dot{\xi}_6 &= u_2 - \lambda_3^2\beta_3 - (\xi_3 + \xi_4 + \xi_5)\sin(8t)\end{aligned} \tag{42b}$$

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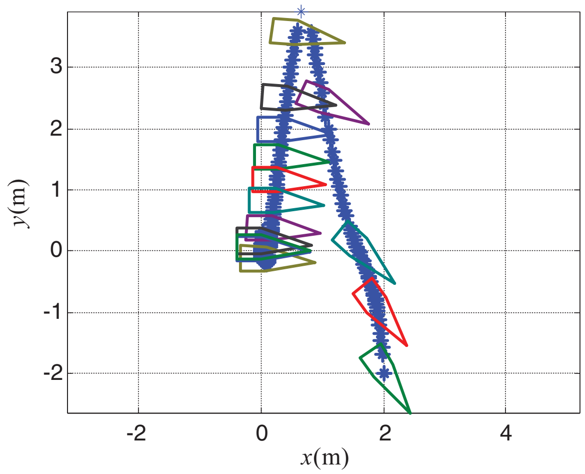

Fig. 6. Movement path of the Hovercraft with data from Fig. 2.

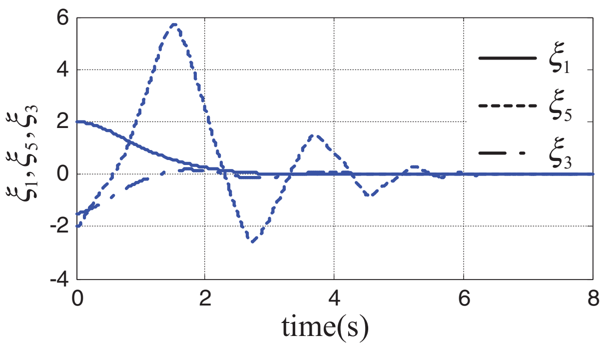

Fig. 7. Trajectories of the position error variables of system (42) in simulation 2.

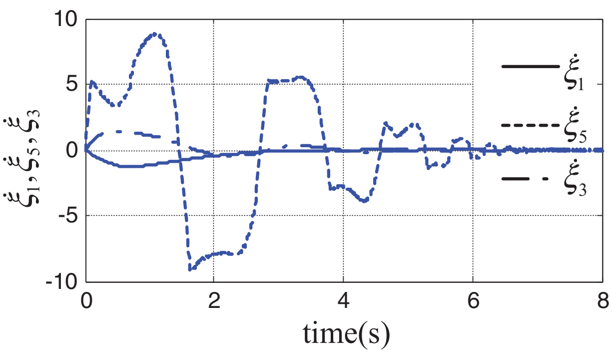

Fig. 8. Trajectories of the speed error variables of system (42) in simulation 2.

Simulation 3: Nominal system (10) with the second kind of conditions

Another initial state is selected to be $\xi = [3\ 0\ 2\ 0\ 0\ 0]^{\mathrm{T}}$, and design the desired trajectory $\beta_2(t) = \exp(-1.25t)$ and $\beta_3(t) = \exp(-t)$, then $\beta_1(t)$ is calculated to be $\beta_1(t) = 25\exp(-0.25t)$ through equality (11). Select the control parameters of (12) to be $(k_1, \ldots, k_6) = (1, 2, 1, 3, 4, 30)$, then the corresponding numerical simulation results are illustrated in Figs. 9–13.

Simulation 4: Disturbed system (42) with the same conditions as simulation 3

This simulation adopts the same disturbed system (42) as in simulation 2 while the control target and control parameters are the same as those in simulation 3. Then the numerical simulation results are plotted in Figs. 14 and 15.

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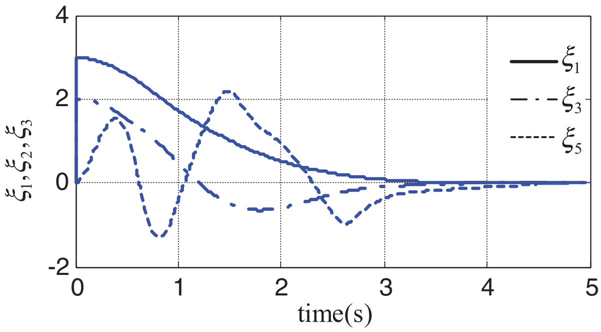

Fig. 9. Trajectories of the position error variables of system (10) in simulation 3.

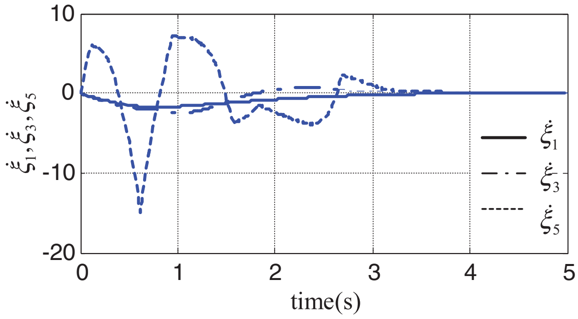

Fig. 10. Trajectories of the speed error variables of system (10) in simulation 3.

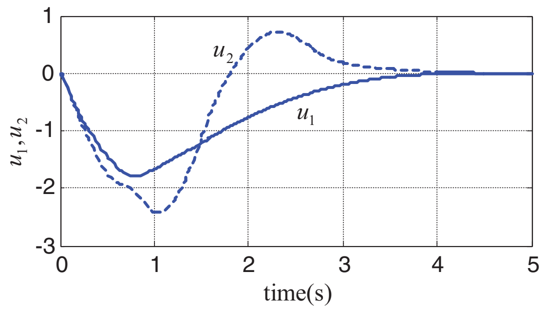

Fig. 11. Trajectories of the virtual inputs u_1 and u_2 of the nominal system (10) in simulation 1.

Comparisons of simulations 1and 2 show that the disturbances extend the settling time and cause more obviously oscillating features in the systems states. Whereas, this conclusion is not always valid due to the comparisons of simulations 3 and 4. In the later case, the disturbances show less influences on the system states.

Since the desired exponentially stable trajectories of the state variables of (8) can be arbitrarily set, the settling-time of the closed-loop system controlled by (12) is far less than that by a controller based on the time-varying smooth feedback. This point had been clearly shown by M'Closkey *et al.* in their works[4,26] since the Hölder continuous feedback was applied there as the work does here. However, a prior known time-varying smooth controller is needed in M'Closkey's approach, which is not required in our approach.

The approach presented in this paper is also different from the switch-based Hölder continuous feedback controller or the controller based on the discontinuous feedback, such as the relevant works of.[25,29,31,35] Even though the state variables during the regulation procedure are nonsmooth, the simulation results illustrated in Figs. 2–15 show the regulation trajectories of the state variables are continuous, thus the impact effects caused by switching the control inputs are considerably reduced, and the actuation forces of the system is smooth (see Figs. 5 and 12).

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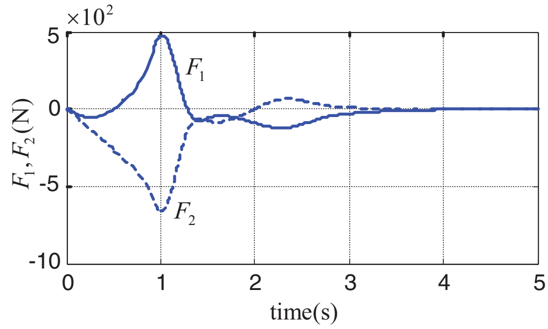

Fig. 12. Trajectories of the actual inputs F_1 and F_2 corresponding to the virtual inputs u_1 and u_2 shown in Fig. 11.

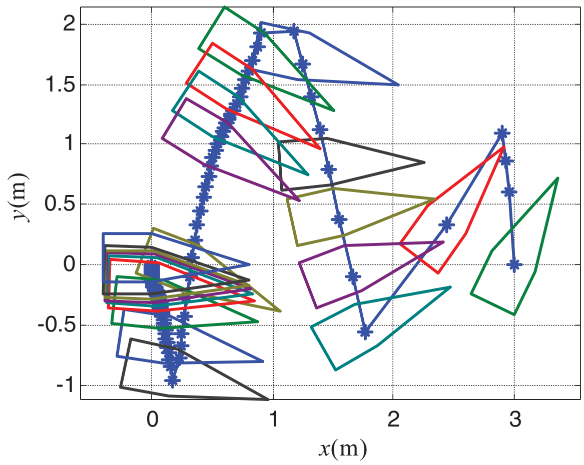

Fig. 13. Movement path of the Hovercraft with data from Fig. 7.

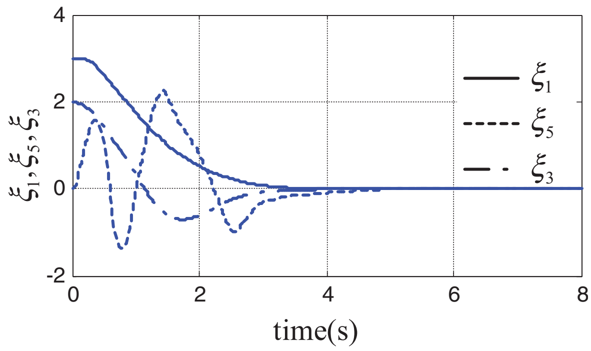

Fig. 14. Trajectories of the position error variables of system (42) in simulation 4.

5. Conclusion

The second-order nonholonomic systems with chained form are a class of typical underactuated nonholonomic systems. To improve the performance of the existing controllers for this class system, in this paper a time-varying controller design method is presented on the basis of nonsmooth but Hölder continuous feedback techniques. By changing an origin stabilization problem to a time-varying trajectory tracking problem, it is shown the second-order nonholonomic chained form system can be stabilized to a given exponentially stable trajectory in finite time, such that the settling-time of the closed-loop system can be arbitrarily setting. Therefore, the presented controller shows a less settling time and avoids the discontinuous inputs that are impractical for some actual systems in engineering.

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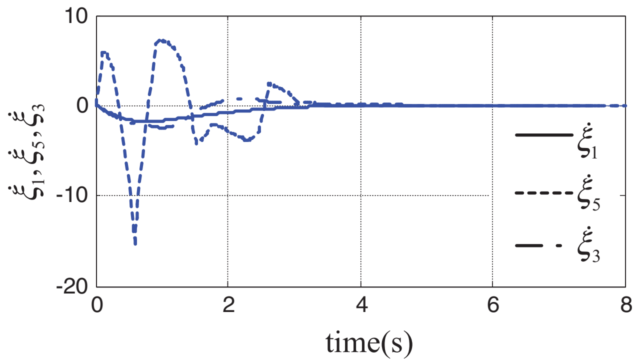

Fig. 15. Trajectories of the speed error variables of system (42) in simulation 4.

It is also worth pointing out that all of the nonsmooth feedback potentially cause the oscillating behaviors of the systems states. This may be undesirable for some applications. Whereas, for highly nonlinear systems without controllable linearization, the nonsmooth feedback might be the unique feasible approach for exponentially stabilizing the systems in large region.

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Appendix A

Proof of the proposition 1

The proof is partitioned to two sections. In the first section, the stability of the subsystem (10a) is proved by a standard back-stepping procedure,[34] in order to clearly show the approach of estimating the bounds of the time derivate of the Lyapunov function candidate by the Lemmas 1 and 2. Then the stability of the overall system (10) is proved in the second section with considering the result of the Section 1.

Section 1:

In the first, let's design a finite-time stabilizing controller for the subsystem (10a). Consider the subsystem ξ_1, and select a Lyapunov function candidate $V_1 = 0.5\xi_1^2$, then the time derivate of V_1 is given by $\dot{V}_1 = \xi_1(\xi_2 - \alpha_1) + \xi_1\alpha_1$. If select the virtual input to be $\alpha_1 = -k_1\xi_1^{7/9}$ with $k_1 > 0$, then

$$\dot{V}_1 = -k_1\xi_1^{16/9} + \xi_1\left(\xi_2 - \alpha_1\right). \tag{A.1}$$

Secondly, consider the subsystem (ξ_1, ξ_2), and select a new Lyapunov function candidate $V_2 = V_1 + W_1$, where $W_1 = \int_{\alpha_1}^{\xi_2}(s_1^{9/7} - \alpha_1^{9/7})^{11/9}ds_1$. Guaranteed by Lemma 3, thus $V_2 = V_1 + W_1 \geq 0$ is

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satisfied. The time derivate of the function V_2 can be written as

$$\dot{V}_2 = \dot{V}_1 + \dot{W}_1 = -k_1\xi_1^{16/9} + \xi_1\left(\xi_2 - \alpha_1\right) + \frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2 + \frac{\partial W_1}{\partial \xi_2}\left(u_1 - \lambda_1^2\beta_1\right). \tag{A.2}$$

Then we will estimate the bounds of the last three terms of the right-hand side of (A2). Using the inequality (4), it follows that

$$\left|\xi_2 - \alpha_1\right| = \left|\left(\xi_2^{9/7}\right)^{7/9} - \left(\alpha_1^{9/7}\right)^{7/9}\right| \le 2\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{7/9}. \tag{A.3}$$

By the inequality (6), then the second term of right-hand side of (A2) satisfies inequality

$$\begin{aligned}\left|\xi_1\left(\xi_2 - \alpha_1\right)\right| &\le 2\left|\xi_1\right|\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{7/9}\\ &\le 2\left[\frac{1}{1+7/9}\left|\xi_1\right|^{\left(1+\frac{7}{9}\right)} + \frac{7/9}{1+7/9}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{\left(1+\frac{7}{9}\right)}\right].\end{aligned}$$

Thus, the last inequality can be simply expressed as

$$\left|\xi_1\left(\xi_2 - \alpha_1\right)\right| \le \delta_1\xi_1^{16/9} + \delta_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{16/9}, \tag{A.4}$$

where δ_1 and δ_2 are two positive constants.

Using the formula (15) of remark 7, it is not difficult to show that

$$\frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1} = \frac{11}{9}k_1^{9/7}\int_{\alpha_1}^{\xi_2}\left(s_1^{9/7} - \alpha_1^{9/7}\right)^{2/9} ds_1.$$

Additionally, it is easy to show the following fact

$$\begin{aligned}\left|\int_{\alpha_1}^{\xi_2}\left(s_1^{9/7} - \alpha_1^{9/7}\right)^{2/9} ds_1\right| &\le \left|\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{2/9}\right|\left|\xi_2 - \alpha_1\right|\\ &\le^{(A(3))} 2\left|\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{2/9}\right|\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{7/9}\\ &= 2\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|,\end{aligned}$$

thus, the third term of right-hand side of (A2) satisfies

$$\begin{aligned}\left|\frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2\right| &\le \frac{22}{9}k_1^{9/7}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|\left|\xi_2\right|\\ &\le \frac{22}{9}k_1^{9/7}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|\left[\left|\xi_2 - \alpha_1\right| + \left|\alpha_1\right|\right]\\ &= \frac{22}{9}k_1^{9/7}\left[\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|\left|\xi_2 - \alpha_1\right| + \left|\xi_2^{9/7} - \alpha_1^{9/7}\right|\left|\alpha_1\right|\right]\\ &\le^{(A(3))} \frac{44}{9}k_1^{9/7}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{(1+7/9)}\\ &+ \frac{22}{9}k_1^{9/7}\left|\xi_2^{9/7} - \alpha_1^{9/7}\right|\left|k_1\xi_1^{7/9}\right|.\end{aligned}$$

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Proceeding along a similar line as that for getting the inequality (A4), it is not difficult to show that

$$\left|\frac{\partial W_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \xi_1}\xi_2\right| \le \delta_3 \xi_1^{16/9} + \delta_4 \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{16/9}, \tag{A.5}$$

where δ_3 and δ_4 are two positive constants.

As to the last term of right-hand side of (A2), using formula (13), it easy to show that

$$\frac{\partial W_1}{\partial \xi_2} = \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{11/9}. \tag{A.6}$$

Substitute (A4)–(A6) into (A2), then (A2) follows that

$$\begin{aligned}\dot{V}_2 \le{}& (\delta_1 + \delta_3 - k_1)\,\xi_1^{16/9} + (\delta_2 + \delta_4)\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{16/9}\\ &+ \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{11/9}\left(u_1 - \lambda_1^2 \beta_1\right).\end{aligned} \tag{A.7}$$

Select the control input to be

$$u_1 = \lambda_1^2 \beta_1 - k_2 \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}, \tag{A.8}$$

where $k_2 = \tilde{k}_2 + \delta_2 + \delta_4 > 0$, $\tilde{k}_2 > 0$, then using the inequality (3), the inequality (A7) satisfies that

$$\begin{aligned}\dot{V}_2 &\le -\tilde{k}_1 \xi_1^{16/9} - \tilde{k}_2 \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{16/9}\\ &\le -\tilde{k}_2 \left[\xi_1^2 + \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^2\right]^{8/9},\end{aligned} \tag{A.9}$$

where $\tilde{k}_1 = k_1 - (\delta_1 + \delta_3) > 0$. On the other hand, it is easy to show the function V_2 satisfies

$$\begin{aligned}V_2 &= 0.5\xi_1^2 + \int_{\alpha_1}^{\xi_2} \left(s_1^{9/7} - \alpha_1^{9/7}\right)^{11/9} ds_1\\ &\le 2\xi_1^2 + \left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{11/9} |\xi_2 - \alpha_1|.\end{aligned}$$

By the inequality (A3), it follows that

$$V_2 \le 2\xi_1^2 + 2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^2. \tag{A.10}$$

From (A9) and (A10), we have

$$\dot{V}_2 \le -\frac{\tilde{k}_2}{2} V_2^{8/9}. \tag{A.11}$$

Thus, according to Lemma 4, for any given exponentially stable trajectories $\beta_1(t)$ there exist positive constants (k_1, k_2), such that the subsystems (10a) can be stabilized to origin $(\xi_1, \xi_2) = (0, 0)$ in finite settling time. This also indicates the input u_1 is bounded. Let's define the upper bound of u_1 as u_{10}, i.e. $|u_1| \le u_{10}$, where $u_{10} > 0$ is a constant. Then as that will be shown in the next, the stability of the closed-loop subsystem (10b) can be proved without any difficulties.

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Section 2:
For the subsystem (10b), define the Lyapunov function candidate

$$V_3 = 0.5\xi_3^2 + W_2 + W_3 + W_4, \tag{A.12}$$

where

$$W_2 = \int_{\alpha_2}^{\xi_4} \left(s_2^{9/7} - \alpha_2^{9/7}\right)^{11/9} ds_2,$$
$$W_3 = \int_{\alpha_3}^{\xi_5} \left(s_3^{9/5} - \alpha_3^{9/5}\right)^{13/9} ds_3,$$

and

$$W_4 = \int_{\alpha_4}^{\xi_6} \left(s_4^3 - \alpha_4^3\right)^{5/3} ds_4.$$

Since u_1 is stable, the stability of u_1 does not depend on the variables ξ_3, ξ_4 and ξ_5, then u_1, ξ_1 and ξ_2 are bounded and could not be zero. For the subsystem (10b), the control input u_1 can be regarded as a variable only depending on time t. Then the time derivate of the function V_3 along the vector field (10b), can be given by

$$\begin{aligned}\dot{V}_3 = \xi_3\xi_4 &+ \frac{\partial W_2}{\partial \xi_{\alpha_2}}\frac{\partial \alpha_2}{\partial \xi_3}\xi_4 + \frac{\partial W_2}{\partial \xi_4}\dot{\xi}_4 \\ &+ \frac{\partial W_3}{\partial \alpha_3}\left[\frac{\partial \alpha_3}{\partial \xi_3}\xi_4 + \frac{\partial \alpha_3}{\partial \xi_4}\dot{\xi}_4 + \frac{\partial \alpha_3}{\partial t}\right] + \frac{\partial W_3}{\partial \xi_5}\xi_6 \\ &+ \frac{\partial W_4}{\partial \alpha_4}\left[\frac{\partial \alpha_4}{\partial \xi_3}\xi_4 + \frac{\partial \alpha_4}{\partial \xi_4}\dot{\xi}_4 + \frac{\partial \alpha_4}{\partial \xi_5}\xi_6 + \frac{\partial \alpha_4}{\partial t}\right] \\ &+ \frac{\partial W_4}{\partial \xi_6}\left(u_2 - \lambda_3^2\beta_3\right).\end{aligned} \tag{A.13}$$

Except the terms about $\dot{\xi}_4$, $\frac{\partial \alpha_3}{\partial t}$ and $\frac{\partial \alpha_4}{\partial t}$ in (A13), the other terms can be estimated by proceeding along similar lines as in Section 1. Thus the terms about $\dot{\xi}_4$, $\frac{\partial \alpha_3}{\partial t}$ and $\frac{\partial \alpha_4}{\partial t}$ in the right-hand side of (A13) just need to be estimated in the following.
Referring Eq. (10b), the time derivate $\dot{\xi}_4$ is given by

$$\dot{\xi}_4 = u_1(\xi_5 + \beta_3) - \lambda_2^2\beta_2. \tag{A.14}$$

Substitute (A8) into (A14), and then it follows that

$$\begin{aligned}\dot{\xi}_4 = \lambda_1^2\beta_1\xi_5 &- k_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}(\xi_5 + \beta_3) \\ &+ \lambda_1^2\beta_1\beta_3 - \lambda_2^2\beta_2.\end{aligned} \tag{A.15}$$

Using the relationship (11), (A15) is simplified as

$$\dot{\xi}_4 = \lambda_1^2\beta_1\xi_5 - k_2\left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}(\xi_5 + \beta_3). \tag{A.16}$$

Since the subsystem (10a) is stabilized in finite settling time by the controller $u_1(t)$ of (A8), then the bounds of the term $(\xi_2^{9/7} - \alpha_1^{9/7})^{5/9}$ in (A16) can be estimated by $|(\xi_2^{9/7} - \alpha_1^{9/7})^{5/9}| \le \varepsilon_0$, where

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$\varepsilon_0 > 0$ is a constant. Thus the bounds of (A16) can be estimated by the following inequality

$$\left|\dot{\xi}_4\right| \le \lambda_1^2 \left|\beta_1 \xi_5\right| + k_2 \varepsilon_0 \left|\xi_5\right| + k_2 \left|\beta_3 \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}\right|. \tag{A.17}$$

With the help of inequality (A17), for instance, the third term in the right-hand side of (A13) satisfies the following inequality

$$\begin{aligned} \left|\frac{\partial W_2}{\partial \xi_4}\dot{\xi}_4\right| &= \left|\xi_4^{9/7} - \alpha_2^{9/7}\right|^{11/9} \times \left(\lambda_1^2 \left|\beta_1 \xi_5\right| + k_2 \varepsilon_0 \left|\xi_5\right| + k_2 \left|\beta_3 \left(\xi_2^{9/7} - \alpha_1^{9/7}\right)^{5/9}\right|\right) \\ &\le \left(\lambda_1^2 \left|\beta_{10}\right| + k_2 \varepsilon_0\right) \left|\xi_4^{9/7} - \alpha_2^{9/7}\right|^{11/9} \left|\xi_5\right| \\ &\quad + \left|\beta_{30}\right| \left|\xi_4^{9/7} - \alpha_2^{9/7}\right|^{11/9} \left|\xi_2^{9/7} - \alpha_1^{9/7}\right|^{5/9}. \end{aligned} \tag{A.18}$$

Since the coefficients $\lambda_1^2|\beta_{10}| + k_2\varepsilon_0$ and $|\beta_{30}|$ of the two terms in right-hand side of (A18) are positive constants, it is straightforward, both of the two terms of (A18) can be estimated by inequality (6) in Lemma 2, such that the inequality (A18) can be written as a final form

$$\begin{aligned} \left|\frac{\partial W_2}{\partial \xi_4}\dot{\xi}_4\right| &\le \delta_5 \xi_3^{16/9} + \delta_6 \left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{16/9} \\ &\quad + \delta_7 \left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^{16/9}. \end{aligned} \tag{A.19}$$

where all of the coefficients δ_i, $i = 5, 6, 7$ are positive constants. Proceeding along similar lines as shown above, the bounds of other terms about $\dot{\xi}_4$ in (A13) can be estimated without any difficulties.

For the term $\frac{\partial W_3}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial t}$ in (A13), it can be shown

$$\frac{\partial W_3}{\partial \alpha_3} = -\frac{13}{5}\alpha_3^{4/5} \int_{\alpha_3}^{\xi_5} \left(s_3^{9/5} - \alpha_3^{9/5}\right)^{4/9} ds_3, \tag{A.20}$$

and

$$\frac{\partial \alpha_3}{\partial t} = -\frac{\lambda_1^3 \beta_1}{u_1^2} k_4 \left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{5/9}. \tag{A.21}$$

In Section 1, it is shown that u_1 is stable in finite time. For a given settling time T of ξ_1 and ξ_2, there exists a constant $\varepsilon > 0$ such that $0 < |u_1| \le \varepsilon$ is satisfied. Using the Lemmas 1 and 2, it is not difficult to show that there always exist sufficient large constants $\delta_8 > 0$ and $\delta_9 > 0$ such that the following inequality is satisfied

$$\left|\frac{\partial W_3}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial t}\right| \le \delta_8 \left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{16/9} + \delta_9 \left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^{16/9}. \tag{A.22}$$

By a similar procedure as above, the following inequality can be obtained

$$\left|\frac{\partial W_4}{\partial \alpha_4}\frac{\partial \alpha_4}{\partial t}\right| \le \delta_{10} \left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{16/9} + \delta_{11} \left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^{16/9} + \delta_{12} \left(\xi_6^3 - \alpha_4^3\right)^{16/9}, \tag{A.23}$$

where δ_{10}, δ_{11} and δ_{12} are positive constants.

By a similar procedure as presented in Section 1, and using the controllers given by (11), it can be shown that there always exists a sufficient large constant $\bar{k} > 0$, such that (A13) satisfies the following

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inequality

$$\dot{V}_3 \leq -\bar{k}\xi_3^{16/9} - \bar{k}\left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^{16/9} - \bar{k}\left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^{16/9} - \bar{k}\left(\xi_6^3 - \alpha_4^3\right)^{16/9}. \quad \text{(A.24)}$$

On the other hand, the function V_3 satisfies

$$V_3 \leq 2\xi_3^2 + 2\left(\xi_4^{9/7} - \alpha_2^{9/7}\right)^2 + 2\left(\xi_5^{9/5} - \alpha_3^{9/5}\right)^2 + 2\left(\xi_6^3 - \alpha_4^3\right)^2. \quad \text{(A.25)}$$

From (A24) and (A25), and by applying the inequality (3) in Lemma 1, it easy to show that

$$\dot{V}_3 \leq -\frac{\bar{k}}{2}V_3^{8/9}. \quad \text{(A.26)}$$

Due to the inequalities (A11) and (A26), if we define $V = V_2 + V_3$ to be the Lyapunov function candidate for overall system (10a)–(10b), then there always exists a positive constant $k \geq \max\{\tilde{k}, \bar{k}\}$ such that

$$\dot{V} = \dot{V}_2 + \dot{V}_3 \leq -\frac{k}{2}V^{8/9}. \quad \text{(A.27)}$$

From the Lemma 4, the settling time of (10) is given by

$$T(\boldsymbol{\xi}_0) \leq \frac{18V(\boldsymbol{\xi}_0)^{1/9}}{k}. \quad \text{(A.28)}$$

where $\boldsymbol{\xi}_0 = (\xi_{10}, \ldots, \xi_{60})^{\mathrm{T}}$ is the initial state of system (10). The global finite-time stability of the claim is ensured by $\lim_{\|\boldsymbol{\xi}_0\| \to \infty} V = \infty$. This completes the proof of proposition 1.

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