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Mathematical Entities without Objects. On Realism in Mathematics and a Possible Mathematization of (Non) Platonism: Does Platonism Dissolve in Mathematics?

THIERRY PAUL

Centre de Mathématiques Laurent Schwartz, CNRS, Ecole polytechnique, 91128, Palaiseau Cedex, France. Email: paul@ljll.math.upmc.fr

By looking at three significant examples in analysis, geometry and dynamical systems, I propose the possibility of having two levels of realism in mathematics: the upper one, the one of entities; and a subordinated ground one, the one of objects. The upper level (entities) is more the one of 'operations', of mathematics in action, of the dynamics of mathematics, whereas the ground floor (objects) is more dedicated to culturally well-defined objects inherited from our perception of the physical or real world. I will show that the upper level is wider than the ground level, therefore foregrounding the possibility of having in mathematics entities without underlying objects. In the three examples treated in this article, this splitting of levels of reality is created directly by the willingness to preserve different symmetries, which take the form of identities or equivalences. Finally, it is proposed that mathematical Platonism is – in fine – a true branch of mathematics in order for mathematicians to avoid the temptation of falling into the Platonist alternative 'everything is real'/'nothing is real'.

Prelude

Repetition, Seriality, Temporality and Reality in Mathematics

It took a long time for mathematicians to realize that symmetry is temporal: something is symmetric because one can act on it repetitively without disturbing it. Musicians were much faster is inventing symmetry, reverse symmetry, dilation, i.e. group theory, just because time is intrinsic to music. So symmetry has to do with temporality; it conforms perfectly to the triad of *temporality*, *repetition* and *seriality*:

being the series of an infinite number of repetitions, which determines whether something is truly symmetric. But temporality in mathematics questions the notion of intrinsic realism: what is real in mathematics? And this refers immediately to Platonism, as I shall develop below.

Symmetry is often thought as identity in mathematics, or more precisely as conservation of identity (two parts of a symmetric object are equal). But this conservation can be seen with respect to very different forms of temporality: exchanges of paradigms, equivalence between objects of different types, symmetry between an equation and its solutions, etc. In other words symmetry in intrinsic mathematics calls necessarily for a precise notion of realism in mathematics.

Repetition, seriality and temporality are three concepts to be considered carefully when looking at how they operate in the field of mathematics. To be more precise, they act, at first glance, more naturally in a field, a space tied to mathematics. This field surrounds them, mostly inspires them, but does not boil down to them.

Repetition refers to the confrontation between identity and non-identity: to repeat means that we repeat the same, the identical. Any change breaks repetition. Seriality questions the possibility of decomposition of sequences, i.e. a succession of, say, objects in their full diversity into identical ones: decomposition of diversity into partial identity. Temporality addresses the phenomenon of successive actions parametrized by a 'time' belonging to an ordered set. The vocabulary used in this description of the three concepts is totally absent from mathematics. It rather belongs, among others and to restrict the view to academic directions, to the natural sciences or the philosophy of them. In other words, it belongs to real situations. In mathematics, there is no intrinsic time, i.e. an ordered set of parameters indexing mathematical objects; the question of identity (together with the strongly related concepts of repetition and seriality) is supposed to have been defined and settled once and for all in the mathematical concept of definition. From that naïve perspective, too, the three concepts of Repetition–Seriality–Temporality do not seem to belong to the core of mathematics. But they do in fact belong to mathematics, and in a fundamental way, to some space of upstream and downstream mathematics. Downstream is their domain of applications, such as physics, chemistry, biology, economy – the list can be long. In short, we are talking of real situations. The field upstream refers to the famous concept of Platonism in mathematics. Very broadly speaking, if one took a Platonistic point of view, the mathematical results would be located somewhere where the mathematician would catch them during his process of research. And it would then be in this area, this 'somewhere' *upstream* from mathematics, that Repetition–Seriality–Temporality in mathematics would be incarnated. That is, upstream mathematics would be a kind of a closet where real objects are placed and ordered by identity or chronology.

In conclusion, one can say, less naively than before, that the mathematical pertinence of Repetition–Seriality–Temporality is definitively tied to the general pertinence of realism in mathematics.

1. Realism and (Different Levels in) Mathematics

When conceiving of the concept of realism in mathematics, one faces immediately the necessity of considering the two actors involved, namely mathematics itself and the mathematician. By mathematics itself, I mean the whole process of thinking, elaborating, proving, stating mathematical results produced by the mathematician at (mathematical) work (and not only the results themselves), and, by the mathematician, the mathematician in his action of looking at realism in mathematics.

What we are talking about by positing the existence of mathematics (itself) independently from the mathematician consists of considering by 'mathematics' the agent (including the mathematician at (mathematical) work for example) responsible for the thinking of the doing of mathematics; and by 'the mathematician' – the thinking of looking at mathematics. These two aspects, although they both concern the mathematician as a human person, are very distinct, especially when realism and Platonism are involved. They are even sometimes opposites, as we shall see later. There are mathematicians who are Platonists in their perspective on mathematics, although their own mathematics develop a non-Platonist view in a sense of an 'internal Platonism' – as we propose to discuss in this text.

Mathematicians and realism: this conjunction appears a bit strange, a tentative conciliation, since mathematicians have the reputation of being external to any trace of reality. However, we will concentrate on realism *inside* mathematics. The question we would like to address is whether there are entities without objects in mathematics in the sense that we shall explain.

It is usually understood (though this may be wrong, in our opinion) that mathematics (once again by this I mean the result of considerations and constructions of the mathematician 'at work') consist of two separate parts: objects (so to speak) and operations, i.e. transformations or a certain 'cooking' with the objects; the important part conceptually being the objects (numbers, manifolds, algebraic structures, etc). There is more: the dynamical operations leading to a mathematical statement have the tendency to disappear at the moment when the statement is definitively settled. Of course definitive, static statements often have a certain aesthetic in mathematics, but what we would like to claim in this article, and prove on the basis of three examples taken from contemporary mathematics, is that this distinction between objects and dynamical operations performed on them vanishes very often. In addition to this, in mathematics, objects *merge* with operations made on them (for example even numbers are numbers that can be divided by two) and these operations are entities themselves (because in mathematics everything is precisely defined so that operations are written in the same language as objects). It is therefore natural, in mathematics, to try to better understand, to progress, by working directly at the level of these entities that are operations on objects, i.e. to perform 'operations on these operations' so to speak. In doing so, however, one sometimes reaches a point where there is no underlying object any more. In other words, new operations without a clear idea of what they are acting on. This might seem paradoxical but we will try to convince the reader that such situations indeed exist in mathematics.

Of course, the history of mathematics is long and there are plenty of instances where the new object corresponding to these new operations was just hidden for a while and eventually appeared in full light.[1] Nevertheless, not being an historian, we would like to concentrate on examples where this happens either intentionally or because of a lack of strength of the technical tools the mathematician has currently at their disposal. In the two cases, there is a mathematician at work who is aware of what is going on. This 'intentionality' will play a key role in the notion of realism we would like to exhibit in mathematics.

And indeed, and this will be a guide for our formulation of a possible 'realism without entities'. The mathematics of the twentieth century, far beyond the aforementioned Platonism, seriously took under consideration this systematic exploration (of operations on objects) through the corpus of what is nowadays called quantum mathematics. But not only that. We will give two other examples from the theory of differential equations.

Our title, 'Mathematical entities without objects', refers to the fact that realism in mathematics can be seen as a feature, a product of the action of doing mathematics. Could it be something else (Benoist 2011)? Mathematics, by the fact of doing something, is real, it is in some reality. But traditionally this is not what is considered as realism, especially from a Platonist point of view: in mathematics one should distinguish between a 'doing', an action, and a 'done', a static result (of this action). And then after this distinction, which is debatable according to us, comes the consideration of the status of these 'done'. Do they exist? Are they real? Or 'just' a product of thinking? A weak solution of a partial differential equation (PDE), as we will see in Section 2.3, is not an object given in its task of being a solution of a PDE. In fact, it solves an equation in the sense that after the action of averaging against test functions (a lot of them and only after this action), the equation is solved. So much work to be done compared with the simple fact of looking at the solution itself, one might say. Well, try to do better – if you can, you win one million dollars.[2]

The philosophical, epistemological scheme of what we just described and which we will further develop in the next section is always the same: a 'classical', (culturally) very well defined, mathematical object (such as an equation, a space, a partition of a space) happens to be *isomorphic to a mathematical entity* of a very different nature (operations, actions of these objects). The latter's slight generalization (for example the suppression of one of its axioms) not only breaks the given isomorphism, but also ruins any attempt to make this modified entity isomorphic to any another 'classical' object.

Think of a second degree equation posed on real numbers. We all know that sometimes a root can be complex, that is, it does not belong to the original space

1. An example is the case of group theory, born out of the study of the transformation of objects and which actually is not restricted to the sciences (think for example of fugues in seventeenth- and eighteenth-century theory and the practice of inversion and dilation of musical themes), before it takes off as a theory of groups per se, that is, transformations of ... nothing.
2. You will win one of the so-called 'Millennium Prizes' for showing the existence and uniqueness for all times of the solutions of the equations of the hydrodynamics. An offer still valid at the time where this article was completed.

where the equation is settled. We know how to solve the problem: we define the 'number' i. Nowadays we can define the complex plane without having in mind the original second-degree equation. But this was not the case when the number i, satisfying $i^2 = -1$, was invented, not being part of any set of numbers, merely a notation. And which equation could be more symmetric than this one, where the unknown is just multiplied by itself? What we learn by this example (and will learn on the coming ones) is that the need of symmetry creates a new paradigm (the complex numbers), a new reality.

The three examples of the next section will be comparable to the creation of $\sqrt{-1}$ outside the framework of complex numbers.

2. Three Key Examples

2.1. Dynamical Systems

Our first example deals with a quite new subject of mathematics: flows with low regularity properties.

The idea that the dynamical movement of rigid objects in our physical space should result from solving differential equations is the great revolutionary discovery of Newton. Let us try and put this in a nutshell as follows. Suppose our rigid body is ideally reduced to a point, a point like the one you get by posing the extremity of your pen on a sheet of paper. If you now draw a curve on the paper, you just draw the successive positions of your ideal rigid body. *Successive* refers to time and *positions* refers to space: at each moment the body occupies *one* point, and when times evolves it follows a trajectory consisting of the different, successive, points of the drawn curve. Exactly in the way you ask the internet the route from one town to another: the answer is a curve on a map.

At each point of the curve you have drawn, you can draw a straight line tangent to the original curve at the point you selected (the tangent to a curve at a chosen point is obtained by taking the straight line passing through two points near the point you chose and letting both points move towards the chosen point). This set of tangents reflects the dynamical process which produced the curve: when you draw in a row a curve, as you do spontaneously with a pencil of a sheet of paper, the curve you obtain is very nice, regular, smooth. On the contrary, if you stop your drawing because your phone rings or somebody touches your hand, the tangent will jump discontinuously and the curve will present at this point an angle, a singularity. The same is true for the online route finder: if you influence the dynamics of the process by asking, for example, not to take highways, or to avoid centres of towns, or to get there the cheapest way, you will find abrupt changes of directions in your route.

The fundamental problem of the theory of dynamical systems, one of the most productive fields in mathematics of the last century, consists of going the inverse way: instead of first drawing the curve and then tracing the tangents at each point, let us suppose that the dynamics is given first, that is, let us be given a straight line at each point of your sheet of paper. Can we start at any point we wish to pose our pen and draw a curve, the tangent at any point of it being the straight line which was given initially? And when this curve exists, another important question arises: is it unique or can we draw two curves having the same distributions of tangents? 'Existence', 'uniqueness', we are entering slowly the vocabulary of mathematics.

The intuition is that these two questions (existence, uniqueness) should have both a positive answer when the distribution of tangents is *continuous*: by this one means that, moving a little bit the point on the sheet, the associated tangent should change its direction just a little bit. The reader can have such an intuition by trying to draw a curve by following a distribution of tangents.

But this intuition is wrong.

It was proven by the end of the nineteenth century that the distribution of tangents must have a stronger property in order to allow the construction of a unique, nice curve. Without defining it for the moment, let us name the extra-property that the distribution of tangent must have the 'Lipschitz continuity'. A natural question arises immediately: what happens when this Lipschitz continuity condition is *not* satisfied by the distribution of tangents?

It took more than one century to have an intimation regarding the answer to this question (DiPerna and Lions 1989; Bouchut 2001; Ambrosio 2004) and we will see that the answer necessitates that one destroy the underlying space. In order to understand the philosophical ideas behind this a priori negative phenomenon, we need to rephrase the preceding discussion in more mathematical language. But the non-mathematician reader should not be afraid of the mathematical 'vocabulary' used below, but should just concentrate on the (changes of) morphology of the syntax, almost at a graphical level, and try to adopt low-level thinking 'à la Teissier' (2005).

In mathematics (and in physics), a flow on a given space is defined, once again similar to a roadmap provided by an online route finder, by a curve (i.e. the route on a map) and a way of assigning to any value t the time a point $X(t)$ in this curve (e.g. 0 hours: departure Paris, 2 hours: Tours, 3 hours: Poitiers, etc.). Such an assignment is called in mathematics a function $t \to X(t)$. When consulting an online route finder, you are asked certain constraints you wish your travel to satisfy. For example, you might decide not to pay any highway fees. The route finder will then decide to change brutally the direction of your trajectory when you are about to meet a highway toll booth: the tangent to your course will be modified and the trajectory will follow this new tangent indication. In other words, a certain trajectory is assigned to a certain dynamics (e.g. avoid fees). Such a dynamics is realized in mathematics by the fact of putting a vector field on your space, that is to say a way of associating with each point X a direction $f(X)$, an (oriented) straight line, supposed to be the tangent of the trajectory at the point X. The problem is then to find a curve with the requirement that at each point the curve is a tangent to the direction assigned by the vector field at this point. It happens that solving this problem consists mathematically of solving a *differential equation* written as

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = f(X(t)), \quad X(0) = X_0 \tag{1}$$

Here, $X(t)$ is the point on the curve at time t and $f(X(t))$ is the direction (vector field) at the point $X(t)$, as seen before. The new ingredients are X_0, the point of departure, which can be chosen in principle anywhere in the space considered, and what is (a bit barbarically) denoted by $\frac{dX}{dt}(t)$, the 'velocity' on the trajectory at time t(see the link with the definition of the tangent expressed earlier), a notion everybody knows intuitively. The equality between the 'velocity' $\frac{dX}{dt}(t)$ and the direction $f(X(t))$ at each point $X(t)$ is, in particular, the formalization of the statement 'the curve is tangent to the vector field'.

As we mentioned earlier, solving this problem *in a unique way and for any initial point* X_0 necessitates that the way $f(X)$ depends on X is regular enough, not too hectic (Lipschitz continuity): otherwise the curve might not exist or the problem might have several solutions.

When the Lipschitz condition is not satisfied and replaced by a weaker hypothesis, 'BV' *regularity*,[3] it happens that equation (1) is still solvable in a unique way, but only for *almost all* initial points X_0, not *all* initial points X_0. When we release the Lipschitz condition, though equation (1) is still very cute, very smooth so that we naively expect as before the solution to be smooth too, the actual solution is often quite hectic: it still exists but only 'almost everywhere', not 'everywhere'. What is meant by 'exists almost everywhere'? It means that if you select by chance an initial point, then almost surely, with probability one, as one says in mathematics, everything will go smoothly, as if the equation were more regular. But nothing prevents you from picking as an initial point, 'by chance', one of these rare points where, for example, two different trajectories (or even worse, none) can be born at the same time.

The flow can then be defined to meet the requirements of the original task, although not on the whole space but rather on an 'almost everywhere' defined space. There is no trouble with that a priori: we know such spaces, real numbers deprived of the rational numbers is such an example. But in our case, the set of remaining points of our space is not known. Or equivalently the bad points are not known (contrary to the rationals imbedded in the reals which are perfectly identifiable). But there is more: the bad points of this 'almost everywhere space' can change if you compose flows, that is, if you stop and restart again. A good point can happen to become bad, and a bad one good. In fact nothing is known about that, except the fact that almost all points are again good when you restart.

By identifying a 'space' by the entity consisting of all the trajectories which are the solutions of a vector field's equation, and not by an 'object' made of points given (platonically?) a priori, you see that the two definitions merge when the vector field is Lipschitz. But when the vector filed has only BV regularity, there is no 'object' counterpart to the 'entity-like' notion of space.

We have the two following diagrams.

3. The definition of BV goes beyond the scope of this text. Let us just mention that the two letters B and V refer to 'bounded variations', which indicate that the vector field could be hectic but not totally crazy.

When the vector field f is Lipschitz,

$$\begin{array}{ccc} \textit{object} & & \textit{space} \\ & & \Updownarrow \\ \textit{entity} & & \textit{equation} \\ & & \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t) = f(X(t)),\, X(0) = X_0, \text{ for all } X_0\right) \end{array}$$

When the vector field f has only BV regularity

$$\begin{array}{ccc} \textit{entity} & & \textit{equation} \\ & & \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t) = f(X(t)),\, X(0) = X_0, \text{ for almost all } X_0\right) \\ & & \Updownarrow \\ \textit{object} & & ? \end{array}$$

There is no underlying object to this new entity.

In the case of Lipschitz vector fields, there was a symmetry/identity between the two spaces where the equation and its solution were sitting: the two spaces were the same. The willingness to preserve this symmetry in the non-Lipschitz situation leads to a new type of space, a new paradigm, a new form of reality.

2.2. Quantum Mathematics

The, nowadays, quite popular notions of quantum groups and quantum (= noncommutative) geometries have no underlying group or geometry-type objects. Nevertheless, they fulfil completely, according to us, the paradigm of realism in the sense that their structures provide an arsenal of study methods, comparable to the ones available in the 'classical' situation.

How are they constructed?

A rigid body, a human body, a landscape, is a geometrical object, a manifold, perfectly understood if one knows enough drawings of it (it can be reconstructed from them). There is no need to explain this fact experienced by all of us. What is a drawing? It is a set of points on a sheet of paper. What does the action of drawing consist of? It consists of assigning to each point of the (surface of the) body a point on the sheet, that is, precisely, a function from the body to the sheet. A single drawing does not capture the body entirely but, and Picasso understood this very well, several drawings do. This beautiful fact turns into a theorem in mathematics: the set of all the functions on a manifold (e.g. the surface of a rigid body) determine the manifold itself.[4] Looking closely at

4. To be more precise: the algebra of continuous functions with values in the complex numbers determines a manifold.

this statement, we see that stating (and using) it consists of building a higher level, as in the preceding section. This theorem refers to a dynamical action ('drawing, looking at') and to a formalism: it belongs definitively to the side of 'entities'.

But identity (again identity) in mathematics allows the semantic shift:

> 'determines' $\Rightarrow$ 'is': a manifold 'is' the algebra of its functions.

This is a dynamical point of view, an operating one. And this algebra of functions inherits from numbers a nice property: one can multiply functions like numbers. The order in which you perform this operation is insignificant: one calls such an algebra a *commutative* algebra since we can commute the different functions/drawings that we multiply without changing the result. (It seems to us that this *multiplication* is truly incarnate in the cubist's portraits. On a single drawing one put/multiplies several different views of the body to be drawn. And, obviously, the order between the different takes is insignificant.)

And, somehow even more importantly, the converse is also true: give me any algebra with this property, that is, any commutative algebra, then I can construct the underlying manifold. This is the Gelfand Theorem (Gelfand 1941). We can start building our diagram: at the lower level, the ground floor, we put what we would like to call, by convention, objects such as a manifold, considered as a set of points. At the upper level we put an alternative 'entity', the algebra of functions on the corresponding object.

We will now remove one property of the algebra, the commutativity. And we will think about this displacement from commutative to noncommutative as a left to right movement. By doing that we leave a commutative algebra, isomorphic to a lower level object, the manifold, and we get a new algebra, noncommutative, with a priori no corresponding object under it, at the lower level. Nevertheless, this right-upper level exists and inherits from the left-upper one all the properties you need, in principle, to generalize the construction made at the lower level. The question is: can we move by staying on the ground floor and construct an object that would be the classical object corresponding to the new upper level? Can we move left/right by remaining at the lower level?

Let us draw a picture.

$$\begin{array}{llcl} \textbf{entities} & \text{commutative algebra} & \longrightarrow & \text{noncommutative algebra} \\ & \qquad\updownarrow & & \qquad\downarrow \\ \textbf{objects} & \qquad\text{manifold} & \longrightarrow & \qquad ? \end{array}$$

The answer is: no!

$$\begin{array}{llcl} \textbf{entities} & \text{commutative algebra} & \longrightarrow & \text{noncommutative algebra} \\ & \qquad\updownarrow & & \qquad\downarrow \\ \textbf{objects} & \qquad\text{manifold} & \longrightarrow & \quad \emptyset,\ \textit{nothing} \end{array}$$

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There is no manifold whose space of functions is noncommutative. But then, what becomes of the arrow at the ground floor? The diagrammatic answer is the following:

$$\begin{tikzcd} \textbf{entities} & \text{commutative algebra} \arrow[r] \arrow[d, leftrightarrow] & \text{noncommutative algebra} \\ \textbf{objects} & \text{manifold} \arrow[ur] & \end{tikzcd}$$

The object ‘manifold’ joins the upper level and becomes the entity ‘noncommutative algebra’, and by another semantic shift one can say:

a noncommutative manifold is a noncommutative algebra.

This presentation might let us think that the noncommutative manifold arrives in the field of mathematics just by a fantasy of generalization that mathematicians are known to be fond of. This is not true, and the most interesting situations where the lower floor disappears are the ones in which this destruction is performed at (by) the lower floor itself. The simplest case comes from the concept of quotient space, which is defined as the set of families of elements of a space, a family being the subset of elements sharing a certain property. For example, take a sheet of paper and fill it completely by drawing straight lines on it. The quotient appears as the set of such lines. The sheet itself is a sweet set of points, each line is itself a collection of points. What about the quotient? Well, it is a fact in mathematics (only in mathematics?) that the set of nice objects included in a nice object might be not nice at all.

This can be metaphorically visualized as follows: when you talk about a packet of spaghetti, you mention in fact two different things. One is the set of the 250 grams of flour its volume contains, the other is the set of 50 spaghetti it contains. The second is the quotient of the first when you regroup the grams of flour sharing the property of belonging to the same spaghetti. When the spaghetti are very well stored in the packet, you can count the spaghetti and easily determine one from the other. But drop the packet of spaghetti on the floor and try to count them without putting them back into the packet.

The reader interested in going a bit further and treating her/himself a very simple example, though totally meaningful, can try the following experiment.

> Let him/her draw on a sheet of paper a (two-dimensional) torus. A torus takes the form of a square whose facing borders are identified, so that each point on the left side is taken as the same of the one on the right side with the same altitude. The same construction should be done for the upper and lower sides. If one now draws an oblique straight line passing by the left-down corner and the middle of the right side, the curve drawn by this method will be closed: one comes back to the original point after two ‘rounds’ around the torus (do not forget that the right and left sides of the square have been identified, so that when the straight line arrives to the right side, it has to be continued by starting again at the middle of the left side). The line will pass two times on the left side: first at its middle, second at the top (which is the same as the bottom after identifying the upper and lower sides). If you now draw a parallel line starting from any point in the lower left half-side, the same argument will apply and the line

> will cross again the left side only at a point in the upper half side this time. Therefore one sees that any such straight line passes through one and only one point in the lower left half-side, and to any such point passes one and only one straight line. That is to say that one can label any such straight lines univocally by one point on the lower left half side: the set of such straight lines is identified with the lower left half-side. Everything goes well, the quotient we were looking at is just a piece of a straight line, a nice geometrical object.[5]

But let us suppose now that one does exactly the same construction but without taking care of how we choose the point where you are going to cross the right side. In other words, let us choose this point randomly. 'In general', the drawn straight lines will not be closed any more: you will not go back to the starting point but you will, in general, miss it narrowly after many rounds (try this!), the line will intersect the left side at an infinite number of points. But there is more: if you continue going around the torus on this line, the set of these intersection points will accumulate everywhere and become dense on the left side. If you now choose another starting point and draw a straight line parallel to the latter, the set of intersections with the left side will look exactly the same as the one for the first straight line, and you will not be able to distinguish which intersection point belongs to which straight line, although each point of the left side belongs to one and only one of the two straight lines.

In mathematical language this means that, in general (namely, for almost all values of the slope of the straight line), the set of straight lines, still ideally well defined as a set, is totally unreachable by any approximation, this last property reflects the 'looks the same' expressed before. Defining the set of drawn straight lines only as 'a set' is tautologically possible. But this view is unsatisfactory, as the simple drawing shows, since one cannot differentiate any line from the other by its trace on the left side. In fact, if one looks at the (algebra of) continuous functions on this set, one can prove the following theorem:

> Any continuous function on the quotient 'space', set of dense points, is constant everywhere (it asserts the same number to all the points).

The set of drawings of our set, the algebra of functions on it, is reduced to trivial ones, the ones making no differences between the points: i.e. a fully black drawing, with not even a texture 'à la Soulages'. Nothing. No classical structure, nothing. But if we lift the whole construction we made on the lower level to the upper one, we find that there is a possibility of describing and 'understanding' this space by identifying it with a noncommutative algebra (Connes 1994). One can calculate, manipulate this set-entity, construct a topology on it, although there is: *no underlying object to this new entity*.

The symmetry/equivalence between manifolds and algebras is lost, but, by disappearing, it creates the incredibly rich mathematical structure of quantum mechanics.

5. The attentive reader might have noticed that if we start the straight line from the very bottom of the lower left half side, this line will cross the segment twice. But this means that the bottom and the middle of the lower left half side are on the same straight line. They therefore have to be identified. And the quotient is thereby not a segment but a segment with the two extremities identified (like in the course of the construction of the torus on the sheet of paper), i.e. a circle, topologically.

We insisted a bit heavily on the preceding construction not because we wanted to torture the reader, but rather because we believe that it gives a quite faithful image of the mathematician at work: tedious drawings, computations, failures, repetitions, etc. – using and pushing to its very extremities a formalism leading to a new paradigm. In a nutshell: the mathematical formalism is a formalism in action (Benoist and Paul 2013a; Benoist and Paul 2013b).

In order to conclude this section, we would like to point out[6] a clear difference between the object–entity dialectic present in this article and the dynamics of 'generalization–extension' so familiar in mathematics, where, though widely generalized, the underlying object never really disappears. An example of this in group theory is already hinted at in Footnote 1: abstract groups are transformations of … nothing. Yet, it happens that a very efficient way of studying groups is to let them 'act' on different types of objects in the framework of representations theory. Let us give another example: *Analysis situs* by Poincaré (1885) views objects as new entities but without removing the lower level. Perhaps the upper level becomes more important, the concepts of fundamental group and simplicial homology becoming even necessary to an understanding of the underlying level, but the underlying object never disappears.

2.3. Partial Differential Equations

Our last example will be a bit more technical and might be skipped by the uninterested reader without affecting the comprehension of the core of this article.

Solving the Navier–Stokes equations, fundamental equations of hydrodynamics, has up to now been limited to defining 'weak' solutions. In this section we would like to show how this concept of weak solutions, which we are going to explain later on, conforms perfectly to the notion of entity as it was defined before: it is an entity considered as the solution (because the important fact here is that there is only one), as a substitute to a 'true' solution; an object which is, for the Navier–Stokes problem, still unknown and might never exist.

Let us be a little bit precise, without too much technicality (once again the reader should not be afraid by the presence of equations whose precise meaning is irrelevant for the purpose developed here). A partial differential equation (PDE) consists of a function u (the unknown), for example a function $x \to u(x)$ as defined in Section 2.1 which sends (real) numbers x to (real) numbers $u(x)$, an operator $P: \ u \to P(u)$, a 'function' with sends the 'function $x \to u(x)$' to another function $P(u): \ x \to P(u)(x)$, and an 'equality to zero':

$$P(u) = 0 \tag{2}$$

What is meant by this is that $P(u)$ is a function. And one looks at a function u such that $P(u)$ is the null function, i.e. the function identically equals to zero: we want to find *the* u such that $P(u)(x) = 0$ *for all* numbers x.

6. This paragraph follows a demand of clarification by J. Brüning, J. Jost and B. Teissier.

We see here a question arising concerning identity: 'identically equal to 0' means what we just wrote. But checking something 'for all numbers' is a long task, a very long task. And doing such a task patiently is boring and the risk of missing 'some x' is high. An alternative way consists in taking averages of $p(u)$ with a given probability distribution. What does this mean? Just that we will add up the numbers $u(x)$ defined out of 'almost' all numbers x by weighting them according to the importance we want to lend to each x – just as insurance or political survey companies do. The reader might argue that we still miss some values of x through the concept of 'almost', intrinsic to the concept of average. The answer is that if we take all the averages with all probability distributions then we determine the value of $P(u)(x)$ for all x. Actually, the reason is very simple: take a probability asserting the maximal value to a number x_0 and the value zero to all others. Obviously, the corresponding average will be equal to $P(u)(x_0)$. What else?

Taking the average of a function f with a probability distribution ϕ is written in mathematics the following way[7]

$$\int \phi(x) f(x) \mathrm{d}s$$

And what we just wrote can be formalized as

$$[P(u)(x) = 0] \Leftrightarrow \left[\int \phi(x) P(u)(x) \mathrm{d}s = 0 \text{ for } \textit{all} \text{ functions } \phi\right] \tag{3}$$

It is a matter of fact (a very unpleasant fact) that in many cases the mathematician finds out that a prototype model of solution u she/he's working on is such that, for some few points y, $P(u)(y)$ is infinite and therefore cannot be properly defined.[8] Of course, in this case, u cannot pretend to be a solution of equation (2) as we just defined, but these 'bad' points y are sometimes so few that, in the interest of proceeding, we would like to be just able to ignore them for the moment. In order to do that we first remark that, usually in these situations, these points are so few and so isolated that they disappear when taking an average of the left hand side of equation (2). But one does not want to take averages with all probabilities since, because of equation (3), this would be equivalent to equation (2). What to do?

7. The reader not familiar with infinitesimal calculus might remember that, when averages are taken over integers numbers, one writes $\sum_i \phi(i) f(i)$. The following notation (it is just a notation) is obtained by formally 'translating' $\sum_i \ldots$ to $\int \ldots \mathrm{d}x$.

8. In many cases, such as for the Navier–Stokes equations, the operator P when acting on u conveys the speed of variation of the number $u(x)$ when x varies just a little bit. This quantity becomes obviously infinite at a point y where, e.g., u is not continuous, that is, where u has jumps. It can jump discontinuously from one value to another one.

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It happens that solving the right-hand side of equation (3) by restricting it to some special functions, called, meaningfully, 'test functions', is easier than solving the left-hand side, namely equation (2).

We will say that u is a weak solution of equation (2) if

$$\int \phi(x)P(u)(x)\mathrm{d}s = 0 \text{ for all '}\textit{test}\text{ functions' } \phi \tag{4}$$

Why do we say that such a function u satisfying equation (4) is a weak solution of equation (2)? Well because if u was a nice solution of equation (2), then equation (4) would be true *for every function, not only a test function* and therefore equation (4) would be equivalent to equation (2), as every function whose integration against every function is equal to 0 is itself equal to 0.

We propose to define as 'objects' the contents of equation (2) and the bracket in the right place in equation (3), and as 'entities' the brackets in the left place in equation (3) and the contents of equation (4). We have

$$\underbrace{[P(u) = 0]}_{\text{object}} \Leftrightarrow \underbrace{\left[\int \phi(x)P(u)(x)\mathrm{d}s = 0 \text{ for } \textit{all} \text{ functions } \phi\right]}_{\text{entity}}$$

and, since test functions are functions, after all,

$$\underbrace{[P(u) = 0]}_{\text{object}} \Rightarrow \underbrace{\left[\int \phi(x)P(u)(x)\mathrm{d}s = 0 \text{ for all '}\textit{test}\text{ functions' } \phi\right]}_{\text{entity}}$$

But as we mentioned before

$$\underbrace{\left[\int \phi(x)P(u)(x)\mathrm{d}s = 0 \text{ for all '}\textit{test}\text{ functions' } \phi\right]}_{\text{entity}} \not\Rightarrow \underbrace{[P(u) = 0]}_{\text{object}}$$

Under this last entity there is no clear existence of a true solution, nothing which plays the role of the complex plane for the second degree equation. Equation (4) is definitively different from equation (2), so:

No underlying object to this new entity[9]

9. Although we will not develop it here, let us make a link with sections 2.1 and 2.2 by mentioning that we believe that we are facing here an epistemological shift: the geometrical space becomes a functional space, i.e. a space of functions. For other studies of mutations of the notion of space, one can consult Paul (2013).

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We get the two following diagrams.

$$
\begin{array}{lccc}
\textbf{entities} & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for all functions } \psi\end{array} & \longrightarrow & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for every } \textbf{test} \textit{ function } \varphi\end{array} \\
& \updownarrow & & \downarrow \\
\textbf{objects} & P(u)=0,\ \text{PDE} & \longrightarrow & \emptyset,\ \textit{nothing} \text{ “}\textit{like a} \text{ PDE ”}
\end{array}
$$

$$
\begin{array}{lccc}
\textbf{entities} & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for all functions } \psi\end{array} & \longrightarrow & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for every } \textbf{test} \textit{ function } \varphi\end{array} \\
& \updownarrow & \nearrow & \\
\textbf{objects} & P(u)=0,\ \text{PDE} & &
\end{array}
$$

Here also we see that it is the need of symmetry, taking the form of an equivalence between entity and object in the good case (column on the left), which leads to an identity: there is only the upper level in the column on the right.

3. The Three Examples Reunified: What Happened?

Let us resume the three diagrams corresponding to the three situations we discussed earlier.

The three unanswered questions (or equivalently wrongly answered by a forced essentialist-type answer: ‘Ø-nothing’)

$$
\begin{array}{lccc}
\textbf{entities} & \begin{array}{c}\frac{dX}{dt}=f(X(t)), X(0)=X_0 \\ \textit{for all initial points } X_0\end{array} & \longrightarrow & \begin{array}{c}\frac{dX}{dt}=f(X(t)), X(0)=X_0 \\ \textit{for } \textbf{almost} \textit{ all initial points } X_0\end{array} \\
& \updownarrow & & \downarrow \\
\textbf{objects} & \text{space } \mathcal{M} & \longrightarrow & \emptyset,\ \textit{nothing} \text{ “}\textit{like a} \text{ space”}
\end{array}
$$

$$
\begin{array}{lccc}
\textbf{entities} & \begin{array}{c}\text{commutative algebra} \\ \textit{of functions on } \mathcal{M}\end{array} & \longrightarrow & \begin{array}{c}\text{noncommutative algebra} \\ \textit{of } \textbf{operators}\end{array} \\
& \updownarrow & & \downarrow \\
\textbf{objects} & \text{manifold} & \longrightarrow & \emptyset,\ \textit{nothing} \text{ “}\textit{like a} \text{ manifold”}
\end{array}
$$

$$
\begin{array}{lccc}
\textbf{entities} & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for all functions } \psi\end{array} & \longrightarrow & \begin{array}{c}\int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for every } \textbf{test} \textit{ function } \varphi\end{array} \\
& \updownarrow & & \downarrow \\
\textbf{objects} & P(u)=0,\ \text{PDE} & \longrightarrow & \emptyset,\ \textit{nothing} \text{ “}\textit{like a} \text{ PDE ”}
\end{array}
$$

were finally answered in an existentialist way by

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$$\begin{tikzcd}
\textbf{entities} & \begin{array}{c} \frac{dX}{dt} = f(X(t)), X(0) = X_0 \\ \textit{for all initial points } X_0 \end{array} \arrow[r] & \begin{array}{c} \frac{dX}{dt} = f(X(t)), X(0) = X_0 \\ \textit{for } \textbf{almost} \textit{ all initial points } X_0 \end{array} \\
\textbf{objects} & \text{space } \mathcal{M} \arrow[u, leftrightarrow] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\textbf{entities} & \begin{array}{c} \text{commutative algebra} \\ \textit{of functions on } \mathcal{M} \end{array} \arrow[r] & \begin{array}{c} \text{noncommutative algebra} \\ \textit{of } \textbf{operators} \end{array} \\
\textbf{objects} & \text{manifold} \arrow[u, leftrightarrow] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\textbf{entities} & \begin{array}{c} \int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for all functions } \psi \end{array} \arrow[r] & \begin{array}{c} \int_{\mathbb{R}} \psi(x)P(u)(x)dx \\ \textit{for every } \textbf{test} \textit{ function } \varphi \end{array} \\
\textbf{objects} & P(u) = 0, \ \text{PDE} \arrow[u, leftrightarrow] \arrow[ur] &
\end{tikzcd}$$

Objects disappear at the benefit of entities.

What Happened?

We claim that one attends here to a sort of mathematization of non-Platonism in the following sense: not only are the mathematical entities somewhere and waiting to be discovered (as being upstairs, over a non-existing ground floor), but they are not even expressible by the standard essentialist language of mathematics available at the time of their creation. And more than that, they influence language right up to its paradoxical extremities: a noncommutative manifold is stricto senso a non-sense as there is nothing in the definition of a manifold to be multiplied, commutatively or not (check the definition on *Wikipedia*).

But how could mathematics be non-Platonist, even sometimes anti-Platonist, in itself without being Platonic when viewed by the mathematicians? More precisely, if mathematics did the job itself – that is, the job of inventing a new realism over the traditional ones – it could just mean that mathematics itself is somewhere, i.e. that it happens to be outside the thinking of mathematicians inside of which it could only be a construction.

The answer to this 'paradox' lies, according to us, in the finality of these entities without objects, which is to compensate for the lack of a classical, standard definition of underlying objects or to compensate for their lack of ability in solving an equation. It is when they face a dead end that mathematics take off to the upper level: this realism is not a new one, it is just the right one.

The realism of an equation lies in its solutions. In the case of a nice, gentle partial differential equation (PDE) the solutions are nice functions on a nice space, and it is to this space, this nice one, that they, the solutions, provide the status of realism. But

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for the Navier–Stokes equations, the solutions exist only in a weak sense (up to nowadays), i.e. without a clear spatial counterpart. But if we set aside this lack of a 'clear spatial counterpart', the situation is the same, one continues to do mathematics, to compute, to estimate: realism = solutions. In fact:

What is space?	It is the repository of a movement.
Do you see space?	NO.
Do you see movement?	YES.

The realism belongs to the movement, identified with a nice space in the good situations, to an 'almost everywhere', a noncommutative one in the bad ones: realism = movement.[10] In the three situations examined in Section 2, realism lies on the upper floor, isomorphic to the 'natural' object for the good situations, isomorphic to itself, and only to itself, in the 'bad' ones. This is the meaning of the north-east oriented arrows in the diagrams.

The essentialist diagrams do not commute, the existentialist ones do.

Intermezzo

Identity: Last Call for Immediate Boarding to Temporality!

Let us return to the theme evoked in the Prelude: Identity–Repetition–Seriality or, more generally and synthetically, identity versus temporality. Indeed, discussing identity together with repetition–seriality seems to us like realizing an attempt to look at identity versus temporality: repetition–seriality refers to identity in a temporality of repetition.

Naively, identity belongs to the ground floor and temporality (namely action, process, dynamics) is located on the upper level. We tried to show in Section 3 that temporality possibly creates a kind of non-existing objects that we named entities. Weak solutions of PDEs, almost everywhere defined as flows, and noncommutative manifolds are examples of such entities without clear underlying *identified* objects. The word *identified* clearly refers to the concept of identity, a concept that disappeared from the lower level in such situations.

But one of the goals of mathematics, under the angle we chose to look at in this article, is precisely to give an identity to these entities. Weak solutions, which appeared first as worse options (or the worst solution), or as the lesser evil, are nowadays perfectly identifiable: they acquired their own identity by themselves.

10. In the good cases, a solution of a PDE is 'pushed' by a flow: that's what you see on a flowing river. One identifies a stream of moving particles with a 'push forward', one identifies a flow with the solution of a PDE. The meaning of this is that one can solve some PDEs as in Section 2.3, by solving some flow equations, that is by the theory of dynamical systems as in Section 2.1. When the motion, the underlying flow is long, fast, chaotic, the movement is unbearable, impossible to see. What is left to be apprehended are some geometric features (invariants), for example eddies. Therefore, in the chaotic situations, the real space for dynamical systems consists of a set of geometrical objects quite similar to the construction on the torus of Section 2.2: a quantum (noncommutative) one. And precisely the solution of the PDE driving the quantum evolution (Schrödinger equation) conveys such a noncommutative space of invariants when looking at large values of the size of the system and time of evolution: realism=space of invariants. This last identification leads to a unified probabilistic view of quantum and classical mechanics where quantum indeterminism and classical unpredictability merge (Paul 2011, 1177–1182; Paul 2010, 219–232; Paul 2009a, 660–669).

Identity jumped from the lower to the upper level.

The temporality of … nothing, happens to be only a transient phenomenon, which, after taking off, creates its own identity. And in the three examples we treated, it was the willing of preserving a type of symmetry/identity/equivalence which was the strong engine responsible for this precious temporality which creates new paradigms, new 'levels' in mathematical reality.

4. Platonism and Realism Revisited

Would we say that this would somehow put Platonism and Realism in duality:

the object belongs to the mathematician,
the entity to mathematics?

What is remarkable in the three examples studied in Section 2 is that there is no a priori willingness behind the disappearance of the object. Nevertheless the replacing, 'standing for' entity is very often (supposed to be) just a tool, something the ontology of which has to be fixed later. And, after all, it often happens that the new entity is as comfortable to handle as the original object and the choice of the floor to sit in is insignificant. More than that: it is insignificant whether there is a lower level or not. The essentialist status of the ground floor does not matter (as did the existence of a god for existentialists in Saint-Germain-des-Prés). A few questions arise.

- Would we therefore put the realism at the entities' level defined earlier? Yes, definitively, because entities are dynamical.
- Is there no need of talking of an (even not existing) underlying object?
- Are not the entities we are talking about just new objects?
- Is it the case in a pure abstract way?
Yes, strictly at a technical level.

But nevertheless, one continues to talk about a 'space', although it has disappeared: the entity replaced the object, but not quite for our mind since one continues to remember the object. We talk about a noncommutative space (a non-sense, stricto senso, as we saw before), a space defined almost everywhere, without any reference to an immutable set and a weak solution, though the equation stricto senso is not anymore solved.

Here again appears the Platonist paradox introduced in Section 3, which we can revisit now: how is it possible to set a question of Platonism inside mathematics, to criticize it, to let the mathematics decide, without being Platonist ourselves, we mathematicians? After all, the mathematics could owe this very Platonist property of being given first, and look nevertheless at themselves in a non-, an anti-Platonist way.

Definitively, an answer consists of overcoming this difficulty by being a non-Platonist as mathematician; the only way, for us, to let go on the creation of these non-Platonic mathematics. Otherwise realism, in the traditional sense of the word, would be everywhere present and, through this, paradoxes would start to proliferate.

More interestingly, can this conception of a form of realism strictly inside mathematics be exported outside of mathematics? The noncommutative space is a space by the fact that topology, among other things, can be defined on it. That is to say, it is its own properties, the look one has on it, the way of indirectly handling it, which transform an object into an entity, on which realism is real. Exporting this outside mathematics, into 'real life' so to speak, would constitute a fantastic issue,[11] to be added to the already long list of services rendered by mathematics to the human community.

Sonate que me veux-tu? Elle veut être écoutée. (A. Boucourechliev, *Essai sur Beethoven*)

5. Synopsis

Five Key Ideas (Kinematics of the Article)

1. Realism inside mathematics leads to the question: is Platonism inside mathematics?
2. Necessity in mathematics to dynamically reinterpret objects: probably one of the lessons of twentieth-century mathematics.
3. Importance in mathematics of the formalism, formalism in action: temporality.
4. Structure: extension without non-extended counterpart.
5. Without object: reference to a classical culture, but 'tradition=trahison' (tradition=treason) and, in mathematical terms, the three diagrams at the beginning of Section 3 never commute (as mathematicians say).

Five Key (E)motions (Dynamics of the Article)

Realism sits down at the (upper) floor of operations,
not at the (lower) floor of objects.
$\Downarrow$
Sometimes you can go down, sometimes not.
$\Downarrow$
The mathematics mathematize this idea
by making the upper floor precise.
$\Downarrow$
The upper floor is the one of entities,
the ground floor the one of objects,
sometimes non-existing.
$\Downarrow$
Entities are in a process of thinking, of operating, and
decide on their own Platonism
(existence or not of underlying objects).

11. For a similar attempt concerning mathematics versus music and mathematics versus quantum mechanics, the reader can consult Paul (2014, 71–77); Paul (2007); and Paul (2009b).

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Postlude

Mathematics versus Philosophy: The Other Way

We believe that this discussion offers the opportunity of considering an example of interaction between mathematics and philosophy which goes 'the other way'. Indeed, the traditional discussion concerning Platonism involves mathematics as outside philosophy, watched by it, and looks at what they do when creativity is in action: do they invent or do they discover?

It seems to us that in the situations described in this article, which belong to recent mathematics, everything works the other way. It is inside mathematics that the question of discovering (something already existing, i.e. at the lower level) or creating (something new, i.e. something which belongs only to the upper level) is considered. A noncommutative manifold does not exist somewhere else than in the framework of 'its' noncommutative algebra of functions, that is, functions … on the (non-existing) manifold itself.

By bringing mathematical Platonism inside their own interior, the mathematics itself may well have closed the debate on its own reality.

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About the Author

Thierry Paul is 'Directeur de recherche' at CNRS and member of the 'Centre de Mathématiques Laurent Schwartz' at the Ecole polytechnique where he is responsible for the team 'Analyse et Équations aux Dérivées Partielles'. His domain of research contains, among other things, analysis, partial differential equations and quantum mechanics. He co-organizes exchanges between mathematicians and philosophers within the Réseau Thématique Pluridisciplinaire 'phenomath' from CNRS.