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THE DRINFELD–GRINBERG–KAZHDAN THEOREM IS FALSE FOR SINGULAR ARCS

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Abstract In this note, we prove that the Drinfeld–Grinberg–Kazhdan theorem on the structure of formal neighborhoods of arc schemes at a nonsingular arc does not extend to the case of singular arcs.

Keywords: arc scheme; curve singularity

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1. Introduction

1.1.

In [4, Theorem 0.1], Drinfeld proved the following statement (which was conjectured, under a weaker form, by Drinfeld himself in private communication; see [5, Introduction]):

Theorem 1.1. *Let k be a field. Let V be an integral k-variety with $\dim(V) \geqslant 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$. If $(\mathscr{L}_\infty(V))_\gamma$ denotes the formal neighborhood of the k-scheme $\mathscr{L}_\infty(V)$ at the point γ, there exist an affine k-scheme S of finite type, with $s \in S(k)$, and an isomorphism of formal k-schemes:*

$$\mathscr{L}_\infty(V)_\gamma \cong S_s \hat{\otimes}_k k[[(T_i)_{i\in\mathbf{N}}]]. \tag{1}$$

This theorem generalizes an earlier result, due to Grinberg and Kazhdan, for fields of characteristic zero (see [5]).

1.2.

In the statement of Theorem 1.1, the assumption that the arc γ is not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$ is a crucial argument of its proof (see [4], or, for example, [1]). The main result of the present article is to prove that such a statement does not extend when $\gamma \in \mathscr{L}_\infty(V_{\mathrm{sing}})$. Let us introduce the following terminology: if V is a k-variety, with $\gamma \in \mathscr{L}_\infty(V)(k)$, then we say that the *pair* (V, γ) *does not satisfy the statement of Theorem 1.1* if there exists no isomorphism such as (1).

In this article, we show that, if $v \in V_{\mathrm{sing}}(k)$ (considered as a constant arc), the pair (V, v) does not satisfy in general the statement of Theorem 1.1.

More precisely, the following statement provides an explicit example of such a pair.

Example 1.1. Let k be a field of characteristic zero which does not contain a root of the equation $T^2+1=0$. Let $f \in k[X, Y]$ be the polynomial X^2+Y^2. Let us denote by $\mathscr{C}_f$ the affine plane curve defined by the datum of f, and by $\mathfrak{o} \in \mathscr{C}_f(k)$ the origin of $\mathbf{A}^2_k$. Then, the pair $(\mathscr{C}_f, \mathfrak{o})$ does not satisfy the statement of Theorem 1.1.

Remark 1.2. The above example gives rise to examples of pairs (V, γ) which do not satisfy the statement of Theorem 1.1 with $\dim(V)$ arbitrary, and the arc γ not necessarily constant. More precisely, if W is a k-variety of arbitrary dimension, with $\gamma_W \in \mathscr{L}_\infty(W)(k)$, the pair $(W \times_k \mathscr{C}_f, (\gamma_W, \mathfrak{o}))$ has the required property.

1.3. Notation

Let k be a field of characteristic zero. As usual, if I is a set, the ring $k[(T_i)_{i\in I}]$ is the ring of polynomials in the indeterminates T_i, with $i \in I$, and with coefficients in k. Let us denote by $k\{X, Y\}$ the k-algebra $k\{X, Y\} := k[(X_i, Y_i)_{i\in\mathbf{N}}]$ endowed with the k-derivation $\Delta : k\{X, Y\} \to k\{X, Y\}$ defined by $\dagger_i \mapsto \dagger_{i+1}$ for every symbol $\dagger \in \{X, Y\}$. It comes equipped with a structure of $k[X, Y]$-algebra by considering the injective morphism of k-algebras defined by $X \mapsto X_0$, $Y \mapsto Y_0$. If S is a subset of $k\{X, Y\}$, the differential ideal of $k\{X, Y\}$ generated by S is denoted $[S]$, and the radical $\sqrt{[S]}$ by $\{S\}$. We define similarly the differential ring $k\{X\}$ and the notation $[S]$, $\{S\}$ for any subset S of $k\{X\}$.

2. Nilpotent elements in the ring $k\{X, Y\}/[X^2+Y^2]$

In this section, we develop key ingredients of the proof of our main statement. In the next statements, we assume that the field k is of characteristic zero and does not contain a root of the equation $T^2+1=0$. Let $f \in k[X, Y]$ be a polynomial. For every integer $n \geqslant 1$, we denote by $Q_n(f)$ the polynomial defined as follows:

$$Q_n(f) := \Delta^{(n)}(f) - (\partial_{X_0}(f)X_n + \partial_{Y_0}(f)Y_n). \tag{2}$$

We observe in particular that, for every integer $n \geqslant 1$, the polynomial $Q_n(f)$ belongs to the ring $k[(X_i, Y_i)_{i\in\{0,\ldots,n-1\}}]$; in other words, the polynomial $Q_n(f)$ is obtained by removing from the expression of $\Delta^{(n)}(f)$ the terms containing X_n or Y_n. We assume from now on that $f = X^2+Y^2$.

Remark 2.1. In the proofs of this section, we will use the following consequence of [9, Lemma 2.7]. For every $g \in k\{X, Y\}$, the following three assertions are equivalent: (1) $g \in \{f\}$, (2) there exists an integer m such that $X_0^m\, g \in \{f\}$, and (3) there exists an integer n such that $Y_0^n\, g \in \{f\}$.

Lemma 2.2. *For every integer $n \geqslant 1$, we have the following properties.*

(1) *The polynomial $Q_n(f)$ belongs to the ideal $\{f\}$.*

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(2) *The polynomial* $X_0X_n + Y_0Y_n$ *belongs to the ideal* $\{f\}$.

(3) *The polynomial* $X_0Y_n - X_nY_0$ *belongs to the ideal* $\{f\}$.

Proof. It follows from definition (2) that $1 \Leftrightarrow 2$.

Let us show $1 \Leftrightarrow 3$. We have

$$\begin{aligned} X_0(X_0Y_n - X_nY_0) &\equiv Y_n(X_0^2 + Y_0^2) + Y_0Q_n(f) \mod [f] \\ &\equiv Y_0Q_n(f) \mod [f]. \end{aligned}$$

Then, we conclude that $X_0Y_n - X_nY_0 \in \{f\}$ if and only if $Y_0Q_n(f) \in \{f\}$, which concludes the proof of $1 \Leftrightarrow 3$ by Remark 2.1.

So, we only have to prove assertion 1 of the lemma. We show it by induction on the integer n. We have $Q_1(f) = 0$; hence, in particular, $-Y_0X_1 + X_0Y_1 \in \{f\}$.

Let $n \geqslant 2$. We assume that, for every integer $1 \leqslant m < n$, the polynomial $Q_m(f)$ belongs to $\{f\}$. By definition, we have

$$\Delta^{(n-1)}(f) = 2(X_{n-1}X_0 + Y_{n-1}Y_0) + Q_{n-1}(f). \tag{3}$$

Differentiating formula (3), we deduce that

$$Q_n(f) = 2(X_{n-1}X_1 + Y_{n-1}Y_1) + \Delta(Q_{n-1}(f)). \tag{4}$$

By the induction hypothesis, we know that $Q_{n-1}(f) \in \{f\}$; hence, $\Delta(Q_{n-1}(f)) \in \{f\}$. So, we only have to prove that the polynomial $T_n := X_{n-1}X_1 + Y_{n-1}Y_1$ belongs to the ideal $\{f\}$. But, setting $U_n := X_0T_n - Y_{n-1}(-Y_0X_1 + X_0Y_1)$, we see that

$$\begin{aligned} U_n &= (X_{n-1}X_0 + Y_{n-1}Y_0)X_1 + X_0Y_1(Y_{n-1} - Y_{n-1}) \\ &= (\Delta^{(n-1)}(f) - Q_{n-1}(f))X_1. \end{aligned}$$

By the induction hypothesis, we conclude that $U_n \in \{f\}$. Since $-Y_0X_1 + X_0Y_1 \in \{f\}$, we have $X_0T_n \in \{f\}$, which concludes the proof by Remark 2.1. □

As a consequence, we have the following key proposition.

Proposition 2.3. *Let* $f \in k[X, Y]$ *be the polynomial* $X^2 + Y^2$. *Then, the subset*

$$\{X_jX_i + Y_jY_i;\ (i, j) \in \mathbf{N}^2\}$$

is included in the ideal $\{f\}$ *of the ring* $k\{X, Y\}$.

Proof. Let us note that, thanks to Lemma 2.2, we have

$$\begin{aligned} X_0(X_iX_j + Y_iY_j) &= (X_0X_i)X_j + Y_i(X_0Y_j) \\ &\equiv (-Y_0Y_i)X_j + Y_i(X_jY_0) \mod \{f\} \\ &\equiv 0 \mod \{f\}. \end{aligned}$$

It concludes the proof by Remark 2.1. □

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3. Proof of our main result

Our main result is crucially based on Theorem 3.1. The fundamental idea behind these statements is to control nilpotence at the level of function rings of arc schemes.

Theorem 3.1. *Let k be a field of characteristic 0. Then, there exist a strictly increasing function $\sigma : \mathbf{N} \to \mathbf{N}$ and a family $(P_n)_{n\in\mathbf{N}}$ of elements of $k\{X\}$ such that for every $n \in \mathbf{N}$ one has $P_n \in [X]^2$ and $P_n^{\sigma(n)} \notin [X^2]$.*

The proof uses the notion of a *strong basis* of a differential ideal, which we now recall according to [10, I/15]. If I is a differential ideal of $k\{X\}$, a strong basis of I is a finite subset B of I such that there exists a positive integer N with the following property: for every element $P \in I$, the element P^N belongs to the differential ideal $[B]$ generated by B.

Proof. Let us assume that the assertion of the statement of Theorem 3.1 does not hold. Then, there exists an integer $N \in \mathbf{N}$ such that, for every polynomial $P \in [X]^2$, $P^N \in [X^2]$. This implies that the singleton $\{X^2\}$ forms a strong basis of the ideal $[X]^2$. But, Kolchin proved in [6, § 5] (see also [10, I/15]) that the ideal $[X]^2$ has no strong basis. That is a contradiction, which concludes the proof. □

The following lemma allows us to deduce the proof of our main result from Theorem 3.1. Let us consider the morphism of differential k-algebras $\theta : k\{X, Y\}/[X^2 + Y^2] \to k\{X\}/[X^2]$ defined by $X_i \mapsto X_i$ and $Y_i \mapsto X_i$ for every integer $i \in \mathbf{N}$.

Lemma 3.2. *Let $P \in [X]^2$ in the ring $k\{X\}$. There exists $Q \in \{X^2 + Y^2\}$ in the ring $k\{X, Y\}$ such that $\theta(Q) = P$.*

Proof. Recall that the differential ideal $[X]$ is generated as an algebraic ideal by $\{X_i;\ i \in \mathbf{N}\}$. Without loss of generality, we may assume that there exist $i, j \in \mathbf{N}$ and $\tilde{P} \in k\{X\}$ such that $P = X_i X_j \tilde{P}$. Let us set $2Q := (X_i X_j + Y_i Y_j) \cdot \tilde{P}$. The polynomial Q has now the required properties by Lemma 2.3. □

We are ready to establish the validity of Example 1.1. Let us assume that the pair $(\mathscr{C}_f, \mathfrak{o})$ satisfies the statement of Theorem 1.1. Let us denote by I the ideal of $k[[(X_i, Y_i)_{i\in\mathbf{N}}]]$ generated by $\{\Delta^{(i)}(f)\}_{i\in\mathbf{N}}$. The existence of isomorphism (1) gives rise to an integer $N \in \mathbf{N}$ such that $(\sqrt{I})^N \subset I$. Indeed, if isomorphism (1) holds, there exist positive integers r, s, polynomials $F_1, \ldots F_s \in k[Z_1, \ldots, Z_r]$, and an isomorphism

$$k[[(X_i, Y_i)_{i\in\mathbf{N}}]]/I \cong k[[Z_1, \ldots, Z_r, (T_i)_{i\in\mathbf{N}}]]/(F_1, \ldots, F_s).$$

One may show, using this presentation, that the nilradical of $k[[(X_i, Y_i)_{i\in\mathbf{N}}]]/I$ is nilpotent (see [1, Theorem 5.1] for a detailed argument). Let us denote by $(\sigma, (P_n)_{n\in\mathbf{N}})$ the pair of objects provided by Lemma 3.1. Since every polynomial P_n belongs to $[X]^2$, there exists a polynomial $Q_n \in \{f\}$ such that $\theta(Q_n) = P_n$ for every integer n (see Lemma 3.2). For every integer n, let us denote by $t(n)$ the smallest integer such that $Q_n^{t(n)} \in [f]$. Because of the very definition of the function σ, we have that $t(n) > \sigma(n)$. So, the sequence $(t(n))_{n\in\mathbf{N}}$ converges to $+\infty$. In particular, there exists an integer $m \in \mathbf{N}$ such that, for every $n \geqslant m$, we have that $t(n) > N$. By Lemma 3.3, that is a contradiction, which concludes the proof.

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Lemma 3.3. *Let k be a field of characteristic 0. Let $f \in k[X, Y]$ be a homogeneous polynomial. Let I be the ideal of $k[[(X_i, Y_i)_{i\in\mathbf{N}}]]$ generated by the family $\{\Delta^{(i)}(f)\}_{i\in\mathbf{N}}$. Then, we have $I \cap k\{X, Y\} = [f]$. In particular, for every polynomial $P \in k\{X, Y\}$ and every integer $n \in \mathbf{N}$, $P^n \in I$ if and only if $P^n \in [f]$.*

In the following, the notation A designates either the ring $k\{X, Y\}$ or the ring $k[[(X_i, Y_i)_{i\in\mathbf{N}}]]$. For every monomial $M = X_{i_1}^{\mu_1} \cdots X_{i_m}^{\mu_m} Y_{j_1}^{\nu_1} \cdots Y_{j_n}^{\nu_n}$ in A, we set

$$\mathrm{weight}(M) = \sum_{\ell=1}^{m} i_\ell \mu_\ell + \sum_{\ell=1}^{n} j_\ell \nu_\ell.$$

Then the ring A is endowed with a graduation defined as follows. For every pair (d, ω) of nonnegative integers, we define the k-vector space $A_{(d,\omega)}$ generated by the monomials M of A such that $\deg(M) = d$ and $\mathrm{weight}(M) = \omega$. An element $Q \in A_{(d,\omega)}$ is said to be *homogeneous of degree d and isobaric of weight ω*. We easily observe that

$$A_{(d,\omega)} \cdot A_{(d',\omega')} \subset A_{(d+d',\omega+\omega')}$$

for every pair of integers $(d, \omega), (d', \omega') \in \mathbf{N}^2$. Besides, for every element $Q \in k\{X, Y\}$ (respectively, $Q \in k[[(X_i, Y_i)_{i\in\mathbf{N}}]]$), there exist a finite number of homogeneous and isobaric polynomials $Q_{d_1,\omega_1} \ldots, Q_{d_n,\omega_n}$ (respectively, an infinite number of homogeneous and isobaric polynomials $(Q_{d_i,\omega_i})_{i\in\mathbf{N}}$) such that

$$Q = \sum_{i=1}^{n} Q_{d_i,\omega_i} \quad \left(\text{respectively, } Q = \sum_{i\in\mathbf{N}} Q_{d_i,\omega_i}\right).$$

Such a decomposition is obviously unique.

Proof. Let $P \in k\{X, Y\}$. We only have to prove that, if we have $P \in I \cap k\{X, Y\}$, then we have $P \in [f]$. Let us assume that there exist an integer $n \in \mathbf{N}$, and power series $R_1, \ldots, R_n$, such that

$$P = \sum_{i=1}^{n} R_i \Delta^{(i)}(f). \tag{5}$$

Then, by the above remark, for every integer $i \in \{1, \ldots, n\}$, there exist two families of homogeneous and isobaric polynomials $(\tilde{R}_{i,j})_{j\in\mathbf{N}}$ and $(P_\ell)_{\ell\in\{1,\ldots,m\}}$ such that

$$R_i = \sum_{j\in\mathbf{N}} \tilde{R}_{i,j} \quad \text{and} \quad P = \sum_{\ell=1}^{m} P_\ell. \tag{6}$$

By gluing formulas (5) and (6), we deduce that

$$\sum_{\ell=1}^{m} P_\ell = \sum_{i=1}^{n} \sum_{j\in\mathbf{N}} \tilde{R}_{i,j} \Delta^{(i)}(f). \tag{7}$$

For every integer $i \in \mathbf{N}$, let us note that the polynomial $\Delta^{(i)}(f)$ is homogeneous and isobaric. We conclude the proof by the uniqueness of decomposition (7). The second assertion directly follows from the first one. □

Remark 3.4. Example 1.1 can be generalized to include every homogeneous polynomial f of degree 2. Let k' be an algebraic closure of the field k. If the image of f in the ring $k'\{X, Y\}$ is denoted by f', we observe that we may assume that $f' \in \{XY, X^2\}$. When $f' = X^2$, we apply Theorem 3.1 to conclude the proof of the assertion. When $f' = XY$, we verify that $\{f'\} = [X] \cap [Y]$ in $k'\{X, Y\}$; hence, $\{X_i Y_j;\ i, j \in \mathbf{N}\} \subset \{f'\}$. Thanks to this remark and the fact that isomorphism (1) is stable under field extension, the arguments of § 3 conclude the proof.

4. Further questions

A next step in the direction of the present work would be to provide an answer to the following questions.

Question 1. *Let k be a field of characteristic zero. If $f = X^3 - Y^2 \in k[X, Y]$, does the pair $(\mathscr{C}_f, \mathfrak{o})$ satisfy the statement of Theorem 1.1?*

Question 2. *Let k be a (perfect) field of arbitrary characteristic. Does there exist a k-curve $\mathscr{C}$, with $x \in \mathscr{C}_{sing}(k)$, such that the pair $(\mathscr{C}, x)$ satisfies the statement of Theorem 1.1?*

A negative answer to Question 2 would in particular imply, for every k-curve $\mathscr{C}$, with $x \in \mathscr{C}(k)$, the equivalence of the following assertions.

(1) The k-curve $\mathscr{C}$ is smooth at x.

(2) The pair $(\mathscr{C}, x)$ satisfies the statement of Theorem 1.1.

In another direction, it would be interesting to obtain methods to compute the nilradical of differential ideals associated with ideals $I \in k[X_1, \ldots, X_n]$. This problem is an open problem both from the theoretic point of view and from the effective point of view. Let us give below two illustrations of this remark which could contribute to shed light on the present picture.

Question 3. *Let k be a field of characteristic zero. Let us consider the ring $k\{X\}/[X^2]$. Does the nilpotence index of X_n equal $n+2$, for every integer $n \in \mathbf{N}$?*

This conjecture has been formulated by O'Keefe in [8]. In particular, a positive answer to this question would improve the qualitative argument used in the proof of Theorem 3.1. By [7, 31/Corollary] or [10, I/26], we easily observe that $X_n^{n+2} = 0$ in the rings $k\{X\}/[X^2]$ and $k[[(X_i)_{i\in\mathbf{N}}]]/(\Delta^{(j)}(X^2))_{j\in\mathbf{N}}$. We can make this assertion more precise from an effective point of view: for every integer $n \leqslant 24$, the monomial X_n^{n+2} belongs to the ideal $J_{2n} = \langle \Delta^{(i)}(X^2); i \in \{0, \ldots, 2n\} \rangle$ of the ring $k[(X_i)_{i\in\{0,\ldots,2n\}}]$. So, we state the following conjecture.

Conjecture 4.1. *Let k be a field of characteristic zero. The monomial X_n^{n+2} belongs to the ideal J_{2n} for every integer $n \in \mathbf{N}$.*

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Example 4.1. Using Sage [11], we checked that Question 3 admits a positive answer for every integer $n \leqslant 8$. In particular, we obtain that

$$X_8^9 \equiv \frac{702464}{30366765} X_0 X_2 X_4 X_6 X_8 X_{10} X_{12} X_{14} X_{16} \mod [X^2],$$

which allows us to conclude the proof of the property for X_8 by [7, 25/corollary] or [10, I/24].

Question 4. *Let k be a field of characteristic zero, and let $f \in k[X, Y]$ be the polynomial $f = X^2 + Y^2$. Is the family $\{(X_j X_i + Y_j Y_i, X_j X_i - Y_j Y_i)_{(i,j)\in\{0,\ldots,n\}^2}\}$ a system of generators (more precisely, a Groebner basis) of the ideal formed by the polynomials $P \in k[(X_i, Y_i)_{i\in\mathbf{N}}]$ such that there exists an integer N which satisfies $X_0^N P \in \langle \Delta^{(i)}(f); i \in \{0, \ldots, n\}\rangle$?*

We note that computations of Groebner bases appear in a slightly different but related context in [3].

Remark 4.2. As pointed out by the referee, it would be interesting to relate the question of the validity Theorem 1.1 for singular arcs to the linearization principle introduced in [2].

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