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Abstract

We prove that a very general smooth cubic fourfold containing a plane can be embedded into an irreducible holomorphic symplectic eightfold as a Lagrangian submanifold. We construct the desired irreducible holomorphic symplectic eightfold as a moduli space of Bridgeland stable objects in the derived category of the twisted K3 surface corresponding to the cubic fourfold containing a plane.

1. Introduction

1.1 Motivation and results

In this paper, we assume that all cubic fourfolds are smooth. Cubic fourfolds have been studied in the contexts of associated irreducible holomorphic symplectic manifolds, relations to K3 surfaces and rationality problems, and so on. For example, Beauville and Donagi [BD85] proved that the Fano variety $F(X)$ of lines on X is an irreducible holomorphic symplectic fourfold which is deformation-equivalent to the Hilbert scheme of two points on a K3 surface. Recently, Lehn *et al.* [LLSvS15] proved that if X is a cubic fourfold *not* containing a plane, then X can be embedded into an irreducible holomorphic symplectic eightfold Z as a Lagrangian submanifold. This Z is constructed as the moduli space of generalized twisted cubics on X [dJS04], and if X is Pfaffian, then Addington and Lehn [AL15] have proved that Z is deformation-equivalent to the Hilbert scheme of four points on a K3 surface. However, if X contains a plane, the argument of Lehn *et al.* does not apply. In this paper, we prove the following theorem.

Theorem 1.1. *Let X be a very general cubic fourfold containing a plane. Then X can be embedded into an irreducible holomorohic symplectic eightfold M as a Lagrangian submanifold. Moreover, M is deformation-equivalent to the Hilbert scheme of four points on a K3 surface.*

Whereas Lehn *et al.* used the moduli space of twisted cubics, we use notions of derived categories and Bridgeland stability conditions in our construction of M. More precisely, the holomorphic symplectic eightfold M is constructed as a moduli space of Bridgeland stable objects in the derived category of the twisted K3 surface (S, α), which corresponds to X. The twisted K3 surface (S, α) was constructed by Kuznetsov [Kuz10, § 4] in the context of his conjecture about K3 surfaces and rationality of cubic fourfolds.

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1.2 Background

Let us recall Kuznetsov's conjecture. The rationality problem of cubic fourfolds is related to K3 surfaces conjecturally. The derived category $D^b(X)$ of coherent sheaves on X has the following semiorthogonal decomposition:

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle. \tag{1}$$

The full triangulated subcategory $\mathcal{A}_X$ is a Calabi–Yau 2-category, i.e. the Serre functor of $\mathcal{A}_X$ is isomorphic to the shift functor [2]. Kuznetsov posed the following conjecture.

Conjecture 1.2 [Kuz10]. A cubic fourfold X is rational if and only if there is a K3 surface S such that $\mathcal{A}_X \simeq D^b(S)$.

Hassett [Has00] introduced the notion of special cubic fourfolds. Cubic fourfolds containing a plane are examples of special cubic fourfolds. Special cubic fourfolds often have associated K3 surfaces Hodge-theoretically [Has00]. Addington and Thomas [AT14] proved that Kuznetsov's and Hassett's relations between cubic fourfolds and K3 surfaces coincide generically. Known examples of rational cubic fourfolds include Pfaffian cubic fourfolds [Tre84, Tre93] and some rational cubic fourfolds containing a plane, which were constructed in [Has99]. Conjecturally, very general cubic fourfolds are irrational. However, so far there are no known examples of irrational cubic fourfolds. Kuznetsov constructed the equivalences between $\mathcal{A}_X$ and the derived categories of coherent sheaves on K3 surfaces for these rational cubic fourfolds. For a general cubic fourfold X containing a plane, Kuznetsov proved the following theorem.

Theorem 1.3 [Kuz10, Theorem 4.3]. *Let X be a general cubic fourfold containing a plane. Then there is a twisted K3 surface (S, α) such that $\mathcal{A}_X \simeq D^b(S, \alpha)$. Moreover, the Brauer class $\alpha \in \mathrm{Br}(S)$ is trivial, i.e. the twisted K3 surface (S, α) is the usual K3 surface S if and only if X is Hassett's rational cubic fourfold containing a plane.*

We say that a general cubic fourfold X containing a plane is very general when the Picard number of S is equal to 1. If a cubic fourfold X containing a plane is very general, then $\mathcal{A}_X$ is not equivalent to derived categories of coherent sheaves on K3 surfaces [Kuz10, Proposition 4.8]. So very general cubic fourfolds containing a plane are irrational conjecturally.

We recall previous work on holomorphic symplectic manifolds associated to cubic fourfolds and derived categories. Using the mutation functors associated to the semiorthogonal decomposition (1), we can define a projection functor $\mathrm{pr}\colon D^b(X) \to \mathcal{A}_X$. The Fano variety $F(X)$ of lines on X and the holomorphic symplectic eightfold Z in [LLSvS15] are related to the projection functor $\mathrm{pr}\colon D^b(X) \to \mathcal{A}_X$. In [KM09], the Fano variety $F(X)$ of lines on X is regarded as a moduli space of objects in $\mathcal{A}_X$ of the form $\mathrm{pr}(\mathcal{O}_{\mathrm{line}}(1))$. For a general cubic fourfold X containing a plane, Macri and Stellari [MS12] constructed Bridgeland stability conditions on $\mathcal{A}_X \simeq D^b(S, \alpha)$ such that all objects of the form $\mathrm{pr}(\mathcal{O}_{\mathrm{line}}(1))$ are stable. So the Fano variety $F(X)$ of lines on a general cubic fourfold X containing a plane is isomorphic to a moduli space of Bridgeland stable objects in $\mathcal{A}_X \simeq D^b(S, \alpha)$. For a general Pfaffian cubic fourfold X not containing a plane, Lehn and Addington [AL15] proved that the holomorphic symplectic eightfold Z is birational to the Hilbert scheme of four points on the K3 surface by considering the projections of ideal sheaves of (generalized) twisted cubics on X and the equivalence between $\mathcal{A}_X$ and the derived category of coherent sheaves on the K3 surface. In particular, the holomorphic symplectic eightfold Z is deformation-equivalent to the Hilbert scheme of four points for a general Paffian cubic fourfold not containing a plane.

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1.3 Strategy for Theorem 1.1

To construct Lagrangian embeddings of cubic fourfolds, we consider the projections of skyscraper sheaves of points on X. First, we illustrate the relation between the projection functor $\mathrm{pr}\colon D^b(X) \to \mathcal{A}_X$ and Lagrangian embeddings of cubic fourfolds. We prove the following proposition in § 4.

Proposition 1.4. *Let X be a cubic fourfold. Take a point $x \in X$. Then the following properties hold.*

- *For $x \neq y \in X$, $\mathrm{pr}(\mathcal{O}_x)$ is not isomorphic to $\mathrm{pr}(\mathcal{O}_y)$.*
- *We have $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) = \mathbb{C}^4$, $\mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}^8$ and $\mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \simeq \mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}$.*
- *The linear map $\mathrm{pr}\colon \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x))$ is injective.*
- *Let*
$$\omega_x\colon \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \times \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \to \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x))$$
be the bilinear form induced by the composition of morphisms in the derived category. Then the bilinear form ω_x vanishes on $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$.

Next, we construct a Lagrangian embedding of a very general cubic fourfold containing a plane using Bridgeland stability conditions σ on the Calabi–Yau 2-category $\mathcal{A}_X$ such that the objects $\mathrm{pr}(\mathcal{O}_x)$ are σ-stable for all $x \in X$. We prove the following proposition.

Proposition 1.5 (Cf. Proposition 3.3). *Let X be a very general cubic fourfold containing a plane and let $\Phi\colon \mathcal{A}_X \xrightarrow{\sim} D^b(S, \alpha)$ be the equivalence as in Corollary 2.12. Let v be the Mukai vector of $\Phi(\mathrm{pr}(\mathcal{O}_x))$. Then there is a stability condition $\sigma \in \mathrm{Stab}(D^b(S, \alpha))$, generic with respect to v, such that $\mathrm{pr}(\mathcal{O}_x)$ is σ-stable for all $x \in X$. In particular, the morphism*

$$X \to M, \quad x \mapsto \Phi(\mathrm{pr}(\mathcal{O}_x))$$

is the Lagrangian embedding. Here M is the moduli space of σ-stable objects with Mukai vector v. So M is deformation-equivalent to the Hilbert scheme of four points on a K3 surface.

To prove Proposition 1.5, we do not need the last statement of Proposition 1.4. Since there are no global holomorphic 2-forms on X, the closed immersion $X \to M$ is automatically Lagrangian. Note that no assumptions about the existence of planes in X are needed for Proposition 1.4. Since, so far, we do not know how to construct stability conditions on $\mathcal{A}_X$ for a general cubic fourfold X, we need to use some kind of geometric description of $\mathcal{A}_X$ to construct Bridgeland stability conditions on $\mathcal{A}_X$. In fact, it is difficult to construct the heart $\mathcal{C}$ of a bounded t-structure on $\mathcal{A}_X$ and a central charge $Z\colon K(\mathcal{A}_X) \to \mathbb{C}$ such that $Z(\mathcal{C}\backslash\{0\})$ is contained in the semiclosed upper half-plane. Moreover, we do not have well-established moduli theory for Bridgeland stable objects in $\mathcal{A}_X$. So we need some (twisted) K3 surfaces in order to use moduli theory for Bridgeland stable objects as in [BM14a, BM14b]. However, if X is a very general cubic fourfold containing a plane, we can construct the desired Bridgeland stability conditions on $\mathcal{A}_X$ using the twisted K3 surface (S, α). Thus, using the moduli theory [BM14a, BM14b] of Bridgeland stable objects on derived categories of twisted K3 surfaces, we obtain the Lagrangian embedding $X \to M$ in Proposition 1.5. Hence we establish Theorem 1.1.

Finally, we comment on two recent papers on cubic fourfolds. One is the work by Toda [Tod16] on Bridgeland stability conditions on $\mathcal{A}_X$. By Orlov's theorem [Orl09], the triangulated category $\mathcal{A}_X$ is equivalent to the triangulated category $\mathrm{HMF}^{\mathrm{gr}}(W)$ of graded matrix factorizations of the defining polynomial W of X. The investigation of Bridgeland stability conditions on $\mathcal{A}_X$ is related to the existence problem of Gepner-type stability conditions on $\mathrm{HMF}^{\mathrm{gr}}(W)$, which is treated in [Tod16]. However, it is also difficult to construct the heart of a bounded t-structure on $\mathrm{HMF}^{\mathrm{gr}}(W)$. The second paper, by Galkin and Shinder [GS14], concerns the rationality problem of cubic fourfolds and Fano varieties of lines. Galkin and Shinder [GS14] proved that rationality of cubic fourfolds is related to birationality of Fano varieties of lines and Hilbert schemes of two points on K3 surfaces if the so-called cancellation conjecture on the Grothendieck ring of varieties holds. Addington [Add16] compared the results in [GS14] with Conjecture 1.2. It would be interesting to study the relationship between Lagrangian embeddings of cubic fourfolds and rationality of cubic fourfolds.

2. Preliminaries

In this section, we recall the notions of twisted K3 surfaces and Bridgeland stability conditions, as well as the relation between cubic fourfolds containing a plane and twisted K3 surfaces.

2.1 Twisted K3 surfaces

We review the definitions of twisted K3 surfaces, twisted sheaves and twisted Mukai lattices.

Definition 2.1 [CĂL00]. A twisted K3 surface is a pair (S, α) consisting of a K3 surface S and an element α of the Brauer group $\mathrm{Br}(S) := H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}}$ of S.

Definition 2.2 [CĂL00]. Let (S, α) be a twisted K3 surface. Taking an analytic open cover $\{U_i\}_{i\in I}$ of S, the Brauer class α can be represented by a Čech cocycle $\{\alpha_{ijk}\}$. An α-twisted coherent sheaf F on S is a collection $(\{F_i\}_{i\in I}, \{\phi_{ij}\}_{i,j\in I})$ where F_i is a coherent sheaf on U_i and $\phi_{ij}|_{U_i\cap U_j} \colon F_i|_{U_i\cap U_j} \to F_j|_{U_i\cap U_j}$ is an isomorphism satisfying the following conditions:

$$\phi_{ii} = \mathrm{id}, \quad \phi_{ij} = \phi_{ji}^{-1}, \quad \phi_{ij}\circ\phi_{jk}\circ\phi_{ki} = \alpha_{ijk}\cdot \mathrm{id}.$$

We denote such a coherent sheaf by $\mathrm{Coh}(S, \alpha)$ and define $D^b(S, \alpha) := D^b(\mathrm{Coh}(S, \alpha))$ to be the category of α-twisted coherent sheaves on S.

Let (S, α) be a twisted K3 surface. For simplicity, we will call $E \in \mathrm{Coh}(S, \alpha)$ a sheaf instead of an α-twisted sheaf.

Take $B \in H^2(S, \mathbb{Q})$ with $\exp(B^{0,2}) = \alpha$. Then B is called a B-field of α. Here $B^{0,2}$ is the $(0, 2)$-part of B in $H^2(S, \mathbb{C})$. We define the twisted Mukai lattice $\widetilde{H}^{1,1}(S, B, \mathbb{Z})$ by

$$\widetilde{H}^{1,1}(S, B, \mathbb{Z}) := e^B\left(\bigoplus_{i=0}^{2} H^{i,i}(S, \mathbb{Q})\right) \cap H^*(S, \mathbb{Z}).$$

The lattice structure is given by the Mukai pairing $\langle -, -\rangle$, where

$$\langle (r, c, d), (r', c', d')\rangle := cc' - rd' - dr'.$$

The twisted Chern character [HS05]

$$\mathrm{ch}^B : K(S, \alpha) \twoheadrightarrow \widetilde{H}^{1,1}(S, B, \mathbb{Z})$$

satisfies the Riemann–Roch formula

$$\chi(E, F) = -\langle v^B(E), v^B(F)\rangle. \tag{2}$$

Here $v^B(E) := \mathrm{ch}^B(E) \cdot \sqrt{\mathrm{td}_S} \in \widetilde{H}^{1,1}(S, B, \mathbb{Z})$ is the (twisted) Mukai vector of $E \in K(S, \alpha)$. We denote by $c_1^B(-)$ the degree-2 part of $v^B(-)$.

Lemma 2.3 [MS12, Lemma 3.1]. *Let d be the order of α. Then the twisted Mukai lattice $\widetilde{H}^{1,1}(S, B, \mathbb{Z})$ is generated by $(d, dB, 0), \mathrm{Pic}(S)$ and $(0,0,1)$ in $H^*(S,\mathbb{Z})$. In particular, the rank of E is divisible by d for all $E \in D^b(S,\alpha)$.*

2.2 Bridgeland stability conditions

Let $\mathcal{D}$ be a triangulated category and $N(\mathcal{D})$ the numerical Grothendieck group of $\mathcal{D}$. Assume that $N(\mathcal{D})$ is finitely generated. If $\mathcal{D}$ is the derived category of a twisted K3 surface, this assumption is satisfied.

Definition 2.4 [Bri07]. A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{C})$ consisting of a group homomorphism (called the central charge) $Z : N(\mathcal{D}) \to \mathbb{C}$ and the heart of a bounded t-structure $\mathcal{C} \subset \mathcal{D}$ on $\mathcal{D}$, which satisfy the following conditions.

- For any $0 \neq E \in \mathcal{C}$, we have $Z(E) \in \{re^{i\pi\phi} \in \mathbb{C} \mid r > 0,\ 0 < \phi \leqslant 1\}$.
- For any $0 \neq E \in \mathcal{C}$, there is a filtration (called the Harder–Narasimhan filtration) in $\mathcal{C}$,

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

  such that $F_i := E_i/E_{i-1}$ is σ-semistable and $\phi(F_i) > \phi(F_{i+1})$ for all $1 \leqslant i \leqslant N-1$.
- Fix a norm $\|\cdot\|$ on $N(\mathcal{D})_{\mathbb{R}}$. Then there is a constant C such that $\|E\| \leqslant C \cdot |Z(E)|$ for any non-zero σ-semistable object $E \in \mathcal{C}$. This property is called the support property.

Here we put $\phi(E) := \arg(Z(E))/\pi \in (0,1]$ for $0 \neq E \in \mathcal{C}$, and $E \in \mathcal{C}$ is σ-stable (respectively, σ-semistable) if the inequality $\phi(F) < \phi(E)$ (respectively, $\phi(F) \leqslant \phi(E)$) holds for any $0 \neq F \subset E$.

Remark 2.5 [Bri07]. We denote by $\mathrm{Stab}(\mathcal{D})$ the set of all stability conditions on $\mathcal{D}$. Then $\mathrm{Stab}(\mathcal{D})$ has a natural topology such that the map

$$\mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(N(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{C}) \mapsto Z$$

is a local homeomorphism. In particular, $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold.

From now on, we focus on stability conditions for derived categories of twisted K3 surfaces. Let (S,α) be a twisted K3 surface and fix a B-field $B \in H^2(S,\mathbb{Q})$ of the Brauer class α. We set $\mathrm{Stab}(S,\alpha) := \mathrm{Stab}(D^b(S,\alpha))$.

Definition 2.6. Fix an ample divisor $\omega \in \mathrm{NS}(S)$ on S. Let $E \in \mathrm{Coh}(S,\alpha)$ be a sheaf. We define the slope $\mu^B(E)$ of E by

$$\mu^B(E) := \frac{c_1^B(E)\cdot\omega}{\mathrm{rk}\, E}.$$

If $\mathrm{rk}\, E = 0$, then we set $\mu^B(E) = \infty$. We say that E is μ^B-stable (respectively, μ^B-semistable) if and only if $\mu^B(F) < \mu^B(E/F)$ (respectively, $\mu^B(F) \leqslant \mu^B(E/F)$) holds for all non-zero subsheaves $F \subset E$.

Note that μ^B-stability admits the Harder–Narasimhan filtrations and Jordan–Hölder filtrations.

Let $\mathrm{Stab}^\dagger(S,\alpha)$ be the distinguished connected component of the space of stability conditions $\mathrm{Stab}(S,\alpha)$; see [Bri08, HMS08]. We will recall how to construct stability conditions on $D^b(S,\alpha)$ in Theorem 6.1.

Remark 2.7 [Bri08, Tod08, BM14b]. Fix a Mukai vector $v \in \widetilde{H}^{1,1}(S, B, \mathbb{Z})$. Then $\mathrm{Stab}^{\dagger}(S, \alpha)$ has a wall and chamber structure which depends only on the choice of v. Upon varying $\sigma \in \mathrm{Stab}^{\dagger}(S, \alpha)$ within a chamber, the set of σ-(semi)stable objects with Mukai vector v does not change. If $\sigma \in \mathrm{Stab}^{\dagger}(S, \alpha)$ is in a chamber, we say that σ is generic with respect to v. If v is primitive, then $\sigma \in \mathrm{Stab}^{\dagger}(S, \alpha)$ is generic with respect to v if and only if all σ-semistable objects with Mukai vector v are σ-stable.

2.3 Moduli spaces of Bridgeland stable complexes on twisted K3 surfaces

We recall some facts about moduli spaces of Bridgeland stable objects on twisted K3 surfaces.

Definition 2.8. An irreducible holomorphic symplectic variety is a simply connected smooth projective variety M with a non-degenerate holomorphic 2-form ω (called a holomorphic symplectic form) such that $H^0(M, \Omega_M^2) = \mathbb{C} \cdot \omega$.

Examples of holomorphic symplectic varieties which will appear later include moduli spaces of Bridgeland stable objects in derived categories of twisted K3 surfaces.

Theorem 2.9 [BM14b]. *Let (S, α) be a twisted K3 surface and $v \in \widetilde{H}^{1,1}(S, B, \mathbb{Z})$ a primitive Mukai vector with $\langle v, v\rangle \geqslant -2$. Let $\sigma \in \mathrm{Stab}^{\dagger}(S, \alpha)$ be a stability condition that is generic with respect to v. Then the coarse moduli space $M_\sigma(v)$ of σ-stable objects with Mukai vector v exists as an irreducible holomorphic symplectic manifold which is deformation-equivalent to the Hilbert scheme of points on a K3 surface, and we have* $\dim M_\sigma(v) = 2 + \langle v, v\rangle$.

2.4 Relation between cubic fourfolds and twisted K3 surfaces

Let X be a cubic fourfold and H a hyperplane section of X. Consider the semiorthogonal decomposition

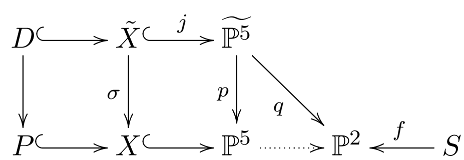
$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle.$$

The full triangulated subcategory

$$\mathcal{A}_X = \{E \in D^b(X) \mid \mathbf{R}\mathrm{Hom}(\mathcal{O}_X(iH), E) = 0,\ i = 0, 1, 2\} \subset D^b(X)$$

is a Calabi–Yau 2-category [Kuz04, Corollary 4.3].

We recall some geometric properties of cubic fourfolds containing a plane [Has99, Kuz10]. Suppose that X contains a plane $P = \mathbb{P}^2$ in $\mathbb{P}^5$. Let $\sigma\colon \tilde{X} \to X$ be the blow-up of X at the plane P and let $p\colon \widetilde{\mathbb{P}^5} \to \mathbb{P}^5$ be the blow-up of $\mathbb{P}^5$ at the plane P. The linear projection from P gives the morphism $q\colon \widetilde{\mathbb{P}^5} \to \mathbb{P}^2$. This is a projectivization of the rank-4 vector bundle $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-h)$ on $\mathbb{P}^2$. Here h is a line in $\mathbb{P}^2$. Let D be the exceptional divisor of σ. Then D is linearly equivalent to $H - h$ on $\tilde{X}$. Set $\pi := q \circ j\colon \tilde{X} \to \mathbb{P}^2$, where $j\colon \tilde{X} \hookrightarrow \widetilde{\mathbb{P}^5}$ is the natural inclusion. Then $\pi\colon \tilde{X} \to \mathbb{P}^2$ is a quadric fibration with degenerate fibres along a plane curve C of degree 6. We assume that fibres of π do not degenerate into the union of two planes. Then C is a smooth curve. Let $f\colon S \to \mathbb{P}^2$ be the double cover ramified along C. Since C is smooth, the surface S is a K3 surface.



We recall Kuznetsov's construction [Kuz10] of the twisted K3 surface (S, α) and the equivalence between $\mathcal{A}_X$ and $D^b(S, \alpha)$.

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The quadric fibration π defines the sheaf of Clifford algebras Cl on $\mathbb{P}^2$. It has even part Cl_0 and odd part Cl_1, which are described as

$$\mathrm{Cl}_0 = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-h)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-2h)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3h),$$
$$\mathrm{Cl}_1 = \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-h)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2h)^{\oplus 3}.$$

Let $\mathrm{Coh}(\mathbb{P}^2, \mathrm{Cl}_0)$ be the category of coherent right Cl_0-modules on $\mathbb{P}^2$. Note that Cl_0 is a spherical object in $D^b(\mathbb{P}^2, \mathrm{Cl}_0)$. Set $D^b(\mathbb{P}^2, \mathrm{Cl}_0) := D^b(\mathrm{Coh}(\mathbb{P}^2, \mathrm{Cl}_0))$.

LEMMA 2.10 [Kuz08, Kuz10]. *There exists a fully faithful functor*

$$\Phi\colon D^b(\mathbb{P}^2, \mathrm{Cl}_0) \hookrightarrow D^b(\tilde{X})$$

with the semiorthgonal decomposition

$$D^b(\tilde{X}) = \langle \Phi(D^b(\mathbb{P}^2, \mathrm{Cl}_0)), \pi^* D^b(\mathbb{P}^2), \pi^* D^b(\mathbb{P}^2)(H)\rangle.$$

The left adjoint functor $\Psi\colon D^b(\tilde{X}) \to D^b(\mathbb{P}^2, \mathrm{Cl}_0)$ of Φ is described as

$$\Psi(-) = \mathbf{R}\pi_*((-) \otimes \mathcal{O}_{\tilde{X}}(h) \otimes \mathcal{E})[2].$$

Here $\mathcal{E}$ is the rank-4 vector bundle on $\tilde{X}$ with the structure of a flat right $\pi^*\mathrm{Cl}_0$-module and the exact sequence

$$0 \to q^*\mathrm{Cl}_1(-h-2H) \to q^*\mathrm{Cl}_0(-H) \to j_*\mathcal{E} \to 0. \tag{3}$$

LEMMA 2.11 [Kuz10]. *The following properties hold.*

- *The functor*
$$\Phi_{\mathbb{P}^2} := \mathbf{R}\sigma_* \mathbf{L}_{\mathcal{O}_{\tilde{X}}(h-H)} \mathbf{R}_{\mathcal{O}_{\tilde{X}}(-h)} \Phi\colon D^b(\mathbb{P}^2, \mathrm{Cl}_0) \to \mathcal{A}_X$$
gives an equivalence.
- *There is a sheaf $\mathcal{B}$ of Azumaya algebras on S such that $f_*\mathcal{B} = \mathrm{Cl}_0$ and $f_*\colon \mathrm{Coh}(S, \mathcal{B}) \to \mathrm{Coh}(\mathbb{P}^2, \mathrm{Cl}_0)$ gives an equivalence.*
- *There exist a Brauer class α of order 2 and a rank-2 vector bundle $\mathcal{U}_0 \in \mathrm{Coh}(S, \alpha)$ such that $\otimes \mathcal{U}_0^\vee\colon \mathrm{Coh}(S, \alpha) \to \mathrm{Coh}(S, \mathcal{B})$ gives an equivalence and $\mathcal{E}nd(\mathcal{U}_0) = \mathcal{B}$.*

COROLLARY 2.12. *The functor $\Phi_S := \Phi_{\mathbb{P}^2} \circ f_* \circ \otimes \mathcal{U}_0^\vee\colon D^b(S, \alpha) \to \mathcal{A}_X$ is an equivalence.*

Remark 2.13. The following holds:

$$\Phi_{\mathbb{P}^2}^{-1} = \Psi \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)} \mathbf{R}_{\mathcal{O}_{\tilde{X}}(h-H)} \mathbf{L}\sigma^*\colon \mathcal{A}_X \to D^b(\mathbb{P}^2, \mathrm{Cl}_0).$$

If X is very general, i.e. $\mathrm{Pic}\, S = \mathbb{Z}$, then α is non-trivial.

PROPOSITION 2.14 [Kuz10, Proposition 4.8]. *If X is very general, then $\mathcal{A}_X$ is not equivalent to $D^b(S')$ for any K3 surface S'. In particular, $\alpha \neq 1$.*

Owing to Lemma 2.3, the condition $\alpha \neq 1$ is a strong constraint. In fact, if $\alpha \neq 1$, then there are no rank-one sheaves on (S, α).

The following lemma will be needed later.

LEMMA 2.15 [MS12, Lemma 2.4]. *The following properties hold.*

- *For any $m \in \mathbb{Z}$, $\Psi(\mathcal{O}_{\tilde{X}}(mh)) = \Psi(\mathcal{O}_{\tilde{X}}(mh - H)) = 0$.*
- *We have $\Psi(\mathcal{O}_{\tilde{X}}(-h+H)) = \mathrm{Cl}_0[2]$ and $\Psi(\mathcal{O}_{\tilde{X}}(h-2H)) = \mathrm{Cl}_1$.*

In the next section, we will look at the construction of the Lagrangian embeddings.

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3. Formulation of the main proposition

In this section, we define the projection functor and formulate Proposition 1.5.

Definition 3.1. Let X be a cubic fourfold and H a hyperplane section of X. We define the projection functor as

$$\mathrm{pr} := \mathbf{R}_{\mathcal{O}_X(-H)}\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{O}_X(H)}[1] \colon D^b(X) \to \mathcal{A}_X. \tag{4}$$

From now on, we use the same notation as in § 2.4.

Definition 3.2. For a point $x \in X$, let $P_x := \Phi_S^{-1}(\mathrm{pr}(\mathcal{O}_x))[-4] \in D^b(S, \alpha)$.

The following proposition is a more precise version of Proposition 1.5.

Proposition 3.3 (Cf. Proposition 1.5). *Assume that X is a very general cubic fourfold containing a plane P. Fix a B-field $B \in H^2(S, \mathbb{Q})$ of the Brauer class α and let $v := v^B(P_x) \in \widetilde{H}^{1,1}(S, B, \mathbb{Z})$ for $x \in X$. Then the following properties hold:*

- (a) *There is a stability condition $\sigma \in \mathrm{Stab}^\dagger(S, \alpha)$, generic with respect to v, such that P_x is σ-stable for each $x \in X$;*
- (b) *$M_\sigma(v)$ is a holomorphic symplectic eightfold;*
- (c) *$X \to M_\sigma(v)$, $x \mapsto P_x$ is a closed immersion;*
- (d) *X is a Lagrangian submanifold of $M_\sigma(v)$.*

In the rest of the paper, we will give a proof of Proposition 3.3. In the proof of property (a), we will construct a family $\{\sigma_\lambda\}$ of stability conditions generic with respect to v such that P_x is σ_λ-stable for each $x \in X$. The construction of stability conditions will take place in § 6. Property (b) is deduced from Theorem 2.9 and $\mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C} \oplus \mathbb{C}^8[-1] \oplus \mathbb{C}[-2]$ or $\langle v, v\rangle = 6$. The Mukai vector v will be calculated in § 5. In proving properties (c) and (d), we will identify the tangent spaces $\mathrm{T}_x X$ and $\mathrm{T}_x M_\sigma(v)$ with $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$ and $\mathrm{Ext}^1(P_x, P_x)$, respectively. Statements (c) and (d) are deduced from Proposition 1.4; this will be done in § 4. Note that in the proof of Proposition 1.4 we will not use K3 surfaces and the plane P in a cubic fourfold X.

4. The projection functor and Lagrangian embeddings

In this section we prove Proposition 1.4. Let X be a cubic fourfold and H a hyperplane section of X. Take a point $x \in X$. Let $I_x \subset \mathcal{O}_X$ be the ideal sheaf of $x \in X$. Considering the restriction of the Koszul complex for a point $x \in X \subset \mathbb{P}^5$, we have the exact sequence

$$0 \to F_x \to \mathcal{O}_X^{\oplus 5} \to \mathcal{O}_X(H) \to \mathcal{O}_x \to 0. \tag{5}$$

Note that $\mathrm{Im}(\mathcal{O}_X^{\oplus 5} \to \mathcal{O}_X(H)) = I_x(H)$. First of all, we collect some facts about cohomology groups which are used in the proof of Proposition 1.4.

Lemma 4.1. *Let $L \in \mathrm{Pic}\, X$ be a line bundle on X. Then:*

- $\mathbf{R}\mathcal{H}om(\mathcal{O}_x, L) = \mathcal{O}_x[-4]$;
- $\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, L) = \mathbb{C}[-4]$.

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Proof. The second assertion can be deduced from the first. So here we prove the first assertion. Let $i_x\colon x \hookrightarrow X$ be the natural inclusion. Using Grothendieck–Verdier duality, we have the isomorphisms

$$\begin{aligned}\mathbf{R}\mathcal{H}om(\mathcal{O}_x, L) &= \mathbf{R}\mathcal{H}om(i_{x*}\mathcal{O}_x, L)\\ &\simeq i_{x*}\mathbf{R}\mathcal{H}om_x(\mathcal{O}_x, i_x^! L)\\ &\simeq i_{x*}\mathbf{R}\mathcal{H}om_x(\mathcal{O}_x, \mathcal{O}_x[-4])\\ &\simeq \mathcal{O}_x[-4].\end{aligned}$$

□

Lemma 4.2. *The following properties hold:*

(a) $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X, F_x) = 0$;
(b) $\mathbf{R}\mathrm{Hom}(F_x, \mathcal{O}_X(-H)) = \mathbb{C}[-2]$;
(c) $\mathbf{R}\Gamma(X, F_x(H)) = \mathbb{C}^{10}$;
(d) $\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X) = \mathbb{C}[-3]$.

Proof. (a), (c) These are deduced from $\mathbf{R}\Gamma(X, I_x(H)) = \mathbb{C}^5$, $\mathbf{R}\Gamma(X, I_x(2H)) = \mathbb{C}^{20}$ and the exact sequence

$$0 \to F_x \to \mathcal{O}_X^{\oplus 5} \to I_x(H) \to 0. \tag{6}$$

(b) Consider the following exact triangles:

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_X(-H)) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), \mathcal{O}_X(-H)) \to \mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H)),$$
$$\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H)) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X^{\oplus 5}, \mathcal{O}_X(-H)) \to \mathbf{R}\mathrm{Hom}(F_x, \mathcal{O}_X(-H)).$$

By Lemma 4.1 and $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), \mathcal{O}_X(-H)) = 0$, the first exact triangle is nothing but

$$\mathbb{C}[-4] \to 0 \to \mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H)).$$

So we obtain $\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H)) = \mathbb{C}[-3]$.

Since $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X^{\oplus 5}, \mathcal{O}_X(-H)) = 0$, the second exact triangle is nothing but

$$\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H)) \to 0 \to \mathbf{R}\mathrm{Hom}(F_x, \mathcal{O}_X(-H)).$$

This implies

$$\begin{aligned}\mathbf{R}\mathrm{Hom}(F_x, \mathcal{O}_X(-H)) &= \mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X(-H))[1]\\ &= \mathbb{C}[-2].\end{aligned}$$

(d) Consider the exact triangle

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X).$$

By Lemma 4.1 and $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), \mathcal{O}_X) = 0$, we obtain

$$\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X) = \mathbb{C}[-3].$$

□

By $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), I_x(H)) = \mathbf{R}\Gamma(X, I_x) = 0$ and Lemma 4.2(a), we have the following remark.

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Remark 4.3. We have

$$\mathbf{L}_{\mathcal{O}_X(H)}(\mathcal{O}_x)[-1] = I_x(H),$$
$$\mathbf{L}_{\mathcal{O}_X}(I_x(H))[-1] = F_x.$$

Moreover, we get the exact triangles

$$I_x(H) \hookrightarrow \mathcal{O}_X(H) \twoheadrightarrow \mathcal{O}_x \xrightarrow{e_1} I_x(H)[1], \tag{7}$$
$$F_x \hookrightarrow \mathcal{O}_X^{\oplus 5} \twoheadrightarrow I_x(H) \xrightarrow{e_2} F_x[1], \tag{8}$$
$$F_x \xrightarrow{c} \mathcal{O}_X(-H)[2] \to \mathrm{pr}(\mathcal{O}_x) \xrightarrow{e_3} F_x \tag{9}$$

by the definition of mutation and Lemma 4.2(b).

The following proposition is the first statement in Proposition 1.4.

Proposition 4.4. *Let $x \neq y \in X$ be distinct points in X. Then $\mathrm{pr}(\mathcal{O}_x)$ is not isomorphic to $\mathrm{pr}(\mathcal{O}_y)$.*

Proof. By (9) we have $\mathcal{H}^0(\mathrm{pr}(\mathcal{O}_x)) = F_x$. So it is sufficient to prove that F_x is not isomorphic to F_y. Therefore we prove that $\mathcal{E}xt^2(F_x, \mathcal{O}_X) \simeq \mathcal{O}_x$.

Applying $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ to the exact triangles (7) and (8), we obtain the isomorphisms

$$\begin{aligned}\mathcal{E}xt^2(F_x, \mathcal{O}_X) &\simeq \mathcal{E}xt^3(I_x(H), \mathcal{O}_X)\\ &\simeq \mathcal{E}xt^4(\mathcal{O}_x, \mathcal{O}_X)\\ &\simeq \mathcal{O}_x.\end{aligned}$$ □

We will prove the remaining statements of Proposition 1.4. To do so, we need some cohomology computations.

Lemma 4.5. *There are the following isomorphisms:*

$$\circ\, e_1 \colon \mathbf{R}\mathrm{Hom}(I_x(H), I_x(H)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(\mathcal{O}_x, I_x(H))[1], \tag{10}$$
$$\circ\, e_2 \colon \mathbf{R}\mathrm{Hom}(F_x, F_x) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(I_x(H), F_x)[1], \tag{11}$$
$$e_3 \circ \colon \mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), F_x)[1]. \tag{12}$$

Proof. Applying $\mathbf{R}\mathrm{Hom}(-, I_x(H))$ to the exact sequence (7), we have the exact triangle

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, I_x(H)) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), I_x(H)) \to \mathbf{R}\mathrm{Hom}(I_x(H), I_x(H)).$$

Since $I_x(H) \in \langle \mathcal{O}_X(H) \rangle^{\perp}$, we have $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X(H), I_x(H)) = 0$. So we obtain the isomorphism

$$\circ\, e_1 \colon \mathbf{R}\mathrm{Hom}(I_x(H), I_x(H)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(\mathcal{O}_x, I_x(H))[1].$$

Similarly, using $F_x \in \langle \mathcal{O}_X \rangle^{\perp}$ and $\mathrm{pr}(\mathcal{O}_x) \in \mathcal{A}_X$, we obtain

$$\circ\, e_2 \colon \mathbf{R}\mathrm{Hom}(F_x, F_x) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(I_x(H), F_x)[1],$$
$$e_3 \circ \colon \mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), F_x)[1].$$ □

Lemma 4.6. *There are the following isomorphisms:*

$$e_1 \circ \colon \mathrm{Ext}^{i-1}(\mathcal{O}_x, \mathcal{O}_x) \xrightarrow{\sim} \mathrm{Ext}^i(\mathcal{O}_x, I_x(H)) \quad (i = 1, 2, 3),$$
$$e_2 \circ \colon \mathrm{Ext}^{i-1}(I_x(H), I_x(H)) \xrightarrow{\sim} \mathrm{Ext}^i(I_x(H), F_x) \quad (i = 1, 2),$$
$$\circ\, e_3 \colon \mathrm{Hom}(F_x, F_x) \xrightarrow{\sim} \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), F_x).$$

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Proof. To prove the first isomorphism, consider the exact triangle

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, I_x(H)) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_X(H)) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x). \tag{13}$$

By Lemma 4.1, the exact triangle (13) is nothing but

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_x, I_x(H)) \to \mathbb{C}[-4] \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x). \tag{14}$$

Taking the long exact sequence, we get the first isomorphism.

Next, we prove the second isomorphism. Consider the exact triangle

$$\mathbf{R}\mathrm{Hom}(I_x(H), F_x) \to \mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X^{\oplus 5}) \to \mathbf{R}\mathrm{Hom}(I_x(H), I_x(H)). \tag{15}$$

By Lemma 4.2(d), the exact triangle (15) is nothing but

$$\mathbf{R}\mathrm{Hom}(I_x(H), F_x) \to \mathbb{C}^5[-3] \to \mathbf{R}\mathrm{Hom}(I_x(H), I_x(H)). \tag{16}$$

Taking the long exact sequence, we get the second isomorphism. Finally, to prove the last isomorphism, consider the exact triangle

$$\mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), F_x) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X(-H)[-2], F_x) \to \mathbf{R}\mathrm{Hom}(F_x, F_x). \tag{17}$$

By Lemma 4.2(c), the exact triangle (17) is nothing but

$$\mathbf{R}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), F_x) \to \mathbb{C}^{10}[-2] \to \mathbf{R}\mathrm{Hom}(F_x, F_x). \tag{18}$$

Taking the long exact sequence, we get the last isomorphism. □

We can prove that the object $\mathrm{pr}(\mathcal{O}_x)$ is simple.

Corollary 4.7. *We have* $\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}$.

Proof. By Lemmas 4.5 and 4.6, we have the isomorphisms

$$\begin{aligned}\mathrm{Hom}(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) &\overset{e_3\circ}{\simeq} \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), F_x)\\ &\overset{\circ e_3}{\simeq} \mathrm{Hom}(F_x, F_x)\\ &\overset{\circ e_2}{\simeq} \mathrm{Ext}^1(I_x(H), F_x)\\ &\overset{e_2\circ}{\simeq} \mathrm{Hom}(I_x(H), I_x(H))\\ &\overset{\circ e_1}{\simeq} \mathrm{Ext}^1(\mathcal{O}_x, I_x(H))\\ &\overset{e_1\circ}{\simeq} \mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x) = \mathbb{C}.\end{aligned}$$

□

Corollary 4.8. *The linear map*

$$\mathrm{pr}\colon \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x))$$

is injective.

Proof. By Lemmas 4.5 and 4.6, the linear map

$$\mathrm{pr}\colon \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x))$$

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can be factorized as follows:

$$\begin{aligned}\mathrm{pr}\colon \mathrm{Ext}^1(\mathcal{O}_x,\mathcal{O}_x) &\overset{e_1\circ}{\simeq} \mathrm{Ext}^2(\mathcal{O}_x, I_x(H))\\ &\overset{\circ e_1}{\simeq} \mathrm{Ext}^1(I_x(H), I_x(H))\\ &\overset{e_2\circ}{\simeq} \mathrm{Ext}^2(I_x(H), F_x)\\ &\overset{\circ e_2}{\simeq} \mathrm{Ext}^1(F_x, F_x)\\ &\overset{\circ e_3}{\hookrightarrow} \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), F_x)\\ &\overset{e_3\circ}{\simeq} \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)).\end{aligned}$$ □

Corollary 4.9. *We have* $\mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}^8$.

Proof. First, we prove that $\mathrm{Ext}^2(F_x, F_x) = \mathbb{C}^7$. By the exact triangle (16), we have the exact sequence

$$\begin{aligned}0 &\to \mathrm{Ext}^2(I_x(H), I_x(H)) \to \mathrm{Ext}^3(I_x(H), F_x) \to \mathbb{C}^5\\ &\to \mathrm{Ext}^3(I_x(H), I_x(H)) \to \mathrm{Ext}^4(I_x(H), F_x) \to 0.\end{aligned}$$

By Lemma 4.5 and the isomorphism (12), we have

$$\begin{aligned}\mathrm{Ext}^4(I_x(H), F_x) &\simeq \mathrm{Ext}^3(F_x, F_x)\\ &\simeq \mathrm{Ext}^4(\mathrm{pr}(\mathcal{O}_x), F_x)\\ &\simeq \mathrm{Ext}^3(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = 0.\end{aligned}$$

By Lemmas 4.5 and 4.6, we have

$$\begin{gathered}\mathrm{Ext}^2(I_x(H), I_x(H)) = \mathbb{C}^6,\\ \mathrm{Ext}^3(I_x(H), I_x(H)) = \mathbb{C}^4,\\ \mathrm{Ext}^2(F_x, F_x) \simeq \mathrm{Ext}^3(I_x(H), F_x).\end{gathered}$$

So the above long exact sequence can be described as

$$0 \to \mathbb{C}^6 \to \mathrm{Ext}^2(F_x, F_x) \to \mathbb{C}^5 \to \mathbb{C}^4 \to 0.$$

Hence $\mathrm{Ext}^2(F_x, F_x) = \mathbb{C}^7$.

By Lemmas 4.5 and 4.6, we have

$$\begin{gathered}\mathrm{Ext}^1(F_x, F_x) = \mathbb{C}^4,\\ \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \simeq \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), F_x).\end{gathered} \tag{19}$$

Moreover, using Lemma 4.5 and Corollary 4.7, we have

$$\mathrm{Ext}^3(\mathrm{pr}(\mathcal{O}_x), F_x) \simeq \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}.$$

Here the last equality is deduced from the Serre duality for $\mathcal{A}_X$. By the exact triangle (18), we obtain the long exact sequence

$$0 \to \mathbb{C}^4 \to \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \to \mathbb{C}^{10} \to \mathbb{C}^7 \to \mathbb{C} \to 0.$$

So $\mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) = \mathbb{C}^8$. □

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Finally, we prove the last statement in Proposition 1.4. Before giving a proof, we recall the definition of the bilinear form on $\mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x))$, which corresponds to the symplectic forms on moduli spaces of Bridgeland stable complexes on twisted K3 surfaces.

DEFINITION 4.10. We define a bilinear form

$$\omega_x \colon \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \times \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \to \mathbb{C}$$

by the composition of morphisms in the derived category.

The following proposition implies Proposition 3.3(d).

PROPOSITION 4.11. *The bilinear form* ω_x *vanishes on* $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \times \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \times \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) & \xrightarrow{\text{composition}} & \mathrm{Ext}^2(\mathcal{O}_x, \mathcal{O}_x) \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \times \mathrm{Ext}^1(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) & \xrightarrow[\omega_x]{} & \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)). \end{array}$$

It is sufficient to prove that

$$\mathrm{pr} \colon \mathrm{Ext}^2(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)) \tag{20}$$

is zero.

The linear map (20) can be factorized as follows:

$$\begin{aligned} \mathrm{pr} \colon \mathrm{Ext}^2(\mathcal{O}_x, \mathcal{O}_x) &\overset{e_1\circ}{\simeq} \mathrm{Ext}^3(\mathcal{O}_x, I_x(H)) \\ &\overset{\circ e_1}{\simeq} \mathrm{Ext}^2(I_x(H), I_x(H)) \\ &\overset{e_2\circ}{\hookrightarrow} \mathrm{Ext}^3(I_x(H), F_x) \\ &\overset{\circ e_2}{\simeq} \mathrm{Ext}^2(F_x, F_x) \\ &\overset{\circ e_3}{\twoheadrightarrow} \mathrm{Ext}^3(\mathrm{pr}(\mathcal{O}_x), F_x) \\ &\overset{e_3\circ}{\simeq} \mathrm{Ext}^2(\mathrm{pr}(\mathcal{O}_x), \mathrm{pr}(\mathcal{O}_x)). \end{aligned}$$

Applying $\mathbf{R}\mathrm{Hom}(-, \mathcal{O}_X)$ to the exact triangle (8), we have the exact triangle

$$\mathbf{R}\mathrm{Hom}(I_x(H), \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(\mathcal{O}_X^{\oplus 5}, \mathcal{O}_X) \to \mathbf{R}\mathrm{Hom}(F_x, \mathcal{O}_X).$$

Since $\mathbf{R}\mathrm{Hom}(\mathcal{O}_X^{\oplus 5}, \mathcal{O}_X) = \mathbb{C}^5$, we have

$$\mathrm{Ext}^2(F_x, \mathcal{O}_X^{\oplus 5}) \overset{\circ e_2}{\simeq} \mathrm{Ext}^3(I_x(H), \mathcal{O}_X^{\oplus 5}). \tag{21}$$

By the isomorphism (21) and the exact triangle (15), we have

$$\begin{aligned} &\mathrm{Im}(\mathrm{Ext}^2(I_x(H), I_x(H)) \hookrightarrow \mathrm{Ext}^2(F_x F_x)) \\ &\quad = \mathrm{Ker}(\mathrm{Ext}^2(F_x, F_x) \to \mathrm{Ext}^2(F_x, \mathcal{O}_X^{\oplus 5})). \end{aligned}$$

Note that this vector space is six-dimensional.

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Recall that $c\colon F_x \to \mathcal{O}_X(-H)[2]$ is the morphism in the exact triangle (9). Taking the long exact sequence of the exact triangle (17), we obtain the exact sequence

$$\begin{aligned} 0 \to \operatorname{Ext}^1(F_x, F_x) &\overset{\circ e_3}{\to} \operatorname{Ext}^2(\operatorname{pr}(\mathcal{O}_x), F_x) \to \operatorname{Ext}^2(\mathcal{O}_X(-H)[2], F_x) \\ &\overset{\circ c}{\to} \operatorname{Ext}^2(F_x, F_x) \overset{\circ e_3}{\twoheadrightarrow} \operatorname{Ext}^3(\operatorname{pr}(\mathcal{O}_x), F_x) \to 0. \end{aligned}$$

Hence we have

$$\begin{aligned} &\operatorname{Ker}(\operatorname{Ext}^2(F_x, F_x) \overset{\circ e_3}{\twoheadrightarrow} \operatorname{Ext}^3(F_x, \operatorname{pr}(\mathcal{O}_x))) \\ &\quad = \operatorname{Im}(\operatorname{Ext}^2(\mathcal{O}_X(-H)[2], F_x) \overset{\circ c}{\to} \operatorname{Ext}^2(F_x, F_x)). \end{aligned}$$

By (19), Lemma 4.2(c), Lemma 4.5 and Corollary 4.9, this vector space is six-dimensional.

So it is enough to prove that

$$\operatorname{Im}(\operatorname{Ext}^2(\mathcal{O}_X(-H)[2], F_x) \overset{\circ c}{\to} \operatorname{Ext}^2(F_x, F_x)) \subset \operatorname{Ker}(\operatorname{Ext}^2(F_x, F_x) \to \operatorname{Ext}^2(F_x, \mathcal{O}_X^{\oplus 5})).$$

Take $\psi \in \operatorname{Im}(\operatorname{Ext}^2(\mathcal{O}_X(-H)[2], F_x) \overset{\circ c}{\to} \operatorname{Ext}^2(F_x, F_x))$.

Then there is a morphism $\eta \in \operatorname{Ext}^2(\mathcal{O}_X(-H)[2], F_x)$ that satisfies the following commutative diagram:

$$\begin{array}{ccc} F_x & \xrightarrow{\ c\ } & \mathcal{O}_X(-H)[2] \\ {\scriptstyle\psi}\downarrow & \swarrow{\scriptstyle\eta} & \\ F_x[2] & \longrightarrow & \mathcal{O}_X^{\oplus 5}[2]. \end{array}$$

Take a hyperplane section H of X such that $x \notin H$. Let $i\colon \mathcal{O}_X(-H) \to \mathcal{O}_X$ be the morphism defining H.

We prove that $i[2] \circ c \neq 0$. Assume that $i[2] \circ c = 0$. Then there is a morphism between exact triangles as follows:

$$\begin{array}{ccccc} F_x & \xrightarrow{\ c\ } & \mathcal{O}_X(-H) & \longrightarrow & \operatorname{pr}(\mathcal{O}_x) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \mathcal{O}_H(1)[1] & \longrightarrow & \mathcal{O}_X(-H)[2] & \xrightarrow{\ i[2]\ } & \mathcal{O}_X[2]. \end{array}$$

Since $x \notin H$, we have

$$0 \to F_x|_H \to \mathcal{O}_H^{\oplus 5} \to \mathcal{O}_H(1) \to 0,$$

which is the restriction of the exact sequence (7).

Applying $\mathbf{R}\operatorname{Hom}(-, \mathcal{O}_H)$ to this exact sequence, we have

$$\mathbf{R}\operatorname{Hom}(F_x, \mathcal{O}_H) = \mathbf{R}\operatorname{Hom}(F_x|H, \mathcal{O}_H) = \mathbb{C}^5.$$

This implies $\operatorname{Ext}^1(F_x, \mathcal{O}_H) = 0$. Since $c \neq 0$, we have a contradiction.

Note that the vector space $\operatorname{Ker}(\operatorname{Ext}^2(\mathcal{O}_X(-H)[2], \mathcal{O}_X) \overset{\circ c}{\twoheadrightarrow} \operatorname{Ext}^2(F_x, \mathcal{O}_X))$ is generated by morphisms $\mathcal{O}_X(-H) \to \mathcal{O}_X$ induced by hyperplane sections of X, which pass through the point x. By the definition of F_x, the composition $\mathcal{O}_X(-H) \overset{\eta}{\to} F_x \to \mathcal{O}_X^{\oplus 5}$ is induced by hyperplane sections of X, which pass through $x \in X$. So the composition $F_x \overset{\psi}{\to} F_x[2] \to \mathcal{O}_X^{\oplus 5}$ is zero. Hence

$$\psi \in \operatorname{Ker}(\operatorname{Ext}^2(F_x, F_x) \to \operatorname{Ext}^2(F_x, \mathcal{O}_X^{\oplus 5})). \qquad \square$$

Thus we have proved Proposition 1.4. In the next section, we will look at properties of the object P_x on the twisted K3 surface, which corresponds to the point $x \in X$.

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5. Description of complexes on twisted K3 surfaces

Let X be a cubic fourfold containing a plane P as in § 2.4, and let (S,α) be the corresponding twisted K3 surface. We use the same notation as in § 2.4.

DEFINITION 5.1. For a point $x \in X$, we define the objects $R_x \in D^b(\mathbb{P}^2, \mathrm{Cl}_0)$ and $P_x \in D^b(S,\alpha)$ as

$$R_x := \Phi_{\mathbb{P}^2}^{-1}(\mathrm{pr}(\mathcal{O}_x))[-4] \in D^b(\mathbb{P}^2, \mathrm{Cl}_0),$$
$$P_x := (\otimes \mathcal{U}_0^\vee \circ f_*)^{-1} R_x \in D^b(S,\alpha).$$

LEMMA 5.2. *Let $x \in X$ be a point. Then:*

(a) $R_x \simeq \Psi(\mathbf{L}\sigma^*(I_x(H)))[-2]$;

(b) *there is the exact triangle*

$$R_x \to \mathrm{Cl}_0(h) \to \Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2]. \tag{22}$$

Proof. (a) Since $\mathbf{L}\sigma^* \colon D^b(\tilde{X}) \to D^b(X)$ is fully faithful, we have

$$\begin{aligned} R_x &= \Phi_{\mathbb{P}^2}^{-1}(\mathrm{pr}(\mathcal{O}_x))[-4] \\ &= \Psi \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)} \mathbf{R}_{\mathcal{O}_{\tilde{X}}(h-H)} \mathbf{L}\sigma^* \mathbf{R}_{\mathcal{O}_X(-H)} \mathbf{L}_{\mathcal{O}_X} \mathbf{L}_{\mathcal{O}_X(H)}(\mathcal{O}_x)[-3] \\ &\simeq \Psi \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)} \mathbf{R}_{\mathcal{O}_{\tilde{X}}(h-H)} \mathbf{R}_{\mathcal{O}_{\tilde{X}}(-H)} \mathbf{L}_{\mathcal{O}_{\tilde{X}}} \mathbf{L}\sigma^*(I_x(H))[-2]. \end{aligned}$$

First, we prove that $\Psi \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)}(E) \simeq \Psi(E)$ for any $E \in D^b(\tilde{X})$. By the definition of mutation functors, there is the exact triangle

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_{\tilde{X}}(-h), E) \otimes \mathcal{O}_{\tilde{X}}(-h) \to E \to \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)}(E).$$

Applying the functor Ψ, we obtain the exact triangle

$$\Psi(\mathbf{R}\mathrm{Hom}(\mathcal{O}_{\tilde{X}}(-h), E) \otimes \mathcal{O}_{\tilde{X}}(-h)) \to \Psi(E) \to \Psi(\mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)}(E)).$$

By Lemma 2.15, we have

$$\Psi(\mathbf{R}\mathrm{Hom}(\mathcal{O}_{\tilde{X}}(-h), E) \otimes \mathcal{O}_{\tilde{X}}(-h)) = \mathbf{R}\mathrm{Hom}(\mathcal{O}_{\tilde{X}}(-h), E) \otimes \Psi(\mathcal{O}_{\tilde{X}}(-h)) = 0.$$

So $\Psi \mathbf{L}_{\mathcal{O}_{\tilde{X}}(-h)}(E) \simeq \Psi(E)$.

Imitating these arguments, we obtain the isomorphism $R_x \simeq \Psi(\mathbf{L}\sigma^*(I_x(H)))[-2]$.

(b) Applying $\Psi(\mathbf{L}\sigma^*(-))[-2]$ to the exact triangle (7), we obtain the exact triangle

$$R_x \to \Psi(\mathbf{L}\sigma\mathcal{O}_X(H))[-2] \to \Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2].$$

By Lemma 2.15, we have the isomorphisms

$$\begin{aligned} \Psi(\mathbf{L}\sigma\mathcal{O}_X(H))[-2] &\simeq \Psi(\mathcal{O}_{\tilde{X}}(H))[-2] \\ &\simeq \Psi(\mathcal{O}_{\tilde{X}}(-h+H))(h)[-2] \\ &\simeq \mathrm{Cl}_0(h). \end{aligned}$$

Thus we obtain the desired exact triangle. □

If $x \in P$, then $\mathcal{H}^{-1}(\mathbf{L}\sigma^*\mathcal{O}_x) = \mathcal{O}_{\sigma^{-1}(x)}(D)$, $\mathcal{H}^0(\mathbf{L}\sigma^*\mathcal{O}_x) = \mathcal{O}_{\sigma^{-1}(x)}$, and the others are all zero. Since $D = H - h$, we have the following lemma.

LEMMA 5.3. *The following properties hold.*

- *If* $x \in X \backslash P$, *then* $\Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2] = \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))}$.
- *If* $x \in P$, *then*

$$\mathcal{H}^0(\Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2]) = \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))},$$
$$\mathcal{H}^{-1}(\Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2]) = \pi_*(\mathcal{E})|_{\pi(\sigma^{-1}(x))},$$

and the others are zero.

LEMMA 5.4. *Let* $x \in X$ *be a point. Then the following properties hold.*

(a) *The object* R_x *is a sheaf.*

(b) *Assume that* $x \in X \backslash P$. *Taking the long exact sequence of the exact triangle (22), we have the exact sequence*

$$0 \to R_x \to \mathrm{Cl}_0(h) \to \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))} \to 0.$$

Here $\pi(\sigma^{-1}(x))$ *is a point in* $\mathbb{P}^2$.

(c) *Assume that* $x \in P$. *Taking the long exact sequence of the exact triangle (22), we have the exact sequence*

$$0 \to (R_x)_{\mathrm{tor}} \to R_x \to \mathrm{Cl}_0(h) \to \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))} \to 0.$$

Note that $\pi(\sigma^{-1}(x))$ *is a line in* $\mathbb{P}^2$ *and* $(R_x)_{\mathrm{tor}}$ *is a one-dimensional pure torsion sheaf.*

Proof. Take a point $x \in X$. By Lemma 5.3, it is sufficient to prove that the morphism $\mathrm{Cl}_0(h) \to \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))}$ is surjective.

Restricting the exact sequence (3) to $\tilde{X}$, we have the surjection $\pi^*\mathrm{Cl}_0(-H) \twoheadrightarrow \mathcal{E}$. So we can obtain the surjective morphism $\pi^*\mathrm{Cl}_0(h-H) \twoheadrightarrow \mathcal{E}(h)$.

Note that $\pi|_{\sigma^{-1}(x)} \colon \sigma^{-1}(x) \to \mathbb{P}^2$ is a closed immersion. By restricting the morphism $\pi^*\mathrm{Cl}_0(h-H) \twoheadrightarrow \mathcal{E}(h)$ to $\sigma^{-1}(x)$ and taking direct images of π, we obtain the surjective morphism $\mathrm{Cl}_0(h-H)|_{\pi(\sigma^{-1}(x))} \twoheadrightarrow \mathcal{E}(h)|_{\pi(\sigma^{-1}(x))}$. Now we can ignore $\otimes\mathcal{O}_X(-H)$; so there is the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Cl}_0(h) & & \\ \downarrow{\scriptstyle \text{restriction}} & \searrow & \\ \mathrm{Cl}_0(h)|_{\pi(\sigma^{-1}(x))} & \twoheadrightarrow & \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))}. \end{array}$$

Therefore the morphism $\mathrm{Cl}_0(h) \to \pi_*(\mathcal{E}(h))|_{\pi(\sigma^{-1}(x))}$ is surjective. □

Considering these exact sequences on the twisted K3 surface (S, α), we have the following proposition.

PROPOSITION 5.5. *Let* $x \in X$ *be a point. Then there is the exact triangle*

$$P_x \to \mathcal{U}_0 \to Q_x. \tag{23}$$

Here $Q_x := (f_*(- \otimes \mathcal{U}_0^\vee))^{-1}(\Psi(\mathbf{L}\sigma^*\mathcal{O}_x)[-2]) \in D^b(S, \alpha)$.

If $x \in X \backslash P$, *then* Q_x *is a zero-dimensional torsion sheaf of length 2 and the exact triangle (23) induces in* $\mathrm{Coh}(S, \alpha)$ *the exact sequence*

$$0 \to P_x \to \mathcal{U}_0 \to Q_x \to 0. \tag{24}$$

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If $x \in P$, then the exact triangle (23) induces in $\mathrm{Coh}(S,\alpha)$ *the exact sequence*

$$0 \to (P_x)_{\mathrm{tor}} \to P_x \to \mathcal{U}_0 \to \mathcal{H}^0(Q_x) \to 0. \tag{25}$$

Note that $(P_x)_{\mathrm{tor}} = \mathcal{H}^0(Q_x)(-h)$ is a one-dimensional pure torsion sheaf.

Definition 5.6. We define the rank-2 α-twisted vector bundle $\mathcal{U}_1 \in \mathrm{Coh}(S,\alpha)$ by

$$\mathcal{U}_1 := (\otimes \mathcal{U}_0^\vee \circ f_*)^{-1} \mathrm{Cl}_0.$$

For $x \in P$, the torsion part $(P_x)_{\mathrm{tor}}$ is related to $\mathcal{U}_1$.

Lemma 5.7. *Let $x \in P$ be a point. Then we have an exact triangle*

$$\mathcal{U}_1(-h) \to \mathcal{U}_0 \to (P_x)_{\mathrm{tor}}. \tag{26}$$

Proof. Take a line $C_x \subset X$ which passes through a point x.

Let $C'_x := \sigma^{-1}(C_x)$ and $l_x := \sigma^{-1}(x)$. Then there are isomorphisms

$$\mathcal{O}_D(-C'_x) \simeq \mathcal{O}_D(-H),$$
$$\mathcal{O}_{C'_x}(-l_x) \simeq \mathcal{O}_{C'_x}(-H).$$

Consider the following exact sequences:

$$0 \to \mathcal{O}_{\tilde{X}}(-D)(= \mathcal{O}_{\tilde{X}}(h-H)) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_D \to 0, \tag{27}$$
$$0 \to \mathcal{O}_D(-C'_x)(= \mathcal{O}_D(-H)) \to \mathcal{O}_D \to \mathcal{O}_{C'_x} \to 0, \tag{28}$$
$$0 \to \mathcal{O}_{C'_x} \to \mathcal{O}_{C'_x}(H) \to \mathcal{O}_{l_x} \to 0. \tag{29}$$

Here the exact sequence (29) is induced by the exact sequence

$$0 \to \mathcal{O}_{C'_x}(-l_x)(= \mathcal{O}_{C'_x}(-H)) \to \mathcal{O}_{C'_x} \to \mathcal{O}_{l_x} \to 0.$$

(Note that $\mathcal{O}_{l_x}(H) \simeq \mathcal{O}_{l_x}$.)

Applying the functor $\Psi(-)[-2]$ to (29), we have the exact triangle

$$\Psi(\mathcal{O}_{C'_x})[-2] \to \Psi(\mathcal{O}_{C'_x}(H))[-2] \to \Psi(\mathcal{O}_{l_x})[-2].$$

If $\Psi(\mathcal{O}_{C'_x})[-2] \simeq \mathrm{Cl}_1$ and $\Psi(\mathcal{O}_{C'_x}(H))[-2] \simeq \mathrm{Cl}_0(h)$, there is an exact triangle

$$\mathcal{U}_1 \to \mathcal{U}_0(h) \to \mathcal{H}^0(Q_x). \tag{30}$$

This implies the statement in the lemma by Proposition 5.5. So we need to prove that $\Psi(\mathcal{O}_{C'_x})[-2] \simeq \mathrm{Cl}_1$ and $\Psi(\mathcal{O}_{C'_x}(H))[-2] \simeq \mathrm{Cl}_0(h)$.

First we prove that $\Psi(\mathcal{O}_{C'_x})[-2] \simeq \mathrm{Cl}_1$. By Lemma 2.15 and the exact triangle (27), we have $\Psi(\mathcal{O}_D) = 0$.

Applying the functor Ψ to the sequence (28), we have the exact triangle

$$\Psi(\mathcal{O}_D(-H)) \to \Psi(\mathcal{O}_D) \to \Psi(\mathcal{O}_{C'_x}).$$

Since $\Psi(\mathcal{O}_D) = 0$, we have $\Psi(\mathcal{O}_{C'_x}) \simeq \Psi(\mathcal{O}_D(-H))[1]$.

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Applying $\Psi(- \otimes \mathcal{O}_{\tilde{X}}(-H))$ to the sequence (27), we have

$$\Psi(\mathcal{O}_{\tilde{X}}(h-2H)) \to \Psi(\mathcal{O}_{\tilde{X}}(-H)) \to \Psi(\mathcal{O}_D(-H)).$$

By Lemma 2.15, we get isomorphisms

$$\begin{aligned}\Psi(\mathcal{O}_{C'_x}) &\simeq \Psi(\mathcal{O}_D(-H))[1]\\ &\simeq \Psi(\mathcal{O}_{\tilde{X}}(h-2H))[2]\\ &\simeq \mathrm{Cl}_1[2].\end{aligned}$$

It remains to prove that $\Psi(\mathcal{O}_{C'_x}(H))[-2] \simeq \mathrm{Cl}_0(h)$. Applying $\Psi(- \otimes \mathcal{O}_{\tilde{X}}(H))$ to the sequence (28), we have

$$\Psi(\mathcal{O}_D) \to \Psi(\mathcal{O}_D(H)) \to \Psi(\mathcal{O}_{C'_x}(H)).$$

Since $\Psi(\mathcal{O}_D) = 0$, we have $\Psi(\mathcal{O}_{C'_x}(H)) \simeq \Psi(\mathcal{O}_D(H))$. Applying $\Psi(- \otimes \mathcal{O}_{\tilde{X}}(H))$ to the sequence (27), we have

$$\Psi(\mathcal{O}_{\tilde{X}}) \to \Psi(\mathcal{O}_{\tilde{X}}(H)) \to \Psi(\mathcal{O}_D(H)).$$

By Lemma 2.15, we get isomorphisms

$$\begin{aligned}\Psi(\mathcal{O}_{C'_x}(H)) &\simeq \Psi(\mathcal{O}_D(H))\\ &\simeq \Psi(\mathcal{O}_{\tilde{X}}(H))\\ &\simeq \mathrm{Cl}_0(h)[2].\end{aligned}$$

□

Next, we calculate Mukai vectors of P_x and $(P_x)_{\mathrm{tor}}$. Fix a B-field $B \in \mathrm{H}^2(S, \frac{1}{2}\mathbb{Z})$ of the Brauer class α.

Lemma 5.8 [Tod16, Lemma 3.2]. *We have* $\mathbf{R}\mathrm{Hom}(\mathrm{Cl}_i, \mathrm{Cl}_i) = \mathbb{C} \oplus \mathbb{C}[-2]$ *for* $i = 0, 1$. *In particular,* $\mathcal{U}_i$ *is spherical.*

Lemma 5.9 [Tod16, Lemma 4.6]. *We can express* $v^B(\mathcal{U}_0) = (2, s, t)$ *such that* $s^2 - 4t = -2$ *and* $s - 2B \in \mathrm{Pic}\, S$.

Proof. Recall that $\mathcal{U}_0$ is the α-twisted vector bundle of rank 2. So we can write $v^B(\mathcal{U}_0) = (2, s, t) \in \tilde{\mathrm{H}}^{1,1}(S, B, \mathbb{Z})$. Since $\mathcal{U}_0$ is spherical, we have $\chi(\mathcal{U}_0, \mathcal{U}_0) = -2$. By the Riemann–Roch formula (2), we have $s^2 - 4t = -2$.

By Lemma 2.3, we have $s - 2B \in \mathrm{Pic}\, S$. □

Toda [Tod16, Corollary 4.4] proved that

$$[\mathrm{Cl}_1] = \tfrac{3}{8}[\mathrm{Cl}_0] + \tfrac{3}{4}[\mathrm{Cl}_0(h)] - \tfrac{1}{8}[\mathrm{Cl}_0(2h)]$$

in $N(D^b(\mathbb{P}^2, \mathrm{Cl}_0))$. Let $\mathcal{U}_1 \in \mathrm{Coh}(S, \alpha)$ be the α-twisted vector bundle corresponding to Cl_1. Using this relation, we can calculate the Mukai vector of $\mathcal{U}_1$ as follows.

Lemma 5.10 [Tod16, Lemma 4.6]. *We have*

$$v^B(\mathcal{U}_1) = e^{h/2} v^B(\mathcal{U}_0) = (2,\, s + h,\, t + \tfrac{1}{2}sh + \tfrac{1}{2}).$$

We now calculate the Mukai vector of P_x.

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Proposition 5.11. *Let $x \in X$ be a point. Then*

$$v^B(P_x) = (2, s + 2h, t + sh). \tag{31}$$

Proof. By Lemma 4.3, the numerical classes of P_x and P_y are the same for any points $x, y \in X$. So we can assume that $x \in X \backslash P$. Since Q_x is a zero-dimensional torsion sheaf of length 2, we have $v^B(Q_x) = (0, 0, 2)$. Using the exact sequence (24), we then have

$$\begin{aligned} v^B(P_x) &= v^B(\mathcal{U}_0(h)) - v^B(Q_x) \\ &= e^h(2, s, t) - (0, 0, 2) \\ &= (2, s + 2h, t + sh). \end{aligned}$$

□

In the next lemma, we calculate the Mukai vector of $(P_x)_{\mathrm{tor}}$.

Lemma 5.12. *Let $x \in P$ be a point. Then we have*

$$v^B((P_x)_{\mathrm{tor}}) = (0, h, \tfrac{1}{2}sh - \tfrac{1}{2}). \tag{32}$$

Proof. By (26) and Lemma 5.10, we have

$$\begin{aligned} v^B((P_x)_{\mathrm{tor}}) &= v^B(\mathcal{U}_0) - e^{-h} v^B(\mathcal{U}_1) \\ &= (0, h, \tfrac{1}{2}sh - \tfrac{1}{2}). \end{aligned}$$

□

Proposition 5.13. *Let $x \in P$ be a point. Then $P_x/(P_x)_{\mathrm{tor}} \simeq \mathcal{U}_1$.*

Proof. Note that $P_x/(P_x)_{\mathrm{tor}}$ and $\mathcal{U}_1$ are μ^B-stable rank-2 torsion-free sheaves. We can calculate the Mukai vector of $P_x/(P_x)_{\mathrm{tor}}$ as follows:

$$\begin{aligned} v^B(P_x/(P_x)_{\mathrm{tor}}) &= v^B(P_x) - v^B((P_x)_{\mathrm{tor}}) \\ &= (2, s + h, t + \tfrac{1}{2}sh + \tfrac{1}{2}) \\ &= v^B(\mathcal{U}_1). \end{aligned}$$

Since the moduli space of stable sheaves with Mukai vector $(0, h, \frac{1}{2}sh - \frac{1}{2})$ is the point, we have $P_x/(P_x)_{\mathrm{tor}} \simeq \mathcal{U}_1$. □

In the next section, we construct stability conditions under which P_x is stable for any points $x \in X$.

6. Construction of stability conditions

We use the same notation as in the previous section. In this section we assume that $\mathrm{Pix}(S) = \mathbb{Z}h$; so we have $\alpha \neq 1$. We will construct stability conditions in the main theorem. First we recall how to construct stability conditions on $D^b(S, \alpha)$.

Theorem 6.1 [Bri08, HS05]. *Take $B' \in \mathrm{NS}(S)_{\mathbb{R}}$ and a real ample class $\omega \in \mathrm{NS}(S)_{\mathbb{R}}$. Let $\tilde{B} := B' + B \in H^2(S, \mathbb{R})$. We define a group homomorphism $Z := Z_{\tilde{B},\omega} \colon N(S, \alpha) \to \mathbb{C}$ by*

$$Z_{\tilde{B},\omega}(E) := \langle v^B(E), e^{\tilde{B} + i\omega} \rangle.$$

We can define a torsion pair $(\mathcal{T}, F)$ on $\mathrm{Coh}(S, \alpha)$ as follows:

- $\mathcal{T} := \langle E \in \mathrm{Coh}(S, \alpha) \mid E$ *is* μ^B*-semistable with* $\mu^B(E) > \tilde{B}\omega \rangle_{\mathrm{ex}}$;
- $\mathcal{F} := \langle E \in \mathrm{Coh}(S, \alpha) \mid E$ *is* μ^B*-semistable with* $\mu^B(E) \leqslant \tilde{B}\omega \rangle_{\mathrm{ex}}$.

Then $\mathcal{C} := \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}} \subset D^b(S, \alpha)$ *is the heart of a bounded t-structure on* $D^b(S, \alpha)$ *induced by the torsion pair* $(\mathcal{T}, \mathcal{F})$*. Here we denote the extension closure by* $\langle - \rangle_{\mathrm{ex}}$*. The pair* $(Z, \mathcal{C})$ *is a stability condition on* $D^b(S, \alpha)$ *if and only if for any spherical twisted sheaf* E *we have* $Z_{\tilde{B},\omega}(E) \notin \mathbb{R}_{\leqslant 0}$*.*

We introduce the candidate of stability conditions in the main theorem.

Definition 6.2. Let

$$\tilde{B} := \tfrac{1}{2}l + \tfrac{1}{4}h + B = \tfrac{1}{2}s + \tfrac{1}{4}h \in H^2(S, \mathbb{Q}).$$

Here $l := s - 2B \in \operatorname{Pic} S$. For $\lambda > 0$, we define the pair $\sigma_\lambda = (Z_\lambda, \mathcal{C}) := (Z_{\tilde{B},\lambda h}, \mathcal{C})$.

By the definition of Z_λ, we can prove the following lemma.

Lemma 6.3. *Let* $E \in D^b(S, \alpha)$ *be an object such that* $v^B(E) = (r, c, d)$*. Then*

$$Z_\lambda(E) = c\tilde{B} - \tfrac{1}{2}r\tilde{B}^2 + r\lambda^2 - d + \lambda i(c - r\tilde{B})h.$$

The pair σ_λ gives the stability conditions.

Lemma 6.4. *If* $\lambda > 1/4$*, then* $\sigma_\lambda = (Z_\lambda, \mathcal{C}) := (Z_{\tilde{B},\lambda h}, \mathcal{C})$ *is a stability condition on* $D^b(S, \alpha)$*.*

Proof. Assume that there is a spherical twisted sheaf E such that $Z_{\tilde{B},\omega}(E) \in \mathbb{R}_{\leqslant 0}$. Let $v^B(E) = (r, c, d)$. Since $\rho(S) = 1$ and E is spherical, we have $r > 0$ and $c^2 - 2rd = -2$, because $\operatorname{Im} Z_\lambda(E) = 0$ and so by Lemma 6.3 we have $c = r\tilde{B}$. Hence

$$r^2\tilde{B}^2 - 2rd = -2. \tag{33}$$

By $\operatorname{Re} Z_{\tilde{B},\omega}(E) \in \mathbb{R}_{\leqslant 0}$, we have the inequality

$$\tfrac{1}{2}r\tilde{B}^2 + r\lambda^2 \leqslant d.$$

Using (33), we have $r^2\lambda^2 \leqslant 1$. Due to $\lambda > 1/4$, we get $r^2 < 16$. Since r is even, we obtain $r = 2$. Hence $c = 2\tilde{B} = s + 1/2h$; but this contradicts $1/2h \notin H^2(S, \mathbb{Z})$. □

For simplicity, we denote the central charge Z_λ by Z. The goal of this section is to establish the following proposition.

Proposition 6.5. *Assume that* X *is very general. If* $\sqrt{3}/4 < \lambda < 3/4$*, then* σ_λ *is generic with respect to* v *and* P_x *is* σ_λ*-stable for all* $x \in X$*.*

As a special case of Lemma 6.4, we have the following result.

Lemma 6.6. *For a point* $x \in X$*, we have*

$$Z(P_x) = 2\lambda^2 + \tfrac{3}{8} + 3\lambda i.$$

For a point $x \in P$*, we have*

$$Z((P_x)_{\mathrm{tor}}) = 1 + 2\lambda i.$$

By Lemma 6.6, we can compare the phases of P_x and $(P_x)_{\mathrm{tor}}$ for $x \in P$.

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Remark 6.7. Let $x \in P$ be a point. Then $\phi((P_x)_{\mathrm{tor}}) < \phi(P_x)$ if and only if $\lambda < 3/4$.

Proof. By Lemma 6.6, the inequality $\phi((P_x)_{\mathrm{tor}}) < \phi(P_x)$ is equivalent to

$$\mathrm{Re}\, Z((P_x)_{\mathrm{tor}}) > \tfrac{2}{3}\mathrm{Re}\, Z(P_x).$$

Solving this inequality, we get $\lambda < 3/4$. □

The following lemma allows us to discuss σ_λ-stability of objects P_x.

Lemma 6.8. *We have $P_x \in \mathcal{C}$ for all $x \in X$.*

Proof. Take $x \in X \backslash P$. Since $\alpha \neq 1$ and P_x is of rank 2 and torsion-free, P_x is μ^B-stable. Because $\mathrm{Im}\, Z(P_x) = 3\lambda > 0$, we have $\mu^B(P_x) > \tilde{B}h$. Hence $P_x \in \mathcal{T}$.

Take $x \in P$. Since $\alpha \neq 1$, $P_x/(P_x)_{\mathrm{tor}}$ is μ^B-stable. Since $\mathrm{Im}\, Z(P_x/(P_x)_{\mathrm{tor}}) = \lambda > 0$, we have $\mu^B(P_x/(P_x)_{\mathrm{tor}}) > \tilde{B}h$. Hence $P_x \in \mathcal{T}$. □

Lemma 6.9. *Let $x \in X$ be a point and let $0 \neq F \subset P_x$ be a subobject in $\mathcal{C}$. Then $F \in \mathcal{T}$ and $\mathrm{Im}\, Z(F) > 0$.*

Proof. Since $P_x \in \mathcal{T}$, we have $\mathcal{H}^{-1}(F) = 0$. So we obtain $F \in \mathcal{T}$.

We prove that $\mathrm{Im}\, Z(F) > 0$. If F is not torsion, then $\mathrm{Im}\, Z(F) > 0$ holds. So we assume that F is torsion.

Consider the exact sequence

$$0 \to F \to P_x \to \mathrm{Coker}(F \to P_x) \to 0$$

in $\mathcal{C}$. Taking the long exact sequence, we have the exact sequence

$$0 \to \mathcal{H}^{-1}(\mathrm{Coker}(F \to P_x)) \to F \to P_x \to \mathcal{H}^0(\mathrm{Coker}(F \to P_x)) \to 0$$

in $\mathrm{Coh}(S, \alpha)$.

Since $\mathcal{H}^{-1}(\mathrm{Coker}(F \to P_x)) \in \mathcal{F}$ and F is torsion, we have

$$\mathcal{H}^{-1}(\mathrm{Coker}(F \to P_x)) = 0.$$

So $F \subset (P_x)_{\mathrm{tor}}$ in $\mathrm{Coh}(S, \alpha)$. If $x \in X \backslash P$, we have $(P_x)_{\mathrm{tor}} = 0$ and $F \neq 0$. This is a contradiction; so we can assume $x \in P$. Since $(P_x)_{\mathrm{tor}}$ is a one-dimensional pure torsion sheaf, F is a one-dimensional torsion sheaf. Hence, $\mathrm{Im}\, Z(F) > 0$ holds. □

The following description of the central charge Z will be useful later.

Lemma 6.10. *Let $E \in D^b(S, \alpha)$ be an object such that $v^B(E) = (r, c, d)$ and set $L := c - r\tilde{B}$ and $m := Lh$. Then $L \in \mathrm{Pic}(S) \otimes \mathbb{Q}$, $m \in \mathbb{Z}$. If $r = 0$, then*

$$Z(E) = -d + c\tilde{B} + i\lambda m \in \tfrac{1}{2}\mathbb{Z} \times 2\lambda\mathbb{Z}. \tag{34}$$

If $r \neq 0$, then

$$Z(E) = -\frac{1}{r}\left(1 + \frac{m^2}{4}\right) + r\lambda^2 + i\lambda m + \frac{2 - \chi(E, E)}{2r}. \tag{35}$$

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Proof. Assume that $r = 0$. Then there is $k \in \mathbb{Z}$ such that $c = kh$ by $\rho(S) = 1$. By Lemma 6.3, we have

$$\begin{aligned} Z(E) &= -d + c\tilde{B} + i\lambda m \\ &= -d + \tfrac{1}{2}ksh + \tfrac{1}{2}k + 2\lambda ki \in \tfrac{1}{2}\mathbb{Z} \times 2\lambda\mathbb{Z}. \end{aligned}$$

The second statement is deduced from Lemma 6.3 and (2). □

Lemma 6.11. *Assume that $\lambda > \sqrt{3}/4$. For a σ_λ-semistable object E with $v^B(E) = v$, we have $\mathcal{H}^{-1}(E) = 0$.*

Proof. Consider the natural exact sequence

$$0 \to \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to 0$$

in $\mathcal{C}$. Suppose that $\mathcal{H}^{-1}(E) \neq 0$. Then $\operatorname{rk} \mathcal{H}^0(E) > 0$ holds. Taking the Harder–Narasimhan filtration and Jordan–Hölder filtration with respect to μ^B-stability, we obtain a μ^B-stable subsheaf $F \subset \mathcal{H}^{-1}(E)$. So we obtain an exact sequence

$$0 \to F[1] \to E \to G \to 0$$

in $\mathcal{C}$. Taking the long exact sequence, we have the exact sequence

$$\begin{aligned} 0 \to F \to \mathcal{H}^{-1}(E) &\to \mathcal{H}^{-1}(G) \\ \to 0 \to \mathcal{H}^0(E) &\to \mathcal{H}^0(G) \to 0. \end{aligned}$$

Since E is σ_λ-semistable, we have $\operatorname{Im} Z(F[1]) > 0$. So we obtain $\operatorname{Im} Z(F[1]) = \lambda$, 2λ or 3λ. Since $\mathcal{H}^{-1}(E)$ is torsion-free, we have $\operatorname{rk} \mathcal{H}^0(E) > 0$.

Suppose that $\operatorname{Im} Z(F[1]) = 3\lambda$. Then we have

$$\operatorname{Im} Z(\mathcal{H}^0(G)) + \operatorname{Im} Z(\mathcal{H}^{-1}(G)[1]) = \operatorname{Im} Z(G) = 0,$$

so we can deduce that $\operatorname{Im} Z(\mathcal{H}^0(E)) = \operatorname{Im} Z(\mathcal{H}^0(G)) = 0$. Since $\operatorname{rk} \mathcal{H}^0(E) > 0$, $\operatorname{Im} Z(\mathcal{H}^0(E))$ must be positive. This is a contradiction; therefore $\operatorname{Im} Z(F[1]) = \lambda$ or 2λ.

Let $v^B(F[1]) = -(r, c, d)$, $L := c - r\tilde{B}$ and $m := Lh$. Note that $r > 0$ and $m = -1$ or -2. By (35), we have

$$Z(F[1]) = \frac{1}{r}\left(1 + \frac{m^2}{4}\right) - r\lambda^2 - i\lambda m - \frac{2 - \chi(F, F)}{2r}.$$

Since F is simple, we have

$$\frac{2 - \chi(F, F)}{2r} \geqslant 0.$$

So, by the σ_λ-semistability of E, we have the following inequality:

$$-\frac{m}{3}\left(2\lambda^2 + \frac{3}{8}\right) \leqslant \operatorname{Re} Z(F[1]) \leqslant \frac{1}{r}\left(1 + \frac{m^2}{4}\right) - r\lambda^2.$$

Note that a function of the form $a/x - bx^2$, where a and b are positive real numbers, is monotone decreasing for $x > 0$. Assume that $m = -1$. Then we have the inequality

$$\tfrac{2}{3}\lambda^2 + \tfrac{1}{8} \leqslant \tfrac{5}{8} - 2\lambda^2.$$

Solving it, we get $\sqrt{3}/4 \geqslant \lambda$. This is a contradiction, so we deduce that $m = -2$. Assume that $r \geqslant 4$. Then we have

$$\tfrac{4}{3}\lambda^2 + \tfrac{1}{4} \leqslant \tfrac{1}{2} - 4\lambda^2.$$

Solving it, we get $\sqrt{3}/8 \geqslant \lambda$. This is a contradiction, so $r = 2$. Since $\rho(S) = 1$, there is $k \in \mathbb{Q}$ such that $L = kh$. By $m = -2$, we have $k = -1$. This implies $c = s - 1/2h$ by the definition of L, which contradicts $1/2h \notin H^2(S, \mathbb{Z})$. □

The following statement is part of the main proposition in this section.

Proposition 6.12. *Assume that $\sqrt{3}/4 < \lambda < 3/4$. Then σ_λ is generic with respect to v.*

Proof. It is sufficient to prove that σ_λ-semistable objects with Mukai vector v are σ_λ-stable. Let $E \in \mathcal{C}$ be a σ_λ-semistable object with Mukai vector v. Suppose that E is not σ_λ-stable. Then there is an exact sequence

$$0 \to F \to E \to G \to 0$$

in $\mathcal{C}$ such that $\phi(E) = \phi(F)$. Now F is also σ_λ-semistable. Taking the Jordan–Hölder filtration of F, we can assume that F is σ_λ-stable. Since $\operatorname{Im} Z(E) = 3\lambda$ and $\phi(F) = \phi(E)$, we have $Z(F) = m(2\lambda^2 + 3/8)/3$, where $v^B(F) = (r, c, d)$, $L := c - r\tilde{B}$ and $m := Lh$. Note that $m = 1$ or 2.

Assume that $r = 0$. Since $\rho(S) = 1$, we have $m = 2$, so

$$\tfrac{1}{2} < \operatorname{Re} Z(F) = \tfrac{2}{3}(2\lambda^2 + \tfrac{3}{8}) < 1.$$

By (34), this is a contradiction. Hence, $r > 0$ holds by Lemma 6.11.

Therefore we have the equality

$$\frac{m}{3}\left(2\lambda^2 + \frac{3}{8}\right) = -\frac{1}{r}\left(1 + \frac{m^2}{4}\right) + r\lambda^2 + \frac{2 - \chi(F, F)}{2r}. \tag{36}$$

Assume that $m = 1$. Suppose that $r \geqslant 4$. Then we have the inequality

$$\tfrac{1}{3}(2\lambda^2 + \tfrac{3}{8}) \geqslant -\tfrac{5}{16} + 4\lambda^2.$$

Solving it, we get $\sqrt{21}/40 > \lambda$. This is a contradiction; hence $r = 2$. By (36), we have

$$\frac{4}{3}\lambda^2 - \frac{3}{4} = \frac{\chi(F, F) - 2}{4}.$$

Since the Mukai lattice is even, the right-hand side is in $\frac{1}{2}\mathbb{Z}$. However, we have $-1/2 < 4\lambda^2/3 - 3/4 < 0$. This is a contradiction, so $m = 2$. Note that $L = h$ due to $\rho(S) = 1$. If $r \geqslant 6$, then we have $1/8 > \lambda^2$ by (36) as in the case of $m = 1$. So we get $r = 2$ or 4. If $r = 2$, then $c = s + 2/3h$. This is a contradiction; hence we must have $r = 4$. By $L = h$ and $r = 4$, we get $c = 2s + 2h$. Since $H^2(S, \mathbb{Z})$ is even, we have $\chi(F, F) = -v^b(F)^2 = -4s^2 - 8sh - 8 + 8d \in 8\mathbb{Z}$. By (36), we have

$$\frac{8}{3}\lambda^2 - \frac{1}{2} = \frac{\chi(F, F)}{8}.$$

The right-hand side is an integer. However, we have $0 < 8\lambda^2/3 - 1/2 < 1$, which is a contradiction. □

We will prove the σ_λ-stability of objects P_x for $x \in X$. We need some cohomology computations for $\mathcal{U}_1$ and P_x.

Lemma 6.13 [Tod16, Lemma 3.7]. *Let $I_P \subset \mathcal{O}_X$ be the ideal sheaf of the plane P in X. Then*

$$\mathbf{R}\sigma_*\Phi(\mathrm{Cl}_1) \simeq I_P \oplus \mathcal{O}_X(-H)^{\oplus 3}.$$

Lemma 6.14. *We have $\mathrm{Hom}(\mathcal{U}_1, P_x) = 0$ for all $x \in X$.*

Proof. There are the following isomorphisms and an inclusion:

$$\begin{aligned}
\mathrm{Hom}(\mathcal{U}_1, P_x) &\simeq \mathrm{Hom}(P_x, \mathcal{U}_1[2]) \\
&\simeq \mathrm{Hom}_{\mathrm{Cl}_0}(\Psi(\mathbf{L}\sigma^* I_x(H))[-2], \mathrm{Cl}_1[2]) \\
&\simeq \mathrm{Hom}_X(I_x(H)[-2], \mathbf{R}\sigma_*\Phi(\mathrm{Cl}_1)[2]) \\
&\simeq \mathrm{Hom}_X(I_x(H)[-2], I_P \oplus \mathcal{O}_X(-H)^{\oplus 3}[2]) \\
&\simeq \mathrm{Hom}_X(I_x(H)[-4], I_P \oplus \mathcal{O}_X(-H)^{\oplus 3}) \\
&\simeq \mathrm{Hom}_X(I_P \oplus \mathcal{O}_X(-H)^{\oplus 3}, I_x(-2H)) \\
&\simeq \mathrm{Hom}_X(I_P \oplus \mathcal{O}_X(-H)^{\oplus 3}, I_x(-2H)) \\
&\simeq \mathrm{Hom}_X(I_P, I_x(-2H)) \oplus \mathrm{Hom}_X(\mathcal{O}_X(-H), I_x(-2H))^{\oplus 3} \\
&\subset \mathrm{Hom}_X(I_P^{\vee\vee}, I_x(-2H)^{\vee\vee}) \oplus \mathrm{Hom}(\mathcal{O}_X(-H)^{\vee\vee}, I_x(-2H)^{\vee\vee})^{\oplus 3} \\
&\simeq \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(-2H)) \oplus \mathrm{Hom}_X(\mathcal{O}_X(-H), \mathcal{O}_X(-2H))^{\oplus 3} \\
&= 0.
\end{aligned}$$

The first isomorphism is given by the Serre duality for $\mathcal{A}_X$. The third isomorphism is deduced from the adjoint property. The fourth isomorphism is given by Lemma 6.13. The sixth isomorphism is given by the Serre duality for $D^b(X)$. So we have $\mathrm{Hom}(\mathcal{U}_1, P_x) = 0$. □

The following proposition completes the proof of the main theorem.

Proposition 6.15. *Assume that $\sqrt{3}/4 < \lambda < 3/4$. Then P_x is σ_λ-stable for any point $x \in X$.*

Proof. Take $x \in X$. By Proposition 6.12, it is sufficient to prove that P_x is σ_λ-semistable. Suppose that P_x is not σ_λ-semistable. Then there is an exact sequence

$$0 \to F \to P_x \to G \to 0$$

in $\mathcal{C}$ such that $\phi(F) > \phi(P_x)$. Taking the Harder–Narasimhan filtration and Jordan–Hölder filtration of F, we can assume that F is σ_λ-stable. Since $P_x \in \mathcal{T}$, the object F is also contained in $\mathcal{T}$.

Let $v^B(F) = (r, c, d)$, $L := c - r\tilde{B}$ and $m := Lh$. First, we prove that $r > 0$. Assume that $r = 0$. Since $\mathcal{H}^{-1}(G) \in \mathcal{F}$, we have $\mathcal{H}^{-1}(G) = 0$. So F is a subsheaf of P_x. Since P_x is torsion-free for $x \in X \backslash P$, it is sufficient to consider the case of $x \in P$. Now $F \subset (P_x)_{\mathrm{tor}}$ and $(P_x)_{\mathrm{tor}}$ is a one-dimensional pure torsion sheaf. So we can write $v^B(F) = (0, h, k)$ for some $k \in \mathbb{Z}$. Since $(P_x)_{\mathrm{tor}}/F$ is a zero-dimensional torsion sheaf, we have $\phi((P_x)_{\mathrm{tor}}) \geqslant \phi(F)$. By Remark 6.7, this is a contradiction. Note that $r > 0$ and $m = 1$ or 2.

By $\phi(F) > \phi(P_x)$ and (35), we have the inequality

$$-\frac{1}{r}\left(1 + \frac{m^2}{4}\right) + r\lambda^2 + \frac{2 - \chi(F, F)}{2r} < \frac{m}{3}\left(2\lambda^2 + \frac{3}{8}\right). \tag{37}$$

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Assume that $m = 1$. If $r \geqslant 4$, we have $\sqrt{21}/40 \geqslant \lambda$ by (37) as in Lemmas 6.11 and 6.12. Hence $r = 2$. By (37), we have $\chi(F, F) > 0$. Since F is simple and $\chi(F, F)$ is even, we get $\chi(F, F) = 2$. By $L = 1/2h$ and $\chi(F, F) = 2$, we have

$$v^B(F) = (2, s + h, t + \tfrac{1}{2}sh + \tfrac{1}{2}) = v^B(\mathcal{U}_1).$$

Since $\rho(S) = 1$ and F is spherical, F must be torsion-free, so F is μ^B-stable. Since the moduli space of stable sheaves with Mukai vector $(2, s + h, t + \frac{1}{2}sh + \frac{1}{2})$ is the point, we have $F \simeq \mathcal{U}_1$. This contradicts Lemma 6.14. If $\operatorname{rk} F = 2$, then $\operatorname{rk} G = 0$. By (34), we have $\operatorname{Im} Z(G) \in 2\mathbb{Z}\lambda \subset \mathbb{R}$. However, $\operatorname{Im} Z(G) = \lambda$ holds. This is a contradiction, so we must have $r \geqslant 4$. However, we have $3/4\sqrt{2} > \lambda$ by (37) again. By Proposition 6.12, we may assume that $3/4\sqrt{2} < \lambda < 3/4$, which is a contradiction. □

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