J. Appl. Probab. **59,** 1009–1025 (2022)
*doi:*10.1017/jpr.2021.101

SCALING LIMITS OF BRANCHING RANDOM WALKS AND BRANCHING-STABLE PROCESSES

JEAN BERTOIN,* ** AND
HAIRUO YANG,* *** *University of Zurich*

Abstract

Branching-stable processes have recently appeared as counterparts of stable subordinators, when addition of real variables is replaced by branching mechanisms for point processes. Here we are interested in their domains of attraction and describe explicit conditions for a branching random walk to converge after a proper magnification to a branching-stable process. This contrasts with deep results obtained during the past decade on the asymptotic behavior of branching random walks and which involve either shifting without rescaling, or demagnification.

Keywords: Branching random walk; scaling limit; branching-stable processes

2020 Mathematics Subject Classification: Primary 60F17; 60J80

1. Introduction

We start by recalling the classical stable limit theorem in the special case of non-negative random variables, referring to the treatises [16] and [18] for the complete story. Consider a random variable $Y \geq 0$ whose tail distribution $\overline{F}(y) = \mathbb{P}(Y > y)$ is regularly varying at infinity with index $-\beta$ for some $\beta \in (0, 1)$, that is, $\lim_{y\to\infty} \overline{F}(ay)/\overline{F}(y) = a^{-\beta}$ for all $a > 0$. Further, let $Y_1, Y_2, \ldots$ denote a sequence of independent and identically distributed (i.i.d.) copies of Y. Then, for any sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} n\overline{F}(a_n) = 1$ (as a consequence, (a_n) varies regularly with index $1/\beta$), the sequence $(Y_1 + \cdots + Y_n)/a_n$ of rescaled partial sums converges in distribution as $n \to \infty$ to a stable law on $\mathbb{R}_+$ with exponent β. The purpose of this paper is to present an analog of this stable limit theorem in the setting of branching random walks. A rather surprising feature is that the counterpart of the index β, denoted here by $-\alpha$, can then be any negative real number.

The study of the asymptotic behavior of branching processes has attracted a lot of attention and effort during many years. First, for light-tailed displacements, let us merely single out the work of Biggins [10, 11] and Bramson [12, 13] amongst the most important earlier contributions. More recently, Aïdékon [1] established a remarkable limit theorem in distribution for the minimum of a super-critical branching random walk. Then Aïdékon *et al.* [2] and Arguin *et al.* [3, 4] showed that the point process formed by a branching Brownian motion seen from its leftmost atom converges in distribution to a random counting measure that can be

Received 12 March 2021; revision received 16 November 2021.
* Postal address: Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland.
** Email address: jean.bertoin@math.uzh.ch
*** Email address: hairuo.yang@math.uzh.ch

constructed from some marked Poisson point process, and is often called a decorated Poisson point process. Finally, Madaule [23] obtained the counterpart of the Brownian results in the framework of branching random walks. For heavy-tailed displacements, weak limit theorems were established in [14] for the location of the rightmost particle, and in [8, 9] for the whole point process. Below we shall recall some of these results, somewhat informally for the sake of simplicity.

First let us introduce some notation that we shall use throughout this text. For every $b>0$ and every counting measure $m=\sum_j \delta_{x_j}$ on a vector space, we write bm for the pushforward of m by the dilation with factor b, $x\mapsto bx$, that is,

$$bm=\sum_j \delta_{bx_j}.$$

In other words, we shall think of a counting measure as a multiset of atoms repeated according to their multiplicities, and in this setting, multiplication by a scalar acts on the location of atoms rather than on the measure itself. We also write $\langle m, f\rangle$ for the integral of a function f with respect to a counting measure m, that is,

$$\langle m, f\rangle=\sum_j f(x_j),$$

whenever this makes sense. In particular, the mass $m(B)$ of a set B is written as $\langle m, \mathbf{1}_B\rangle$, with $\mathbf{1}_B$ denoting the indicator function of B.

Assume that $(\mathbf{Z}(n))_{n\geq 0}$ is a branching random walk on $\mathbb{R}$, where $\mathbf{Z}(n)$ is the point process induced by the locations of the particles at generation n; suppose also for simplicity that $\mathbf{Z}(0)=\delta_0$ is the Dirac point mass at 0. We refer to the lecture notes [25] for the necessary background, terminology, and of course much more. The fundamental assumption in [23] is that for the functions $\mathbf{1}(x)=1$, $f(x)=\mathrm{e}^{-x}$, $g(x)=x\mathrm{e}^{-x}$, and $h(x)=x^2\mathrm{e}^{-x}$,

$$\begin{aligned} &\mathbb{E}(\langle \mathbf{Z}(1), \mathbf{1}\rangle)>1, \quad \mathbb{E}(\langle \mathbf{Z}(1), h\rangle)<\infty, \\ &\mathbb{E}(\langle \mathbf{Z}(1), f\rangle)=1, \quad \mathbb{E}(\langle \mathbf{Z}(1), g\rangle)=0. \end{aligned} \tag{1.1}$$

This may look stringent, but in practice a simple linear map transforms many branching random walks into another branching random walk that satisfies (1.1). Also taking for granted some further technical requirements, the point process obtained by shifting the atoms of $\mathbf{Z}(n)$ by $-\frac{3}{2}\log n-\log D_\infty$, where D_∞ denotes the terminal value of the so-called derivative martingale, then converges in distribution as $n\to\infty$. It is remarkable that this weak limit theorem involves shifting but not rescaling. Moreover, the limiting point process can be described as a decorated Poisson point process.

More recently, Bhattacharya *et al.* [8] considered branching random walks obtained by superposing i.i.d. heavy-tailed displacements onto a supercritical Galton–Watson tree. Specifically, one supposes there that the first generation has the form

$$\mathbf{Z}(1)=\sum_{j=1}^{N}\delta_{Y_j},$$

where $Y_1,\ldots$ is a sequence of i.i.d. real random variables and N an independent integer-valued random variable with $\mathbb{E}(N)=\mu>1$ and $\mathbb{E}(N\log N)<\infty$. Assuming further that the tail distribution $\mathbb{P}(|Y_1|>x)$ is regularly varying at infinity with index $-\beta<0$ and a tail balanced

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

condition, there is a sequence (b_n) of positive real numbers that grows roughly like $\mu^{n/\beta}$ such that, conditionally on non-extinction, $b_n^{-1}\mathbf{Z}(n)$ converges weakly to a so-called Cox cluster process. See Theorem 2.1 of [8] for a precise statement. The same authors extended their result and replaced the assumption that the sequence $Y_1, \ldots$ is i.i.d. by a weaker condition involving regular variation in the sense of [17] and [22]; see Theorem 2.6 of [9].

We shall now present, again informally, the main result of the present work. Henceforth we consider a branching random walk $(\mathbf{Z}(n))_{n\geq 0}$ on the non-negative half-line $\mathbb{R}_+ = [0, \infty)$, and assume as before that $\mathbf{Z}(0) = \delta_0$. We suppose that $\mathbf{Z}(1)$ has a single atom at the origin a.s., and we write

$$0 < X_1 \leq X_2 \leq \cdots \leq \infty$$

for the sequence of atoms of $\mathbf{Z}(1)$ on $(0, \infty)$, ranked in non-decreasing order and repeated according to their multiplicities, with the convention that $X_j = \infty$ if and only if $\mathbf{Z}(1)$ has less than j atoms in $(0, \infty)$. In other words,

$$\mathbf{Z}(1) = \delta_0 + \sum_{j\geq 1} \delta_{X_j},$$

where we implicitly agree to discard atoms at ∞ in the series on the right-hand side. Strictly speaking, after each unit of time, every individual dies and simultaneously gives birth to children among which a single one occupies the same location as its parent. If instead we view this child as the same individual as its parent (which thus survives after reaching age 1), we may think of this branching random walk as describing a spatial population model with static immortal individuals, such that at each generation, each individual gives birth to children located at its right and at distances given by independent copies of $X_1, X_2, \ldots$.

Under these assumptions, the log-Laplace transform of the intensity measure of $\mathbf{Z}(1)$,

$$\psi(t) = \log \mathbb{E}(\langle \mathbf{Z}(1), \mathrm{e}^{-t\bullet}\rangle) = \log\left(1 + \sum_{j\geq 1} \mathbb{E}(\mathrm{e}^{-tX_j})\right) \in (0, \infty],$$

fulfills $\psi(t) > 0$ and $\psi'(t) < 0$ for all $t \geq 0$ in the domain of ψ. As a consequence, no linear transform of $\mathbf{Z}(1)$ can satisfy (1.1).

We next introduce the three assumptions on the point process $\mathbf{Z}(1)$ under which we shall establish a scaling limit theorem for the branching random walk. The first is that if $F_1(t) = \mathbb{P}(X_1 \leq t)$ denotes the cumulative distribution function of the first positive atom X_1, then for some $\alpha > 0$,

$$F_1 \text{ is regularly varying at } 0+ \text{ with index } \alpha. \tag{1.2}$$

The second is the existence of a scaling limit for the conditional distribution of

$$\mathbf{Z}^*(1) = \mathbf{Z}(1) - \delta_0 = \sum_{j\geq 1} \delta_{X_j}$$

given that X_1 is small. Specifically, recall the notation bm for the pushforward of a measure m by the dilation with factor b, and view $\mathbf{Z}^*(1)$ as a random variable on the space $\mathcal{M}$ of locally finite counting measures on $\mathbb{R}_+$ endowed with the vague topology (see e.g. Appendix A.2 of [21]). Our second assumption is as follows:

$$\text{The conditional law of } t^{-1}\mathbf{Z}^*(1) \text{ given } X_1 \leq t \text{ has a weak limit as } t \to 0+. \tag{1.3}$$

Our final assumption is that the log-Laplace transform ψ of the intensity measure of $\mathbf{Z}(1)$ fulfills

$$\sup_{n\geq 1} n\,\psi(1/a_n) < \infty, \tag{1.4}$$

where $(a_n)_{n\geq 1}$ is a sequence of positive real numbers such that

$$\lim_{n\to\infty} nF_1(a_n) = 1.$$

Note that (a_n) is then regularly varying with index $-1/\alpha$.

Our first two assumptions are given explicitly in terms of the point process $Z(1)$; however, the interpretation of the third assumption may be less clear. One has to recall that the function $n\psi$ is the log-Laplace transform of the intensity measure of the branching random walk at the nth generation, in particular

$$\mathbb{E}\big(\langle a_n^{-1}\mathbf{Z}(n), \mathrm{e}^{-\bullet}\rangle\big) = \mathbb{E}\big(\langle \mathbf{Z}(n), \mathrm{e}^{-a_n^{-1}\bullet}\rangle\big) = \exp(n\psi(1/a_n)).$$

Thus (1.4) should be viewed as a natural condition to ensure that on average, the rescaled branching random walk $a_n^{-1}\mathbf{Z}(n)$ remains locally bounded. We also refer to the forthcoming Remark 3.1 for further comments on these assumptions.

Under these assumptions, the sequence of rescaled processes in continuous time

$$\big(a_n^{-1}\mathbf{Z}(\lfloor nt\rfloor)\big)_{t\geq 0}$$

converges in distribution as $n\to\infty$, on a space of rcll (right continuous with left limits) functions with values on a certain space of counting measures. The limit is a branching-stable process $\mathbf{S} = (\mathbf{S}(t))_{t\geq 0}$ introduced in [7]. In words, $\mathbf{S}$ is a branching process in continuous time which is self-similar with scaling exponent $-\alpha < 0$, in the sense that for every $c > 0$, there is identity in distribution, that is,

$$(\mathbf{S}(ct))_{t\geq 0} \stackrel{(d)}{=} \big(c^{-1/\alpha}\mathbf{S}(t)\big)_{t\geq 0}.$$

The law of $\mathbf{S}$ is characterized by the exponent α and the limiting distribution appearing in (1.3).

Although our result bears somewhat the same flavor as those of [8, 9] mentioned above (notably, our assumptions (1.2) and (1.3) resemble the hypothesis of regular variation for the distribution of $\mathbf{Z}(1)$ in [9]), there are major differences. The most obvious one is that [8, 9] work with a demagnification $b_n^{-1}\mathbf{Z}(n)$ with $b_n \approx c^n \gg 1$, whereas here, in contrast, we consider a magnification $a_n^{-1}\mathbf{Z}(n)$ with $a_n \approx n^{-1/\alpha} \ll 1$. Roughly speaking, extreme value theory and the so-called principle of a single big jump (see [14]) lie at the heart of the approach in [8, 9], whereas our result depends on Markov chain approximations of Feller processes. Another significant difference is that we obtain a weak limit theorem for processes depending on time, whereas [8, 9] consider the branching random walk $\mathbf{Z}$ at one given generation n only. Last but not least, branching-stable processes with negative indices only exist in the one-sided framework (i.e. on a half-line; see Lemma 2.2 of [7]), and hence one should not expect a two-sided version as in [8, 9].

The rest of this article is organized as follows. Section 2 is devoted to preliminaries. We shall first provide some background on a family of branching processes in continuous time which were introduced by Uchiyama [26], and point out that under appropriate assumptions these arise as the weak limits of certain families of branching random walks in discrete time.

Then we shall recall some features on branching-stable processes and their trimmed versions, and show that the latter belong to the family considered by Uchiyama. Our main result will then be properly stated and proved in Section 3. We will need to work with various spaces of counting measures, and for the reader's convenience, we gather their definition and notation in an appendix.

2. Preliminaries

2.1. Weak convergence to Uchiyama's branching process

Even though this work is mainly concerned with branching processes living on the positive half-line, in this section we shall consider the d-dimensional setting more generally.

Uchiyama [26] introduced a family of branching processes that can be thought of as analogs of branching random walks in continuous time; they can be described informally as follows. Fix $r > 0$ and let Π denote a probability measure on the space of finite counting measures on $\mathbb{R}^d$. We write $r \cdot \Pi$ for the ordinary scalar multiplication of the measure Π by r (to avoid possible confusion with the notation bm defined in the Introduction); in particular, the total mass of $r \cdot \Pi$ equals r. Imagine a particle system with no interactions on $\mathbb{R}^d$, where each particle, say located at x, dies at rate r and does not move during its lifetime. At the time of its death, it gives birth to children whose locations relative to x are given by a point process distributed according to Π, independently of the other particles. The process $\mathbf{U} = (\mathbf{U}(t))_{t\geq 0}$, which records the locations of particles alive as a function of time, is a branching process considered by Uchiyama. The finite measure $r \cdot \Pi$ on the space of finite counting measures characterizes the evolution of $\mathbf{U}$; it will be referred to as the reproduction rate. The purpose of this section is to point out that $\mathbf{U}$ arises as the weak limit of certain sequences of branching random walks, in much the same way that compound Poisson processes appear as weak limits of certain sequences of discrete-time random walks with rare non-zero steps. In this direction, we shall first describe $\mathbf{U}$ as a Feller process and determine its infinitesimal generator.

We write $\mathcal{M}_f$ for the set of finite counting measures on $\mathbb{R}^d$, endowed with the Lévy–Prokhorov distance. A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{x}$ in $\mathcal{M}_f$ if and only if $\lim_{n\to\infty} \langle \mathbf{x}_n, f\rangle = \langle \mathbf{x}, f\rangle$ for every $f \in \mathcal{C}_b(\mathbb{R}^d)$ (i.e. $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded). It is seen from Prokhorov's theorem that a subset $\mathcal{S} \subset \mathcal{M}_f$ is relatively compact if and only if both the total mass remains bounded, namely

$$\sup_{\mathbf{x}\in\mathcal{S}} \langle \mathbf{x}, \mathbf{1}\rangle < \infty,$$

and there are no atoms out of some compact subset of $\mathbb{R}^d$, that is, there exists some $b > 0$ such that

$$\left\langle \mathbf{x}, \mathbf{1}_{B_b^c}\right\rangle = 0 \quad \text{for all } \mathbf{x} \in \mathcal{S},$$

where B_b^c denotes the complement of the closed ball $B_b = \{x \in \mathbb{R}^d : |x| \leq b\}$. Hence $\mathcal{M}_f$ is a locally compact metric space with one-point compactification $\overline{\mathcal{M}_f} = \mathcal{M}_f \cup \{\partial\}$, and a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_f$ converges to ∂ as $n \to \infty$ if and only if either

$$\lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{1}\rangle = \infty$$

or

$$\liminf_{n\to\infty} \left\langle \mathbf{x}_n, \mathbf{1}_{B_b^c}\right\rangle \geq 1 \quad \text{for all } b > 0.$$

We also write $\mathcal{C}_0(\mathcal{M}_f)$ for the space of continuous maps $\varphi : \mathcal{M}_f \to \mathbb{R}$ with $\lim_{\mathbf{x}\to\partial} \varphi(\mathbf{x}) = 0$.

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

A random variable with values in $\mathcal{M}_f$ is called a finite point process. Recall that Π is a probability measure on $\mathcal{M}_f$ which determines the statistics of the point processes induced by the children in $\mathbf{U}$. We assume throughout this section that no particles die without offspring and the number of children has a finite expectation, that is,

$$\Pi(\langle \mathbf{x}, \mathbf{1}\rangle = 0) = 0 \quad \text{and} \quad \int_{\mathcal{M}_f} \langle \mathbf{x}, \mathbf{1}\rangle \Pi(\mathrm{d}\mathbf{x}) < \infty. \tag{2.1}$$

The process $\langle \mathbf{U}(t), \mathbf{1}\rangle$ that counts the number of particles alive at time $t \geq 0$ is a one-dimensional continuous-time Markov branching process in the sense of Chapter III of [5], and (2.1) ensures that $\langle \mathbf{U}(\cdot), \mathbf{1}\rangle$ remains finite (no explosion). In particular, this enables us to view $\mathbf{U}$ as a Markov process with values in $\mathcal{M}_f$. In order to analyze its semigroup, we need to introduce some notation.

First, for every $\mathbf{x} \in \mathcal{M}_f$ and $y \in \mathbb{R}^d$, we write $y + \mathbf{x}$ for the finite counting measure obtained by translating each and every atom of $\mathbf{x}$ by y. Equivalently, $y + \mathbf{x}$ is the pushforward measure of $\mathbf{x}$ by the translation $x \mapsto y + x$; in particular, $y + \delta_x = \delta_{x+y}$. The map $(y, \mathbf{x}) \mapsto y + \mathbf{x}$ is continuous from $\mathbb{R}^d \times \mathcal{M}_f$ to $\mathcal{M}_f$. Next, for any finite sequence $\mathbf{x}^1, \ldots, \mathbf{x}^k$ in $\mathcal{M}_f$, we write $\mathbf{x}^1 \sqcup \cdots \sqcup \mathbf{x}^k$ for the sum of those counting measures, so that the family of atoms of $\mathbf{x}^1 \sqcup \cdots \sqcup \mathbf{x}^k$ is the multiset which results from the superposition of the families of atoms of $\mathbf{x}^1, \ldots, \mathbf{x}^k$. This enables us to express the branching property of $\mathbf{U}$ as follows. Consider a finite counting measure $\mathbf{x} = \sum_{j=1}^k \delta_{x_j}$ with atoms $x_1, \ldots, x_k \in \mathbb{R}^d$. Let $\mathbf{U}^1, \ldots, \mathbf{U}^k$ be independent copies of $\mathbf{U}$, all started from the Dirac point mass at the origin. Then the process

$$\left(x_1 + \mathbf{U}^1(t)\right) \sqcup \cdots \sqcup \left(x_k + \mathbf{U}^k(t)\right), \quad t \geq 0$$

is a version of $\mathbf{U}$ started from $\mathbf{x}$. Recall that $r > 0$ is the rate of death of particles; we can now state the following.

Lemma 2.1. *An Uchiyama branching process* $\mathbf{U}$ *with reproduction rate* $r \cdot \Pi$ *satisfying* (2.1) *is a Feller process on* $\mathcal{M}_f$*. Its infinitesimal generator* $\mathcal{A}$ *has full domain* $\mathcal{C}_0(\mathcal{M}_f)$ *and is given for every* $\mathbf{x} = \sum_{j=1}^k \delta_{x_j} \in \mathcal{M}_f$ *and* $\varphi \in \mathcal{C}_0(\mathcal{M}_f)$ *by*

$$\mathcal{A}\varphi(\mathbf{x}) = r \sum_{j=1}^{k} \int_{\mathcal{M}_f} \varphi\left(\mathbf{x}_j^* \sqcup (x_j + \mathbf{y})\right) \Pi(\mathrm{d}\mathbf{y}) - rk\varphi(\mathbf{x}),$$

where $\mathbf{x}_j^* = \sum_{i \neq j} \delta_{x_i}$ *denotes the counting measure obtained by removing the atom* x_j *from* $\mathbf{x}$.

Remark 2.1. The first assumption in (2.1) that each particle has a non-empty child is crucial, and the Feller property always fails otherwise. To see why, let φ denote the indicator function of the zero measure $\varnothing$ (no atom), which is a continuous function on $\mathcal{M}_f$ with compact support. Consider also a sequence (x_n) in $\mathbb{R}^d$ which tends to ∞, so δ_{x_n} tends to ∂ in $\mathcal{M}_f$. Clearly, if the probability that a particle dies without progeny is strictly positive, then

$$\mathbb{E}(\varphi(\mathbf{U}(1)) \mid \mathbf{U}(0) = \delta_{x_n}) = \mathbb{P}(\mathbf{U}(1) = \varnothing \mid \mathbf{U}(0) = \delta_{x_n})$$

does not converge to 0 as $n \to \infty$.

Proof. Let $\mathbf{x} = \sum_{j=1}^k \delta_{x_j}$ be a finite counting measure. In the notation introduced before the lemma, the probability that $\mathbf{U}^1(t) = \cdots = \mathbf{U}^k(t) = \delta_0$ tends to 1 as $t \to 0+$. Therefore

$$\lim_{t \to 0+} \left(x_1 + \mathbf{U}^1(t)\right) \sqcup \cdots \sqcup \left(x_k + \mathbf{U}^k(t)\right) = \mathbf{x} \quad \text{in probability},$$

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

and for any function $\varphi \in \mathcal{C}_0(\mathcal{M}_f)$ we have

$$\lim_{t\to 0+} \mathbb{E}\big(\varphi\big(\big(x_1 + \mathbf{U}^1(t)\big) \sqcup \cdots \sqcup \big(x_k + \mathbf{U}^k(t)\big)\big)\big) = \varphi(\mathbf{x}).$$

Next consider any sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_f$ that converges to $\mathbf{x}$. Recall that for n sufficiently large, the number of atoms $\langle \mathbf{x}_n, \mathbf{1}\rangle$ of $\mathbf{x}_n$ coincides with k, and then we can enumerate those atoms, that is, we can write $\mathbf{x}_n = \sum_{j=1}^{k} \delta_{x_{n,j}}$ in such a way that the sequence $((x_{n,1}, \ldots, x_{n,k}))_{n\in\mathbb{N}}$ converges to $(x_1, \ldots, x_k)$ in $\mathbb{R}^{d\times k}$ as $n \to \infty$. Since φ is continuous and bounded, we easily conclude that the map

$$\mathbf{x} \to \mathbb{E}\big(\varphi\big(\big(x_1 + \mathbf{U}^1(t)\big) \sqcup \cdots \sqcup \big(x_k + \mathbf{U}^k(t)\big)\big)\big)$$

is continuous on $\mathcal{M}_f$. To check that this map also has limit 0 as $\mathbf{x} \to \partial$, since φ is bounded and has limit 0 at ∂, we simply need to verify that

$$\lim_{\mathbf{x}\to\partial} \big(x_1 + \mathbf{U}^1(t)\big) \sqcup \cdots \sqcup \big(x_k + \mathbf{U}^k(t)\big) = \partial \quad \text{in probability.}$$

So let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_f$ that tends to ∂. In the case where the total mass of $\mathbf{x}_n$ goes to infinity, the assumption that no particles die without offspring makes the above claim obvious. In the opposite case, there exists $k \geq 1$ and a subsequence $(\mathbf{x}'_n)_{n\in\mathbb{N}}$ extracted from $(\mathbf{x}_n)_{n\in\mathbb{N}}$ with $\langle \mathbf{x}'_n, \mathbf{1}\rangle = k$ for all n. Since $\lim_{n\to\infty} \mathbf{x}'_n = \partial$, the largest atom of $\mathbf{x}'_n$ tends to ∞ as $n \to \infty$, and the claim above follows again from the fact that $\mathbf{U}(t) \neq \varnothing$ a.s.

This completes the proof of the Feller property. The formula for the infinitesimal generator $\mathcal{A}$ should then be plain from the dynamics of Uchiyama branching processes and the alarm clock lemma. □

We write $\mathcal{D}(\mathcal{M}_f)$ for the space of rcll functions $\omega : \mathbb{R}_+ \to \mathcal{M}_f$, which we endow with the Skorokhod J_1-topology (we refer e.g. to Appendix A2 of [21] for quick background). The Feller property ensures the existence of a version of $\mathbf{U}$ with sample paths in $\mathcal{D}(\mathcal{M}_f)$ a.s., and henceforth we shall always work with such a version.

We now arrive at the main purpose of this section, namely the observation that $\mathbf{U}$ arises as the weak limit of certain sequences of branching random walks on $\mathbb{R}^d$.

Lemma 2.2. *Let $r > 0$ and let Π be a probability measure on $\mathcal{M}_f$ satisfying (2.1). For each $n \in \mathbb{N}$, let $\mathbf{Z}^n = (\mathbf{Z}^n(k))_{k\in\mathbb{N}}$ be a branching random walk on $\mathbb{R}^d$ started from δ_0. Assume that:*

(i) $\langle \mathbf{Z}^n(1), \mathbf{1}\rangle \geq 1$ *a.s. and* $\mathbb{E}(\langle \mathbf{Z}^n(1), \mathbf{1}\rangle) < \infty$ *for each* $n \in \mathbb{N}$,

(ii) $\mathbb{P}(\mathbf{Z}^n(1) \neq \delta_0) \sim r/n$ *as* $n \to \infty$,

(iii) $\lim_{n\to\infty} \mathbb{P}(\mathbf{Z}^n(1) \in \cdot \mid \mathbf{Z}^n(1) \neq \delta_0) = \Pi(\cdot)$ *in the sense of weak convergence for distributions on* $\mathcal{M}_f$.

Then we have

$$\lim_{n\to\infty} (\mathbf{Z}^n(\lfloor tn \rfloor))_{t\geq 0} = (\mathbf{U}(t))_{t\geq 0}$$

in the sense of weak convergence on $\mathcal{D}(\mathcal{M}_f)$, where in the right-hand side, $\mathbf{U}$ denotes an Uchiyama branching process with reproduction rate $r \cdot \Pi$ started from δ_0.

Proof. We shall establish the claim by verifying that the conditions of a well-known convergence theorem for Markov chains are satisfied for a killed version of the processes, where the

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

killing occurs at the time when the total mass exceeds some fixed threshold. Then letting this threshold tend to infinity will complete the proof.

To start with, the first assumption ensures that every particle in $\mathbf{Z}^n$ has at least one child at the next generation and that $\langle \mathbf{Z}^n(k), \mathbf{1}\rangle < \infty$ a.s. for all $k \in \mathbb{N}$. In particular, the branching random walks $\mathbf{Z}^n$ can be thought of as Markov chains with values in $\mathcal{M}_f$. Fix some $\ell \geq 1$ and write

$$\mathcal{P}_\ell = \{\mathbf{x} \in \mathcal{M}_f : \langle \mathbf{x}, \mathbf{1}\rangle \leq \ell\}$$

for the open subset of counting measures with at most ℓ atoms. The map

$$K_\ell : \mathcal{M}_f \to \overline{\mathcal{P}}_\ell = \mathcal{P}_\ell \cup \{\partial\}, \qquad K_\ell(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \langle \mathbf{x}, \mathbf{1}\rangle \leq \ell, \\ \partial & \text{otherwise,} \end{cases}$$

is continuous.

Since the total mass $\langle \mathbf{Z}^n(k), \mathbf{1}\rangle$ is non-decreasing in the variable $k \geq 0$ a.s., $K_\ell(\mathbf{Z}^n(\cdot))$ describes the branching random walk $\mathbf{Z}^n(\cdot)$ killed when its total mass exceeds ℓ, and is also a Markov chain. Similarly, $K_\ell(\mathbf{U}(\cdot))$ is still a Feller process (recall that K_ℓ is continuous) on the compact metric space $\overline{\mathcal{P}}_\ell$. We deduce from Lemma 2.1 that its infinitesimal generator $\mathcal{A}_\ell$ has full domain $\mathcal{C}(\overline{\mathcal{P}}_\ell)$ and is given by

$$\mathcal{A}_\ell \varphi(\mathbf{x}) = \mathcal{A}(\varphi \circ K_\ell)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathcal{P}_\ell \text{ and } \mathcal{A}_\ell\varphi(\partial) = 0.$$

Let $\mathbf{x} \in \mathcal{P}_\ell$ with $\mathbf{x} = \sum_{j=1}^k \delta_{x_j}$ for some $k \leq \ell$, and let $\mathbf{z}_1^n, \ldots, \mathbf{z}_k^n$ be i.i.d. copies of $\mathbf{Z}^n(1)$. By the branching property, the point process

$$\boldsymbol{\zeta}^n(\mathbf{x}) = (x_1 + \mathbf{z}_1^n) \sqcup \cdots \sqcup (x_k + \mathbf{z}_k^n)$$

has the distribution of $\mathbf{Z}^n(1)$ started from $\mathbf{x}$. The events $\{\mathbf{z}_j^n \neq \delta_0\}$ for $j = 1, \ldots, k$ are independent, and by the second assumption of the statement, each has probability $\sim r/n$.

Consider any $\varphi \in \mathcal{C}(\overline{\mathcal{P}}_\ell)$ and recall that φ is uniformly continuous since $\overline{\mathcal{P}}_\ell$ is a compact metric space. We deduce from the above that if we let $\mathbf{Y}^n$ denote a point process with the law of $\mathbf{Z}^n(1)$ conditioned on $\mathbf{Z}^n(1) \neq \delta_0$, then we have the bound

$$\left| \mathbb{E}(\varphi \circ K_\ell(\boldsymbol{\zeta}^n(\mathbf{x})) - \varphi(\mathbf{x})) - rn^{-1} \sum_{j=1}^k \mathbb{E}\big(\varphi \circ K_\ell\big(\mathbf{x}_j^* \sqcup (x_j + \mathbf{Y}^n)\big) - \varphi(\mathbf{x})\big) \right| \leq cn^{-2},$$

where $c > 0$ is some constant depending on ℓ and φ but not on $\mathbf{x}$.

On the other hand, we readily deduce from the third assumption, Skorokhod's representation theorem, and again the uniform continuity of φ, that there exists a sequence $(\varepsilon(n))_{n\in\mathbb{N}}$ converging to 0 and depending only on ℓ and φ, such that for all $\mathbf{x} \in \mathcal{M}_f$ and all $j = 1, \ldots, k = \langle \mathbf{x}, \mathbf{1}\rangle$,

$$\left| \mathbb{E}\big(\varphi \circ K_\ell\big(x_j^* \sqcup (x_j + \mathbf{Y}^n)\big)\big) - \int_{\mathcal{M}_f} \varphi \circ K_\ell\big(x_j^* \sqcup (x_j + \mathbf{y})\big)\Pi(\mathrm{d}\mathbf{y}) \right| \leq \varepsilon(n).$$

Putting the pieces together, we now see that

$$\lim_{n\to\infty} n\mathbb{E}(\varphi \circ K_\ell(\boldsymbol{\zeta}^n(\mathbf{x})) - \varphi(\mathbf{x})) = \mathcal{A}_\ell \varphi(\mathbf{x}) \quad \text{uniformly on } \overline{\mathcal{P}}_\ell.$$

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

We conclude from Theorem 19.28 of [21] (see also Theorem 6.5 in Chapter 1 of [15]) that there is weak convergence on $\mathcal{D}(\overline{\mathcal{P}}_\ell)$:

$$\lim_{n\to\infty} (K_\ell(\mathbf{Z}^n(\lfloor tn\rfloor)))_{t\geq 0} = (K_\ell(\mathbf{U}(t)))_{t\geq 0}. \tag{2.2}$$

The proof will be complete if, for every $T > 0$, we can show that

$$\lim_{n\to\infty} (\mathbf{Z}^n(\lfloor tn\rfloor))_{0\leq t\leq T} = (\mathbf{U}(t))_{0\leq t\leq T} \tag{2.3}$$

in the sense of weak convergence on the space $\mathcal{D}([0, T], \overline{\mathcal{M}_f})$ of rcll paths from $[0, T]$ to $\overline{\mathcal{M}_f}$. For this purpose, we fix a continuous functional $\Phi : \mathcal{D}([0, T], \overline{\mathcal{M}_f}) \to [0, 1]$. Again by the fact that the total mass of $\mathbf{Z}^n(\lfloor tn\rfloor)$ is non-decreasing in t, we have the inequality

$$\Phi((\mathbf{Z}^n(\lfloor tn\rfloor))_{0\leq t\leq T}) \geqslant \Phi((K_\ell(\mathbf{Z}^n(\lfloor tn\rfloor)))_{0\leq t\leq T})\mathbf{1}_{\{K_\ell(\mathbf{Z}^n(\lfloor Tn\rfloor))\in\mathcal{P}_\ell\}}.$$

By the continuity of mapping $(\mathbf{x}_t)_{0\leq t\leq T} \mapsto \mathbf{1}_{\{K_\ell(\mathbf{x}_T)\in\mathcal{P}_\ell\}}$ on $\mathcal{D}([0, T], \overline{\mathcal{M}_f})$ and weak convergence (2.2), we have

$$\liminf_{n\to\infty} \mathbb{E}(\Phi((\mathbf{Z}^n(\lfloor tn\rfloor))_{0\leq t\leq T})) \geqslant \mathbb{E}(\Phi((K_\ell(\mathbf{U}(t))_{0\leq t\leq T})\mathbf{1}_{\{K_\ell(\mathbf{U}(T))\in\mathcal{P}_\ell\}}).$$

Sending ℓ to infinity gives

$$\liminf_{n\to\infty} \mathbb{E}(\Phi((\mathbf{Z}^n(\lfloor tn\rfloor))_{0\leq t\leq T})) \geqslant \mathbb{E}(\Phi((\mathbf{U}(t)_{0\leq t\leq T})).$$

For the upper bound, replace Φ by $1-\Phi$ in the above reasoning. This shows (2.3) and completes the proof. □

2.2. Branching-stable processes and their trimmed versions

In this section we provide some background on the construction of branching-stable processes in Section 2 of [7] and some of their properties. Our presentation is tailored to our purposes; beware also that the present notation sometimes differs from that of [7].

As a quick summary, we start from a self-similar measure Λ^* on a space of counting measures $\mathbf{x}$ on $\mathbb{R}_+^* = (0, \infty)$, where self-similarity means that the pushforward image of Λ^* by the map $\mathbf{x} \mapsto c\mathbf{x}$ is $c^{-\alpha}\Lambda^*$ for every $c > 0$. The atoms of a Poisson point process $\mathbf{N}$ with intensity $\mathrm{d}t \otimes \Lambda^*(\mathrm{d}\mathbf{x})$ yield a point process $\mathbf{W}(1)$ on the upper quadrant $(0, \infty)^2$ that describes the progeny of an immortal and motionless ancestor located at 0. More precisely, an atom at (t, x) of $\mathbf{W}(1)$ is interpreted as a birth event occurring at the time when the ancestor has age t with the newborn child located at distance x to the right of the ancestor. We iterate for the next generation, just as for general branching processes [20], by considering $\mathbf{W}(1)$ as the first generation of a branching random walk $(\mathbf{W}(n))_{n\geq 0}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ started from $\mathbf{W}(0) = \delta_{(0,0)}$. A branching-stable process $\mathbf{S}(t)$ at time $t \geq 0$ then arises by restricting $\bigsqcup_{n\geq 0} \mathbf{W}(n)$ (i.e. the family of all the atoms appearing in the branching random walk $(\mathbf{W}(n))_{n\geq 0}$, possibly repeated according to their multiplicities) to the strip $[0, t] \times \mathbb{R}_+$.

We denote the space of locally finite counting measures on $\mathbb{R}_+$ by $\mathcal{M}$, endowed with the topology of vague convergence and its Borel σ-algebra. Possible atoms at 0 play a special role, and it is convenient also to introduce the notation $\mathcal{M}^*$ for the subspace of non-zero counting measures $\mathbf{x} \in \mathcal{M}$ with no atom at 0. Just as in the Introduction, we write $(x_j)_{j\geq 1}$ for the ordered

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

sequence of the atoms of $\mathbf{x} \in \mathcal{M}^*$, i.e. $\mathbf{x} = \sum_{j\geq 1} \delta_{x_j}$, where $x_j \in (0, \infty]$ for all $j \geq 1$ and $x_1 < \infty$. We also write

$$\mathcal{M}^1 = \{\mathbf{x} \in \mathcal{M}^* : x_1 = 1\},$$

so that any $\mathbf{x} \in \mathcal{M}^*$ has a 'polar' representation in the form $\mathbf{x} = r\mathbf{y}$ with $r = x_1 \in \mathbb{R}_+^*$ and $\mathbf{y} \in \mathcal{M}^1$.

Let $\alpha > 0$; we first consider some finite measure $\boldsymbol{\lambda}$ on $\mathcal{M}^1$ such that

$$\int_{\mathcal{M}^1} \langle \mathbf{y}, \bullet^{-\alpha}\rangle \boldsymbol{\lambda}(\mathrm{d}\mathbf{y}) < \infty, \quad \text{where } \langle \mathbf{y}, \bullet^{-\alpha}\rangle = \sum_{j\geq 1} y_j^{-\alpha}. \tag{2.4}$$

We then define a self-similar measure Λ^* on $\mathcal{M}^*$ as the image of $r^{\alpha-1}\,\mathrm{d}r \otimes \boldsymbol{\lambda}(\mathrm{d}\mathbf{y})$ by the map $(r, \mathbf{y}) \mapsto r\mathbf{y}$. In other words, for every measurable functional $\varphi : \mathcal{M}^* \to \mathbb{R}_+$,

$$\int_{\mathcal{M}^*} \varphi(\mathbf{x})\Lambda^*(\mathrm{d}\mathbf{x}) = \int_0^\infty r^{\alpha-1} \int_{\mathcal{M}^1} \varphi(r\mathbf{y})\boldsymbol{\lambda}(\mathrm{d}\mathbf{y})\mathrm{d}r. \tag{2.5}$$

We next introduce a Poisson point process $\mathbf{N}$ on $(0, \infty) \times \mathcal{M}^*$ with intensity $\mathrm{d}t \times \Lambda^*(\mathrm{d}\mathbf{x})$ and represent each atom $(t, \mathbf{x})$ of $\mathbf{N}$ as a sequence of atoms $((t, x_j))_{j\geq 1}$ on the fiber $\{t\} \times (0, \infty]$. Discarding as usual any atoms (t, ∞), this induces a point process $\mathbf{W}(1)$ on the quadrant $(0, \infty)^2$ such that

$$\langle \mathbf{W}(1), f\rangle := \int_{(0,\infty)\times\mathcal{M}_+^*} \langle \mathbf{x}, f(t, \bullet)\rangle \mathbf{N}(\mathrm{d}t,\ \mathrm{d}\mathbf{x}),$$

where $f : (0, \infty)^2 \to \mathbb{R}_+$ is a generic measurable function. In turn $\mathbf{W}(1)$ inherits the scaling property, namely, for every $c > 0$, its image by the map $(t, x) \mapsto (c^\alpha t, cx)$ has the same distribution as $\mathbf{W}(1)$.

We consider $\mathbf{W}(1)$ as the first generation of a branching random walk $(\mathbf{W}(n))_{n\geq 0}$ on $\mathbb{R}_+^2$ started as usual from a single atom at the origin. Finally, for every $t \geq 0$, we write $\mathbf{S}(t)$ for the point process on $\mathbb{R}_+$ defined by

$$\langle \mathbf{S}(t), g\rangle = \sum_{n=0}^\infty \langle \mathbf{W}(n), g_t\rangle,$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ stands for a generic measurable function and $g_t(s, x) = \mathbf{1}_{[0,t]}(s)g(x)$. According to Theorem 2.1 of [7], $(\mathbf{S}(t))_{t\geq 0}$ is a branching-stable process, that is, a branching process in continuous time which is self-similar with exponent $-\alpha$, in the sense that for every $c > 0$, the processes $(\mathbf{S}(c^{-\alpha}t))_{t\geq 0}$ and $(c\mathbf{S}(t))_{t\geq 0}$ have the same law.

By construction, the law of the branching-stable process $\mathbf{S} = (\mathbf{S}(t))_{t\geq 0}$ is determined by the self-similar measure Λ^* on the space $\mathcal{M}^*$. It is convenient to introduce a closely related measure Λ, now on the space $\mathcal{M}$, which is given by the pushforward of Λ^* by the map $\mathcal{M}^* \to \mathcal{M}$, $\mathbf{x} \mapsto \mathbf{x} = \delta_0 \sqcup \mathbf{x}$ that adds an atom at 0 to $\mathbf{x}$. Plainly Λ is also self-similar and obviously determines Λ^*. One calls Λ the Lévy measure of $\mathbf{S}$ as it bears the same relation to $\mathbf{S}$ viewed as a branching Lévy process as the classical Lévy measure does to a stable subordinator; see [6].

We shall now recall a trimming transformation, which was introduced more generally in [6] for so-called branching Lévy processes under the name censoring, and which allows us to represent $(\mathbf{S}(t))_{t\geq 0}$ as the increasing limit of a sequence of Uchiyama branching processes. Trimming is better understood when one recalls the interpretation of the construction of $\mathbf{S}(t)$

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

as a general branching process that we sketched at the beginning of this section. Fix some threshold $b > 0$ and write $\mathbf{W}^{[b]}(1)$ for the point process obtained from $\mathbf{W}(1)$ by restriction to the strip $\mathbb{R}_+ \times [0, b]$; in other words, we delete all the atoms (t, x) with $x > b$. Just as above, we see $\mathbf{W}^{[b]}(1)$ as the first generation of a branching random walk $(\mathbf{W}^{[b]}(n))_{n\geq 0}$ on $\mathbb{R}^2_+$, and define

$$\langle \mathbf{S}^{[b]}(t), g \rangle = \sum_{n=0}^{\infty} \langle \mathbf{W}^{[b]}(n), g_t \rangle.$$

In words, the trimmed process $(\mathbf{S}^{[b]}(t))_{t\geq 0}$ is obtained from $\mathbf{S}$ by killing at every birth event every child born at distance greater than b from its parent, of course together with its descendants.

We write $\mathbf{x} \mapsto \mathbf{x}^{[b]} = \mathbf{1}_{[0,b]}\mathbf{x}$ for the cut-off map from $\mathcal{M}$ to the space of finite counting measures $\mathcal{M}_f$ on $[0, \infty)$ (beware that the trimmed process $\mathbf{S}^{[b]}$ is not the image of $\mathbf{S}$ by the cut-off map; the latter would instead be denoted by $(\mathbf{S}(t)^{[b]})_{t\geq 0}$), and $\Lambda^{[b]}$ for the image of the Lévy measure Λ restricted to point processes $\mathbf{x} \in \mathcal{M}$ having two or more atoms on $[0, b]$ (recall that by construction $\mathbf{x}$ has exactly one atom at 0, Λ-almost everywhere) by this map. Note that for $\mathbf{x} \in \mathcal{M}^*$, $\mathbf{x}^{[b]} = \varnothing$ is the zero point mass if and only if $x_1 > b$, and therefore

$$\Lambda^{[b]}(\mathcal{M}) = \Lambda^*(\mathbf{x} \in \mathcal{M}^* : x_1 \leq b) = \alpha^{-1} b^{\alpha} \boldsymbol{\lambda}(\mathcal{M}^1) < \infty.$$

Hence $\Lambda^{[b]}$ is a finite measure on $\mathcal{M}_f$; it can be viewed as the reproduction rate of some Uchiyama branching process. We also stress that $\Lambda^{[b]}$ gives no mass to the zero point process $\varnothing$ and is carried by the subspace of finite counting measures having a single atom at 0.

Lemma 2.3. *The branching-stable process trimmed at threshold b,* $\mathbf{S}^{[b]} = (\mathbf{S}^{[b]}(t))_{t\geq 0}$*, of a branching-stable process* $\mathbf{S}$ *with Lévy measure* Λ*, is an Uchiyama branching process on* $\mathbb{R}_+$ *with reproduction rate* $\Lambda^{[b]}$*. Moreover, the latter fulfills* (2.1)*.*

Proof. The statement is closely related to Section 5.2 of [6]; we shall nonetheless present an independent proof. The trimmed process $\mathbf{S}^{[b]}$ is a general branching process with progeny described by the cut-off version $\mathbf{N}^{[b]}$ of the Poisson point process $\mathbf{N}$. By the image property of Poisson point processes, the latter is a Poisson point process on $\mathbb{R}_+ \times \mathcal{M}_f$ with intensity $\mathrm{d}t \otimes \Lambda^{*[b]}(\mathrm{d}\mathbf{x})$, where $\Lambda^{*[b]}$ denotes the pushforward measure of Λ^* restricted to $\{\mathbf{x} \in \mathcal{M}^* : x_1 \leq b\}$ by the cut-off map $\mathbf{x} \mapsto \mathbf{x}^{[b]}$.

As a consequence, the first instant at which the ancestor gives birth,

$$\tau^{[b]} = \inf\{t > 0 : \mathbf{N}^{[b]}([0, t] \times \mathcal{M}_f) > 0\},$$

has the exponential distribution with parameter $\alpha^{-1} b^{\alpha} \boldsymbol{\lambda}(\mathcal{M}^1)$ and its offspring at that time has the normalized law $\alpha b^{-\alpha} \boldsymbol{\lambda}(\mathcal{M}^1)^{-1} \Lambda^{*[b]}$, independently of $\tau^{[b]}$. Moreover, the point process of the progeny shifted at time is again a Poisson point process with intensity $\mathrm{d}t \otimes \Lambda^{*[b]}(\mathrm{d}\mathbf{x})$, and is independent of the preceding quantities.

Imagine now that we decide to kill the ancestor at time $\tau^{[b]}$ and simultaneously add a child located at 0 to its progeny. Since the new child is born at the same location as the ancestor and precisely at the time when the ancestor is killed, this has no effect on the process itself. This should clarify the claim that $\mathbf{S}^{[b]}$ evolves like an Uchiyama branching process with reproduction rate given the image of $\Lambda^{*[b]}$ by the map $\mathbf{x} \mapsto \delta_0 \sqcup \mathbf{x}$ which adds an atom at 0 to the finite counting measure $\mathbf{x}$. The latter is precisely $\Lambda^{[b]}$.

Finally, the cut-off $\mathbf{x} \mapsto \mathbf{x}^{[b]}$ preserves atoms at 0, so $\Lambda^{[b]}$ verifies the first assertion of (2.1). The second assertion is plain from the Markov inequality $\langle \mathbf{x}^{[b]}, \mathbf{1} \rangle \leq b^{\alpha} \langle \mathbf{x}, \bullet^{-\alpha} \rangle$ and (2.4). □

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

3. A branching-stable limit theorem

The purpose of this section is to establish our main result, which was presented only informally in the Introduction; let us briefly recall the setting. We consider a branching random walk $\mathbf{Z} = (\mathbf{Z}(n))_{n\geq 0}$ on $\mathbb{R}_+$ started from $\mathbf{Z}(0) = \delta_0$. We suppose that $\mathbf{Z}(1)$ has a single atom at the origin a.s. and write

$$0 < X_1 \leq X_2 \leq \cdots \leq \infty$$

for the ordered sequence of positive locations of atoms of $\mathbf{Z}(1)$ repeated according to their multiplicities, and then set

$$\mathbf{Z}^*(1) = \mathbf{Z}(1) - \delta_0 = \sum_{j\geq 1} \delta_{X_j}.$$

We shall also need the following basic fact.

Lemma 3.1. *Assume* (1.2)*,* (1.3)*, and* (1.4)*. Then the limit distribution in* (1.3) *is the law of a point process that can be expressed in the form* $V^{1/\alpha}\mathbf{Y}$*, with* $\mathbf{Y}$ *a point process in* $\mathcal{M}^1$ *and* V *a uniform random variable on* $(0, 1)$ *independent of* $\mathbf{Y}$*. Furthermore, the distribution* $\boldsymbol{\rho}$ *of* $\mathbf{Y}$ *satisfies* (2.4)*.*

Proof. The first part of the claim belongs to the folklore on multidimensional regular variation; see for instance Chapter 5 of [24]. For the reader's convenience, we recall the argument.

Denote the point process arising as the weak limit in (1.3) by $\mathbf{L}$, and write L_1 for the location of its leftmost atom. We deduce immediately from (1.2) that $V = L_1^\alpha$ has the uniform distribution on [0, 1], and $\mathbf{Y} = L_1^{-1}\mathbf{L}$ is a point process in $\mathcal{M}^1$. So $\mathbf{L} = V^{1/\alpha}\mathbf{Y}$ and we shall now argue that V and $\mathbf{Y}$ are independent.

We get from the continuous mapping theorem and (1.3) that as $t \to 0+$, the conditional distribution of the pair

$$\big(X_1/t, X_1^{-1}\mathbf{Z}^*(1)\big) = \big(X_1/t, (t/X_1)(t^{-1}\mathbf{Z}^*(1))\big)$$

conditioned on $X_1 \leq t$ converges weakly to that of $(V^{1/\alpha}, \mathbf{Y})$. For any continuous and bounded function $f : \mathcal{M}^1 \to \mathbb{R}$ and $u \in (0, 1]$, we have

$$\lim_{t\to 0+} \mathbb{E}\big(f\big(X_1^{-1}\mathbf{Z}^*(1)\big)\mathbf{1}_{\{X_1/t\leq u\}} \mid X_1 \leq t\big) = \mathbb{E}\big(f(\mathbf{Y})\mathbf{1}_{\{V\leq u^\alpha\}}\big).$$

We can also write

$$\begin{aligned}&\mathbb{E}\big(f\big(X_1^{-1}\mathbf{Z}^*(1)\big)\mathbf{1}_{\{X_1/t\leq u\}} \mid X_1 \leq t\big)\\ &\quad = \mathbb{E}\big(f\big(X_1^{-1}\mathbf{Z}^*(1)\big)\mathbf{1}_{\{X_1\leq ut\}} \mid X_1 \leq ut\big)\mathbb{P}(X_1 \leq ut \mid X_1 \leq t).\end{aligned}$$

So letting $t \to 0$, we get

$$\mathbb{E}\big(f(\mathbf{Y})\mathbf{1}_{\{V\leq u^\alpha\}}\big) = \mathbb{E}(f(\mathbf{Y}))u^\alpha,$$

which shows that V and $\mathbf{Y}$ are independent.

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

Finally, recall that $(a_n)_{n\geq 1}$ is a sequence of positive real numbers such that $\mathbb{P}(X_1 \leq a_n) \sim 1/n$ and ψ is the log-Laplace transform of the intensity measure of $\mathbf{Z}(1)$. From (1.4) and the bound

$$\mathbb{E}\left(\sum_{j=1}^{\infty} \exp(-a_n^{-1} X_j) \mid X_1 \leq a_n\right) \leq \frac{1}{\mathbb{P}(X_1 \leq a_n)}\big(\mathrm{e}^{\psi(1/a_n)} - 1\big),$$

we deduce that

$$\limsup_{n\to\infty} \sum_{j=1}^{\infty} \mathbb{E}\big(\exp(-a_n^{-1} X_j) \mid X_1 \leq a_n\big) < \infty.$$

We write $\mathbf{Y} = \sum_{j\geq 1} \delta_{Y_j}$, and recall from the above that for every $j \geq 1$, the conditional law of $a_n^{-1} X_j$ given $X_1 \leq a_n$ converges weakly to that of Y_j on $[1, \infty]$. Fatou's lemma now entails that

$$\alpha \sum_{j=1}^{\infty} \mathbb{E}\left(\int_0^1 \mathrm{e}^{-tY_j} t^{\alpha-1}\,\mathrm{d}t\right) = \mathbb{E}\left(\sum_{j=1}^{\infty} \exp(-V^{1/\alpha} Y_j)\right) < \infty.$$

Since there is some $c_\alpha > 0$ such that

$$c_\alpha y^{-\alpha} \leq \int_0^1 \mathrm{e}^{-ty} t^{\alpha-1}\,\mathrm{d}t \quad \text{for all } y \geq 1,$$

our last claim follows. □

Next we introduce for every $r > 0$ the space $\mathcal{M}_{r,f}$ of counting measures $\mathbf{x}$ on $\mathbb{R}_+$ such that $\langle \mathbf{x}, \mathrm{e}^{-r\bullet}\rangle < \infty$. We associate to each $\mathbf{x} \in \mathcal{M}_{r,f}$ the finite measure $m_{r,\mathbf{x}}$ on $\mathbb{R}_+$ which has density $\mathrm{e}^{-r\bullet}$ with respect to $\mathbf{x}$. In words, assuming for simplicity that the counting measure $\mathbf{x}$ is simple, $m_{r,\mathbf{x}}$ is a purely atomic measure, the locations of its atoms are the same as for $\mathbf{x}$, and the mass of an atom at x is e^{-rx}. We then define $d_r(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{M}_{r,f}$ as the Lévy–Prokhorov distance between $m_{r,\mathbf{x}}$ and $m_{r,\mathbf{y}}$; this makes $\mathcal{M}_{r,f}$ a locally compact metric space. We write $\mathcal{D}(\mathcal{M}_{r,f})$ for the space of rcll functions $\omega : \mathbb{R}_+ \to \mathcal{M}_{r,f}$, endowed with the Skorokhod J_1 topology.

We may now state rigorously the main result of this work.

Theorem 3.1. *Assume* (1.2), (1.3), *and* (1.4), *and let* $\mathbf{S}$ *be a branching-stable process with Lévy measure* Λ, *such that* Λ^* *is given by* (2.5) *for* $\boldsymbol{\lambda} = \alpha \cdot \boldsymbol{\rho}$, *and* $\boldsymbol{\rho}$ *is the probability measure on* $\mathcal{M}^1$ *that arises in Lemma* 3.1.

Then, for every $r > 0$, *we have*

$$\lim_{n\to\infty} \big(a_n^{-1} \mathbf{Z}(\lfloor tn \rfloor)\big)_{t\geq 0} = (\mathbf{S}(t))_{t\geq 0}$$

in the sense of weak convergence on $\mathcal{D}(\mathcal{M}_{r,f})$.

Remark 3.1. Assume (1.2) holds; then we have by a Tauberian theorem

$$\mathbb{E}(\mathrm{e}^{-tX_1}) \sim \Gamma(1+\alpha) F_1(1/t) \quad \text{as } t \to \infty.$$

Since $\mathbb{E}(\mathrm{e}^{-tX_j}) \leq \mathbb{E}(\mathrm{e}^{-tX_1})$ for all $t > 0$ and all $j \geq 1$, we see that (1.4) follows from (1.2) whenever the total mass of $\mathbf{Z}(1)$ is bounded, say $X_j = \infty$ a.s. whenever $j \geq k$. Indeed, we then have

$$\psi(t) \leq \log\big(1 + k\mathbb{E}(\mathrm{e}^{-tX_1})\big) \sim k\Gamma(1+\alpha) F_1(1/t).$$

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

Moreover, if $\mathbf{Z}(1)$ actually has at most two atoms a.s., i.e. $k=1$ above, then (1.3) also holds, and more precisely the limiting distribution there is that of the sequence $(V^{1/\alpha}, \infty, \infty, \dots)$ with V a uniform random variable on [0, 1].

Throughout the rest of this section we assume without further mention that (1.2), (1.3), and (1.4) hold, and we shall further use the notation in Lemma 3.1 and Theorem 3.1. We first set some further notation relevant to the proof of Theorem 3.1.

For every $n \geq 1$ and $b > 0$, we introduce the rescaled branching random walk

$$\mathbf{Z}^{[n,b]} = \big(\mathbf{Z}^{[n,b]}(k)\big)_{k\geq 0}$$

that results from $\mathbf{Z}$ by first rescaling with a factor a_n^{-1} and then trimming at threshold b (i.e. the children born at distance greater than b from their parents are killed). In words, the first generation is given by

$$\mathbf{Z}^{[n,b]}(1) = \big(a_n^{-1}\mathbf{Z}(1)\big)^{[b]} = \delta_0 + \sum_{j\geq 1} \mathbf{1}_{[0,b]}\delta_{X_j/a_n}.$$

We start by checking that for every fixed $b > 0$, this sequence of branching random walks fulfills the assumptions of Lemma 2.2. The first assumption there is straightforward and we focus on the second and third.

Lemma 3.2. *We have*

$$\mathbb{P}\big(\mathbf{Z}^{[n,b]}(1) \neq \delta_0\big) \sim b^\alpha/n \quad \text{as } n \to \infty$$

and, in the notation introduced above Lemma 2.3,

$$\lim_{n\to\infty} \mathbb{P}\big(\mathbf{Z}^{[n,b]}(1) \in \cdot \mid \mathbf{Z}^{[n,b]}(1) \neq \delta_0\big) = \Pi^{[b]}(\cdot)$$

in the sense of weak convergence for distributions on $\mathcal{M}_f$, where $\Pi^{[b]}$ denotes the law of the finite point process

$$\delta_0 \sqcup b(V^{1/\alpha}\mathbf{Y})^{[1]} = \delta_0 + \sum_{j\,:\, V^{1/\alpha}Y_j \leq 1} \delta_{bV^{1/\alpha}Y_j}.$$

Proof. The events $\big\{\mathbf{Z}^{[n,b]}(1) \neq \delta_0\big\}$ and $\{X_1 \leq a_n b\}$ coincide, and the first estimate is then plain from (1.2). Next, write

$$0 < X_1^{[n,b]} \leq X_2^{[n,b]} \cdots \leq \infty$$

for the ordered sequence of atoms of $\mathbf{Z}^{[n,b]}(1)$ (discarding as usual the atom at the origin), and fix some $\ell \geq 1$ and $x_j \in [0, b]$ for $j = 1, \dots, \ell$. We then write

$$\begin{aligned}
&\mathbb{P}\big(X_1^{[n,b]} \leq x_1, \dots, X_\ell^{[n,b]} \leq x_\ell, X_{\ell+1}^{[n,b]} = \infty\big) \\
&\quad = \mathbb{P}(X_1 \leq a_n x_1, \dots, X_\ell \leq a_n x_\ell, X_{\ell+1} > a_n b) \\
&\quad = \mathbb{P}\left(\frac{X_1}{a_n b} \leq \frac{x_1}{b}, \dots, \frac{X_\ell}{a_n b} \leq \frac{x_\ell}{b}, \frac{X_{\ell+1}}{a_n b} > 1 \mid X_1 \leq a_n b\right)\mathbb{P}(X_1 \leq a_n b).
\end{aligned}$$

Recall on the one hand from (1.2) that $\mathbb{P}(X_1 \leq a_n b) \sim b^\alpha/n$, and on the other hand, by Lemma 3.1, that the first term of the product in the last displayed quantity converges as $n \to \infty$ to

$$\mathbb{P}(bV^{1/\alpha}Y_1 \leq x_1, \ldots, bV^{1/\alpha}Y_\ell \leq x_\ell, V^{1/\alpha}Y_{\ell+1} > 1).$$

This entails our second claim. □

Lemma 3.2 immediately entails the following version of Theorem 3.1 for the trimmed processes.

Corollary 3.1. *Under the same assumptions and notation as in Theorem 3.1, we have for every $b > 0$*

$$\lim_{n\to\infty} \big(\mathbf{Z}^{[n,b]}(\lfloor tn \rfloor)\big)_{t\geq 0} = (\mathbf{S}^{[b]}(t))_{t\geq 0}$$

in the sense of weak convergence on $\mathcal{D}(\mathcal{M}_f)$.

Proof. It suffices to observe first the easy identity $\alpha \cdot \Pi^{[b]} = \Lambda^{[b]}$, and then to combine Lemmas 2.2, 2.3, and 3.2. □

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Recall that if m and m' are two measures with $m \geq m'$, then the Lévy–Prokhorov distance between m and m' is bounded from above by the total mass of the positive measure $m - m'$. Recalling also the definition of the distance d_r on $\mathcal{M}_{r,f}$, this yields for every fixed $b > 0$ the bound

$$d_r\big(\mathbf{S}(s), \mathbf{S}^{[b]}(s)\big) \leq \big\langle \mathbf{S}(s) - \mathbf{S}^{[b]}(s), \mathrm{e}^{-r\bullet}\big\rangle \quad \text{for all } s \geq 0.$$

Moreover, the map $s \mapsto \big(\mathbf{S}(s) - \mathbf{S}^{[b]}(s)\big)$ with values in the space of positive measures is non-decreasing. Working on the time-interval $[0, t]$ for some fixed $t > 0$, we get

$$\sup_{0\leq s\leq t} d_r\big(\mathbf{S}(s), \mathbf{S}^{[b]}(s)\big) \leq \big\langle \mathbf{S}(t) - \mathbf{S}^{[b]}(t), \mathrm{e}^{-r\bullet}\big\rangle.$$

On the other hand, we plainly have $\mathbf{1}_{[0,b]}\mathbf{S}(t) \leq \mathbf{S}^{[b]}(t)$, so

$$\mathbf{S}(t) - \mathbf{S}^{[b]}(t) \leq \mathbf{1}_{(b,\infty)}\mathbf{S}(t),$$

and since $\langle \mathbf{S}(t), \mathrm{e}^{-r\bullet}\rangle) < \infty$ a.s. (see e.g. Proposition 3.1(ii) of [7]), we conclude that

$$\lim_{b\to\infty} \sup_{0\leq s\leq t} d_r\big(\mathbf{S}(s), \mathbf{S}^{[b]}(s)\big) = 0 \quad \text{a.s.} \tag{3.1}$$

From (3.1), Corollary 3.1, and the fact that the Prokhorov distance dominates d_r on $\mathcal{M}_f$, we can find a sequence (b_n) of positive numbers which grows to ∞ sufficiently slowly, such that

$$\lim_{n\to\infty} \big(\mathbf{Z}^{[n,b_n]}(\lfloor sn \rfloor)\big)_{0\leq s\leq t} = (\mathbf{S}(s))_{0\leq s\leq t} \tag{3.2}$$

in the sense of weak convergence on the space $\mathcal{D}([0, t], \mathcal{M}_{r,f})$ of rcll paths from $[0, t]$ to $\mathcal{M}_{r,f}$.

By the same argument as in the first paragraph of the proof, we also have for each $n \geq 1$

$$\sup_{0\leq s\leq t} d_r\big(a_n^{-1}\mathbf{Z}(\lfloor ns \rfloor), \mathbf{Z}^{[n,b_n]}(\lfloor ns \rfloor)\big) \leq \big\langle a_n^{-1}\mathbf{Z}(\lfloor nt \rfloor), \mathbf{1}_{(b_n,\infty)}\,\mathrm{e}^{-r\bullet}\big\rangle.$$

https://doi.org/10.1017/jpr.2021.101 Published online by Cambridge University Press

Thanks to the Markov inequality, the right-hand side is bounded from above by

$$\mathrm{e}^{-b_n(r-\widetilde{r})}\langle a_n^{-1}\mathbf{Z}(\lfloor nt\rfloor),\, \mathrm{e}^{-\widetilde{r}\bullet}\rangle = \mathrm{e}^{-b_n(r-\widetilde{r})}\langle \mathbf{Z}(\lfloor nt\rfloor),\, \mathrm{e}^{-a_n^{-1}\widetilde{r}\bullet}\rangle$$

where $0<\widetilde{r}<r$. Recall that ψ denotes the log-Laplace transform of the intensity measure of $\mathbf{Z}(1)$, so the expectation of the right-hand side equals

$$\mathbb{E}\big(\langle \mathbf{Z}(\lfloor nt\rfloor),\, \mathrm{e}^{-a_n^{-1}\widetilde{r}\bullet}\rangle\big) = \exp(\lfloor nt\rfloor \psi(\widetilde{r}x/a_n)).$$

This quantity remains bounded as $n\to\infty$ by assumption (1.4) and the fact that (a_n) is regularly varying. Putting the pieces together, we have shown that

$$\lim_{n\to\infty} \mathbb{E}\bigg(\sup_{0\le s\le t} d_r\big(a_n^{-1}\mathbf{Z}(\lfloor ns\rfloor),\, \mathbf{Z}^{[n,b_n]}(\lfloor ns\rfloor)\big)\bigg) = 0. \tag{3.3}$$

Applying the argument of Lemma VI. 3.31 of [19] in the setting of metric spaces rather than $\mathbb{R}^d$, we conclude from (3.2) and (3.3) that

$$\lim_{n\to\infty} \big(a_n^{-1}\mathbf{Z}(\lfloor sn\rfloor)\big)_{0\le s\le t} = (\mathbf{S}(s))_{0\le s\le t}$$

in the sense of weak convergence on $\mathcal{D}([0,\,t],\,\mathcal{M}_{r,f})$, and the proof is complete. □

Appendix: Some spaces of counting measures

We list below the notation for several spaces of counting measures which appear in this text.

- $\mathcal{M}$ denotes the space of locally finite counting measures on $\mathbb{R}_+$ equipped with the topology of vague convergence and its Borel σ-algebra.
- $\mathcal{M}_f$ denotes the space of finite counting measures (first on $\mathbb{R}^d$ in Section 2.1 and then on $\mathbb{R}_+$ in the rest of the article), endowed with the Lévy–Prokhorov distance.
- $\overline{\mathcal{M}_f} = \mathcal{M}_f \cup \{\partial\}$ is the one-point compactification of $\mathcal{M}_f$ for the Lévy–Prokhorov metric.
- $\mathcal{M}_\ell$, for some $\ell\ge 1$, denotes the space of counting measures in $\mathcal{M}_f$ with total mass at most ℓ (first on $\mathbb{R}^d$ in Section 2.1, and then on $\mathbb{R}_+$ in the rest of the article), endowed with the Lévy–Prokhorov distance.
- $\mathcal{M}^*$ denotes the subspace of non-zero counting measures in $\mathcal{M}$ with no atom at 0.
- $\mathcal{M}^1$ is the subspace of counting measures in $\mathcal{M}^*$ with leftmost atom located at 1.
- $\mathcal{M}_{r,f}$, for some $r>0$, denotes the subspace of counting measures $\mathbf{x}\in\mathcal{M}$ with $\langle \mathbf{x},\, \mathrm{e}^{-r\bullet}\rangle<\infty$.

Acknowledgements

The authors are grateful to two anonymous referees for their careful reading of an earlier version of this paper and their insightful comments which helped to improve this work.

Funding information

The work was supported by the Swiss National Science Foundation, grant 188693.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Prob.* **41**, 1362–1426.
[2] AÏDÉKON, E., BERESTYCKI, J., BRUNET, E. AND SHI, Z. (2013). Branching Brownian motion seen from its tip. *Prob. Theory Related Fields* **157**, 405–451.
[3] ARGUIN, L.-P., BOVIER, A. AND KISTLER, N. (2012). Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Prob.* **22**, 1693–1711.
[4] ARGUIN, L.-P., BOVIER, A. AND KISTLER, N. (2013). The extremal process of branching Brownian motion. *Prob. Theory Related Fields* **157**, 535–574.
[5] ATHREYA, K. B. AND NEY, P. E. (2004). *Branching Processes*. Dover Publications.
[6] BERTOIN, J. AND MALLEIN, B. (2019). Infinitely ramified point measures and branching Lévy processes. *Ann. Prob.* **47**, 1619–1652.
[7] BERTOIN, J., CORTINES, A. AND MALLEIN, B. (2018). Branching-stable point measures and processes. *Adv. Appl. Prob.* **50**, 1294–1314.
[8] BHATTACHARYA, A., HAZRA, R. S. AND ROY, P. (2017). Point process convergence for branching random walks with regularly varying steps. *Ann. Inst. H. Poincaré Prob. Statist.* **53**, 802–818.
[9] BHATTACHARYA, A., HAZRA, R. S. AND ROY, P. (2018). Branching random walks, stable point processes and regular variation. *Stoch. Process. Appl.* **128**, 182–210.
[10] BIGGINS, J. D. (1977). Martingale convergence in the branching random walk. *J. Appl. Prob.* **14**, 25–37.
[11] BIGGINS, J. D. (1992). Uniform convergence of martingales in the branching random walk. *Ann. Prob.* **20**, 137–151.
[12] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Commun. Pure Appl. Math.* **31**, 531–581.
[13] BRAMSON, M. D. (1983). *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves* (Mem. Amer. Math. Soc. **44**). American Mathematical Society.
[14] DURRETT, R. (1983). Maxima of branching random walks. *Z. Wahrscheinlichkeitsth.* **62**, 165–170.
[15] ETHIER, S. N. AND KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence* (Wiley Series in Probability and Mathematical Statistics). John Wiley.
[16] GNEDENKO, B. V. AND KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, MA.
[17] HULT, H. AND LINDSKOG, F. (2006). Regular variation for measures on metric spaces. *Publ. Inst. Math. N.S.* **80**, 121–140.
[18] IBRAGIMOV, I. A. AND LINNIK, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff Publishing, Groningen.
[19] JACOD, J. AND SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes* (Grundlehren der Mathematischen Wissenschaften **288**), 2nd edn. Springer, Berlin.
[20] JAGERS, P. (1989). General branching processes as Markov fields. *Stoch. Process. Appl.* **32**, 183–212.
[21] KALLENBERG, O. (2002). *Foundations of Modern Probability* (Probability and its Applications (New York)), 2nd edn. Springer, New York.
[22] LINDSKOG, F., RESNICK, S. I. AND ROY, J. (2014). Regularly varying measures on metric spaces: hidden regular variation and hidden jumps. *Prob. Surv.* **11**, 270–314.
[23] MADAULE, T. (2017). Convergence in law for the branching random walk seen from its tip. *J. Theoret. Prob.* **30**, 27–63.
[24] RESNICK, S. I. (2007). *Heavy-tail Phenomena: Probabilistic and Statistical Modeling* (Springer Series in Operations Research and Financial Engineering). Springer, New York.
[25] SHI, Z. (2015). *Branching Random Walks: Lecture Notes From the 42nd Probability Summer School, Saint Flour 2012* (Lecture Notes Math. **2151**). Springer, Cham.
[26] UCHIYAMA, K. (1982). Spatial growth of a branching process of particles living in $\mathbf{R}^d$. *Ann. Prob.* **10**, 896–918.