106.25 Three discs for the incentre

The orthocentroidal disc, the Brocard disc and the incentre

Let ABC be a triangle with incentre I, orthocentre H, centroid G and symmedian point K. The last centre K, sometimes referred to as the Lemoine point after the French civil engineer and geometer Emile Lemoine (1840-1912), is the point of concurrency of the three symmedians. Symmedian is a cevian which is a reflection of the median in the corresponding angle bisector. The theory of the symmedian point is discussed in several books [1], [2], the best being [3].

Since Euler's time it is well-known that the incentre I is always within the orthocentroidal disc $\mathcal{D}_{GH}$ − the disc with diameter GH. The symmedian point K lies in $\mathcal{D}_{GH}$ as well. The proof given in [4] uses areal coordinates. In [5] we proved that the incentre is interior to the Brocard disc $\mathcal{D}_{OK}$. This second disc for I with diameter OK is named after the French geometer Henri Brocard (1845-1922), known in triangle geometry for the Brocard points and the Brocard angle (see [3]).

The symmedicentroidal disc and the incentre

The incentre I lives in yet another, third, disc $\mathcal{D}_{GK}$. This disc with diameter GK we shall call the symmedicentroidal disc. Being confined to three discs, the space for the incentre I is pretty well bounded. See Figure 1. The image of a point within a disc (half-disc, to be precise) corresponds, in the world of inequalities, to a squared diameter being at least the sum of the squared distances of the point to the diametrically opposite points. In our case for the symmedicentroidal disc, $I \in \mathcal{D}_{GK}$, that is

$$GK^2 \geqslant GI^2 + KI^2. \qquad (1)$$

The diameter of the symmedicentroidal disc

In order to prove (1), we need to find the distances between the triangle centres G, K and I. Let ABC be triangle with sides a, b, c, semiperimeter s, circumradius R and inradius r. We begin with finding the diameter of $\mathcal{D}_{GK}$.

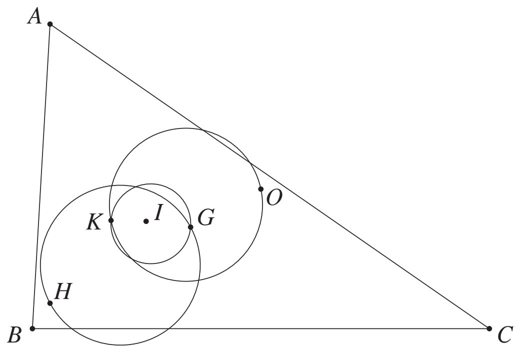

FIGURE 1: The three discs, $I \in \mathcal{D}_{GH} \cap \mathcal{D}_{OK} \cap \mathcal{D}_{GK}$

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Theorem 1

The distance between the centroid G and the symmedian point K is

$$GK^2 = \frac{s^6 - 3r(4R - 3r)s^4 - 3r^2(20R^2 + 8Rr + 3r^2)s^2 - r^3(4R + r)^3}{9(s^2 - r(4R + r))^2} \qquad (2)$$

Proof

To calculate GK^2 we use the Leibniz formula

$$GP^2 = \frac{1}{3}\sum AP^2 - \frac{a^2 + b^2 + c^2}{9},$$

where P is an arbitrary point in the plane of ABC.

$$AK^2 = \frac{b^2c^2(-a^2 + 2b^2 + 2c^2)}{(a^2 + b^2 + c^2)^2}$$

follows from the equation for the length of the symmedian

$$s_a^2 = AK_A^2 = \frac{b^2c^2(-a^2 + 2b^2 + 2c^2)}{(b^2 + c^2)^2}$$

and

$$\frac{AK}{KK_A} = \frac{b^2 + c^2}{a^2},$$

see [3, p. 102]. Hence from the Leibniz formula

$$\begin{aligned} GK^2 &= \frac{1}{3}\sum AK^2 - \frac{a^2 + b^2 + c^2}{9} \\ &= \frac{-15a^2b^2c^2 + 3\sum a^2b^4 - \sum a^6}{9(a^2 + b^2 + c^2)^2}. \end{aligned} \qquad (3)$$

To find the sums $\sum a^2b^4$ and $\sum a^6$, we use the identity

$$\sum a^3b^3 = s^6 - 3r(4R - r)s^4 + 3r^4s^2 + r^3(4R + r)^3.$$

This follows from

$$\sum a^3b^3 = \left(\sum ab\right)^3 - 3abc\sum ab^2 - 6a^2b^2c^2, \sum ab = s^2 + r(4R + r),$$

$$abc = 4Rrs \text{ and } \sum ab^2 = 2s(s^2 - 2Rr + r^2)$$

(see [6]). Hence

$$\begin{aligned} \sum a^2b^4 &= \left(\sum ab^2\right)^2 - 2abc\sum ab^2 - 2\sum a^3b^3 - 2abc\sum a^3 - 6a^2b^2c^2 \\ &= 2\left[s^6 - r(12R - r)s^4 + r^2(24R^2 + 8Rr - r^2)s^2 - r^3(4R + r)^3\right], \end{aligned} \qquad (4)$$

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where we used the known identity $\sum a^3 = 2s\left(s^2 - 6Rr - 3r^2\right)$. The other sum is

$$\sum a^6 = \left(\sum a^3\right)^2 - 2\sum a^3b^3$$

$$= 2\left[s^6 - 3r(4R + 5r)s^4 + 3r^2\left(24R^2 + 24Rr + 5r^2\right)s^2 - r^3(4R + r)^3\right]. \quad (5)$$

Putting (4) and (5) in (3) gives (2).

The distance between the centroid G and the incentre I is [7]

$$9GI^2 = s^2 - 16Rr + 5r^2. \quad (6)$$

Immediate consequence from $GI^2 \geqslant 0$ is the Gerretsen's inequality

$$s^2 \geqslant 16Rr - 5r^2, \quad (7)$$

which is of utmost importance in the theory of triangle inequalities. In our proof of the next theorem it will deliver the decisive blow to the inequality we need to show.

The distance between the symmedian point K and the incentre I is obtained in [5]

$$KI^2 = \frac{8Rr^2(4R + r)}{a^2 + b^2 + c^2} - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}$$

$$= \frac{4Rr^2\left((R + r)s^2 - r(4R + r)^2\right)}{(s^2 - r(4R + r))^2}. \quad (8)$$

The proof

Now we have collected all the ingredients to prove the following:

Theorem 2

The incentre I lies within the symmedicentroidal disc $\mathscr{D}_{GK}$.

Proof

It is sufficient to prove (1). By (2), (6) and (8), $GK^2 - GI^2 - KI^2 \geqslant 0$ is equivalent to

$$(2R + r)s^4 - 10Rr(4R + r)s^2 + r^2(4R + r)^2(8R - r) \geqslant 0. \quad (9)$$

This fifth degree inequality doesn't look very nice. Transforming the left-hand side of (9), we obtain

$$(2R + r)s^4 - 10Rr(4R + r)s^2 + r^2(4R + r)^2(8R - r)$$

$$= (2R + r)\left(s^2 - 16Rr + 5r^2\right)^2 + 2r\left(12R^2 + Rr - 5r^2\right)\left(s^2 - 16Rr + 5r^2\right)$$

$$+ 12r^3(2R - r)(R - 2r) \geqslant 0.$$

The first term is non-negative. The second term is non-negative by Gerretsen's inequality (7) and the third term is non-negative by Euler's inequality $R \geqslant 2r$. The proof is complete.

To finish, we ask as a problem for research to look for another, fourth, disc, formed from the classical triangle centres in which the incentre I is contained. We note that the centroid G and the symmedian point K are isogonal conjugates and conjecture that for any interior point P with isogonal conjugate P^*, the incentre I is within the disc with diameter PP^*.

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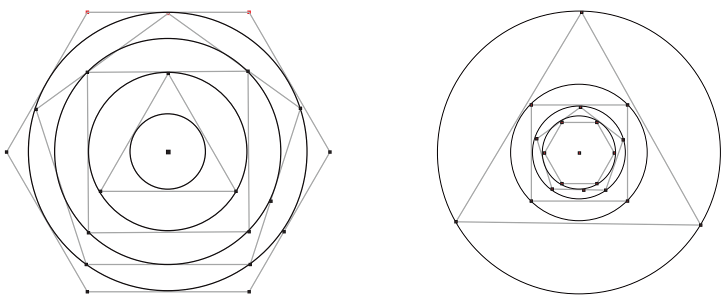
106.26 The nested polygons problem revisited

The title refers to the following puzzle popularised recently by Ian Stewart in [1]; his account is based on [2].

A circle of unit radius is circumscribed by an equilateral triangle which is then circumscribed by another circle. Repeat, but on successive stages use a square, regular pentagon, and so on. Does the figure (shown in Figure 1(a)) become arbitrarily large or does it remain bounded in size?



(a) (b)

FIGURE 1: (a) Nested outward polygons; (b) nested inward polygons