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ON RELATIVE AGEING OF COHERENT SYSTEMS WITH DEPENDENT IDENTICALLY DISTRIBUTED COMPONENTS

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Abstract

Relative ageing describes how one system ages with respect to another. The ageing faster orders are used to compare the relative ageing of two systems. Here, we study ageing faster orders in the hazard and reversed hazard rates. We provide some sufficient conditions for one coherent system to dominate another with respect to ageing faster orders. Further, we investigate whether the active redundancy at the component level is more effective than that at the system level with respect to ageing faster orders, for a coherent system. Furthermore, a used coherent system and a coherent system made out of used components are compared with respect to ageing faster orders.

Keywords: Coherent system; dual distortion/domination function; k-out-of-n system; redundancy; residual lifetime; stochastic orders

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1. Introduction and preliminaries

Ageing is a common phenomenon experienced by both living organisms and mechanical systems. It largely describes how a system or living organism improves or deteriorates with age. The study of stochastic ageing has received considerable attention from researchers in the past few decades. In the literature, different types of stochastic ageing concepts (e.g. increasing failure rate (IFR), increasing failure rate on average (IFRA), etc.) have been developed to describe different ageing characteristics of a system. There are broadly three types of ageing, namely positive ageing, negative ageing, and no ageing. A brief discussion of different ageing concepts can be found in [4] and [31]. Similar to these ageing concepts, there is another useful notion of ageing, called relative ageing, which describes how one system ages relative to another.

The proportional hazard (PH) rate model, commonly known as Cox's PH model (see [10]), is widely used to analyze the failure time data in reliability and survival analysis. Subsequently other models were introduced, namely the proportional mean residual lifetime model, the proportional reversed hazard rate model, the proportional odds model, etc. (see [18], [31], and [35]). In many real-life scenarios, the phenomenon of crossing hazards and mean residual lives has been observed (see [9], [34], and [50]). Motivated by this, Kalashnikov and Rachev [26]

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introduced a stochastic order (called the ageing faster order in the hazard rate) based on the concept of relative ageing. Indeed, this approach could be considered as a reasonable alternative to the PH model. A detailed study of this order has been provided by Sengupta and Deshpande [54], who also introduced ageing faster stochastic orders based on the cumulative hazard and cumulative reversed hazard rate functions. Later, Finkelstein [17] proposed a stochastic order based on mean residual lifetime functions, and Rezaei [52] introduced a similar stochastic order in terms of the reversed hazard rate functions. Some generalized orderings in this direction were proposed by Hazra and Nanda [24].

The basic structures of most real-life systems match the so-called coherent system. A system is called coherent if its all components are relevant and its structure function (see [4] for the definition) is monotonically non-decreasing in each argument (which means that an improvement in performance of a component cannot decrease the lifetime of the system). The well-known k-out-of-n system is a special case of coherent systems. A system of n components is said to be a k-out-of-n system if it functions as long as at least k of its n components function. Two extreme cases of k-out-of-n systems are the 1-out-of-n system (called the parallel system) and the n-out-of-n system (called the series system). Further, there is a one-to-one correspondence between a k-out-of-n system and an $(n-k+1)$th order statistic (of lifetimes of n components). Thus the study of k-out-of-n systems is essentially the same as the study of order statistics.

Stochastic comparisons of coherent systems are considered to be one of the important problems in reliability theory. A comprehensive list of results so far developed on various stochastic comparisons of k-out-of-n systems with independent components can be found in [3], [25], [49], [51], and the references therein. Further, stochastic comparisons of general coherent systems have been considered in [1], [6], [16], [30], [42], [43], [44], [45], [47], and [53], to name a few. Note that all these results were developed under different stochastic orders, namely, usual stochastic order, hazard rate order, likelihood ratio order, etc. However, the study of coherent systems using ageing faster orders has not yet been adequately completed. Misra and Francis [36], Li and Li [32], and Ding and Zhang [14] developed some results on behavior of k-out-of-n systems under ageing faster orders. Later, Ding, Fang and Zhao [15] gave some sufficient conditions, in terms of signature, for comparison of the lifetimes of two coherent systems (with independent components) with respect to ageing faster orders. However, there is no such result where the sufficient conditions are given in terms of reliability functions. Furthermore, coherent systems with dependent components have not yet been considered. Thus one of the major goals of our paper is to provide some sufficient conditions (in terms of reliability functions) under which one coherent system dominates another with respect to ageing faster orders.

One of the effective ways to enhance the lifetime of a system is to incorporate spares (or redundant components) into the system. Then the key question is how to allocate spares into the system so that the system's lifetime will be optimal in some stochastic sense. Barlow and Proschan [4] showed that the allocation of active redundancy at the component level (of a coherent system) is superior to that at the system level with respect to the usual stochastic order. Later, many other researchers studied this problem in different directions (see [8], [11], [21], [38], [40], [56], [57], and the references therein). However, to the best of our knowledge, this problem using ageing faster orders has not yet been studied. Thus another goal of this paper is to derive some necessary and sufficient conditions under which the lifetime of a coherent system with active redundancy at the component level is larger (smaller) than that at the system level with respect to ageing faster orders.

Real-life systems are made from either new components or used components. Consider two coherent systems, namely a used coherent system (i.e. a coherent system made from new components, and which has been used for time $t>0$) and a coherent system of used components (i.e. a coherent system made from a set of components which have already been used for time $t>0$). It is a fact that a coherent system of new components does not always have a longer lifetime than a coherent system made out of used components (see [46]). Similarly, a used coherent system may or may not perform better than a coherent system of used components. Stochastic comparisons between these two systems have been made in numerous papers; see e.g. [19], [20], [23], and [33], to name a few. However, to the best of our knowledge, the ageing faster orders have not yet been used as a tool for comparing these two systems. Thus the study of stochastic comparisons between a used coherent system and a coherent system of used components is another approach we will focus on. We now describe a real-life situation where such a comparison is meaningful. In today's competitive world, the demand for highly reliable systems is ever increasing. A common approach to enhancing the reliability of a system is to plan some maintenance and replacement strategies. Consider a coherent system consisting of n components, whose random lifetimes are denoted by random variables $X_1, \ldots, X_n$. For any fixed $\boldsymbol{x}=(x_1, \ldots, x_n)\in(0, \infty)^n$, let $\tau(\boldsymbol{x})$ denote the lifetime of the system when the observed component lifetimes are $x_1, \ldots, x_n$. Then the random variable $\tau(\boldsymbol{X})$ denotes the lifetime of the systems consisting of brand new components, and let $(\tau(\boldsymbol{X}))_t=(\tau(\boldsymbol{X})-t\,|\,\tau(\boldsymbol{X})>t)$ denote its residual lifetime after it has lived to age t. To test the reliability of components to be used in the system, samples of these components are tested in an ideal environment without interaction effects from other components. These tested components of various ages are commonly used as spares for the replacement of components in the system. Let the residual lifetime of the ith spare, which was tested for time t, be denoted by $(X_i)_t \overset{\text{def}}{=} (X_i-t\,|\,X_i>t)$, $i=1, \ldots, n$. If a new system is assembled out of the spares of age t, then $\tau(\boldsymbol{X}_t)$ denotes the lifetime of the corresponding system, where $\boldsymbol{X}_t=((X_1)_t, \ldots, (X_n)_t)$. Suppose that $\boldsymbol{X}$ and $\boldsymbol{X}_t$ share the same copula. One replacement policy might be to replace all the components of a system that have lived up to age t with corresponding spares of age t. In order to judge the effectiveness of the replacement strategy, it may be of interest to compare two lifetimes $(\tau(\boldsymbol{X}))_t$ (lifetime of used system of age t) and $\tau(\boldsymbol{X}_t)$ (lifetime of system of used components of age t).

In what follows, we introduce some notation that will be used throughout the paper. For a random variable W (with absolutely continuous cumulative distribution function), we denote its probability density function (PDF) by $f_W(\cdot)$, the cumulative distribution function (CDF) by $F_W(\cdot)$, the hazard rate function by $r_W(\cdot)$, the reversed hazard rate function by $\tilde{r}_W(\cdot)$, and the survival/reliability function by $\bar{F}_W(\cdot)$; here $\bar{F}_W(\cdot)=1-F_W(\cdot)$.

Let us consider a coherent system, with lifetime $\tau(\boldsymbol{X})$, made from n components having dependent and identically distributed (d.i.d.) lifetime vector $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$, where the X_i's are identically distributed, say $X_i \overset{\text{d}}{=} X$, $i=1, 2, \ldots, n$, for some non-negative random variable X; here $\overset{\text{d}}{=}$ means equality in distribution. Then the joint reliability function of $\boldsymbol{X}$ is given by

$$\begin{aligned}\bar{F}_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n) &= \mathbb{P}(X_1>x_1, X_2>x_2, \ldots, X_n>x_n)\\ &= K(\bar{F}_X(x_1), \bar{F}_X(x_2), \ldots, \bar{F}_X(x_n)), \qquad \boldsymbol{x}=(x_1, x_2, \ldots, x_n)\in\mathbb{R}^n,\end{aligned}$$

where $K(\cdot, \cdot, \ldots, \cdot)$ is a survival copula describing the dependency structure among components of the system and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Indeed, this representation is well known via Sklar's theorem (see [48]). In the literature, many different types of survival copulas have been studied in order to describe different dependency structures

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among components. Some of the widely used copulas are the Farlie–Gumbel–Morgenstern (FGM) copula, the Archimedean copula with different generators, the Clayton–Oakes (CO) copula, etc. We refer the reader to [48] for a detailed discussion of the copula theory and its various applications. In what follows, we give a lemma that describes a fundamental bridge between a system and its corresponding components through the domination function.

Lemma 1.1. (Navarro *et al.* [44]) *Let $\tau(\boldsymbol{X})$ be the lifetime of a coherent system made from n d.i.d. components with lifetime vector $\boldsymbol{X}=(X_1, X_2, \ldots, X_n)$. Then the reliability function of $\tau(\boldsymbol{X})$ can be written as*

$$\bar{F}_{\tau(\boldsymbol{X})}(x)=h(\bar{F}_X(x)),$$

where $h(\cdot)\colon [0, 1]\to[0, 1]$, called the domination (or dual distortion) function, depends on the structure function $\phi(\cdot)$ (see [4] for a definition) and on the survival copula K of $X_1, X_2, \ldots, X_n$. Furthermore, $h(\cdot)$ is an increasing continuous function in $[0, 1]$ such that $h(0)=0$ and $h(1)=1$.

Below we give an example (borrowed from [44]) that illustrates the result given in the above lemma.

Example 1.1. Let $\tau(\boldsymbol{X})=\min\{X_1, \max\{X_2, X_3\}\}$, where the reliability function of $\boldsymbol{X}=(X_1, X_2, X_3)$ is described by the FGM copula (see Nelsen [48])

$$K(p_1, p_2, p_3)=p_1p_2p_3(1+\theta(1-p_1)(1-p_2)(1-p_3)), \quad p_i\in(0, 1),\ i=1, 2, 3,$$

where $\theta\in[-1, 1]$. Further, let X_1, X_2, and X_3 be identically distributed with $X_i\overset{d}{=}X$, $i=1, 2, 3$, for some non-negative random variable X. Then the minimal path sets (see [4]) of $\tau(\boldsymbol{X})$ are given by $\{1, 2\}$ and $\{1, 3\}$. Let $X_{\{1,2\}}$, $X_{\{1,3\}}$, and $X_{\{1,2,3\}}$ be the lifetimes of the path sets $\{1, 2\}$, $\{1, 3\}$, and $\{1, 2, 3\}$, respectively. Then the reliability function of $\tau(\boldsymbol{X})$ can be written as

$$\begin{aligned}\bar{F}_{\tau(\boldsymbol{X})}(x)&=\mathbb{P}(\{X_{\{1,2\}}>x\}\cup\{X_{\{1,3\}}>x\})\\&=\mathbb{P}(X_{\{1,2\}}>x)+\mathbb{P}(X_{\{1,3\}}>x)-\mathbb{P}(X_{\{1,2,3\}}>x)\\&=\bar{F}_{\boldsymbol{X}}(x, x, 0)+\bar{F}_{\boldsymbol{X}}(x, 0, x)-\bar{F}_{\boldsymbol{X}}(x, x, x)\\&=K(\bar{F}_X(x), \bar{F}_X(x), 1)+K(\bar{F}_X(x), 1, \bar{F}_X(x))-K(\bar{F}_X(x), \bar{F}_X(x), \bar{F}_X(x))\\&=h(\bar{F}_X(x)),\end{aligned}$$

where, for $\theta\in[-1, 1]$,

$$\begin{aligned}h(p)&=K(p, p, 1)+K(p, 1, p)-K(p, p, p)\\&=2p^2-p^3-\theta p^3(1-p)^3, \quad p\in(0, 1).\end{aligned}$$

The theory of stochastic orders is an effective tool for comparing two random variables (or two sets of random variables) stochastically. Stochastic orders have been extensively studied in the literature due to their applications in different branches of science and engineering. Encyclopedic information on this topic is nicely encapsulated in the book by Shaked and Shanthikumar [55] (see also [7]). For the sake of completeness, we give the following definitions of the stochastic orders that are used in our paper.

Definition 1.1. Let X and Y be two absolutely continuous random variables with cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, supported on $[0, \infty)$. Then X is said to be smaller than Y in:

(a) the hazard rate (hr) order, denoted by $X \leq_{hr} Y$, if

$$\bar{F}_Y(x)/\bar{F}_X(x) \text{ is increasing in } x \in [0, \infty);$$

(b) the reversed hazard rate (rhr) order, denoted by $X \leq_{rhr} Y$, if

$$F_Y(x)/F_X(x) \text{ is increasing in } x \in [0, \infty).$$

Similar to the above discussed stochastic orders, there are two more sets of stochastic orders that are useful for describing the relative ageing of two systems. The first set of stochastic orders, known as transform orders (namely, convex transform order, quantile mean inactivity time order, star-shaped order, super-additive order, DMRL order, s-IFR order, etc.), describe whether one system is ageing faster than another in terms of the increasing failure rate, the increasing failure rate on average, the new better than used, etc. A detailed discussion of these orders can be found in [2], [4], [5], [12], [29], [41], and the references therein. The second set of stochastic orders, called ageing faster orders, are defined based on monotonicity of ratios of some reliability measures, namely, hazard rate function, reversed hazard rate function, mean residual lifetime function, etc. For motivation and usefulness of these orders, we refer the reader to [13], [17], [24], [26], [28], [37], [39], [52], and [54]. Below we give the definitions of the ageing faster orders that are used in our paper.

Definition 1.2. Let X and Y be two absolutely continuous random variables with failure rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, respectively, and reversed failure rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$, respectively. Then X is said to be ageing faster than Y in:

(a) the failure rate, denoted by $X \underset{c}{\prec} Y$, if

$$r_X(x)/r_Y(x) \text{ is increasing in } x \in [0, \infty);$$

(b) the reversed failure rate, denoted by $X \underset{b}{\prec} Y$, if

$$\tilde{r}_X(x)/\tilde{r}_Y(x) \text{ is decreasing in } x \in [0, \infty).$$

The theory of totally positive functions has various applications in different areas of probability and statistics (see [27]). Below we give the definitions of TP_2 and RR_2 functions. Different properties of these functions are used in proving the main results of our paper.

Definition 1.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two linearly ordered sets. Then a real-valued function $\kappa(\cdot, \cdot)$ defined on $\mathcal{X} \times \mathcal{Y}$ is said to be TP_2 (resp. RR_2) if

$$\kappa(x_1, y_1)\kappa(x_2, y_2) \geq (\text{resp. } \leq)\ \kappa(x_1, y_2)\kappa(x_2, y_1),$$

for all $x_1 < x_2$ and $y_1 < y_2$.

Throughout the paper, increasing and decreasing, as usual, mean non-decreasing and non-increasing, respectively. Similarly, positive and negative mean non-negative and non-positive, respectively. Assume that all random variables considered in this paper are absolutely continuous and non-negative (i.e. distributional support is $[0, \infty)$). By $a \overset{\text{sgn}}{=} b$, we mean that a and b have the same sign, whereas $a \overset{\text{def}}{=} b$ means that a is defined by b. Further, we use bold symbols to represent vectors. We use the acronyms i.i.d. and d.i.d. for 'independent and identically

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distributed' and 'dependent and identically distributed', respectively. For any positive integers r and s $(1 \leq r \leq s)$, we write $\tau_{r|s}(\boldsymbol{W})$ to represent the lifetime of an r-out-of-s system made from components having the lifetime vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_s)$.

The rest of the paper is organized as follows. In Section 2 we discuss some useful lemmas which are intensively used in the proofs of the main results. In Section 3 we provide some sufficient conditions under which the lifetime of one coherent system is larger than that of another system with respect to ageing faster orders in terms of the hazard and reversed hazard rates. In Section 4 we discuss a redundancy allocation problem in a coherent system. We derive some necessary and sufficient conditions under which the allocation of active redundancy at the component level (of a coherent system) is superior to that at the system level with respect to ageing faster orders. Stochastic comparisons of a used coherent system and a coherent system made from used components are discussed in Section 5. Some concluding remarks are given in Section 6.

The proofs of various lemmas and theorems, wherever given, are deferred to the Appendix.

2. Useful lemmas

In this section we discuss some lemmas which will be used in proving the main results of this paper. In the first lemma we discuss the sign change property of the integral of a function. The following lemma is adopted from [27, Theorem 11.2, pp. 324–325] and [22, Lemma 3.5].

Lemma 2.1. *Let $\kappa(x, y) > 0$, defined on $\mathcal{X} \times \mathcal{Y}$, be* RR_2 *(resp.* TP_2*), where $\mathcal{X}$ and $\mathcal{Y}$ are subsets of the real line. Assume that a function $f(\cdot, \cdot)$ defined on $\mathcal{X} \times \mathcal{Y}$ is such that the following hold.*

(i) *For each $x \in \mathcal{X}$, $f(x,y)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as y traverses $\mathcal{Y}$.*

(ii) *For each $y \in \mathcal{Y}$, $f(x,y)$ is increasing (resp. decreasing) in $x \in \mathcal{X}$.*

(iii) *$\omega(x) = \int_{\mathcal{Y}} \kappa(x, y) f(x, y)\, \mathrm{d}\mu(y)$ exists absolutely and defines a continuous function of x, where μ is a sigma-finite measure.*

Then $\omega(x)$ changes sign at most once and, if the change of sign does occur, it is from negative (resp. positive) to positive (resp. negative), as x traverses $\mathcal{X}$.

In the following lemma we state an equivalent condition of a monotonic function. The proof is straightforward, and hence omitted.

Lemma 2.2. *Let $f(\cdot)$ and $g(\cdot)$ be two non-negative and real-valued functions defined on $(a, b) \subseteq (0, \infty)$. Then $f(x)/g(x)$ is increasing (resp. decreasing) in x if and only if, for any real number c, the difference $f(x) - cg(x)$ changes sign at most once and, if the change of sign does occur, it is from negative (resp. positive) to positive (resp. negative), as x traverses from a to b.*

Some properties of the reliability functions of k-out-of-n and l-out-of-m systems are discussed in the next two lemmas. Lemma 2.3(i) is given in [16], whereas Lemma 2.4(i) is given in [42]. The other proofs are deferred to the Appendix.

Lemma 2.3. *Let $h_{k|n}(\cdot)$ and $h_{l|m}(\cdot)$, respectively, be the reliability functions of the k-out-of-n and l-out-of-m systems with i.i.d. component lifetimes, where $1 \leq k \leq n$ and $1 \leq l \leq m$. Further,*

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let $H_{k|n}(p) = ph'_{k|n}(p)/h_{k|n}(p)$ *and* $H_{l|m}(p) = ph'_{l|m}(p)/h_{l|m}(p)$, $p \in (0, 1)$, $1 \le k \le n$, $1 \le l \le m$. *Then the following results hold.*

(i) $H_{k|n}(p)$ *is decreasing in* $p \in (0, 1)$, *for all* $1 \le k \le n$.

(ii) $H_{k|n}(p)/H_{l|m}(p)$ *is decreasing in* $p \in (0, 1)$, *for all* $l - k \ge \max\{0, m - n\}$.

(iii) $(1 - p)H'_{k|n}(p)/H_{k|n}(p)$ *is decreasing in* $p \in (0, 1)$, *for all* $1 \le k \le n$.

Lemma 2.4. *Let* $h_{k|n}(\cdot)$ *and* $h_{l|m}(\cdot)$, *respectively, be the reliability functions of the k-out-of-n and l-out-of-m systems with i.i.d. component lifetimes, where* $1 \le k \le n$ *and* $1 \le l \le m$. *Further, let* $R_{k|n}(p) = (1 - p)h'_{k|n}(p)/(1 - h_{k|n}(p))$ *and* $R_{l|m}(p) = (1 - p)h'_{l|m}(p)/(1 - h_{l|m}(p))$, $p \in (0, 1)$, $1 \le k \le n$, $1 \le l \le m$. *Then the following results hold.*

(i) $R_{k|n}(p)$ *is increasing in* $p \in (0, 1)$, *for all* $1 \le k \le n$.

(ii) $R_{k|n}(p)/R_{l|m}(p)$ *is increasing in* $p \in (0, 1)$, *for all* $k - l \ge \max\{0, n - m\}$.

(iii) $pR'_{k|n}(p)/R_{k|n}(p)$ *is decreasing in* $p \in (0, 1)$, *for all* $1 \le k \le n$.

3. Stochastic comparisons of two coherent systems

In this section we will compare two coherent systems with respect to ageing faster orders based on the failure and the reversed failure rates. We will show that the proposed results hold for the k-out-of-n and l-out-of-m systems with i.i.d. components.

Let $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ (resp. $\tau_{k|n}(\boldsymbol{X})$ and $\tau_{l|m}(\boldsymbol{Y})$) be the lifetimes of two coherent systems (resp. k-out-of-n and l-out-of-m systems) made from two different sets of d.i.d. components with lifetime vectors $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_m)$, respectively. For simplicity of notation, let us assume that $X_i \overset{d}{=} X$, $i = 1, 2, \dots, n$, for some non-negative random variable X, and $Y_j \overset{d}{=} Y$, $j = 1, 2, \dots, m$, for some non-negative random variable Y. Further, let $h_1(\cdot)$ and $h_2(\cdot)$ be the domination functions of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$, respectively. In what follows, we use the following notation. For $p \in (0, 1)$,

$$H_i(p) = \frac{ph'_i(p)}{h_i(p)}, \quad i = 1, 2 \tag{3.1}$$

and

$$R_i(p) = \frac{(1 - p)h'_i(p)}{1 - h_i(p)}, \quad i = 1, 2. \tag{3.2}$$

In the following theorem we provide a set of sufficient conditions to show that $\tau_1(\boldsymbol{X})$ ages faster than $\tau_2(\boldsymbol{Y})$ in terms of the failure rate.

Theorem 3.1. *Suppose that the following conditions hold.*

(i) $H_1(p)$ *and* $H_1(p)/H_2(p)$ *are decreasing in* $p \in (0, 1)$.

(ii) $(1 - p)H'_1(p)/H_1(p)$ *or* $(1 - p)H'_2(p)/H_2(p)$ *is decreasing in* $p \in (0, 1)$.

(iii) $X \underset{c}{\prec} Y$ *and* $Y \le_{rh} X$.

Then $\tau_1(\boldsymbol{X}) \underset{c}{\prec} \tau_2(\boldsymbol{Y})$.

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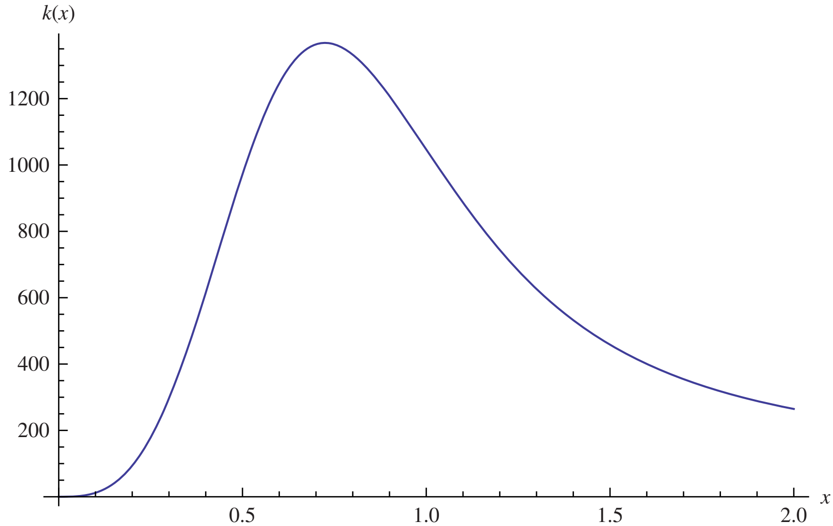

FIGURE 1: Plot of $k(x)$ against $x \in (0, 2)$.

The next corollary follows from Theorem 3.1 and Lemma 2.3. It is worth mentioning here that Theorem 3.1(a) of [36] is a particular case of this corollary ($k = l$ and $m = n$).

Corollary 3.1. *Suppose that the X_i's are i.i.d., and that the Y_j's are i.i.d. If $X \underset{c}{\prec} Y$ and $Y \le_{rh} X$, then $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|m}(\boldsymbol{Y})$, for $l - k \ge \max\{0, m - n\}$.*

Remark 3.1. Under the assumptions of Corollary 3.1, the following statements hold true.

(i) $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|n}(\boldsymbol{Y})$, for $1 \le k \le l \le n$.

(ii) $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{k|m}(\boldsymbol{Y})$, for $1 \le k \le m \le n$.

(iii) $\tau_{l-r|m-r}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|m}(\boldsymbol{Y})$, for $1 \le r \le l \le m$.

One natural question that arises here is whether the result stated in Theorem 3.1 can be proved without the condition $Y \le_{rh} X$. Below we cite a counterexample that shows that this condition cannot be relaxed.

Example 3.1. Consider two coherent systems

$$\tau_1(\boldsymbol{X}) = \max\{X_1, X_2, X_3\}, \qquad \tau_2(\boldsymbol{Y}) = \max\{Y_1, Y_2, Y_3\},$$

where the X_i's are i.i.d. with the common reliability function given by $\bar{F}_X(x) = \exp\{-2x^3\}$, $x > 0$, and the Y_i's are i.i.d. with the common reliability function given by $\bar{F}_Y(x) = \exp\{-0.1x^2\}$, $x > 0$. Then it can easily be verified that $X \underset{c}{\prec} Y$ but $Y \not\le_{rh} X$ (in fact $Y \not\le_{st} X$). Now, by writing $k(x) = r_{\tau_1(\boldsymbol{X})}(x)/r_{\tau_2(\boldsymbol{Y})}(x)$, we have

$$k(x) = 30x\, \mathrm{e}^{-(2x^3 - 0.1x^2)} \left[\frac{1 - (1 - \mathrm{e}^{-0.1x^2})^3}{1 - (1 - \mathrm{e}^{-2x^3})^3}\right] \left[\frac{(1 - \mathrm{e}^{-2x^3})^2}{(1 - \mathrm{e}^{-0.1x^2})^2}\right], \qquad x > 0.$$

Figure 1 shows that $k(x)$ is not monotone on $(0, \infty)$, and hence $\tau_1(\boldsymbol{X}) \underset{c}{\not\prec} \tau_2(\boldsymbol{Y})$.

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Henceforth $\tau_1(\boldsymbol{X})$ denotes the lifetime of a coherent system made from components having lifetimes $X_1, X_2, \ldots, X_n$, whereas $\tau_2(\boldsymbol{X})$ denotes the lifetime of another coherent system made from components having lifetimes $X_1, X_2, \ldots, X_m$. Note that the failure rate function of $\tau_i(\boldsymbol{X})$ can be written as

$$r_{\tau_i(\boldsymbol{X})}(x) = r_X(x) H_i(\bar{F}_X(x)), \quad x > 0, \; i = 1, 2,$$

where $H_i(\cdot)$ (given in (3.1)) is a function defined in terms of the domination function. Then we have the following proposition, where we provide the necessary and sufficient condition under which $\tau_1(\boldsymbol{X})$ ages faster (resp. slower) than $\tau_2(\boldsymbol{X})$ in terms of the hazard rate. The proof of the proposition is immediate on using the above expressions and Definition 1.2(a).

Proposition 3.1. *Let the X_i's be identically distributed. Then $\tau_1(\boldsymbol{X}) \underset{c}{\prec}$ (resp. $\underset{c}{\succ}$) $\tau_2(\boldsymbol{X})$ if and only if*

$$H_1(p)/H_2(p) \text{ is decreasing (resp. increasing) in } p \in (0, 1).$$

The following result proved in [36] (see Theorem 2.1) follows from Proposition 3.1 and Lemma 2.3.

Corollary 3.2. *Suppose that the X_i's are i.i.d. Then $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|m}(\boldsymbol{X})$, for $l - k \geq \max\{0, m - n\}$.*

Remark 3.2. Under the assumption of Corollary 3.2, the following statements hold true.

(i) $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|n}(\boldsymbol{X})$, for $1 \leq k \leq l \leq n$.

(ii) $\tau_{k|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{k|m}(\boldsymbol{X})$, for $1 \leq k \leq m \leq n$.

(iii) $\tau_{l-r|m-r}(\boldsymbol{X}) \underset{c}{\prec} \tau_{l|m}(\boldsymbol{X})$, for $1 \leq r \leq l \leq m$.

The next corollary, proved in [14], follows from Proposition 3.1. It shows that a parallel system ages faster (in terms of the hazard rate) as its number of components increases, whereas the reverse scenario is observed for the series system.

Corollary 3.3. *Suppose that the X_i's are d.i.d. with the joint distribution function described by the Archimedean copula with generator $\phi(\cdot)$. If $x \ln{}'[-\phi'(x)/(1-\phi(x))]$ is decreasing in $x > 0$, then*

(i) $\tau_{1|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{1|m}(\boldsymbol{X})$, *for* $1 \leq m \leq n$,

(ii) $\tau_{n|n}(\boldsymbol{X}) \underset{c}{\prec} \tau_{m|m}(\boldsymbol{X})$, *for* $1 \leq n \leq m$.

Below we give an example that illustrates an application of Proposition 3.1.

Example 3.2. Consider two coherent systems $\tau_1(\boldsymbol{X}) = \min\{X_1, \max\{X_2, X_3\}\}$ and $\tau_2(\boldsymbol{X}) = \min\{X_1, X_2, X_3\}$, each of which is made up of three components having d.i.d. lifetimes X_1, X_2, and X_3. Further, let the joint distribution function of (X_1, X_2, X_3) be described by the FGM copula

$$K(p_1, p_2, p_3) = p_1 p_2 p_3 (1 + \theta(1-p_1)(1-p_2)(1-p_3)), \quad 0 < p_i < 1, \; i = 1, 2, 3,$$

where $\theta \in [-1, 1]$. Then the domination functions of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{X})$ are, respectively, given by

$$h_1(p) = 2p^2 - p^3 - \theta p^3 (1-p)^3, \quad 0 < p < 1$$

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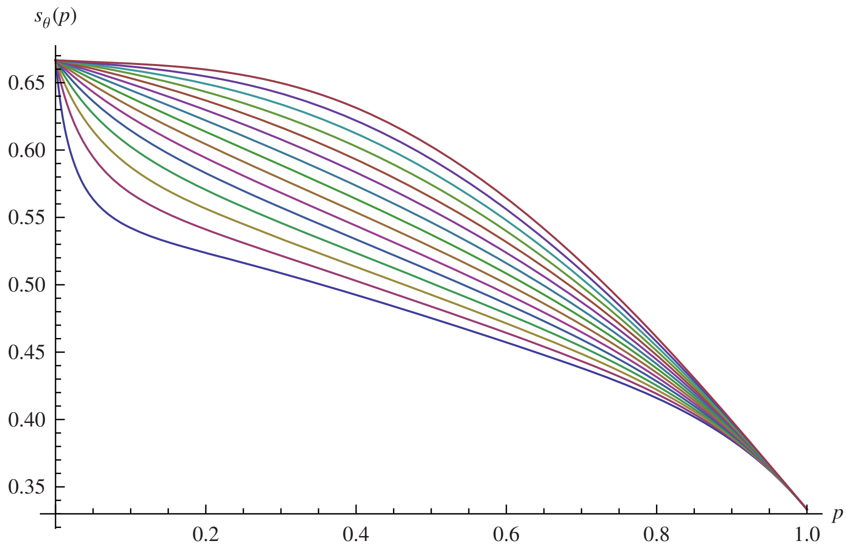

FIGURE 2: Plot of $s_\theta(p)$ against $p \in (0, 1)$, for $\theta = -0.9, -0.8, \ldots, 0.4, 0.5$ (from bottom to top).

and

$$h_2(p) = p^3 + \theta p^3(1-p)^3, \quad 0 < p < 1.$$

These give

$$H_1(p) = \frac{ph_1'(p)}{h_1(p)} = \frac{4p^2 - 3(1+\theta)p^3 + 12\theta p^4 - 15\theta p^5 + 6\theta p^6}{2p^2 - (1+\theta)p^3 + 3\theta p^4 - 3\theta p^5 + \theta p^6}, \quad 0 < p < 1$$

and

$$H_2(p) = \frac{ph_2'(p)}{h_2(p)} = \frac{3(1+\theta)p^3 - 6\theta p^6 - 12\theta p^4 + 15\theta p^5}{(1+\theta)p^3 - \theta p^6 - 3\theta p^4 + 3\theta p^5}, \quad 0 < p < 1.$$

Writing $s_\theta(p) = H_1(p)/H_2(p)$, we have, for $\theta \in [-1, 1]$ and $0 < p < 1$,

$$s_\theta(p) = \frac{(4p^2 - 3(1+\theta)p^3 + 12\theta p^4 - 15\theta p^5 + 6\theta p^6)((1+\theta)p^3 - \theta p^6 - 3\theta p^4 + 3\theta p^5)}{(2p^2 - (1+\theta)p^3 + 3\theta p^4 - 3\theta p^5 + \theta p^6)(3(1+\theta)p^3 - 6\theta p^6 - 12\theta p^4 + 15\theta p^5)}.$$

Now consider the following two cases.

Case I. Let $\theta = -0.9, -0.8, \ldots, 0.4, 0.5$. Then Figure 2 shows that $s_\theta(p) = H_1(p)/H_2(p)$ is decreasing in $p \in (0, 1)$, and hence $\tau_1(\boldsymbol{X}) \underset{c}{\prec} \tau_2(\boldsymbol{X})$, using Proposition 3.1.

Case II. Let $\theta = 0.75, 0.80, \ldots, 0.95, 1$. Then Figure 3 shows that $s_\theta(p) = H_1(p)/H_2(p)$ is not monotone on (0,1). Using Proposition 3.1, we conclude that neither $\tau_1(\boldsymbol{X}) \underset{c}{\prec} \tau_2(\boldsymbol{X})$ nor $\tau_1(\boldsymbol{X}) \underset{c}{\succ} \tau_2(\boldsymbol{X})$ holds.

In the following theorem we compare $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ with respect to the ageing faster order in the reversed hazard rate.

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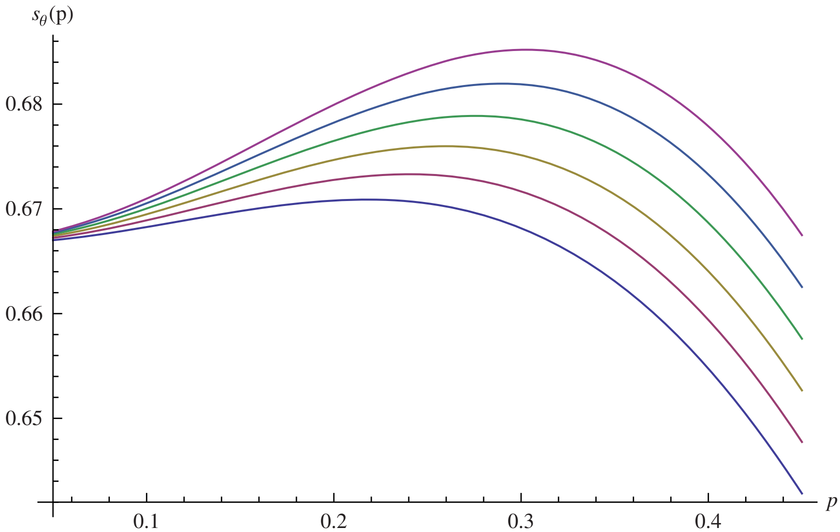

FIGURE 3: Plot of $s_\theta(p)$ against $p \in [0.05, 0.45]$, for $\theta = 0.75, 0.80, \ldots, 0.95, 1$ (from bottom to top).

Theorem 3.2. *Suppose that the following conditions hold.*

(i) $R_1(p)$ *and* $R_1(p)/R_2(p)$ *are increasing in* $p \in (0, 1)$.

(ii) $pR_1'(p)/R_1(p)$ *or* $pR_2'(p)/R_2(p)$ *is decreasing in* $p \in (0, 1)$.

(iii) $X \underset{b}{\prec} Y$ *and* $X \leq_{hr} Y$.

Then $\tau_1(\boldsymbol{X}) \underset{b}{\prec} \tau_2(\boldsymbol{Y})$.

The next corollary immediately follows from Theorem 3.2 and Lemma 2.4. Note that Theorem 3.1(b) of [36] is a particular case of this corollary ($k = l$ and $m = n$).

Corollary 3.4. *Suppose that the* X_i*'s are i.i.d., and that the* Y_j*'s are i.i.d. If* $X \underset{b}{\prec} Y$ *and* $X \leq_{hr} Y$, *then* $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{l|m}(\boldsymbol{Y})$, *for* $k - l \geq \max\{0, n - m\}$.

Remark 3.3. Under the assumptions of Corollary 3.4, the following statements hold true.

(i) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{l|n}(\boldsymbol{Y})$, for $1 \leq l \leq k \leq n$.

(ii) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{k|m}(\boldsymbol{Y})$, for $1 \leq k \leq n \leq m$.

(iii) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{k-r|n-r}(\boldsymbol{Y})$, for $1 \leq r \leq k \leq n$.

The following counterexample shows that the result given in Theorem 3.2 may not hold in the absence of the condition $X \leq_{hr} Y$.

Example 3.3. Consider the coherent systems $\tau_1(\boldsymbol{X}) = \min\{X_1, X_2\}$ and $\tau_2(\boldsymbol{Y}) = \min\{Y_1, Y_2\}$, where the X_i's are i.i.d. with the common cumulative distribution function given by $F_X(x) =$

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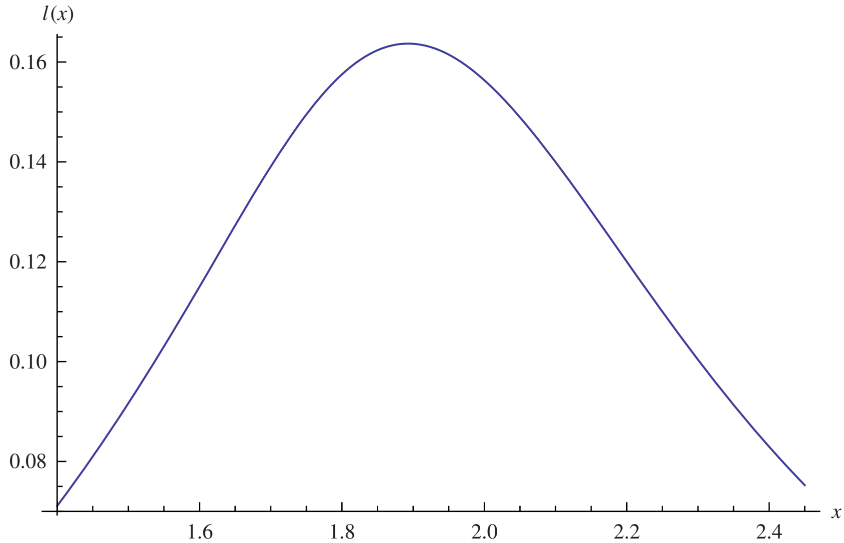

FIGURE 4: Plot of $l(x)$ against $x \in [1.4, 2.45]$.

$\exp\{-(2.1/x)^7\}$, $x > 0$, and the Y_i's are i.i.d. with the cumulative distribution function given by $F_Y(x) = \exp\{-(2/x)^3\}$, $x > 0$. Then it is easy to verify that $X \underset{b}{\prec} Y$ but $X \nleq_{hr} Y$ (in fact $X \nleq_{st} Y$). Now, by writing $l(x) = \tilde{r}_{\tau_2(\boldsymbol{Y})}(x)/\tilde{r}_{\tau_1(\boldsymbol{X})}(x)$, we have

$$l(x) = \left[\frac{24x^4\,\mathrm{e}^{-(2/x)^3}(1-\mathrm{e}^{-(2/x)^3})}{7\times 2.1^7\,\mathrm{e}^{-(2.1/x)^7}(1-\mathrm{e}^{-(2.1/x)^7})}\right]\left[\frac{1-(1-\mathrm{e}^{-(2.1/x)^7})^2}{1-(1-\mathrm{e}^{-(2/x)^3})^2}\right], \qquad x > 0.$$

Figure 4 shows that $l(x)$ is not monotone on $(0, \infty)$, and hence $\tau_{2|2}(\boldsymbol{X}) \underset{b}{\nprec} \tau_{2|2}(\boldsymbol{Y})$.

In the following proposition we discuss an analog of Proposition 3.1 for the ageing faster order in the reversed hazard rate. Note that the reversed hazard rate function of $\tau_i(\boldsymbol{X})$ can be written as

$$\tilde{r}_{\tau_i(\boldsymbol{X})}(x) = \tilde{r}_X(x) R_i(\bar{F}_X(x)), \qquad x > 0, \ \ i = 1, 2,$$

where $R_i(\cdot)$ (given in (3.2)) is a function defined in terms of the domination function. The proof of the proposition is immediate on using the above expressions and Definition 1.2(b).

Proposition 3.2. *Let the X_i's be identically distributed. Then $\tau_1(\boldsymbol{X}) \underset{b}{\prec}$ (resp. $\underset{b}{\succ}$) $\tau_2(\boldsymbol{X})$ if and only if*

$$R_1(p)/R_2(p) \text{ is increasing (resp. decreasing) in } p \in (0, 1).$$

The following result proved in [36] (see Theorem 2.2) follows from Proposition 3.2 and Lemma 2.4.

Corollary 3.5. *Suppose that the X_i's are i.i.d. Then $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{l|m}(\boldsymbol{X})$, for $k - l \geq \max\{0, n-m\}$.*

Remark 3.4. Under the assumption of Corollary 3.5, the following statements hold true.

(i) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{l|n}(\boldsymbol{X})$, for $1 \le l \le k \le n$.

(ii) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{k|m}(\boldsymbol{X})$, for $1 \le k \le n \le m$.

(iii) $\tau_{k|n}(\boldsymbol{X}) \underset{b}{\prec} \tau_{k-r|n-r}(\boldsymbol{X})$, for $1 \le r \le k \le n$.

The following corollary proved in [14] immediately follows from Proposition 3.2. It shows that, under certain conditions, a parallel system ages faster (resp. slower) in terms of the reversed hazard rate as its number of components decreases (resp. increases), whereas the reverse scenario is observed for the series system.

Corollary 3.6. *Suppose that the X_i's are d.i.d. with the joint distribution function described by the Archimedean copula with generator $\phi(\cdot)$. If $x \ln{}' [-\phi'(x)/\phi(x)]$ is decreasing (resp. increasing) in $x > 0$, then we have the following.*

(i) $\tau_{1|n}(\boldsymbol{X}) \underset{b}{\succ}$ *(resp.* $\underset{b}{\prec}$*)* $\tau_{1|m}(\boldsymbol{X})$ *for* $1 \le m \le n$.

(ii) $\tau_{n|n}(\boldsymbol{X}) \underset{b}{\succ}$ *(resp.* $\underset{b}{\prec}$*)* $\tau_{m|m}(\boldsymbol{X})$ *for* $1 \le n \le m$.

The result stated in Proposition 3.2 is revealed via the following example.

Example 3.4. Consider two coherent systems which are discussed in Example 3.2. Then

$$R_1(p) = \frac{(1-p)h_1'(p)}{1-h_1(p)} = \frac{4p-(7+3\theta)p^2+3(1+5\theta)p^3-27\theta p^4+21\theta p^5-6\theta p^6}{1-2p^2+(1+\theta)p^3-3\theta p^4+3\theta p^5-\theta p^6}, \quad 0<p<1$$

and

$$R_2(p) = \frac{(1-p)h_2'(p)}{1-h_2(p)} = \frac{3(1+\theta)p^2-3(1+5\theta)p^3+27\theta p^4-21\theta p^5+6\theta p^6}{1-(1+\theta)p^3+\theta p^6+3\theta p^4-3\theta p^5}, \quad 0<p<1.$$

Writing $v_\theta(p) = R_2(p)/R_1(p)$, we have

$$v_\theta(p) = \frac{3(1+\theta)p^2-3(1+5\theta)p^3+27\theta p^4-21\theta p^5+6\theta p^6}{1-(1+\theta)p^3+\theta p^6+3\theta p^4-3\theta p^5} \times \frac{1-2p^2+(1+\theta)p^3-3\theta p^4+3\theta p^5-\theta p^6}{4p-(7+3\theta)p^2+3(1+5\theta)p^3-27\theta p^4+21\theta p^5-6\theta p^6}, \quad 0<p<1.$$

In Figure 5 we plot $v_\theta(p)$ against $p \in (0, 1)$, for $\theta = -1, -0.8, \ldots, 0.8, 1$. This shows that $v_\theta(p)$ is increasing in $p \in (0, 1)$. Hence $\tau_1(\boldsymbol{X}) \underset{b}{\succ} \tau_2(\boldsymbol{X})$, using Proposition 3.2.

4. Stochastic comparisons of coherent systems with active redundancy at the component level versus the system level

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a vector of d.i.d. random variables representing the lifetimes of n components, where $X_i \overset{d}{=} X$, $i = 1, 2, \ldots, n$, for some non-negative random variable X. Further, let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m$ be a collection of n-dimensional random vectors representing the

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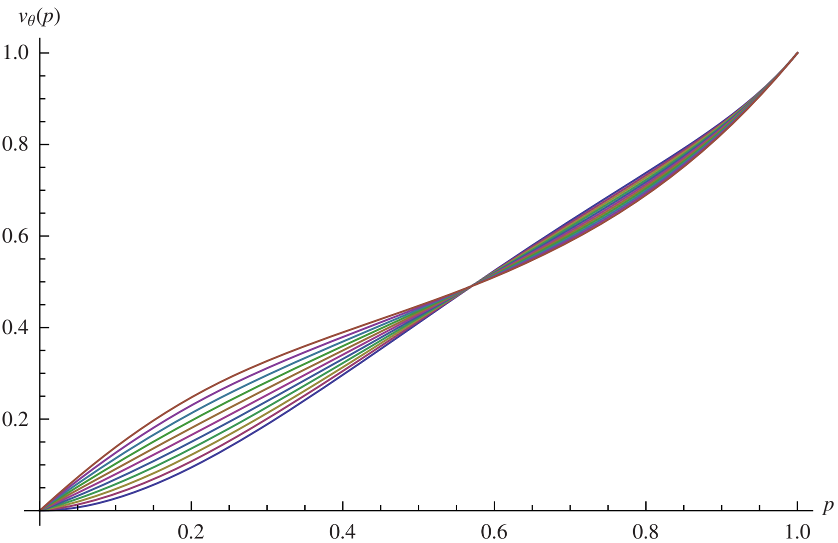

FIGURE 5: Plot of $v_\theta(p)$ against $p \in (0, 1)$, for $\theta = -1, -0.8, \ldots, 0.8, 1$ (from bottom to top).

lifetimes of m sets of n spares (or redundant components) each, where $\boldsymbol{Y}_j \stackrel{d}{=} \boldsymbol{X}$, $j = 1, 2, \ldots, m$, and $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m$ and $\boldsymbol{X}$ are independent. We write $T_C = \tau(\boldsymbol{X} \vee \boldsymbol{Y_1} \vee \boldsymbol{Y_2} \vee \ldots \vee \boldsymbol{Y_m})$ to denote the lifetime of a coherent system with active redundancies at the component level, where, for $\boldsymbol{Y}_j = (Y_{j1}, Y_{j2}, \ldots, Y_{jn})$, $j = 1, 2, \ldots, m$, the symbol $\boldsymbol{X} \vee \boldsymbol{Y_1} \vee \boldsymbol{Y_2} \vee \ldots \vee \boldsymbol{Y_m}$ stands for an n-tuple vector $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$ such that $Z_i = X_i \vee Y_{1i} \vee \cdots \vee Y_{mi}$ represents the lifetime of a parallel system made from $(m+1)$ independent components having lifetimes $\{X_i, Y_{1i}, \ldots, Y_{mi}\}$, $i = 1, 2, \ldots, n$; here the symbol $\vee$ stands for maximum. Further, we write $T_S = \tau(\boldsymbol{X}) \vee \tau(\boldsymbol{Y}_1) \vee \tau(\boldsymbol{Y}_2) \vee \cdots \vee \tau(\boldsymbol{Y}_m)$ to denote the lifetime of a coherent system with active redundancies at the system level. We let $h(\cdot)$ denote the domination function of $\tau(\boldsymbol{X})$ (and hence also that of the $\tau(\boldsymbol{Y}_j)$). In what follows, we use the notation $R(p) = (1-p)h'(p)/(1-h(p))$, $p \in (0, 1)$.

In the following theorem, we provide a necessary and sufficient condition for allocation of redundancy at the component level to be better/worse than that at the system level with respect to the ageing faster order in terms of the hazard rate.

Theorem 4.1. *For any positive integer m, $T_S \underset{c}{\prec}$ (resp. $\underset{c}{\succ}$) T_C holds if and only if*

$$\left(\frac{(1-h(p))^m h'(p)}{1-(1-h(p))^{m+1}}\right)\left(\frac{h(1-(1-p)^{m+1})}{(1-p)^m h'(1-(1-p)^{m+1})}\right) \tag{4.1}$$

is decreasing (resp. increasing) in $p \in (0, 1)$.

The following corollary (for $m = 1$) follows from Theorem 4.1.

Corollary 4.1. *If all the X_i's and Y_{ji}'s are i.i.d., then $\tau_{n|n}(\boldsymbol{X}) \vee \tau_{n|n}(\boldsymbol{Y_1}) \underset{c}{\succ} \tau_{n|n}(\boldsymbol{X} \vee \boldsymbol{Y_1})$.*

In the next theorem we discuss an analog of Theorem 4.1 for the ageing faster order in terms of the reversed hazard rate.

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Theorem 4.2. *For any positive integer* m, $T_S \prec_b T_C$ *holds if and only if*

$$\frac{R(p)}{R(1-(1-p)^{m+1})} \text{ is increasing in } p \in (0, 1).$$

The condition given in Theorem 4.2 may sometimes be difficult to verify. In the following proposition we suggest a sufficient condition that is easy to verify.

Proposition 4.1. *If* $pR'(p)/R(p)$ *is decreasing and positive for all* $p \in (0, 1)$*, then* $T_S \prec_b T_C$.

The next corollary follows from Proposition 4.1 and Lemma 2.4.

Corollary 4.2. *Suppose that all the* X_i*'s and* Y_{ji}*'s are i.i.d. Then, for* $1 \le k \le n$,

$$\tau_{k|n}(\boldsymbol{X}) \vee \tau_{k|n}(\boldsymbol{Y_1}) \vee \cdots \vee \tau_{k|n}(\boldsymbol{Y_m}) \prec_b \tau_{k|n}(\boldsymbol{X} \vee \boldsymbol{Y_1} \vee \ldots \vee \boldsymbol{Y_m}).$$

Below we provide an application of Proposition 4.1.

Example 4.1. Let $m = 1$. Consider a coherent system $\tau(\boldsymbol{X}) = \min\{X_1, X_2, \ldots, X_n\}$ made from n components having d.i.d. lifetimes. Further, let $\{X_1, X_2, \ldots, X_n\}$ have the Gumbel–Hougard copula given by

$$K(p_1, p_2, \ldots, p_n) = \exp\left\{-\left(\sum_{i=1}^{n}(-\ln p_i)^{\theta}\right)^{1/\theta}\right\}, \quad 0 < p_i < 1, \ i = 1, 2, \ldots, n,$$

where $\theta \in [1, \infty)$. Then the domination function of $\tau(\boldsymbol{X})$ is given by $h(p) = p^a$, where $a = n^{1/\theta}$ (≥ 1). This gives

$$R(p) = \frac{(1-p)h'(p)}{(1-h(p))} = \frac{a(p^{a-1}-p^a)}{1-p^a}, \quad 0 < p < 1$$

and

$$\frac{pR'(p)}{R(p)} = \frac{a-1-ap+p^a}{1-p-p^a+p^{a+1}}, \quad 0 < p < 1.$$

Since $((1-p^a)/(1-p)) \le a$, for all $a \ge 1$ and $p \in (0, 1)$, we have $pR'(p)/R(p) \ge 0$, for all $p \in (0, 1)$. Further,

$$\left[\frac{pR'(p)}{R(p)}\right]' = \frac{\gamma_1(p)}{(1-p-p^a+p^{a+1})^2}, \quad 0 < p < 1,$$

where

$$\gamma_1(p) = a^2p^{a-1} + 2(1-a^2)p^a + a^2p^{a+1} - p^{2a} - 1, \quad 0 < p < 1.$$

Clearly

$$\gamma_1'(p) = p^{a-2}\gamma_2(p), \quad 0 < p < 1,$$

where

$$\gamma_2(p) = a^2(a-1) - 2a(a^2-1)p + a^2(a+1)p^2 - 2ap^{a+1}, \quad 0 < p < 1.$$

Differentiating $\gamma_2(p)$ twice, we get

$$\gamma_2'(p) = -2a(a^2-1) + 2a^2(a+1)p - 2a(a+1)p^a, \quad 0 < p < 1$$

and

$$\gamma_2''(p) = 2a^2(a+1)(1-p^{a-1}) \geq 0, \quad 0 < p < 1.$$

Thus we have $\gamma_2'(p) \leq \gamma_2'(1) = 0$, for all $p \in (0, 1)$, and $\gamma_2(p) \geq \gamma_2(1) = 0$, for all $p \in (0, 1)$. Consequently, $\gamma_1'(p) \geq 0$, for all $p \in (0, 1)$, and $\gamma_1(p) \leq \gamma_1(1) = 0$, for all $p \in (0, 1)$. Hence $pR'(p)/R(p)$ is decreasing in $p \in (0, 1)$. Thus $T_S \underset{b}{\prec} T_C$ follows from Proposition 4.1.

5. Stochastic comparisons of a used coherent system and a coherent system of used components

Let X be a random variable representing the lifetime of a component or system. Then its residual lifetime at time t (> 0) is denoted by X_t and is defined by

$$X_t = (X - t \mid X > t).$$

We call X_t a used component or system. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a vector of random variables representing the lifetimes of n d.i.d. components. Then we write

$$\boldsymbol{X}_t = ((X_1)_t, (X_2)_t, \ldots, (X_n)_t), \quad t > 0$$

to represent a vector of n used components $\{(X_1)_t, (X_2)_t, \ldots, (X_n)_t\}$, $t > 0$. Consequently, we write $\tau(\boldsymbol{X}_t)$ to denote the lifetime of a coherent system made from a set of components with lifetime vector $\boldsymbol{X}_t$. Further, by $(\tau(\boldsymbol{X}))_t = (\tau(\boldsymbol{X}) - t | \tau(\boldsymbol{X}) > t)$, we mean the lifetime of a used coherent system made from a set of components with lifetime vector $\boldsymbol{X}$. For simplicity we assume that all the X_i's are identically distributed with $X_i \overset{d}{=} X$, $i = 1, 2, \ldots, n$, for some non-negative random variable X. In what follows, we denote the reliability function of $\tau(\boldsymbol{X})$ by $h(\cdot)$, and we write $H(p) = ph'(p)/h(p)$, $0 < p < 1$.

In the following theorem we derive a necessary and sufficient condition for a coherent system of used components to be ageing faster than a used coherent system in terms of the hazard rate.

Theorem 5.1. *For any fixed* $t \geq 0$, $\tau(\boldsymbol{X}_t) \underset{c}{\prec} (\tau(\boldsymbol{X}))_t$ *holds if and only if*

$$pH'(p)/H(p) \text{ is decreasing in } p \in (0, 1).$$

As an immediate consequence of Theorem 5.1, we have the following proposition.

Proposition 5.1. *For any fixed* $t \geq 0$, $\tau(\boldsymbol{X}_t) \underset{c}{\prec} (\tau(\boldsymbol{X}))_t$ *holds if*

$$(1-p)H'(p)/H(p) \text{ is decreasing and negative in } p \in (0, 1).$$

The next corollary follows from Proposition 5.1 and Lemma 2.3.

Corollary 5.1. *If the* X_i*'s are i.i.d., then* $\tau_{k|n}(\boldsymbol{X}_t) \underset{c}{\prec} (\tau_{k|n}(\boldsymbol{X}))_t$, *for any fixed* $t \geq 0$, *and* $1 \leq k \leq n$.

The following theorem is an analog of Theorem 5.1 for the ageing faster order in terms of the reversed hazard rate.

Theorem 5.2. *For any fixed* $t \geq 0$, $\tau(\boldsymbol{X}_t) \underset{b}{\prec}$ *(resp.* $\underset{b}{\succ}$*)* $(\tau(\boldsymbol{X}))_t$ *holds if and only if, for all* $q \in (0, 1)$,

$$\left[\frac{h'(p/q)}{h'(p)}\right]\left[\frac{h(q)-h(p)}{1-h(p/q)}\right] \text{ is increasing (resp. decreasing) in } p \in (0, q). \tag{5.1}$$

As a consequence of Theorem 5.2, we have the following corollary.

Corollary 5.2. *If the the* X_i*'s are i.i.d., then* $\tau_{1|n}(\boldsymbol{X}_t) \underset{b}{\prec} (\tau_{1|n}(\boldsymbol{X}))_t$, *for any fixed* $t \geq 0$.

6. Concluding remarks

In this paper we study ageing faster orders (in terms of the hazard and reversed hazard rates) which are useful for comparing the relative ageing of two systems. To be more specific, we provide sufficient conditions under which one coherent system is ageing faster than another with respect to the hazard and reversed hazard rates. Further, we consider a problem of allocation of redundancies into a coherent system. We show that, under some necessary and sufficient conditions, the allocation of active redundancy at the component level of a coherent system is superior (inferior) to that at the system level with respect to ageing faster orders. Furthermore, a used coherent system and a coherent system made out of used components are compared with respect to these ageing faster orders. Apart from these, we also show that most of our developed results hold for the well-known k-out-of-n and l-out-of-m systems. Nevertheless, we provide a list of examples to illustrate the proposed results. Some counterexamples are also given wherever needed.

Even though there is a vast literature on the study of usual stochastic orders, there is a limited literature on the ageing faster orders. Since the ageing faster orders compare the relative ageing of two systems, and ageing is a common phenomenon experienced by any system, the study of ageing faster orders deserves adequate attention. We believe that our study not only enriches the literature on ageing faster orders but also has applications.

Similar to the problems considered in this paper, the study of other stochastic orders (as discussed in the introduction), which describe the relative ageing of two systems, is under investigation, and will be reported in the future.

Appendix A

Proof of Lemma 2.3(ii). Note that, for $1 \leq k \leq n$,

$$h_{k|n}(p) = \frac{1}{B(k, n-k+1)} \int_0^p u^{k-1}(1-u)^{n-k}\,\mathrm{d}u, \quad 0 < p < 1, \tag{A.1}$$

where $B(\cdot, \cdot)$ is the beta function. Then, for $1 \leq k \leq n$ and $1 \leq l \leq m$,

$$\frac{1}{H_{k|n}(p)} = \int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k} \mathrm{d}u, \quad 0 < p < 1 \tag{A.2}$$

and

$$\frac{H_{l|m}(p)}{H_{k|n}(p)} = \frac{\displaystyle\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k} \mathrm{d}u}{\displaystyle\int_0^1 u^{l-1}\left(\frac{1-up}{1-p}\right)^{m-l} \mathrm{d}u}, \quad 0 < p < 1.$$

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For any fixed real constant c, consider

$$H_{l|m}(p) - cH_{k|n}(p) \overset{\text{sgn}}{=} \int_0^1 \xi_1(u, p)\eta_1(u, p)\, \mathrm{d}u, \quad 0 < p < 1,$$

where

$$\xi_1(u, p) = u^{l-1}\left(\frac{1-up}{1-p}\right)^{m-l}, \quad 0 < u < 1,\ 0 < p < 1$$

and

$$\eta_1(u, p) = u^{k-l}\left(\frac{1-up}{1-p}\right)^{n-k-m+l} - c, \quad 0 < u < 1,\ 0 < p < 1.$$

Note that

$$\xi_1(u, p) \text{ is } \mathrm{RR}_2 \text{ in}(u, p) \in (0, 1) \times (0, 1), \tag{A.3}$$

and

$$\eta_1(u, p) \text{ is increasing in } p \in (0, 1), \tag{A.4}$$

for all $u \in (0, 1)$. Further, since $l - k \geq \max\{0, m - n\}$,

$$u^{k-l}\left(\frac{1-up}{1-p}\right)^{n-k-m+l} \quad \text{is decreasing in } u \in (0, 1),$$

for all $p \in (0, 1)$. Using Lemma 2.2 we have that, for all $p \in (0, 1)$, $\eta_1(u, p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as u traverses from 0 to 1. Also, using this together with (A.3) and (A.4) in Lemma 2.1, we find that $H_{l|m}(p) - cH_{k|n}(p)$ changes sign at most once and, if the change of sign does occur, it is from negative to positive, as p traverses from 0 to 1. Now, using Lemma 2.2, we conclude that $H_{l|m}(p)/H_{k|n}(p)$ is increasing in $p \in (0, 1)$, that is, $H_{k|n}(p)/H_{l|m}(p)$ is decreasing in $p \in (0, 1)$. Hence the result is proved. □

Proof of Lemma 2.3(iii). Differentiating both sides of (A.2), we get

$$-\frac{H'_{k|n}(p)}{[H_{k|n}(p)]^2} = \frac{n-k}{1-p}\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k-1}\left(\frac{1-u}{1-p}\right)\mathrm{d}u, \quad 0 < p < 1,$$

that is,

$$(1-p)\frac{H'_{k|n}(p)}{H_{k|n}(p)} = -\frac{(n-k)\displaystyle\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k-1}\left(\frac{1-u}{1-p}\right)\mathrm{d}u}{\displaystyle\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k}\mathrm{d}u}, \quad 0 < p < 1.$$

To prove the result it suffices to show that

$$\frac{N_1(p)}{D_1(p)} \overset{\text{def}}{=} \frac{\displaystyle\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k-1}\left(\frac{1-u}{1-p}\right)\mathrm{d}u}{\displaystyle\int_0^1 u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k}\mathrm{d}u} \text{ is increasing in } p \in (0, 1).$$

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For any fixed real constant α, consider

$$N_1(p) - \alpha D_1(p) \overset{\text{sgn}}{=} \int_0^1 \xi_2(u, p)\eta_2(u, p)\,\mathrm{d}u, \quad 0 < p < 1,$$

where

$$\xi_2(u, p) = u^{k-1}\left(\frac{1-up}{1-p}\right)^{n-k}, \quad 0 < u < 1,\ 0 < p < 1$$

and

$$\eta_2(u, p) = \left(\frac{1-u}{1-up}\right) - \alpha, \quad 0 < u < 1,\ 0 < p < 1.$$

Note that

$$\xi_2(u, p) \text{ is } \mathrm{RR}_2 \text{ in } (u, p) \in (0, 1) \times (0, 1) \tag{A.5}$$

and

$$\eta_2(u, p) \text{ is increasing in } p \in (0, 1), \tag{A.6}$$

for all $u \in (0, 1)$. Further, for every fixed $p \in (0, 1)$,

$$\left(\frac{1-u}{1-up}\right) \text{ is decreasing in } u \in (0, 1).$$

Then, using Lemma 2.2 it follows that, for every fixed $p \in (0, 1)$, $\eta_2(u, p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as u traverses from 0 to 1. Using this together with (A.5) and (A.6) in Lemma 2.1, we conclude that $N_1(p) - \alpha D_1(p)$ changes sign at most once and, if the change of sign does occur, it is from negative to positive, as p traverses from 0 to 1. Now, using Lemma 2.2, we get that $N_1(p)/D_1(p)$ is increasing in $p \in (0, 1)$, and hence the result is proved. □

Proof of Lemma 2.4(ii). From (A.1), we have, for $1 \le k \le n$ and $1 \le l \le m$,

$$\frac{1}{R_{k|n}(p)} = \int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-1} \mathrm{d}u, \quad 0 < p < 1 \tag{A.7}$$

and

$$\frac{R_{l|m}(p)}{R_{k|n}(p)} = \frac{\displaystyle\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-1} \mathrm{d}u}{\displaystyle\int_0^1 u^{m-l}\left(\frac{1-u(1-p)}{p}\right)^{l-1} \mathrm{d}u}, \quad 0 < p < 1.$$

For any fixed real constant β, consider

$$R_{l|m}(p) - \beta R_{k|n}(p) \overset{\text{sgn}}{=} \int_0^1 \xi_3(u, p)\eta_3(u, p)\,\mathrm{d}u, \quad 0 < p < 1,$$

where

$$\xi_3(u, p) = u^{m-l}\left(\frac{1-u(1-p)}{p}\right)^{l-1}, \quad 0 < u < 1,\ 0 < p < 1$$

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and

$$\eta_3(u, p) = u^{n-k-m+l}\left(\frac{1-u(1-p)}{p}\right)^{k-l} - \beta, \quad 0 < u < 1,\ 0 < p < 1.$$

Note that

$$\xi_3(u, p) \text{ is TP}_2 \text{ in } (u, p) \in (0, 1) \times (0, 1) \tag{A.8}$$

and

$$\eta_3(u, p) \text{ is decreasing in } p \in (0, 1), \tag{A.9}$$

for all $u \in (0, 1)$. Further, for any fixed $p \in (0, 1)$,

$$u^{n-k-m+l}\left(\frac{1-u(1-p)}{p}\right)^{k-l} \text{ is decreasing in } u \in (0, 1).$$

Using Lemma 2.2 we infer that, for any fixed $p \in (0, 1)$, $\eta_3(u, p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as u traverses from 0 to 1. Also, using this together with (A.8) and (A.9) in Lemma 2.1, we get that $R_{l|m}(p) - \beta R_{k|n}(p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as p traverses from 0 to 1. Now, using Lemma 2.2, we conclude that $R_{l|m}(p)/R_{k|n}(p)$ is decreasing in $p \in (0, 1)$, that is, $R_{k|n}(p)/R_{l|m}(p)$ is increasing in $p \in (0, 1)$. Hence the result is proved. □

Proof of Lemma 2.4(iii). On differentiating both sides of (A.7), we get

$$\frac{R'_{k|n}(p)}{[R_{k|n}(p)]^2} = \frac{k-1}{p}\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-2}\left(\frac{1-u}{p}\right)\mathrm{d}u, \quad 0 < p < 1,$$

or equivalently

$$p\frac{R'_{k|n}(p)}{R_{k|n}(p)} = \frac{(k-1)\displaystyle\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-2}\left(\frac{1-u}{p}\right)\mathrm{d}u}{\displaystyle\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-1}\mathrm{d}u}, \quad 0 < p < 1.$$

To prove the result it suffices to show that

$$\frac{N_2(p)}{D_2(p)} \overset{\text{def}}{=} \frac{\displaystyle\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-2}\left(\frac{1-u}{p}\right)\mathrm{d}u}{\displaystyle\int_0^1 u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-1}\mathrm{d}u} \text{ is decreasing in } p \in (0, 1).$$

For any fixed real constant γ, consider

$$N_2(p) - \gamma D_2(p) \overset{\text{sgn}}{=} \int_0^1 \xi_4(u, p)\eta_4(u, p)\,\mathrm{d}u, \quad 0 < p < 1,$$

where

$$\xi_4(u, p) = u^{n-k}\left(\frac{1-u(1-p)}{p}\right)^{k-1}, \quad 0 < u < 1,\ 0 < p < 1$$

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and

$$\eta_4(u, p) = \left(\frac{1-u}{1-u(1-p)}\right) - \gamma, \quad 0 < u < 1, \ 0 < p < 1.$$

Note that

$$\xi_4(u, p) \text{ is TP}_2 \text{ in } (u, p) \in (0, 1) \times (0, 1) \tag{A.10}$$

and, for every fixed $u \in (0, 1)$,

$$\eta_4(u, p) \text{ is decreasing in } p \in (0, 1). \tag{A.11}$$

Moreover, for any fixed $p \in (0, 1)$,

$$\frac{1-u}{1-u(1-p)} \text{ is decreasing in } u \in (0, 1).$$

Using Lemma 2.2 we get that, for any fixed $p \in (0, 1)$, $\eta_4(u, p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as u traverses from 0 to 1. Also, using this together with (A.10) and (A.11) in Lemma 2.1, we get that $N_2(p) - \alpha D_2(p)$ changes sign at most once and, if the change of sign does occur, it is from positive to negative, as p traverses from 0 to 1. Now, using Lemma 2.2, we conclude that $N_2(p)/D_2(p)$ is decreasing in $p \in (0, 1)$, and hence the result is proved. □

Proof of Theorem 3.1. Note that

$$\bar{F}_{\tau_1(\boldsymbol{X})}(x) = h_1(\bar{F}_X(x)) \quad \text{and} \quad \bar{F}_{\tau_2(\boldsymbol{Y})}(x) = h_2(\bar{F}_Y(x)), \quad x > 0,$$

which gives failure rates of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ as

$$r_{\tau_1(\boldsymbol{X})}(x) = \frac{f_X(x)h_1'(\bar{F}_X(x))}{h_1(\bar{F}_X(x))} = r_X(x)H_1(\bar{F}_X(x)), \quad x > 0$$

and

$$r_{\tau_2(\boldsymbol{Y})}(x) = \frac{f_Y(x)h_2'(\bar{F}_Y(x))}{h_2(\bar{F}_X(x))} = r_Y(x)H_2(\bar{F}_Y(x)), \quad x > 0,$$

respectively. Then $\tau_1(\boldsymbol{X}) \underset{c}{\prec} \tau_2(\boldsymbol{Y})$ holds if and only if

$$\frac{r_{\tau_1(\boldsymbol{X})}(x)}{r_{\tau_2(\boldsymbol{X})}(x)} = \left[\frac{r_X(x)}{r_Y(x)}\right]\left[\frac{H_1(\bar{F}_X(x))}{H_2(\bar{F}_Y(x))}\right] \text{ is increasing in } x > 0.$$

To prove the theorem it suffices to show that

$$\frac{r_X(x)}{r_Y(x)} \text{ is increasing in } x > 0 \tag{A.12}$$

and

$$\frac{H_1(\bar{F}_X(x))}{H_2(\bar{F}_Y(x))} \text{ is increasing in } x > 0. \tag{A.13}$$

Note that (A.12) holds as $X \underset{c}{\prec} Y$. Further, (A.13) holds if and only if

$$\tilde{r}_Y(x)\left[(1-\bar{F}_Y(x))\frac{H_2'(\bar{F}_Y(x))}{H_2(\bar{F}_Y(x))}\right] \geq \tilde{r}_X(x)\left[(1-\bar{F}_X(x))\frac{H_1'(\bar{F}_X(x))}{H_1(\bar{F}_X(x))}\right] \quad \text{for all } x > 0. \tag{A.14}$$

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Since $Y \leq_{rh} X$, we have

$$\tilde{r}_Y(x) \leq \tilde{r}_X(x) \quad \text{and} \quad \bar{F}_Y(x) \leq \bar{F}_X(x) \quad \text{for all } x > 0. \tag{A.15}$$

Now consider the following two cases.

Case I. Let $(1-p)H_1'(p)/H_1(p)$ be decreasing in $p \in (0, 1)$. Then

$$(1-\bar{F}_Y(x))\frac{H_2'(\bar{F}_Y(x))}{H_2(\bar{F}_Y(x))} \geq (1-\bar{F}_Y(x))\frac{H_1'(\bar{F}_Y(x))}{H_1(\bar{F}_Y(x))} \geq (1-\bar{F}_X(x))\frac{H_1'(\bar{F}_X(x))}{H_1(\bar{F}_X(x))} \quad \text{for all } x > 0,$$

where the first inequality follows from condition (i) and the second inequality follows from (A.15) and condition (ii).

Case II. Let $(1-p)H_2'(p)/H_2(p)$ be decreasing in $p \in (0, 1)$. Then

$$(1-\bar{F}_Y(x))\frac{H_2'(\bar{F}_Y(x))}{H_2(\bar{F}_Y(x))} \geq (1-\bar{F}_X(x))\frac{H_2'(\bar{F}_X(x))}{H_2(\bar{F}_X(x))} \geq (1-\bar{F}_X(x))\frac{H_1'(\bar{F}_X(x))}{H_1(\bar{F}_X(x))} \quad \text{for all } x > 0,$$

where the first inequality follows from (A.15) and (ii) and the second inequality follows from (i). Now, from Cases I and II, we obtain

$$-(1-\bar{F}_Y(x))\frac{H_2'(\bar{F}_Y(x))}{H_2(\bar{F}_Y(x))} \leq -(1-\bar{F}_X(x))\frac{H_1'(\bar{F}_X(x))}{H_1(\bar{F}_X(x))} \quad \text{for all } x > 0. \tag{A.16}$$

Further, (i) implies that

$$-(1-\bar{F}_X(x))\frac{H_1'(\bar{F}_X(x))}{H_1(\bar{F}_X(x))} \geq 0 \quad \text{for all } x > 0. \tag{A.17}$$

Combining (A.15), (A.16), and (A.17), we get (A.14). Hence the result is proved. □

Proof of Theorem 3.2. Note that

$$F_{\tau_1(\boldsymbol{X})}(x) = 1 - h_1(\bar{F}_X(x)) \quad \text{and} \quad F_{\tau_2(\boldsymbol{Y})}(x) = 1 - h_2(\bar{F}_Y(x)), \quad x > 0,$$

which gives reversed failure rates of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ as

$$\tilde{r}_{\tau_1(\boldsymbol{X})}(x) = \frac{f_X(x)h_1'(\bar{F}_X(x))}{1-h_1(\bar{F}_X(x))} = \tilde{r}_X(x)R_1(\bar{F}_X(x)), \quad x > 0$$

and

$$\tilde{r}_{\tau_2(\boldsymbol{Y})}(x) = \frac{f_Y(x)h_2'(\bar{F}_Y(x))}{1-h_2(\bar{F}_X(x))} = \tilde{r}_Y(x)R_2(\bar{F}_Y(x)), \quad x > 0,$$

respectively. Then $\tau_1(\boldsymbol{X}) \underset{b}{\prec} \tau_2(\boldsymbol{Y})$ holds if and only if

$$\frac{\tilde{r}_{\tau_1(\boldsymbol{X})}(x)}{\tilde{r}_{\tau_2(\boldsymbol{X})}(x)} = \left[\frac{\tilde{r}_X(x)}{\tilde{r}_Y(x)}\right]\left[\frac{R_1(\bar{F}_X(x))}{R_2(\bar{F}_Y(x))}\right] \text{ is decreasing in } x > 0.$$

To prove the theorem it suffices to establish that

$$\frac{\tilde{r}_X(x)}{\tilde{r}_Y(x)} \text{ is decreasing in } x > 0 \tag{A.18}$$

and

$$\frac{R_1(\bar{F}_X(x))}{R_2(\bar{F}_Y(x))} \text{ is decreasing in } x > 0. \tag{A.19}$$

Note that (A.18) holds as $X \underset{b}{\prec} Y$. Further, (A.19) holds if and only if

$$r_Y(x)\left[\bar{F}_Y(x)\frac{R_2'(\bar{F}_Y(x))}{R_2(\bar{F}_Y(x))}\right] \le r_X(x)\left[\bar{F}_X(x)\frac{R_1'(\bar{F}_X(x))}{R_1(\bar{F}_X(x))}\right] \quad \text{for all } x > 0. \tag{A.20}$$

Since $X \le_{hr} Y$, we have

$$r_Y(x) \le r_X(x) \quad \text{and} \quad \bar{F}_X(x) \le \bar{F}_Y(x) \quad \text{for all } x > 0. \tag{A.21}$$

Now consider the following two cases.

Case I. Let $pR_1'(p)/R_1(p)$ be decreasing in $p \in (0, 1)$. Then

$$\bar{F}_Y(x)\frac{R_2'(\bar{F}_Y(x))}{R_2(\bar{F}_Y(x))} \le \bar{F}_Y(x)\frac{R_1'(\bar{F}_Y(x))}{R_1(\bar{F}_Y(x))} \le \bar{F}_X(x)\frac{R_1'(\bar{F}_X(x))}{R_1(\bar{F}_X(x))} \quad \text{for all } x > 0,$$

where the first inequality follows from (i) and the second inequality follows from (A.21) and (ii).

Case II. Let $pR_2'(p)/R_2(p)$ be decreasing in $p \in (0, 1)$. Then

$$\bar{F}_Y(x)\frac{R_2'(\bar{F}_Y(x))}{R_2(\bar{F}_Y(x))} \le \bar{F}_X(x)\frac{R_2'(\bar{F}_X(x))}{R_2(\bar{F}_X(x))} \le \bar{F}_X(x)\frac{R_1'(\bar{F}_X(x))}{R_1(\bar{F}_X(x))} \quad \text{for all } x > 0,$$

where the first inequality follows from (A.21) and (ii) and the second inequality follows from (i). Now, from Cases I and II, we obtain

$$\bar{F}_Y(x)\frac{R_2'(\bar{F}_Y(x))}{R_2(\bar{F}_Y(x))} \le \bar{F}_X(x)\frac{R_1'(\bar{F}_X(x))}{R_1(\bar{F}_X(x))} \quad \text{for all } x > 0. \tag{A.22}$$

Further, (i) implies that

$$\bar{F}_X(x)\frac{R_1'(\bar{F}_X(x))}{R_1(\bar{F}_X(x))} \ge 0 \quad \text{for all } x > 0. \tag{A.23}$$

Combining (A.21), (A.22), and (A.23), we get (A.20). Hence the result is proved. □

Proof of Theorem 4.1. We have

$$\bar{F}_{T_C}(x) = h(1-(1-\bar{F}_X(x))^{m+1}), \quad x > 0$$

and

$$\bar{F}_{T_S}(x) = 1-(1-h(\bar{F}_X(x)))^{m+1}, \quad x > 0.$$

Consequently the failure rates of T_C and T_S are given by

$$r_{T_C}(x) = (m+1)f_X(x)(1-\bar{F}_X(x))^m \frac{h'(1-(1-\bar{F}_X(x))^{m+1})}{h(1-(1-\bar{F}_X(x))^{m+1})}, \quad x > 0$$

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and

$$r_{T_S}(x) = (m+1) f_X(x)(1 - h(\bar{F}_X(x)))^m \frac{h'(\bar{F}_X(x))}{1 - (1 - h(\bar{F}_X(x)))^{m+1}}, \qquad x > 0,$$

respectively. Then $T_S \underset{c}{\prec}$ (resp. $\underset{c}{\succ}$) T_C holds if and only if

$$\frac{r_{T_S}(x)}{r_{T_C}(x)} = \left(\frac{(1 - h(\bar{F}_X(x)))^m h'(\bar{F}_X(x))}{1 - (1 - h(\bar{F}_X(x)))^{m+1}} \right) \left(\frac{h(1 - (1 - \bar{F}_X(x))^{m+1})}{(1 - \bar{F}_X(x))^m h'(1 - (1 - \bar{F}_X(x))^{m+1})} \right)$$

is increasing (resp. decreasing) in $x > 0$, that is, if and only if

$$\left(\frac{(1 - h(p))^m h'(p)}{1 - (1 - h(p))^{m+1}} \right) \left(\frac{h(1 - (1 - p)^{m+1})}{(1 - p)^m h'(1 - (1 - p)^{m+1})} \right)$$

is decreasing (resp. increasing) in $p \in (0, 1)$. Hence the result is proved. □

Proof of Corollary 4.1. The reliability function of an n-out-of-n system with i.i.d. component lifetimes is given by $h(p) = p^n$. Thus, to prove the result it suffices to show that (4.1) holds for $h(p) = p^n$ with $m = 1$ and $n \geq 2$. Thus proving the result boils down to showing that

$$\frac{(2 - p)(1 - p^n)}{(1 - p)(2 - p^n)} \text{ is increasing in } p \in (0, 1),$$

or equivalently

$$1 + \zeta_1(p) \text{ is increasing in } p \in (0, 1),$$

where

$$\zeta_1(p) = \frac{p - p^n}{2 - 2p - p^n + p^{n+1}}, \qquad 0 < p < 1.$$

We have

$$\zeta_1'(p) \overset{\text{sgn}}{=} 2 - \zeta_2(p), \qquad 0 < p < 1,$$

where

$$\zeta_2(p) = 2np^{n-1} - 3(n-1)p^n + np^{n+1} - p^{2n}, \qquad 0 < p < 1.$$

Clearly

$$\zeta_2'(p) = np^{n-2} \zeta_3(p), \qquad 0 < p < 1,$$

where

$$\begin{aligned} \zeta_3(p) &= 2(n-1) - 3(n-1)p + (n+1)p^2 - 2p^{n+1} \\ &\geq (n-1)(2 - 3p + p^2) \\ &= (n-1)(2-p)(1-p) \geq 0, \qquad 0 < p < 1. \end{aligned}$$

From the above we conclude that $\zeta_2(p)$ is increasing in $p \in (0, 1)$ with $\zeta_2(0) = 0$ and $\zeta_2(1) = 2$. Consequently $0 \leq \zeta_2(p) \leq 2$. This, in turn, implies that $\zeta_1(p)$ is increasing in $p \in (0, 1)$, and hence the result is proved. □

Proof of Theorem 4.2. We have

$$F_{T_C}(x) = 1 - h(1 - (1 - \bar{F}_X(x))^{m+1}), \qquad x > 0$$

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and

$$F_{T_S}(x) = (1 - h(\bar{F}_X(x)))^{m+1}, \quad x > 0.$$

Consequently the reversed failure rates of T_C and T_S are given by

$$\tilde{r}_{T_C}(x) = (m+1) f_X(x)(1 - \bar{F}_X(x))^m \frac{h'(1 - (1 - \bar{F}_X(x))^{m+1})}{1 - h(1 - (1 - \bar{F}_X(x))^{m+1})}, \quad x > 0$$

and

$$\tilde{r}_{T_S}(x) = (m+1) f_X(x) \frac{h'(\bar{F}_X(x))}{1 - h(\bar{F}_X(x))}, \quad x > 0,$$

respectively. Then $T_S \underset{b}{\prec} T_C$ holds if and only if

$$\frac{\tilde{r}_{T_S}(x)}{\tilde{r}_{T_C}(x)} = \left(\frac{h'(\bar{F}_X(x))}{1 - h(\bar{F}_X(x))}\right) \left(\frac{1 - h(1 - (1 - \bar{F}_X(x))^{m+1})}{(1 - \bar{F}_X(x))^m h'(1 - (1 - \bar{F}_X(x))^{m+1})}\right) \text{ is decreasing in } x > 0,$$

or equivalently

$$\left(\frac{(1-p)h'(p)}{1 - h(p)}\right) \left(\frac{1 - h(1 - (1-p)^{m+1})}{(1 - (1 - (1-p)^{m+1}))h'(1 - (1-p)^{m+1})}\right) \text{ is increasing in } p \in (0, 1),$$

that is, if and only if

$$\frac{R(p)}{R(1 - (1-p)^{m+1})} \text{ is increasing in } p \in (0, 1),$$

and hence the result is proved. □

Proof of Proposition 4.1. Since $pR'(p)/R(p)$ is decreasing in $p \in (0, 1)$, and $p \le 1 - (1 - p)^{m+1}$, for all $p \in (0, 1)$, we have

$$p\frac{R'(p)}{R(p)} \ge (1 - (1-p)^{m+1}) \frac{R'(1 - (1-p)^{m+1})}{R(1 - (1-p)^{m+1})} \quad \text{for all } p \in (0, 1). \tag{A.24}$$

Further, it can be easily checked that, for all $p \in (0, 1)$,

$$1 - (1-p)^{m+1} \ge (m+1)p(1-p)^m.$$

Since $pR'(p)/R(p)$ is positive for all $p \in (0, 1)$, we get from the above inequality that, for all $p \in (0, 1)$,

$$(1 - (1-p)^{m+1}) \frac{R'(1 - (1-p)^{m+1})}{R(1 - (1-p)^{m+1}} \ge (m+1)p(1-p)^m \frac{R'(1 - (1-p)^{m+1})}{R(1 - (1-p)^{m+1})}. \tag{A.25}$$

Combining (A.24), and (A.25), we get

$$p\frac{R'(p)}{R(p)} \ge (m+1)p(1-p)^m \frac{R'(1 - (1-p)^{m+1})}{R(1 - (1-p)^{m+1}} \quad \text{for all } p \in (0, 1),$$

or equivalently

$$\frac{R(p)}{R(1 - (1-p)^{m+1})} \text{ is increasing in } p \in (0, 1).$$

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Now the result follows using Theorem 4.2. □

Proof of Theorem 5.1. Note that, for any fixed $t > 0$,

$$\bar{F}_{\tau(\boldsymbol{X}_t)}(x) = h\left(\frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right), \quad \bar{F}_{(\tau(\mathbf{X}))_t}(x) = \frac{h(\bar{F}_X(t+x))}{h(\bar{F}_X(t))}, \quad x > 0,$$
$$r_{\tau(\boldsymbol{X}_t)}(x) = \frac{f_X(t+x)h'\left(\dfrac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)}{\bar{F}_X(t)h\left(\dfrac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)} = r_X(t+x)H\left(\frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right), \quad x > 0, \tag{A.26}$$

and

$$r_{(\tau(\boldsymbol{X}))_t}(x) = \frac{f_X(t+x)h'(\bar{F}_X(t+x))}{h(\bar{F}_X(t+x))} = r_X(t+x)H(\bar{F}_X(t+x)), \quad x > 0.$$

Then $\tau(\boldsymbol{X}_t) \underset{c}{\prec} (\tau(\boldsymbol{X}))_t$ holds if and only if

$$\frac{r_{\tau(\boldsymbol{X}_t)}(x)}{r_{(\tau(\boldsymbol{X}))_t}(x)} = \frac{H(\bar{F}_X(t+x)/\bar{F}_X(t))}{H(\bar{F}_X(t+x))} \text{ is increasing in } x > 0,$$

which is equivalent to the fact that, for every fixed $q \in (0,\ 1)$,

$$\frac{H(p/q)}{H(p)} \text{ is decreasing in } p \in (0,\ q).$$

Further, this holds if and only if

$$\frac{p}{q}\left(\frac{H'(p/q)}{H(p/q)}\right) \le p\frac{H'(p)}{H(p)} \quad \text{for all } 0 < p \le q < 1,$$

or equivalently

$$p\frac{H'(p)}{H(p)} \text{ is decreasing in } p \in (0,\ 1).$$

Hence the result is proved. □

Proof of Theorem 5.2. Let $t > 0$ be fixed. From (A.26) we have

$$\tilde{r}_{\tau(\boldsymbol{X}_t)}(x) = \frac{f_X(t+x)h'\left(\dfrac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)}{\bar{F}_X(t)\left(1 - h\left(\dfrac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)\right)}, \quad x > 0$$

and

$$\tilde{r}_{(\tau(\boldsymbol{X}))_t}(x) = \frac{f_X(t+x)h'(\bar{F}_X(t+x))}{h(\bar{F}_X(t)) - h(\bar{F}_X(t+x))}, \quad x > 0.$$

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Then $\tau_1(\boldsymbol{X}_t) \underset{b}{\prec}$ (resp. $\underset{b}{\succ}$) $(\tau_2(\boldsymbol{X}))_t$ holds if and only if

$$\frac{\tilde{r}_{\tau(\boldsymbol{X}_t)}(x)}{\tilde{r}_{(\tau(\boldsymbol{X}))_t}(x)} = \left[\left(h'\left(\frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)\right) \Big/ \left(\bar{F}_X(t)\left(1-h\left(\frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}\right)\right)\right)\right]\left[\frac{h(\bar{F}_X(t))-h(\bar{F}_X(t+x))}{h'(\bar{F}_X(t+x))}\right]$$

is decreasing (resp. increasing) in $x>0$, which is equivalent to (5.1). Hence the result is proved. □

Proof of Corollary 5.2. The reliability function of a 1-out-of-n system is given by $h(p)=1-(1-p)^n,\ 0<p<1$. Thus, to prove the result it suffices to show that (5.1) holds for $h(p)=1-(1-p)^n$, $0<p<1$, that is, the result will be proved if we can show that, for every fixed $q\in(0,1)$,

$$\frac{(1-p)^n-(1-q)^n}{(1-p)^{n-1}(q-p)} \text{ is increasing in } p\in(0,q),$$

or equivalently

$$\zeta_5(y) \stackrel{\text{def}}{=} \frac{y^n-1}{(y-1)y^{n-1}} \text{ is decreasing in } y>1.$$

We have

$$\zeta_5'(y) \stackrel{\text{sgn}}{=} y^{n-2}\zeta_6(y), \qquad y>1,$$

where

$$\zeta_6(y) = -y^n + ny - (n-1), \qquad y>1.$$

Note that $\zeta_6(\cdot)$ is a decreasing function with $\zeta_6(1)=0$, and hence $\zeta_6(y)\le 0$ for all $y>1$. Further, this implies that $\zeta_5(y)$ is decreasing in $y>1$. Hence the result is proved. □

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