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GAUSSIAN CURVATURE AND UNICITY PROBLEM OF GAUSS MAPS OF VARIOUS CLASSES OF SURFACES

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Abstract. In this article, we establish a new estimate for the Gaussian curvature of open Riemann surfaces in Euclidean three-space with a specified conformal metric regarding the uniqueness of the holomorphic maps of these surfaces. As its applications, we give new proofs on the unicity problems for the Gauss maps of various classes of surfaces, in particular, minimal surfaces in Euclidean three-space, constant mean curvature one surfaces in the hyperbolic three-space, maximal surfaces in the Lorentz–Minkowski three-space, improper affine spheres in the affine three-space and flat surfaces in the hyperbolic three-space.

§1. Introduction

One of the well-known problems in minimal surface theory is to understand the global behavior of the Gauss map. In 1988, Fujimoto [8] proved Nirenberg's conjecture that if M is a complete nonflat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most four points in the unit two-sphere $\mathbb{S}^2$, and there are a number of examples showing that the bound is sharp. Later, Fujimoto improved the previous result by giving a curvature bound for a minimal surface, which is not necessarily complete, when all of the multiple values of the Gauss map are totally ramified. Here, a value α of a map or function g is said to be totally ramified if the equation $g = \alpha$ has no simple roots. He proved the following theorem.

THEOREM 1.1. (See [9]) *Let $x : M \to \mathbb{R}^3$ be a minimal surface immersed in $\mathbb{R}^3$ with its Gauss map $g : M \to \overline{\mathbb{C}}$. Let $\{a_j\}_{j=1}^q$ be q distinct points in $\overline{\mathbb{C}}$. Suppose that g is ramified over a_j with multiplicity at least m_j for each j and*

$$\sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) > 4.$$

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Then there exists a constant C, *depending on the set of points* $\{a_j\}_{j=1}^q$ *but not the surface, such that*

$$|K(p)|^{1/2}d(p) \leqslant C,$$

where $K(p)$ *is the Gaussian curvature of the surface at* p *and* $d(p)$ *is the geodesic distance from* p *to the boundary of* M.

After that, the relations of the omitted properties or ramifications of the Gauss map and the Gaussian curvature of minimal surfaces have been studied (see [14, 18, 20, 30, 34, 36] for some newest results).

On the other hand, Fujimoto [10] gave some uniqueness theorems for the Gauss maps of minimal surfaces, which are analogous to the Nevanlinna unicity theorem [33] for meromorphic functions on the complex plane $\mathbb{C}$. Precisely, he proved the following theorem.

THEOREM 1.2. (See [10]) *Let* M *and* $\widehat{M}$ *be two nonflat minimal surfaces in* $\mathbb{R}^3$ *with their Gauss maps* g *and* $\widehat{g}$, *respectively. Suppose that there is a conformal diffeomorphism* Ψ *from* M *onto* $\widehat{M}$ *and there are* q *distinct points* $\alpha_1, \alpha_2, \ldots, \alpha_q$ *in* $\overline{\mathbb{C}}$ *such that* $g^{-1}(\alpha_j) = (\widehat{g} \circ \Psi)^{-1}(\alpha_j)$ *for every* $1 \leqslant j \leqslant q$. *Then we have necessarily* $g \equiv \widehat{g} \circ \Psi$ *if* $q \geqslant 7$ *and either* M *or* $\widehat{M}$ *is complete.*

He also gave an example to show that number 7 is the best possible. Recently, many results on the unicity problems of the Gauss maps of minimal surfaces were introduced (see [11, 15, 17, 20, 22, 35, 37] for more details).

A natural question is whether there is a relation between the Gaussian curvature and the unicity problem of the Gauss maps of minimal surfaces of a nonflat minimal surface in $\mathbb{R}^3$. In this paper, we will give an affirmative answer to that question.

Moreover, there exist several classes of immersed surfaces whose Gauss maps have these function-theoretic properties. For instance, Yu [43] showed that the hyperbolic Gauss map of a nonflat complete constant mean curvature one surface in hyperbolic three-space $\mathbf{H}^3$ can omit at most four values, Kawakami and Nakajo [23] obtained that the maximal number of omitted values of the Lagrangian Gauss map of weakly complete improper affine spheres (or improper affine fronts) in the affine three-space $\mathbb{R}^3$ is 3, unless it is an elliptic paraboloid, or Kawakami [19] gave similar results for flat fronts in $\mathbf{H}^3$. Furthermore, Kawakami [18, 20] elucidated the geometric interpretation of function-theoretic properties for the Gauss maps of several classes of immersed surfaces in three-dimensional space forms, for example, minimal surfaces in Euclidean three-space, constant mean curvature one surfaces in the hyperbolic three-space, maximal surfaces in the Lorentz–Minkowski three-space, improper affine spheres in the affine three-space and flat surfaces in hyperbolic three-space (also see the review article [21] for instance).

We first want to show the relation of the Gaussian curvature and the unicity problem of the Gauss maps of minimal surfaces. But, motivated by the recent works of Kawakami (which are mentioned above), we would like to study the same situations for the various classes of surfaces. The main purpose of this article is to construct a new estimate for the Gaussian curvature of several classes of immersed surfaces in three-dimensional space forms regarding the uniqueness of the holomorphic maps of these surfaces. After that, we use it to give new proofs on the unicity problems for the Gauss maps of those surfaces. The paper is organized as follows. In Section 2, we first give a curvature bound for the conformal metrics $ds^2 = (1+|g|^2)^m|\omega|^2$, $d\widehat{s}^2 = (1+|\widehat{g}|^2)^m|\widehat{\omega}|^2$ on open Riemann surfaces $M, \widehat{M}$, respectively, where m is a positive integer, ω and $\widehat{\omega}$ are holomorphic 1-forms, and g and $\widehat{g}$ are holomorphic maps into $\overline{\mathbb{C}}$ on M and $\widehat{M}$, respectively (Theorem 2.5). After that, we give some examples to show that our main result is optimal (Examples 2.6 and 2.7). As a corollary of it, we give a unicity theorem (Theorem 2.8) for g on M with the complete metric ds^2. We will prove the main result in Section 3. In Section 4, we recall the backgrounds of the several classes of immersed surfaces in three-dimensional space forms based on the terminology in [20, 21]: minimal surfaces in $\mathbb{R}^3$ (Section 4.1), constant mean curvature one surfaces in $\mathbf{H}^3$ (Section 4.2), maximal surfaces in the Lorentz–Minkowski three-space $\mathbf{R}^3_1$ (Section 4.3), improper affine spheres in $\mathbb{R}^3$ (Section 4.4) and flat surfaces in hyperbolic three-space $\mathbf{H}^3$ (Section 4.5). The reason is the convenience of the reader to realize that our main results can give some unicity theorems of Kawakami in [20]. We thus show some value-distribution-theoretic properties for the Gauss maps of the following classes of surfaces as applications of our main results.

§2. Statements of the main results

2.1 Curvature bound for specified conformal metrics on open Riemann surfaces

Let M be a Riemann surface with a metric ds^2 which is conformal, namely, represented as

$$ds^2 = \lambda_z^2 |dz|^2$$

with a positive $\mathbb{C}^\infty$ function λ_z in terms of a holomorphic local coordinate.

DEFINITION 2.1. (See [12]) For each point $p \in M$, we define the Gaussian curvature of the metric ds^2 of M at p by

$$K \equiv K_{ds^2} := -\frac{\Delta_z \log \lambda_z}{\lambda_z^2}.$$

DEFINITION 2.2. (See [5]) A curve $p(t)$, $0 \leqslant t < 1$, on a Riemann surface M is called divergent if for every compact subset K on M, there exists $t_0 < 1$ such that $p(t) \notin K$ for every $t > t_0$.

DEFINITION 2.3. (See [5]) The Riemann surface M with a metric ds^2 is complete if the length of every divergent curve on M is infinite.

DEFINITION 2.4. Let $M, \widehat{M}$ be two open Riemann surfaces with the conformal metrics $ds^2, d\widehat{s}^2$, respectively. The map $\Psi\colon M \to \widehat{M}$ is called a conformal diffeomorphism if Ψ is biholomorphic and there exists a (local) nowhere zero holomorphic function ζ such that $ds^2 = |\zeta|^2 \Psi^*(d\widehat{s}^2)$ on coordinate charts.

The main theorem of this article is the following:

THEOREM 2.5. *Let $M, \widehat{M}$ be two open Riemann surfaces with the conformal metrics*

$$ds^2 = (1 + |g|^2)^m |\omega|^2, \qquad d\widehat{s}^2 = (1 + |\widehat{g}|^2)^m |\widehat{\omega}|^2$$

where ω and $\widehat{\omega}$ are holomorphic 1-forms, g and $\widehat{g}$ are holomorphic maps into $\overline{\mathbb{C}}$ on M and $\widehat{M}$, respectively, and m is a positive integer. We assume that there exists a conformal diffeomorphism $\Psi\colon M \to \widehat{M}$ and $g, \widehat{g}$ are nonconstant. Suppose that there exist $q(\geqslant 5 + m)$ distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $g^{-1}(\alpha_j) = (\widehat{g} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$. Then there exists a constant C, depending on m and $\alpha_1, \ldots, \alpha_q$ but not the surface, such

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that for all $p \in M$, we have

$$|K_{ds^2}(p)|^{1/2} \cdot d(p) \cdot |g(p), \widehat{g} \circ \Psi(p)| \leqslant C \tag{1}$$

where $d(p)$ is the geodesic distance from p to the boundary of M, that is, the infimum of the lengths of the divergent curves in M emanating from p and $|\alpha, \beta|$ is the chordal distance between two values in the Riemann sphere $\overline{\mathbb{C}}$.

Now we give an example to show that we cannot remove the part $|g(p), \widehat{g} \circ \Psi(p)|$ in (1):

EXAMPLE 2.6. For an arbitrarily given $\epsilon > 0$, we give an example of a family of minimal surfaces which shows that there is no positive constant C, not depending on the minimal surfaces, which satisfies the following condition:

$$|K_{ds^2}(p)|^{1/2} \cdot d(p) \leqslant C.$$

Consider Enneper surface $M \equiv \widehat{M}$ whose domain of definition is restricted to the disc of radius R. Namely, for the functions $f(z) \equiv \widehat{f}(z) \equiv 1$ and $g(z) \equiv \widehat{g}(z) = z$ on the disc $\Delta_R := \{z; |z| < R\}$, setting

$$x_1 := \mathrm{Re} \int_0^z f(1 - g^2)\, dz, \qquad x_2 := \mathrm{Re} \int_0^z \sqrt{-1} f(1 + g^2)\, dz,$$
$$x_3 := 2\mathrm{Re} \int_0^z fg\, dz,$$

we define the surface $x = (x_1, x_2, x_3) : \Delta_R \to \mathbb{R}^3$ in $\mathbb{R}^3$. Then, this is a minimal surface immersed in $\mathbb{R}^3$ whose Gauss map is the function g and whose metric is given by $ds^2 = (1 + |z|^2)^2 |dz|^2$. Consider the quantities $K(p)$ and $d(p)$ as in the main theorem at the point $p = 0$. We have

$$d(0) = \int_0^R (1 + x^2)\, dx = R + \frac{1}{3} R^3$$

and

$$|K(0)|^{1/2} = \frac{2|g'(0)|}{|f(0)|(1 + |g(0)|^2)^2} = 2.$$

So $|K(0)|^{1/2} d(0) = 2(R + (1/3)R^3)$, which converges to ∞ as R tends to ∞. Therefore, there is no positive constant C satisfying condition (1) without $|g(p), \widehat{g} \circ \Psi(p)|$ which does not depend on the minimal surfaces.

We also remark that the number $5+m$ in Theorem 2.5 is optimal because there exist the following examples.

EXAMPLE 2.7. For an even positive integer m, we take $m/2+1$ distinct points $p, \alpha_1, \ldots, \alpha_{m/2}$ in $\Delta_R \setminus \{0, \pm 1\}$. Let M be either the complex disk Δ_R punctured at $m+1$ distinct points $0, \alpha_1, \ldots, \alpha_{m/2}, 1/\alpha_1, \ldots, 1/\alpha_{m/2}$ or the universal covering of the punctured disk. We set that

$$\omega = \frac{dz}{z\prod_{i=1}^{m/2}(z-\alpha_i)(\alpha_i z - 1)}$$

and the map $g(z) = z$. In a similar manner, we set

$$\widehat{\omega}(=\omega) = \frac{dz}{z\prod_{i=1}^{m/2}(z-\alpha_i)(\alpha_i z - 1)}$$

and the map $\widehat{g} = 1/z$. We can easily show that the identity map $\Psi\colon M \to M$ is a conformal diffeomorphism and the metric $ds^2 = (1+|g|^2)^m|\omega|^2$ is complete. It is also easy to see that the maps g and $\hat{g}$ share the $m+4$ distinct values

$$0, \infty, 1, -1, \alpha_1, \ldots, \alpha_{m/2}, 1/\alpha_1, \ldots, 1/\alpha_{m/2}$$

and $g(p) \neq \widehat{g}(p)$. On the other hand, the Gaussian curvature K_{ds^2} of the metric

$$ds^2 = (1+|g|^2)^m|\omega|^2 = (1+|g|^2)^m|\omega_z|^2|dz|^2$$

is given by

$$K_{ds^2}(p) = -\frac{2m|g'_z|^2}{(1+|g|^2)^{m+2}|\omega_z|^2}(p) = -\frac{2m(p\prod_{i=1}^{m/2}(p-\alpha_i)(\alpha_i p - 1))^2}{(1+|p|^2)^{m+2}}.$$

Now for any a divergent curve Γ_p in M emanating from p, it must tend to one of the points $0, \alpha_1, \ldots, \alpha_{m/2}$, $1/\alpha_1, \ldots, 1/\alpha_{m/2}$ or boundary of Δ_R. Thus, we have

$$d(p) = \int_{\Gamma_p}(1+|g|^2)^{m/2}|\omega| = \int_{\Gamma_p}\frac{(1+|g|^2)^{m/2}dz}{z\prod_{i=1}^{m/2}(z-\alpha_i)(\alpha_i z - 1)} \sim \log R \to \infty$$

when $R \to \infty$.

These show that if g and $\hat{g}$ share only the $m+4$ distinct values, then we cannot show a constant C, depending on m and $\alpha_1, \ldots, \alpha_q$ but not the surface, such that for all $p \in M$, (1) is correct.

2.2 Unicity problem of the holomorphic maps

Applying Theorem 2.5, we get the following result on the unicity problem of the holomorphic maps on open Riemann surfaces.

THEOREM 2.8. [20, Theorem 2.9] *Let $M, \widehat{M}$ be two open Riemann surfaces with the conformal metrics*

$$ds^2 = (1 + |g|^2)^m |\omega|^2, \qquad d\widehat{s}^2 = (1 + |\widehat{g}|^2)^m |\widehat{\omega}|^2$$

where ω and $\widehat{\omega}$ are holomorphic 1-forms, g and $\widehat{g}$ are holomorphic maps into $\overline{\mathbb{C}}$ on M and $\widehat{M}$, respectively, and m is a positive integer. We assume that there exists a conformal diffeomorphism $\Psi\colon M \to \widehat{M}$ and $g, \widehat{g}$ are nonconstant. Suppose that there exist q distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $g^{-1}(\alpha_j) = (\widehat{g} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$. If $q \geqslant 5 + m$ and either ds^2 or $d\widehat{s}^2$ is complete, then $g \equiv \widehat{g} \circ \Psi$.

Proof of Theorem 2.8. Since ds^2 is complete, we may set $d(p) = \infty$ for all $p \in M$. Set $A = \{p \in M | g(p) - \widehat{g}(p) \neq 0\}$, then A is an open subset in M. By (1),

$$|K_{ds^2}(p)|^{1/2} \cdot |g(p), \widehat{g}(p)| = 0.$$

Thus, $K_{ds^2}(p) = 0$ for all $p \in A$. So we get $g'(p) = (\widehat{g}_\circ \Psi)'(p) = 0$ for all $p \in A$ by (9). By Identity Theorem (see [7, Theorem 1.11] for example), we get that $g'(p) = (\widehat{g}_\circ \Psi)'(p)$ for all $p \in M$. This implies that $g - \widehat{g}_\circ \Psi$ is a constant function on M. On the other hand, $g^{-1}(\alpha_j) = (\widehat{g} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$; we thus get $g \equiv \widehat{g}_\circ \Psi$. Theorem 2.8 is proved. □

REMARK 2.9. We note that the number $5 + m$ in Theorem 2.8 is also optimal (see [18]).

§3. Proof of the main theorem

We first recall the notion of chordal distance between two distinct values in the Riemann sphere $\overline{\mathbb{C}}$. For each $\alpha, \beta \in \overline{\mathbb{C}}$ we define

$$|\alpha, \beta| = \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2}\sqrt{1 + |\beta|^2}}$$

if $\alpha \neq \infty$ and $\beta \neq \infty$, and $|\alpha, \beta| = |\beta, \alpha| = 1/\sqrt{1 + |\alpha|^2}$ if $\beta = \infty$.

PROPOSITION 3.1. [10, Proposition 2.1] *Let f and $\widehat{f}$ be mutually distinct nonconstant meromorphic functions on a Riemann surface M and q distinct points $\alpha_1, \ldots, \alpha_q$ $(q > 4)$. Assume that $f^{-1}(\alpha_j) = \widehat{f}^{-1}(\alpha_j)$ $(1 \leqslant j \leqslant$*

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q). For $a_0 > 0$ and ϵ with $q-4 > q\epsilon > 0$, set

$$\lambda := \left(\prod_{j=1}^{q} |f, \alpha_j| \cdot \log\left(\frac{a_0}{|f, \alpha_j|^2}\right)\right)^{-1+\epsilon},$$

$$\widehat{\lambda} := \left(\prod_{j=1}^{q} |\widehat{f}, \alpha_j| \cdot \log\left(\frac{a_0}{|\widehat{f}, \alpha_j|^2}\right)\right)^{-1+\epsilon}$$

$$d\tau^2 := |f, \widehat{f}|^2 \lambda\widehat{\lambda} \cdot \frac{|f'|}{1+|f|^2} \cdot \frac{|\widehat{f}'|}{1+|\widehat{f}|^2} \cdot |dz|^2 \tag{2}$$

outside the set $E := \bigcup_{j=1}^{q} f^{-1}(\alpha_j)$ and $d\tau^2 = 0$ on E. Then, for a suitably chosen a_0, $d\tau^2$ is continuous on M and has strictly negative curvature on the set $\{d\tau^2 \neq 0\}$.

LEMMA 3.2. (See [1]) *If a continuous nonnegative function v on Δ_R is of class $\mathbb{C}^2$ on the set $\{z \in \Delta_R; v(z) > 0\}$ and satisfies the condition*

$$\Delta \log v \geqslant v^2,$$

then

$$v(z) \leqslant \frac{2R}{R^2 - |z|^2} \quad (z \in \Delta_R).$$

LEMMA 3.3. *Let f and $\widehat{f}$ be mutually distinct nonconstant meromorphic functions on a Riemann surface M satisfying the same assumption as in Proposition 3.1. Then, for the metric $d\tau^2$ defined by (2), there is a positive real number C such that*

$$d\tau^2 \leqslant C\left(\frac{2R}{R^2 - |z|^2}\right)^2 |dz|^2.$$

Proof. This is an immediate consequence of Proposition 3.1 and Lemma 3.2. □

LEMMA 3.4. [12, Lemma 1.6.7] *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface M. Then for every point $p \in M$, there is a holomorphic and locally biholomorphic map Φ of a disk (possibly with radius ∞) $\Delta_{R_0} := \{w : |w| < R_0\}$ $(0 < R_0 \leqslant \infty)$ onto an open neighborhood of p with $\Phi(0) = p$ such that Φ is a local isometry, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat) metric on Δ_{R_0}, and for some point a_0 with*

$|a_0|=1$, the Φ-image of the curve

$$L_{a_0} : w := a_0 \cdot s \quad (0 \leqslant s < R_0)$$

is divergent in M (i.e., for any compact set $K \subset M$, there exists an $s_0 < R_0$ such that the Φ-image of the curve $L_{a_0} : w := a_0 \cdot s$ $(s_0 \leqslant s < R_0)$ does not intersect K).

Proof of Theorem 2.5. For each holomorphic local coordinate z defined on a simply connected open set U of M, we can find a nowhere zero holomorphic function ζ_z such that

$$\begin{aligned} ds^2 &= |\zeta_z|^2 \Psi^*(d\widehat{s}^2) \\ \Rightarrow |h|^2(1+|g|^2)^m|dz|^2 &= |\zeta_z|^2|\widehat{h}\circ\Psi|^2(1+|\widehat{g}\circ\Psi|^2)^m|dz|^2 \\ \Rightarrow |h|(1+|g|^2)^{m/2} &= |\zeta_z||\widehat{h}\circ\Psi|(1+|\widehat{g}\circ\Psi|^2)^{m/2}. \end{aligned} \tag{3}$$

We denote the functions $\widehat{g}\circ\Psi, \widehat{h}\circ\Psi$ by $\widehat{g}, \widehat{h}$, respectively, for brevity. Therefore, by (3), for each holomorphic local coordinate z defined on a simply connected open set U, we can find a nowhere zero holomorphic function $k^2 = h\cdot\widehat{h}\cdot\zeta$ such that

$$\begin{aligned} ds^2 &= |h|^2(1+|g|^2)^m|dz|^2 = |\zeta|^2|\widehat{h}|^2(1+|\widehat{g}|^2)^m|dz|^2 \\ &= |k|^2(1+|g|^2)^{m/2}(1+|\widehat{g}|^2)^{m/2}|dz|^2. \end{aligned} \tag{4}$$

Taking a positive real number η with

$$\frac{q-4-m}{q} > \eta > \max\left\{\frac{q-4-m}{q+1}; \frac{q-4-2m}{q}\right\},$$

we set

$$\tau := \frac{m}{q-4-q\eta}.$$

Then

$$\frac{1}{2} < \tau < 1 \qquad \text{and} \qquad \frac{\tau}{1-\tau} > 1, \qquad \frac{\eta\tau}{1-\tau} > 1, \tag{5}$$

and define the pseudometric

$$d\sigma^2 := |k|^{2/(1-\tau)}\left(\frac{\prod_{j=1}^q(|g-\alpha_j||\widehat{g}-\alpha_j|)^{1-\eta}}{|g-\widehat{g}|^2|g'||\widehat{g}'|\prod_{j=1}^q(1+|\alpha_j|^2)^{1-\eta}}\right)^{\tau/(1-\tau)}|dz|^2, \tag{6}$$

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which does not depend on a choice of holomorphic local coordinate z and so well-defined on $M_1 = M - E$, where

$$E := \{z \in M; g'(z) = 0 \text{ or } \widehat{g}'(z) = 0 \text{ or } g(z) = \widehat{g}(z)\}.$$

Take an arbitrary point p in M_1. Using the fact that $d\sigma^2$ is flat on M_1, by Lemma 3.4, there exists a local isometry Φ satisfying $\Phi(0) = p$ from a disk $\Delta_R = \{z \in \mathbb{C}; |z| < R\}$ $(0 < R \leqslant \infty)$ with the standard metric ds^2_{Euc} onto an open neighborhood of p in M_1 with the metric $d\sigma^2$ such that, for a point w_0 with $|w_0| = 1$, the $\Phi-$image Γ_{w_0} of the curve $L_{w_0} = \{w := w_0 s; 0 < s < R\}$ is divergent in M_1. For brevity, we denote the functions $g \circ \Phi, \widehat{g} \circ \Phi$ on Δ_R by $g, \widehat{g}$, respectively, in the following. On the other hand, from Lemma 3.3, we have

$$|g, \widehat{g}|^2 \lambda \widehat{\lambda} \cdot \frac{|g'|}{1 + |g|^2} \cdot \frac{|\widehat{g}'|}{1 + |\widehat{g}|^2} \leqslant C_0 \left(\frac{2R}{R^2 - |z|^2} \right)^2$$

for some positive real number C_0.

This implies that

$$R^2 \leqslant \frac{4C_0(1 + |g(0)|^2)(1 + |\widehat{g}(0)|^2)}{|g(0), \widehat{g}(0)|^2 \lambda(0) \widehat{\lambda}(0) |g'(0)||\widehat{g}'(0)|} < \infty. \tag{7}$$

Hence,

$$L_{d\sigma}(\Gamma_{w_0}) = \int_{\Gamma_{w_0}} d\sigma = R < \infty$$

where $L_{d\sigma}(\Gamma_{w_0})$ denotes the length of Γ_{w_0} with respect to the metric $d\sigma^2$.

Now we prove that Γ_{w_0} is divergent in M. Indeed, if not, then Γ_{w_0} must tend to a point $p_0 \in E$ because Γ_{w_0} is divergent in M_1 and $L_{d\sigma}(\Gamma_{w_0}) < \infty$. Then we consider the following two possible cases:

Case 1. $g(p_0) = \widehat{g}(p_0)$.

If $g(p_0) = \alpha_j$ for some j, then $g(p_0) = \widehat{g}(p_0) = \alpha_j$. Combining with $g'(p_0) = (g - \alpha_j)'(p_0)$ and $\widehat{g}'(p_0) = (\widehat{g} - \alpha_j)'(p_0)$, the function

$$\lambda(z) = |k|^{2/(1-\tau)} \left(\frac{\prod_{j=1}^q (|g - \alpha_j||\widehat{g} - \alpha_j|)^{1-\eta}}{|g - \widehat{g}|^2 |g'||\widehat{g}'| \prod_{j=1}^q (1 + |\alpha_j|^2)^{1-\eta}} \right)^{\tau/(1-\tau)}$$

has a pole of order at least $2\eta\tau/(1 - \tau)$ at p_0. Otherwise, the function $\lambda(z)$ has a pole of order at least $2\tau/(1 - \tau)$ at p_0. Taking a local complex coordinate ζ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write the metric

$d\sigma^2$ as

$$d\sigma^2 = |\zeta|^{-2u}\gamma|d\zeta|^2$$

with some positive function γ and $u \geqslant \min\{\eta\tau/(1-\tau); \tau/(1-\tau)\} > 1$ by (5); we thus have the following:

$$R = \int_{\Gamma_{a_0}} d\sigma > C_1 \int_{\Gamma_{a_0}} |d\zeta|/|\zeta|^u = \infty,$$

for some positive constant C_1. This contradicts that R is finite.

Case 2. $g'(p_0)\widehat{g}'(p_0) = 0$.

Without loss of generality, we may assume that $g'(p_0) = 0$. From (9), we have $\widehat{g}'(p_0) = 0$. Taking a local complex coordinate $\zeta := g'$ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write the metric $d\sigma^2$ as

$$d\sigma^2 = |\zeta|^{-2\tau/(1-\tau)}\gamma|d\zeta|^2$$

with some positive function γ. Since $\tau/(1-\tau) > 1$, we have

$$R = \int_{\Gamma_{a_0}} d\sigma > C_1 \int_{\Gamma_{a_0}} |d\zeta|/|\zeta|^{\tau/(1-\tau)} = \infty,$$

for some positive constant C_2. This also contradicts that R is finite.

So we get that Γ_{w_0} is divergent in M.

On the other hand, since Φ is a local isometric, we may take the coordinate w as a holomorphic local coordinate on M_1 and we may write $d\sigma^2 = |dw|^2$. By (6), we obtain

$$|k|^2 = \left(\frac{|g-\widehat{g}|^2|g'||\widehat{g}'|\prod_{j=1}^{q}(1+|\alpha_j|^2)^{1-\eta}}{\prod_{j=1}^{q}(|g-\alpha_j||\widehat{g}-\alpha_j|)^{1-\eta}}\right)^{\tau}.$$

According to (4), we have

$$\begin{aligned} ds^2 &= |k|^2(1+|g|^2)^{m/2}(1+|\widehat{g}|^2)^{m/2}|dw|^2 \\ &= \left(\frac{|g-\widehat{g}|^2|g'||\widehat{g}'|(1+|g|^2)^{m/2\tau}(1+|\widehat{g}|^2)^{m/2\tau}\prod_{j=1}^{q}(1+|\alpha_j|^2)^{1-\eta}}{\prod_{j=1}^{q}(|g-\alpha_j||\widehat{g}-\alpha_j|)^{1-\eta}}\right)^{\tau}|dw|^2 \\ &= \left(\mu^2\prod_{j=1}^{q}(|g,\alpha_j|\cdot|\widehat{g},\alpha_j|)^{\epsilon}\cdot\left(\log\frac{a_0}{|g,\alpha_j|^2}\log\frac{a_0}{|\widehat{g},\alpha_j|^2}\right)^{1-\epsilon}\right)^{\tau}|dw|^2, \end{aligned}$$

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where μ is the function with $d\tau^2 = \mu^2|dw|^2$ as in (2) and $\epsilon := \eta/2$. Since the function $x^\epsilon \log^{1-\epsilon}(k/x^2)(0 < x \leqslant 1)$ is bounded, we obtain that

$$ds^2 \leqslant C_3\left(|g,\widehat{g}|^2\lambda\widehat{\lambda}\cdot\frac{|g'|}{1+|g|^2}\cdot\frac{|\widehat{g}'|}{1+|\widehat{g}|^2}\right)^\tau\cdot|dw|^2$$

for some positive constant C_3. Moreover, using Lemma 3.3, we have

$$|g,\widehat{g}|^2\lambda\widehat{\lambda}\cdot\frac{|g'|}{1+|g|^2}\cdot\frac{|\widehat{g}'|}{1+|\widehat{g}|^2}\leqslant C_4\cdot\left(\frac{2R}{R^2-|w|^2}\right)^2.$$

Thus, we obtain

$$\Phi^* ds \leqslant C_5\cdot\left(\frac{2R}{R^2-|w|^2}\right)^\tau|dw|$$

where C_5 is a positive real number. This yields that

$$\begin{aligned} d(p) = d_{\Gamma_{a_0}} &\leqslant \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}}\Phi^* ds \leqslant C_5\cdot\int_{L_{a_0}}\left(\frac{2R}{R^2-|w|^2}\right)^\tau|dw| \\ &= C_5\int_0^R\left(\frac{2R}{R^2-x^2}\right)^\tau dx = C_6\cdot R^{1-\tau} \end{aligned}$$

because $0 < \tau < 1$, and $d_{\Gamma_{a_0}}$ denotes the distance of the divergent curve Γ_{a_0} in M. Combining with (7), we obtain

$$d(p) \leqslant C_7\left(\frac{(1+|g(0)|^2)(1+|\widehat{g}(0)|^2)}{|g(0),\widehat{g}(0)|^2\lambda(0)\widehat{\lambda}(0)|g'(0)||\widehat{g}'(0)|}\right)^{(1-\tau)/2}. \tag{8}$$

On the other hand, the Gaussian curvature K_{ds^2} of the metric $ds^2 = |h|^2(1+|g|^2)^m|dz|^2$ is given by

$$\begin{aligned} K_{ds^2} &= -\frac{2m|g'_z|^2}{|h|^2(1+|g|^2)^{m+2}} = -\frac{2m|\widehat{g}'_z|^2}{|\zeta\cdot\widehat{h}|^2(1+|\widehat{g}|^2)^{m+2}} \quad (9)\\ &= -\frac{2m|g'||\widehat{g}'|}{|k|^2(1+|g|^2)^{\frac{m+2}{2}}(1+|\widehat{g}|^2)^{\frac{m+2}{2}}} \\ &= -\frac{2m|g'||\widehat{g}'|}{\left(\frac{|g-\widehat{g}|^2|g'||\widehat{g}'|\prod_{j=1}^q(1+|\alpha_j|^2)^{1-\eta}}{\prod_{j=1}^q(|g-\alpha_j||\widehat{g}-\alpha_j|)^{1-\eta}}\right)^\tau(1+|g|^2)^{\frac{m+2}{2}}(1+|\widehat{g}|^2)^{\frac{m+2}{2}}}. \quad (10) \end{aligned}$$

Combining with (8) and (10), we get

$$|K_{ds^2}(p)|d(p)^2|g(p),\widehat{g}(\Psi(p))|^2 \leqslant \frac{C_4}{(\lambda(0)\widehat{\lambda}(0))^{1-\tau}}.$$

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Using the property that the function $x\log(k/x^2)(0<x\leqslant 1)$ is bounded, we have

$$|K_{ds^2}(p)|^{1/2}d(p)|g(p),\widehat{g}(\Psi(p))|\leqslant C$$

with a constant C depending on $\alpha_1,\ldots,\alpha_q$ but not the surface. The main theorem is proved. ∎

§4. Applications of the main results

4.1 Minimal surfaces in $\mathbb{R}^3$

We first introduce some basic background on minimal surface. Let $x=(x_1,x_2,x_3):M\to\mathbb{R}^3$ be a nonflat minimal surface in $\mathbb{R}^3$, or more precisely, a connected oriented minimal surface in $\mathbb{R}^3$. By definition, the Gauss map G of M is the map which maps each point $p\in M$ to the unit normal vector $G(p)\in\mathbb{S}^2$ of M at p. Instead of G, we study the map $g:=\pi\circ G:M\to\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}=\mathbb{P}^1(\mathbb{C})$ for the stereographic projection π of $\mathbb{S}^2$ onto $\mathbb{P}^1(\mathbb{C})$. Therefore, we also tell that g is the Gauss map of M. The surface M is canonically considered as an open Riemann surface with a conformal metric and g is a meromorphic function on M because of the minimal property of M.

Set $\phi_i:=\partial x/\partial z$ $(i=1,2,3)$ and $h:=\phi_1-\sqrt{-1}\phi_2$. Then, the Gauss map $g:M\to\mathbb{P}^1(\mathbb{C})$ is given by

$$g=\frac{\phi_3}{\phi_1-\sqrt{-1}\phi_2},$$

and the metric on M induced from $\mathbb{R}^3$ is given by

$$ds^2=|h|^2(1+|g|^2)^2|dz|^2. \tag{11}$$

Using Theorem 2.5 for $m=2$, we get the following theorem.

THEOREM 4.1. *Let $X:M\to\mathbb{R}^3$ and $\widehat{X}:\widehat{M}\to\mathbb{R}^3$ be two nonflat minimal surfaces, and assume that there exists a conformal diffeomorphism $\Psi:M\to\widehat{M}$. Let g and $\widehat{g}$ be the Gauss maps of $X(M)$ and $\widehat{X}(\widehat{M})$, respectively. Suppose that there are $q\geqslant 7(=5+2)$ distinct points $\alpha_1,\ldots,\alpha_q$ in $\overline{\mathbb{C}}$ such that $g^{-1}(\alpha_j)=(\widehat{g}\circ\Psi)^{-1}(\alpha_j)$ for every $1\leqslant j\leqslant q$. Then there exists a constant C, depending on $\alpha_1,\ldots,\alpha_q$ but not the surface, such that for all $p\in M$, we have*

$$|K_{ds^2}(p)|^{1/2}\cdot d(p)\cdot|g(p),\widehat{g}(p)|\leqslant C\ (p\in M),$$

where $K_{ds^2}(p)$ is the Gaussian curvature of the metric ds^2 at p and $d(p)$ is the geodesic distance from p to the boundary of M.

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Repeating the proof of Theorem 2.8, we can see that Theorem 1.2 is a corollary of Theorem 4.1.

4.2 The constant mean curvature one surfaces in $\mathbf{H}^3$

The hyperbolic three-space $\mathbf{H}^3$ is the simply connected Riemannian three-manifold with constant sectional curvature -1, which is represented as

$$\mathbf{H}^3 = SL(2,\mathbb{C})/SU(2) = \{aa^*; a \in SL(2,C)\}(a^* := {}^t\,\overline{a}).$$

As an analogy of the Enneper–Weierstrass representation formula in minimal surface theory, we have the representation formula for constant mean curvature one (CMC-1, for short) surfaces in $\mathbf{H}^3$ as follows:

THEOREM 4.2. [2, 39] *Let $\widetilde{M}$ be a simply connected Riemann surface with a base point $z_0 \in \widetilde{M}$ and let (g, ω) be a pair consisting of a meromorphic function and a holomorphic 1-form on $\widetilde{M}$ such that*

$$ds^2 = (1 + |g|^2)^2|\omega|^2 \tag{12}$$

gives a (positive definite) Riemannian metric on $\widetilde{M}$. Take a holomorphic immersion $F = (F_{ij}) : \widetilde{M} \to SL(2,\mathbb{C})$ satisfying $F(z_0) = \mathrm{id}$ and

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega. \tag{13}$$

Then $f : \widetilde{M} \to \mathbf{H}^3$ defined by

$$f = FF^* \tag{14}$$

is a CMC-1 surface and the induced metric of f is ds^2. Moreover, the second fundamental form h and the Hopf differential Q of f are given by

$$h = -Q - \overline{Q} + ds^2, \qquad Q = \omega dg.$$

Conversely, for any CMC-1 surface $f : \widetilde{M} \to \mathbf{H}^3$, there exist a meromorphic function g and a holomorphic 1-form ω on $\widetilde{M}$ such that the induced metric of f is given by (12), and (14) holds, where the map $F : \widetilde{M} \to SL(2,\mathbb{C})$ is a holomorphic null ("null" means that $\det(F^{-1}dF) = 0$) immersion satisfying (13).

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Following the terminology of [39], g is called a secondary Gauss map of f. The pair (g,ω) is called *Weierstrass data* of f. Let $f:M\to\mathbf{H}^3$ be a CMC-1 surface on a (not necessarily simply connected) Riemann surface M. Then the map F is defined only on its universal covering surface $\widetilde{M}$. Thus, although the pair (ω,g) is not single-valued on M, the hyperbolic Gauss map of f defined by

$$G=\frac{dF_{11}}{dF_{21}}=\frac{dF_{12}}{dF_{22}},\quad \text{where } F(z)=\begin{pmatrix}F_{11}(z) & F_{12}(z)\\ F_{21}(z) & F_{22}(z)\end{pmatrix}$$

is a single-valued meromorphic function on M. Because we can identify the ideal boundary S^2_∞ of $\mathbf{H}^3$ with the Riemann sphere $\overline{\mathbb{C}}$, the hyperbolic Gauss map G seems to send each $p\in M$ to the point $G(p)$ at S^2_∞ reached by the oriented normal geodesics emanating from the surface [2]. The inverse matrix F^{-1} is also a holomorphic null immersion and produce a new CMC-1 surface $f^\sharp:=F^{-1}(F^{-1})^*:\widetilde{M}\to\mathbf{H}^3$ which is called the dual of f [40]. Then, the Weierstrass data $(g^\sharp,\omega^\sharp)$, the Hopf differential $Q^\sharp$ and the hyperbolic Gauss map $G^\sharp$ of $f^\sharp$ are given by following formulas:

$$g^\sharp=G,\qquad \omega^\sharp=-\frac{Q}{dG},\qquad Q^\sharp=-Q,\qquad G^\sharp=g. \tag{15}$$

By Theorem 4.2 and (15), the induced metric $ds^{2\sharp}$ of $f^\sharp$ is given by

$$ds^{2\sharp}=(1+|g^\sharp|^2)^2|\omega^\sharp|^2=(1+|G|^2)^2|\frac{Q}{dG}|^2. \tag{16}$$

We call the metric $ds^{2\sharp}$ the dual metric of f. The relationship between the metric ds^2 and the dual metric $ds^{2\sharp}$ is given by the following:

THEOREM 4.3. [40, 43] *The metric ds^2 is complete (resp. nondegenerate) if and only if the dual metric $ds^{2\sharp}$ is complete (resp. nondegenerate).*

Applying Theorem 2.5 to the dual metric $ds^{2\sharp}$, we get the following theorem.

THEOREM 4.4. *Let $f:M\to\mathbf{H}^3$, $\widehat{f}:\widehat{M}\to\mathbf{H}^3$ be two nonflat CMC-1 surfaces, and assume that there exists a conformal diffeomorphism $\Psi:M\to\widehat{M}$. Let $G:M\to\overline{\mathbb{C}}$ and $\widehat{G}:\widehat{M}\to\overline{\mathbb{C}}$ be the hyperbolic Gauss maps of $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. Suppose that there exist $q\geqslant 7(=5+2)$ distinct values $\alpha_1,\ldots,\alpha_q\in\overline{\mathbb{C}}$ such that $G^{-1}(\alpha_j)=(\widehat{G}\circ\Psi)^{-1}(\alpha_j)$ $(j=1,\ldots,q)$. Then there exists a constant C, depending on $\alpha_1,\ldots,\alpha_q$ but not the surface, such that for all $p\in M$, we have*

$$|K_{ds^{2\sharp}}(p)|^{1/2}\cdot d(p)\cdot|G(p),\widehat{G}\circ\Psi(p)|\leqslant C,$$

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where $K_{ds^{2\sharp}}(p)$ is the Gaussian curvature of the metric $ds^{2\sharp}$ at p and $d(p)$ is the geodesic distance from p to the boundary of M.

As a corollary of Theorem 4.4 or Theorem 2.8, we have the following unicity theorem:

THEOREM 4.5. [20, Theorem 4.12] *Let $f : M \to \mathbf{H}^3$, $\widehat{f} : \widehat{M} \to \mathbf{H}^3$ be two nonflat CMC-1 surfaces, and assume that there exists a conformal diffeomorphism $\Psi : M \to \widehat{M}$. Let $G : M \to \overline{\mathbb{C}}$ and $\widehat{G} : \widehat{M} \to \overline{\mathbb{C}}$ be the hyperbolic Gauss maps of $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. If $G \not\equiv \widehat{G} \circ \Psi$ and either $f(M)$ or $\widehat{f}(\widehat{M})$ is complete, then G and $\widehat{G} \circ \Psi$ share at most $6(= 2 + 4)$ distinct values.*

4.3 Maximal surfaces in the Lorentz–Minkowski three-space $\mathbf{R}^3_1$

As introduced by Umehara and Yamada [41], maxfaces are maximal surfaces with some admissible singularities. It should be remarked that maxfaces, nonbranched generalized maximal surfaces in the sense of [6] and nonbranched generalized maximal maps in the sense of [16] are all the same class of maximal surfaces. The Lorentz–Minkowski three-space $\mathbf{R}^3_1$ is the affine three-space $\mathbb{R}^3$ with the inner product

$$\langle , \rangle = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

where (x^1, x^2, x^3) is the canonical coordinate system of $\mathbb{R}^3$. We consider a fibration

$$p_L : (\xi^1, \xi^2, \xi^3)(\in \mathbb{C}^3) \to \mathrm{Re}(-\sqrt{-1}\xi^1, \xi^2, \xi^3) \in \mathbf{R}^3_1.$$

The projection of null holomorphic immersions into $\mathbf{R}^3_1$ by p_L gives maxfaces. Here, a holomorphic map $F = (F_1, F_2, F_3) : M \to \mathbb{C}^3$ is said to be null if $\{(F_1)'_z\}^2 + \{(F_2)'_z\}^2 + \{(F_3)'_z\}^2$ vanishes identically, where $' = d/dz$ denotes the derivative with respect to a local complex coordinate z of M. Maxfaces in $\mathbf{R}^3_1$ have some properties closely related to minimal surfaces in $\mathbb{R}^3$. The following result shows that a maxface can be represented by a formula, which is an analogue of the Enneper–Weierstrass representation formula for a minimal surface (see also [24]).

THEOREM 4.6. [41, Theorem 2.6] *Let M be a Riemann surface and (g, ω) a pair consisting of a meromorphic function and a holomorphic 1-form on M such that*

$$d\sigma^2 := (1 + |g|^2)^2|\omega|^2 \tag{17}$$

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gives a (positive definite) Riemannian metric on M, and $|g|$ is not identically 1. Assume that

$$\mathrm{Re}\int_{\gamma}(-2g, 1+g^2, \sqrt{-1}(1-g^2))\omega = 0$$

for all loops γ in M. Then,

$$f = \mathrm{Re}\int_{z_0}^{z}(-2g, 1+g^2, \sqrt{-1}(1-g^2))\omega \tag{18}$$

is well-defined on M and gives a maxface in $\mathbf{R}^3_1$, where $z_0 \in M$ is a base point. Moreover, all maxfaces are obtained in this manner. The induced metric $ds^2 := f^*\langle , \rangle$ is given by $ds^2 = (1-|g|^2)^2|\omega|^2$, and the point $p \in M$ is a singular point of f if and only if $|g(p)| = 1$.

We call g the *Lorentzian Gauss map* of f. If f has no singularities, then g coincides with the composition of the Gauss map (i.e., (Lorentzian) unit normal vector) $n : M \to \mathbf{H}^2_{\pm}$ into the upper or lower connected component of the two-sheet hyperboloid $\mathbf{H}^2_{\pm} = \mathbf{H}^2_{+} \cup \mathbf{H}^2_{-}$ in $\mathbf{R}^3_1$, where

$$\mathbf{H}^2_{+} := \{n = (n^1, n^2, n^3) \in \mathbf{R}^3_1; \langle n, n\rangle = -1, n^1 > 0\},$$
$$\mathbf{H}^2_{-} := \{n = (n^1, n^2, n^3) \in \mathbf{R}^3_1; \langle n, n\rangle = -1, n^1 < 0\},$$

and the stereographic projection from the north pole $(1, 0, 0)$ of the hyperboloid onto the Riemann sphere $\overline{\mathbb{C}}$ (see [41, Section 1] for more details). A maxface is said to be *weakly complete* if the metric $d\sigma^2$ as in (17) is complete. We also remark that $(1/2)\, d\sigma^2$ coincides with the pull-back of the standard metric on $\mathbb{C}^3$ by the null holomorphic immersion of f (see [41, Section 2]).

Applying Theorem 2.5 to the metric $d\sigma^2$, we can get the following theorem.

THEOREM 4.7. *Let $f : M \to \mathbf{R}^3_1$, $\widehat{f} : \widehat{M} \to \mathbf{R}^3_1$ be two nonflat maxfaces, and assume that there exists a conformal diffeomorphism $\Psi : M \to \widehat{M}$. Let $g : M \to \overline{\mathbb{C}}$ and $\widehat{g} : \widehat{M} \to \overline{\mathbb{C}}$ be the Lorentzian Gauss maps of $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. Suppose that there exist $q \geqslant 7 (= 5 + 2)$ distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $g^{-1}(\alpha_j) = (\widehat{g} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$. Then there exists a constant C, depending on $\alpha_1, \ldots, \alpha_q$ but not the surface, such that for all $p \in M$, we have*

$$|K_{d\sigma^2}(p)|^{1/2} \cdot d(p) \cdot |g(p), \widehat{g} \circ \Psi(p)| \leqslant C,$$

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where $K_{d\sigma^2}(p)$ *is the Gaussian curvature of the metric* $d\sigma^2$ *at* p *and* $d(p)$ *is the geodesic distance from* p *to the boundary of* M.

Using the same proof of Theorem 2.8 and Theorem 4.7, we can get the following result:

THEOREM 4.8. [20, Theorem 4.18] *Let* $f: M \to \mathbf{R}^3_1$, $\widehat{f}: \widehat{M} \to \mathbf{R}^3_1$ *be two nonflat maxfaces, and assume that there exists a conformal diffeomorphism* $\Psi: M \to \widehat{M}$. *Let* $g: M \to \overline{\mathbb{C}}$ *and* $\widehat{g}: \widehat{M} \to \overline{\mathbb{C}}$ *be the Lorentzian Gauss maps of* $f(M)$ *and* $\widehat{f}(\widehat{M})$, *respectively. If* $g \not\equiv \widehat{g} \circ \Psi$ *and either* $f(M)$ *or* $\widehat{f}(\widehat{M})$ *is weakly complete, then* g *and* $\widehat{g} \circ \Psi$ *share at most* $6 (= 2 + 4)$ *distinct values.*

4.4 Improper affine spheres in $\mathbb{R}^3$

Improper affine spheres in the affine three-space $\mathbb{R}^3$ also have similar properties to that of minimal surfaces in Euclidean three-space (e.g., see [3]). In 2005, Martínez [31] discovered the correspondence between improper affine spheres and smooth special Lagrangian immersions in the complex two-space $\mathbb{C}^2$ and introduced the notion of improper affine fronts, that is, a class of (locally strongly convex) improper affine spheres with some admissible singularities in $\mathbb{R}^3$. We note that this class is called improper affine maps in [31], but here, we call this class improper affine fronts, following Kawakami and Nakajo [23]; the reason is that all of the improper affine maps are wave fronts in $\mathbb{R}^3$ [32, 42]. We also can find more differential geometry properties of wave fronts in [38]. Moreover, Martínez also gave the following holomorphic representation for this class.

THEOREM 4.9. [31, Theorem 3] *Let* M *be a Riemann surface and* (F, G) *a pair of holomorphic functions on* M *such that* $\mathrm{Re}(F dG)$ *is exact and* $|dF|^2 + |dG|^2$ *is positive definite. Then the induced map* $f: M \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ *given by*

$$f := \left(G + \overline{F}; \frac{|G|^2 - |F|^2}{2} + \mathrm{Re}\left(GF - 2 \int F dG \right) \right)$$

is an improper affine front. Conversely, any improper affine front is given in this way. Moreover, we set $x := G + \bar{F}$ *and* $n := \overline{F} - G$. *Then,* $L_f := x + \sqrt{-1} n : M \to \mathbb{C}^2$ *is a special Lagrangian immersion whose induced metric* $d\tau^2$ *from* $\mathbb{C}^2$ *is given by*

$$d\tau^2 = 2(|dF|^2 + |dG|^2).$$

In addition, the affine metric h *of* f *is expressed as* $h := |dG|^2 - |dF|^2$, *and the singular points of* f *correspond to the points where* $|dF| = |dG|$.

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We remark that Nakajo [32] constructed a representation formula for indefinite improper affine spheres with some admissible singularities. The nontrivial part of the Gauss map of $L_f : M \to \mathbb{C}^2 \cong \mathbb{R}^4$ (see [4]) is the meromorphic function $\nu : M \to \overline{\mathbb{C}}$ given by $\nu := dF/dG$, which is called the *Lagrangian Gauss map* of f. An improper affine front is said to be *weakly complete* if the induced metric $d\tau^2$ is complete. On the other hand, we have

$$d\tau^2 = 2(|dF|^2 + |dG|^2) = 2(1 + |\nu|^2)|dG|^2.$$

Now, applying Theorem 2.5 to the metric $d\tau^2$, we can get the following theorem.

THEOREM 4.10. *Let $f : M \to \mathbb{R}^3$, $\widehat{f} : \widehat{M} \to \mathbb{R}^3$ be two improper affine fronts, and assume that there exists a conformal diffeomorphism $\Psi : M \to \widehat{M}$. Let $\nu : M \to \overline{\mathbb{C}}$ and $\widehat{\nu} : \widehat{M} \to \overline{\mathbb{C}}$ be the Lagrangian Gauss maps of $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. Suppose that there exist $q \geqslant 6(= 5 + 1)$ distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $\nu^{-1}(\alpha_j) = (\widehat{\nu} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$. Then there exists a constant C, depending on $\alpha_1, \ldots, \alpha_q$ but not the surface, such that for all $p \in M$, we have*

$$|K_{d\tau^2}(p)|^{1/2} \cdot d(p) \cdot |\nu(p), \widehat{\nu} \circ \Psi(p)| \leqslant C,$$

where $K_{d\tau^2}(p)$ is the Gaussian curvature of the metric $d\tau^2$ at p and $d(p)$ is the geodesic distance from p to the boundary of M.

As a corollary of Theorem 4.10 or Theorem 2.8, we provide the following unicity theorem for the Lagrangian Gauss maps of weakly complete improper affine fronts in $\mathbb{R}^3$.

THEOREM 4.11. [20, Theorem 4.24] *Let $f : M \to \mathbb{R}^3$, $\widehat{f} : \widehat{M} \to \mathbb{R}^3$ be two improper affine fronts, and assume that there exists a conformal diffeomorphism $\Psi : M \to \widehat{M}$. Let $\nu : M \to \overline{\mathbb{C}}$ and $\widehat{\nu} : \widehat{M} \to \overline{\mathbb{C}}$ be the Lagrangian Gauss maps of $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. Suppose that there exist $q \geqslant 6(= 5 + 1)$ distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $\nu^{-1}(\alpha_j) = (\widehat{\nu} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$ and either $f(M)$ or $\widehat{f}(\widehat{M})$ is weakly complete, then either $\nu \equiv \widehat{\nu} \circ \Psi$ or ν and $\widehat{\nu}$ are both constant, that is, $f(M)$ and $\widehat{f}(\widehat{M})$ are both elliptic paraboloids.*

4.5 Flat fronts in $\mathbf{H}^3$

Flat fronts in $\mathbf{H}^3$ are flat surfaces in $\mathbf{H}^3$ with some admissible singularities (see [26, 29] for more details). Let M be a simply connected Riemann surface

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and let $\mathcal{L}: M \to SL(2,\mathbb{C})$ be a holomorphic Legendrian immersion. The projection

$$f := \mathcal{L}\mathcal{L}^* : M \to \mathbf{H}^3$$

gives a flat front in $\mathbf{H}^3$. We call $\mathcal{L}$ the holomorphic lift of f. Since $\mathcal{L}$ is a holomorphic Legendrian map, $\mathcal{L}^{-1}d\mathcal{L}$ is off-diagonal (see [13, 28, 29]). Now, if we set

$$\mathcal{L}^{-1}d\mathcal{L} = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix},$$

then the pull-back of the canonical Hermitian metric of $SL(2,\mathbb{C})$ by $\mathcal{L}$ is represented as

$$ds^2_{\mathcal{L}} := |\omega|^2 + |\theta|^2$$

for holomorphic 1-forms ω and θ on M. A flat front f is said to be *weakly complete* if the metric $ds^2_{\mathcal{L}}$ is complete (see [27, 42]). We define a meromorphic function on M by the ratio of canonical forms

$$\rho := \frac{\theta}{\omega}.$$

We note that a point $p \in M$ is a singular point of f if and only if $|\rho(p)| = 1$ [25]. Now we have

$$ds^2_{\mathcal{L}} = |\omega|^2 + |\theta|^2 = (1 + |\rho|^2)|\omega|^2.$$

Applying Theorem 2.5 to the metric $ds^2_{\mathcal{L}}$, we can get the following result.

THEOREM 4.12. *Let $f : M \to \mathbf{H}^3$, $\widehat{f} : \widehat{M} \to \mathbf{H}^3$ be two flat fronts on simply connected Riemann surfaces, and assume that there exists a conformal diffeomorphism $\Psi : M \to \widehat{M}$. Let $\rho : M \to \overline{\mathbb{C}}$ and $\widehat{\rho} : \widehat{M} \to \overline{\mathbb{C}}$ be the ratios of canonical forms $f(M)$ and $\widehat{f}(\widehat{M})$, respectively. Suppose that there exist $q \geqslant 6 (= 5 + 1)$ distinct values $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ such that $\rho^{-1}(\alpha_j) = (\widehat{\rho} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$. Then there exists a constant C, depending on $\alpha_1, \ldots, \alpha_q$ but not the surface, such that for all $p \in M$, we have*

$$|K_{ds^2_{\mathcal{L}}}(p)|^{1/2} \cdot d(p) \cdot |\rho(p), \widehat{\rho} \circ \Psi(p)| \leqslant C,$$

where $K_{ds^2_{\mathcal{L}}}(p)$ is the Gaussian curvature of the metric $ds^2_{\mathcal{L}}$ at p and $d(p)$ is the geodesic distance from p to the boundary of M.

By applying Theorem 4.12, we can get the following unicity theorem for the ratios of canonical forms of weakly complete flat fronts in $\mathbf{H}^3$.

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THEOREM 4.13. [20, Theorem 4.29] *Let* $f: M \to \mathbf{H}^3$, $\widehat{f}: \widehat{M} \to \mathbf{H}^3$ *be two flat fronts on simply connected Riemann surfaces, and assume that there exists a conformal diffeomorphism* $\Psi: M \to \widehat{M}$. *Let* $\rho: M \to \overline{\mathbb{C}}$ *and* $\widehat{\rho}: \widehat{M} \to \overline{\mathbb{C}}$ *be the ratios of canonical forms* $f(M)$ *and* $\widehat{f}(\widehat{M})$, *respectively. Suppose that there exist* $q \geqslant 6(=5+1)$ *distinct values* $\alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}}$ *such that* $\rho^{-1}(\alpha_j) = (\widehat{\rho} \circ \Psi)^{-1}(\alpha_j)$ $(j = 1, \ldots, q)$ *and either* $f(M)$ *or* $\widehat{f}(\widehat{M})$ *is weakly complete, then either* $\rho \equiv \widehat{\rho} \circ \Psi$ *or* ρ *and* $\widehat{\rho}$ *are both constant.*

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