have groups {1, 9, 11, 19 }, {1, 9, 11, 19, 21, 29, 31, 39}, and {1, 9, 11, 19, 21, 29, 31, 39, 41, 49} under multiplication modulo 20, modulo 40, and modulo 50, respectively; but {1, 9, 11, 19, 21, 29} is not a group under multiplication modulo $30 = 3 \times 10$ since 3 is not a divisor of 10.

We have discussed some ways to construct multiplicative groups in modular arithmetic. Are there other different ways to construct such groups? One can try to do the following exercise.

Exercise: Find new constructions for multiplicative groups in modular arithmetic, which maybe contain elements of sequences other than geometric sequences or arithmetic sequences.

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103.34 Indefinite integration by parts as a translation of functions

Introduction

We propose a geometrical representation of the formula of integration by parts as a translation in the vector space of functions to gain some understanding of the role played by integration constants.

It can serve two purposes: firstly, offering a visual model, possibly easier to handle by students than algebraic abstraction; secondly, showing how the interplay between concepts learnt under different subjects, elementary calculus and basic vector algebra and geometry, can help solving a problem.

Geometrical representation of indefinite integration by parts

We know that the sum of two functions is again a function, and the product of a function by a constant is a function.

Since such properties also apply to vectors, and are distinctive of them, we will see functions as elements belonging to an infinite-dimensional vector space, say $\mathcal{S}$. Indefinite integration can be seen as a linear mapping in $\mathcal{S}$. That is, if f and g are functions, and h and k are constants, we know that:

$$\int \left[hf(x) + kg(x)\right] dx = h\int f(x)\,dx + k\int g(x)\,dx.$$

On the other hand we know that indefinite integration does not yield a single function, because, if F is a completely determined primitive of f, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant [1, Section 5.6]. Geometrically this implies that an indefinite integral is not a single point in the vector space $\mathcal{S}$, but a line. Let us name $\mathbf{1}$ the vector representing the function that maps any number x to 1, and $\mathbf{F}$ the vector representing the function F. Then we will read the equation $\int f(x)dx = \mathbf{F} + C\mathbf{1}$ as the parametric equation of a straight line in $\mathcal{S}$, for varying C. Compare it to the equation $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ for uniform rectilinear motion, where the parameter is time, t, instead of C.

Now let us consider the formula of integration by parts:

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx,$$

[1, Section 5.9], and rewrite it, after setting $\int fg'dx = \mathbf{U} + C\mathbf{1}$, $fg = \mathbf{V}$ and $-\int f'gdx = \mathbf{W} + C'\mathbf{1}$, as

$$\mathbf{U} + C\mathbf{1} = \mathbf{V} + \mathbf{W} + C'\mathbf{1}. \tag{1}$$

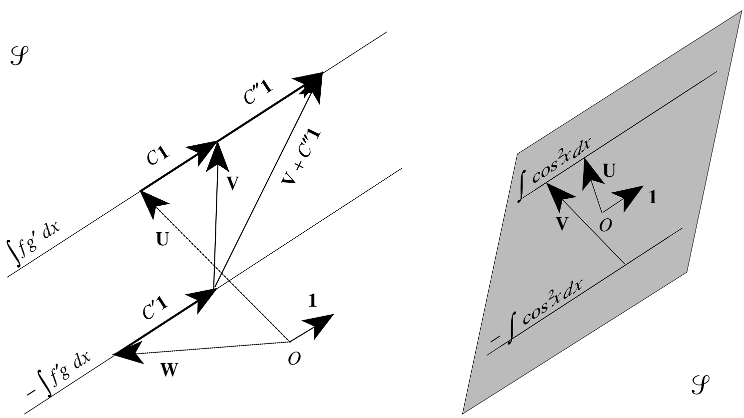

FIGURE 1: *Left*: Indefinite integration by parts as a translation of lines by a vector, **V**. Note that O does not belong in the same plane as the two lines, in general. *Right*: A translation working in a plane containing O.

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In Figure 1 left, we see the head of vector $\mathbf{U} + C\mathbf{1}$ drawing the upper line, as C varies. The vector $-\mathbf{V}$ *translates* this line to the lower line, drawn by $\mathbf{W} + C'\mathbf{1}$. Denoting the translation by $\mathbf{V}$ or $-\mathbf{V}$ is just a conventional matter. Let us observe that any vector $\mathbf{V} + C''\mathbf{1}$ performs the same translation as $\mathbf{V}$ does, independently of C''. Which point on the starting line is mapped to which point on the arrival line depends on C'', but the arrival line depends only on the components of $\mathbf{V}$ not parallel to $\mathbf{1}$. That is why we do not usually write the formula of integration by parts with a constant. Let us further remark that we can sum all vectors parallel to $\mathbf{1}$, in equation (1):

$$\mathbf{U} = \mathbf{V} + \mathbf{W} + (C' - C)\mathbf{1}.$$

Some examples

In an indefinite integration by parts, it may happen that $\mathbf{V}$ is parallel to $\mathbf{1}$, when fg is constant. For example, take $f(x) = \frac{1}{x}$ and $g(x) = x$. Then integration by parts is not useful, because it does not reduce the starting integral to a different one, thus providing new information for its solution, but it yields the same integral as before. We can understand this geometrically by noting that, if $\mathbf{V}$ is parallel to $\mathbf{1}$, the translation maps the line $\mathbf{U} + C\mathbf{1}$ to itself.

In other cases, integration by parts can relate an indefinite integral not to itself, but to a multiple, and thus it succeeds in bringing a solution. Let us consider the example

$$\int \cos^2 x\, dx = \sin x \cos x + x - \int \cos^2 x\, dx,$$

and write it in vector form as

$$\mathbf{U} = \mathbf{V} - \mathbf{U} + C\mathbf{1}.$$

The above equation shows that $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{1}$ are linearly dependent. Thus the translation occurs in the plane of the starting line and the origin, that is the two-dimensional vector subspace spanned by $\mathbf{U}$ and $\mathbf{1}$. See Figure 1 right. Geometrically, if $\mathbf{V}$ belongs in the plane of two independent vectors, $\mathbf{U}$ and $\mathbf{1}$, then also the unknown $\mathbf{U}$ belongs in the plane of the known vectors $\mathbf{V}$ and $\mathbf{1}$. Hence it can be expressed as a linear combination of them, and the integral is solved.

Let us further explore the geometry of such planes, and their role in indefinite integration. If f is a function and k is a constant, we know that

$$\int kf(x)\, dx = k \int f(x)\, dx, \tag{2}$$

by linearity. As a practical rule, (2) means that integrating the function kf is *exactly as difficult* as integrating f. Algebraically the mapping defined by $\mathbf{F} \to k\mathbf{F}$, takes a linear combination of $\mathbf{U}$ and $\mathbf{1}$ to another linear combination of $\mathbf{U}$ and $\mathbf{1}$. Geometrically the line $\mathbf{U} + C\mathbf{1}$ is mapped to a parallel line $k\mathbf{U} + C'\mathbf{1}$ in the plane of the line r and O, by a *central dilatation**. See Figure 2 left.

* In the elementary geometry of point space, a central dilatation is the transformation

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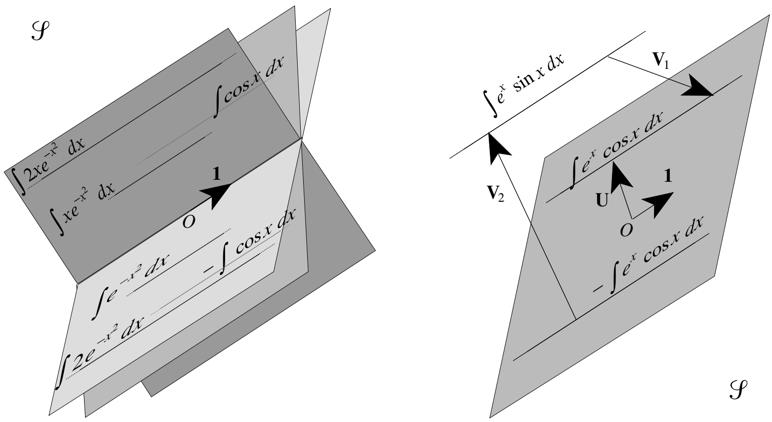

FIGURE 2: *Left*: Equation (2) is represented geometrically by lines spanning a plane. *Right*: The geometry of equation (3).

Each plane through the origin containing the line of an indefinite integral is an invariant two-dimensional subspace with respect to dilatations. It is a *locus* of integrals *exactly as difficult* as a given one, and simply related to it.

There are cases where a double integration by parts first takes off from such a plane, and finally lands back in it. For example:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \quad (3)$$

The first translation vector, $\mathbf{V_1} = e^x \sin x$, does not lie in the plane of $\mathbf{U} = \frac{1}{2}e^x(\sin x + \cos x)$ and $\mathbf{1}$, nor does the second translation vector, $\mathbf{V_2} = e^x \cos x$, but their sum does. See Figure 2 right.

Conclusion

Henri Poincaré, one of the greatest mathematicians of last century, in a work of his [3, Chapter II], wrote that

Mathematics is the art of giving the same name to different things.

As a matter of fact, after giving the name *vector* to a function, and *line* to an indefinite integral, we could easily understand some applications of the formula of integration by parts geometrically.

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that we need besides isometries to build similarities. Simply imagine what happens to a balloon while it is inflated. The dilatation with centre in C maps each point P to the point Q so that C, P and Q are aligned, and $\overline{CQ} = k \cdot \overline{CP}$, that is each point is sent far away from the centre in proportion to its initial distance from it and to a constant factor, k. See, for example, [2, Section 13.2].

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103.35 Hölder's inequality revisited

Based on the same idea provided by Razminia [1], we are going to present a new proof for Hölder's inequality.

Lemma 1: For all positive numbers a_1, a_2, b_1 and b_2, we have

$$a_1b_1 + a_2b_2 \leqslant \left(a_1^p + a_2^p\right)^{1/p}\left(b_1^q + b_2^q\right)^{1/q}, \tag{1}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ with $p, q > 1$. Moreover, the equality holds when $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q}$.

Proof: Let us define the following function

$$f(x) = \left(1 + x^p\right)^{1/p}\left(1 + y^q\right)^{1/q} - 1 - xy, \tag{2}$$

with a given y. Clearly $f(x)$ is continuous and differentiable, so the critical points of $f(x)$ can be obtained by solving the following equation

$$\frac{d}{dx}f(x) = x^{p-1}\left(1 + x^p\right)^{\frac{1}{p}-1}\left(1 + y^q\right)^{\frac{1}{q}} - y = 0. \tag{3}$$

Moving y to the right-hand side of (3), and raising both sides of the equation to the power q, we obtain

$$x^{(p-1)q}\left(1 + x^p\right)^{\left(\frac{1}{p}-1\right)q}\left(1 + y^q\right) = y^q.$$

Now, using the fact that $\frac{1}{p} - 1 = -\frac{1}{q}$ and $q(p - 1) = p$, we obtain

$$x^p\left(1 + x^p\right)^{-1}\left(1 + y^q\right) = y^q \text{ or } \left(1 + y^q\right) = y^q\left(1 + x^{-p}\right),$$

from which we conclude that $1 = y^q x^{-p}$ or, equivalently, $x^p = y^q$ $\left(x = y^{\frac{q}{p}}\right)$.

On the other hand, by taking the derivative of (3), which gives

$$\left[(p - 1)x^{p-2}\left(1 + x^p\right)^{\frac{1}{p}-1} + (1 - p)x^{2p-2}\left(1 + x^p\right)^{\frac{1}{p}-2}\right]\left(1 + y^q\right)^{\frac{1}{q}},$$

and after factorisation, it becomes

$$\frac{d^2}{dx^2}f(x) = (p - 1)x^{p-2}\left(1 + x^p\right)^{\frac{1}{p}-2}\left(1 + y^q\right)^{\frac{1}{q}},$$

which is clearly positive and, consequently, $f(x)$ is convex. Based on

$$\left.\frac{df}{dx}\right|_{\left(x = y^{q/p}\right)} = f\left(y^{\frac{q}{p}}\right) = 0$$

and the convexity of $f(x)$, we can conclude that $f(x)$ is always non-negative.