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ON BAD SUPERNILPOTENT RADICALS

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Abstract

A supernilpotent radical α is called *bad* if the class $\pi(\alpha)$ of all prime and α-semisimple rings consists of the one-element ring 0 only. We construct infinitely many bad supernilpotent radicals which form a generalization of Ryabukhin's example of a supernilpotent nonspecial radical. We show that the family of all bad supernilpotent radicals is a sublattice of the lattice of all supernilpotent radicals and give examples of supernilpotent radicals that are not bad.

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1. Introduction

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. A ring A is called *Boolean* if $a^2 = a$ for every $a \in A$. The fundamental definitions and properties of radicals can be found in [2] and [8]. A class μ of rings is called hereditary if μ is closed under ideals. If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. For a radical α the class of all α-semisimple rings is denoted by $\mathcal{S}(\alpha)$. Also, π denotes the class of all prime rings and $\beta = \mathcal{U}(\pi)$ the prime radical. An ideal I of a ring R is called essential if $I \cap J \neq 0$ for any nonzero ideal J of R. A hereditary class μ of prime rings is called special if μ is closed under essential extensions, that is, if $I \in \mu$ is an essential ideal of a ring R, then $R \in \mu$. The upper radical $\mathcal{U}(\mu)$ generated by a special class μ is called a *special radical. A* hereditary radical that contains β is called a *supernilpotent radical*. We call a supernilpotent radical α *bad* if the class $\pi(\alpha) = \pi \cap \mathcal{S}(\alpha)$ consists of the one-element ring 0 only. A radical α that is not the class of all associative rings is called nontrivial. The α-radical of a ring R is denoted by $\alpha(R)$.

Since special radicals are hereditary and contain β, every special radical is supernilpotent. Therefore, Andrunakievich [1] asked whether every supernilpotent radical is special. Examples of nonspecial supernilpotent radicals were given in [3, 4, 6, 7, 12, 14, 15]. Since a supernilpotent radical α is special if and only if $\alpha = \mathcal{U}(\pi(\alpha))$ [2, 8], nontrivial bad supernilpotent radicals provide the most natural counterexamples to Andrunakievich's question. The first such example was constructed by Ryabukhin [11] who showed that the upper radical generated by the class of all Boolean rings which do not contain an ideal which is a prime field with two elements is a supernilpotent but nonspecial radical. We will now give a generalization of Ryabukhin's construction which will allow us to build infinitely many nontrivial bad supernilpotent radicals. Moreover, we will show that the family of all bad supernilpotent radicals is a sublattice of the lattice of all supernilpotent radicals. Also, we will show that there exist supernilpotent radicals that are not bad.

2. Main results

Let ε denote the class of all *prime essential* rings [7], that is, semiprime rings R such that no nonzero ideal of R is a prime ring. Let $*$ denote the class of all $*$-rings, that is, semiprime rings R such that $R/I \in \beta$ for every nonzero ideal I of R. A special class σ of rings is called *subdirectly closed* if $\pi(\mathcal{U}(\sigma)) = \sigma$. For example, for every natural number n and any finite field F, the class $\{F_n\} \subseteq *$ consisting of the ring F_n of all $n \times n$ matrices with entries from F is such a class [9].

Theorem 1. *For every nonzero subdirectly closed special class $\sigma \subseteq *$, the radical $\alpha = \mathcal{U}(\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon)$, is a nontrivial bad supernilpotent radical. Thus α is not a special radical.*

Proof. To prove that α is nontrivial, it suffices to show that $\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon \neq \{0\}$. To build a nonzero ring in $\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon$, we will adopt the construction described in [7, Lemma 1, p. 234]. Let R be a nonzero element of σ, let κ be a cardinal number greater than the cardinality of R and let $W(\kappa)$ be the set of all finite words made from a (well-ordered) alphabet of cardinality κ, lexicographically ordered. Then $W(\kappa)$ is a semigroup with multiplication defined by $xy = \max\{x, y\}$ and it follows from [7] that the semigroup ring $A = R(W(\kappa))$ is prime essential and a subdirect sum of copies of $R \in \sigma$. Thus $0 \neq A \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon$, which shows that α is nontrivial.

It follows from [7] that ε is a weakly special class. But, since $\mathcal{U}(\sigma)$ is a special (and so supernilpotent) radical, it follows that $\mathcal{S}(\mathcal{U}(\sigma))$ is a weakly special class, too. Thus $\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon$ is a weakly special class. Therefore the radical $\alpha = \mathcal{U}(\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon)$ is a supernilpotent radical.

We will now show that α is bad. Suppose that it is not and let R be a nonzero ring in $\pi(\alpha)$. Then R is α -semisimple and, since α is a supernilpotent radical determined by the weakly special class $\mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon$, it follows that R is a subdirect sum of rings $R_\lambda \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon \subseteq \mathcal{S}(\mathcal{U}(\sigma))$. But, being a semisimple class, $\mathcal{S}(\mathcal{U}(\sigma))$ is closed under subdirect sums. Thus $R \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi = \sigma$. But, as $\sigma \subseteq *$, it follows that R is a $*$-ring. On the other hand, since R is a subdirect sum of the rings R_λ, there exists an ideal I_λ of R such that $I_\lambda \neq R$ and $R/I_\lambda \cong R_\lambda \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \varepsilon \subseteq \mathcal{S}(\beta)$. But, since R is a $*$-ring,

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we then must have $I_\lambda = 0$ which implies that $R \in \varepsilon \cap \pi = \{0\}$, a contradiction. Thus $\pi(\alpha) = \{0\}$, which means that α is bad.

Since $\pi(\alpha) = \{0\}$, $\mathcal{U}(\pi(\alpha))$ is a trivial radical. But α is not, so $\alpha \neq \mathcal{U}(\pi(\alpha))$, which implies that α is not a special radical. □

Proposition 2. *For every natural number n and every finite field F the class $\mathcal{S}(\mathcal{U}(\{F_n\})) \cap \varepsilon$ consists precisely of rings that do not contain ideals isomorphic to F_n and are subdirect sums of copies of F_n.*

Proof. Let $R \in \mathcal{S}(\mathcal{U}(\{F_n\})) \cap \varepsilon$. Then $R \in \mathcal{S}(\mathcal{U}(\{F_n\}))$ and so R is isomorphic to a subdirect sum of copies of F_n. But we also have that $R \in \varepsilon$ so, since $F_n \in \pi$, R cannot contain an ideal isomorphic to F_n. Conversely, suppose that R is isomorphic to a subdirect sum of copies of F_n and does not contain ideals isomorphic to F_n. Then $R \in \mathcal{S}(\mathcal{U}(\{F_n\}))$. If R contained a nonzero ideal $I \in \pi$, then I would be a member of $\mathcal{S}(\mathcal{U}(\{F_n\})) \cap \pi = \{F_n\}$ because $\mathcal{S}(\mathcal{U}(\{F_n\}))$, being a semisimple class, is hereditary. This gives a contradiction. Thus R is prime essential, which implies that $R \in \mathcal{S}(\mathcal{U}(\{F_n\})) \cap \varepsilon$. □

It is well known [10, Theorem 3.16, p. 58] that a ring A is Boolean if and only if A is a subdirect sum of copies of the two-element field $\mathbb{Z}_2$. Thus, taking $\sigma = \{\mathbb{Z}_2\}$ in Theorem 1, we have the following corollary.

Corollary 3 [11]. *The upper radical generated by the class of all Boolean rings which do not contain an ideal which is a prime field with two elements is a supernilpotent but nonspecial radical.*

It is well known [13] that the family $\mathbb{K}$ of all supernilpotent radicals is a lattice with respect to inclusion. Minimal elements of $\mathbb{K}$ are called *supernilpotent atoms.* Examples of supernilpotent atoms can be found in [5]. We have the following theorem.

Theorem 4. *The family $\mathbb{B}$ of all bad supernilpotent radicals is a sublattice of $\mathbb{K}$.*

Proof. Let $\alpha, \gamma \in \mathbb{B}$. Then $\pi(\alpha) = \{0\}$ and $\pi(\gamma) = \{0\}$. Then, since $\mathcal{S}(\alpha \vee \gamma) = \mathcal{S}(\alpha) \cap \mathcal{S}(\gamma)$, it follows that $\pi(\alpha \vee \gamma) = \{0\}$, which means that $\alpha \vee \gamma \in \mathbb{B}$.

To show that $\alpha \wedge \gamma \in \mathbb{B}$, suppose that $0 \neq R \in \pi(\alpha \wedge \gamma)$. If $\alpha(R) = 0$ then, since $R \in \pi$, it follows that $R \in \pi(\alpha)$, which is impossible since $\pi(\alpha) = \{0\}$. Thus $\alpha(R)$ is a nonzero ideal of R. Similarly, $\gamma(R)$ is a nonzero ideal of R. So, since $R \in \pi$, it follows that $\alpha(R)\gamma(R)$ is a nonzero ideal of R. Moreover, since both α and γ are hereditary radicals, it follows that $\alpha(R)\gamma(R) \in \alpha \wedge \gamma$, which contradicts the fact that $R \in \mathcal{S}(\alpha \wedge \gamma)$. Thus $\pi(\alpha \wedge \gamma) = \{0\}$, which means that $\alpha \wedge \gamma \in \mathbb{B}$. □

We do not know whether $\mathbb{B}$ is a complete sublattice of $\mathbb{K}$. To answer this question in the negative, it would suffice to show that $\wedge \alpha_p \notin \mathbb{B}$, where p is a prime, $\alpha_p = \mathcal{U}(\mathcal{S}(\mathcal{U}(\{\mathbb{Z}_p\})) \cap \varepsilon)$ and $\mathbb{Z}_p$ is the p-element field.

Our final result shows examples of supernilpotent radicals which are not bad.

Theorem 5. *If α is a supernilpotent atom, then α is not bad.*

Proof. Let S be a nonzero simple prime ring. Then either $S \in \mathcal{S}(\alpha)$ or $S \in \alpha$. In the first case $S \in \pi(\alpha)$, which makes α not bad. In the second case we have $\beta \subsetneq \bar{l}_A \subseteq \alpha$, where $\bar{l}_A$ denotes the smallest supernilpotent radical containing A. So, since α is a supernilpotent atom, we must have $\alpha = \bar{l}_A$. But then every nonzero simple prime ring R which is not isomorphic to S is in $\pi(\alpha)$, which shows that α is not bad. □

References

[1] V. A. Andrunakievich, 'Radicals of associative rings I', *Mat. Sb.* **44** (1958), 179–212 (in Russian).
[2] V. A. Andrunakievich and Yu. M. Ryabukhin, *Radicals of Algebras and Structure Theory* (Nauka, Moscow, 1979), (in Russian).
[3] K. I. Beidar and K. Salavova, 'Some examples of supernilpotent nonspecial radicals', *Acta Math. Hungar.* **40** (1982), 109–112.
[4] K. I. Beidar and R. Wiegandt, 'Radicals induced by the total of rings', *Beiträge Algebra Geom.* **38** (1997), 149–159.
[5] H. France-Jackson, 'On atoms of the lattice of supernilpotent radicals', *Quaest. Math.* **10** (1987), 251–256.
[6] H. France-Jackson, 'On prime essential rings', *Bull. Aust. Math. Soc.* **47** (1993), 287–290.
[7] B. J. Gardner and P. N. Stewart, 'Prime essential rings', *Proc. Edinb. Math. Soc.* **34** (1991), 241–250.
[8] B. J. Gardner and R. Wiegandt, *Radical Theory of Rings* (Marcel Dekker, New York, 2004).
[9] K. K. Krachilov, 'Coatoms of the lattice of special radicals', *Mat. Issled. Kishinev* **49** (1979), 80–86 (in Russian).
[10] N. H. McCoy, *The Theory of Rings* (Chelsea, New York, 1973).
[11] Yu. M. Ryabukhin, 'On overnilpotent and special radicals', *Issled. Alg. Mat. Anal., Kishinev* (1965), 65–72 (in Russian).
[12] Yu. M. Ryabukhin, 'Supernilpotent and special radicals', *Mat. Issled. Kishinev* **48** (1978), 80–83 (in Russian).
[13] R. L. Snider, 'Lattices of radicals', *Pac. J. Math.* **40** (1972), 207–220.
[14] S. Tumurbat, 'Some issues in the theory of supernilpotent radicals', *Bul. Acad. Stiinte Rep. Mold.* **1**(38) (2002), 3–43.
[15] L. C. A. van Leeuwen and T. L Jenkins, 'A supernilpotent non-special radical class', *Bull. Aust. Math. Soc.* **9** (1973), 343–348.

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