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Real hypersurfaces in the complex quadric with Killing normal Jacobi operator

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We introduce the notion of Killing normal Jacobi operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The Killing normal Jacobi operator implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with Killing normal Jacobi operator.

Keywords: killing normal Jacobi operator; $\mathfrak{A}$-isotropic; $\mathfrak{A}$-principal; Kähler structure; complex conjugation; complex quadric.

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1. Introduction

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to the Ricci tensor and the structure Jacobi operator were investigated by Kimura [**4**, **5**], Pérez [**9**] and Pérez and Suh [**11**, **12**], Pérez and Santos [**10**], and Pérez, Santos and Suh [**13**, **14**], respectively. When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [**17**–**19**]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

The classification problems of real hypersurfaces in the complex 2-plane Grassmannian $SU_{m+2}/S(U_2U_m)$ with certain geometric conditions were mainly investigated in Jeong and Suh [**2**], Jeong, Machado, Pérez and Suh [**3**, **8**], Pérez [**9**], and Suh [**17**–**19**], where the classification of *commuting shape operator*, *parallelism of normal and structure Jacobi operators*, *contact hypersurfaces*, *parallel Ricci tensor*, and *harmonic curvature* for real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [**19**], we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

As another kind of Hermitian symmetric space with rank 2 of the compact type different from the above ones, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Klein [6], and Smyth [16]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of the compact type with rank 2 (see Kobayashi and Nomizu [7]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geqslant 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of the compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [6] and Reckziegel [15]).

In addition to the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m as follows:

$$\mathcal{Q} = \{X \in T_zM | AX \in T_zM \quad \text{for all} \quad A \in \mathfrak{A}\}.$$

Moreover, the derivative of the complex conjugation A on Q^m is defined by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M and q denotes a certain 1-form defined on M.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of *singular* tangent vectors for the complex quadric Q^m:

(1) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

(2) If there exists a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

When we consider a real hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties the unit normal vector field N of M in Q^m can be either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [20, 22]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [20] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [22]).

Jacobi fields along geodesics of a given Riemannian manifold $\bar{M}$ satisfy a well-known differential equation. Naturally, this classical differential equation inspires the so-called *Jacobi operator*. That is, if $\bar{R}$ is the curvature operator of $\bar{M}$, the Jacobi operator with respect to X at $z \in M$, is defined by

$$(\bar{R}_X Y)(z) = (\bar{R}(Y, X)X)(z)$$

for any $Y \in T_z\bar{M}$. Then $\bar{R}_X \in \mathrm{End}(T_z\bar{M})$ becomes a symmetric endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. Clearly, each tangent vector field X to $\bar{M}$ provides a Jacobi operator with respect to X (see Pérez and Santos [**10**], and Pérez, Santos and Suh [**13**, **14**]).

From such a viewpoint, in the complex quadric Q^m the *normal Jacobi operator* $\bar{R}_N$ is defined by

$$\bar{R}_N = \bar{R}(\cdot, N)N \in End\ (T_zM), \quad z \in M$$

for a real hypersurface M in Q^m with unit normal vector field N, where $\bar{R}$ denotes the curvature tensor of the complex quadric Q^m. Of course, the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of M in Q^m (see Jeong, Machado, Pérez and Suh [**3**] and [**8**]).

The Reeb vector field ξ is *Killing* on M in Q^m if and only if $g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) = 0$ for any vector fields X and Y on M. This means that the Reeb flow of the Reeb vector field ξ is isometric. That is, the Reeb vector field has an *isometric Reeb flow*.

As a generalization of such a notion of isometric Reeb flow, first Yano [**26**] defined the notion of *Killing tensor*. A skew symmetric tensor $T_{i_1\cdots i_r}$ is called a *Killing tensor* of order r if it satisfies

$$\nabla_{i_1}T_{i_2\cdots i_{r+1}} + \nabla_{i_2}T_{i_1\cdots i_{r+1}} = 0.$$

Next Blair [**1**] has applied the notion of Killing tensor to a tensor field of T type $(1, 1)$ on a Riemannian manifold and a geodesic γ on M. If we denote by γ' the tangent vector of the geodesic γ, then $T\gamma'$ is parallel along the geodesic γ for the Killing tensor field T. Geometrically, this means that $(\nabla_{\gamma'}T)\gamma' = 0$ along a geodesic γ on M. If this is the case for any geodesic on M, we have

$$(\nabla_XT)X = 0 \quad \text{or equivalently} \quad (\nabla_XT)Y + (\nabla_YT)X = 0$$

for any vector fields X and Y on M. In this case, we say that the tensor T a *Killing tensor field of type* $(1, 1)$.

The normal Jacobi operator $\bar{R}_N$ of M in Q^m is said to be *Killing* if the operator $\bar{R}_N$ satisfies

$$(\nabla_X\bar{R}_N)Y + (\nabla_Y\bar{R}_N)X = 0$$

for any $X, Y \in T_zM$, $z \in M$. The equation is equivalent to $(\nabla_X\bar{R}_N)X = 0$ for any $X \in T_zM$, $z \in M$, because of linearization. Moreover, we can give the geometric meaning of Killing Jacobi tensor as follows:

When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot{\gamma}(0) = X$. Then the transformed vector field $\bar{R}_N\dot{\gamma}$ is Levi–Civita *parallel* along the geodesic γ of the vector field X (see Blair [**1**] and Tachibana [**25**]).

In the study of real hypersurfaces in the complex quadric Q^m, we considered the notion of parallel Ricci tensor, that is, $\nabla\text{Ric} = 0$ (see Suh [**22**]). But from the assumption of Ricci parallel, it was difficult for us to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [**22**], we gave a classification with the further assumption of $\mathfrak{A}$-isotropic. But fortunately, when we consider Killing normal Jacobi operator, first we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows:

Theorem 1. *Let M be a Hopf real hypersurface in Q^m, $m\geqslant 3$, with Killing normal Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Then motivated by such a result, next we give a complete classification for real hypersurfaces in the complex quadric Q^m with Killing normal Jacobi operator as follows:

Theorem 2. *There do not exist any Hopf real hypersurfaces in Q^m, $m\geqslant 3$ with Killing normal Jacobi operator.*

Usually, Killing normal Jacobi operator is a generalization of parallel normal Jacobi operator $\bar{R}_N$ of M in Q^m, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M. The parallelism of normal Jacobi operator has a geometric meaning that every eigen space of the normal Jacobi operator $\bar{R}_N$ is parallel along any direction on M in Q^m. Then naturally, by theorem 2 above, we give the following

Corollary [24]. *There do not exist any Hopf real hypersurfaces in Q^m, $m\geqslant 3$ with parallel normal Jacobi operator.*

2. The complex quadric

For more background to this section, we refer to [6, 7, 15, 20–23]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini-Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini-Study metric $\bar{g}$ is defined by $\bar{g}(X, Y) = \Phi(JX, Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \ldots, z^{m+1}] \in \mathbb{C}P^{m+1} | z^j \neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = ((z^k)/(z^j))$ for $j, k = 0, \ldots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason, we will assume $m \geqslant 3$ from now on.

In another way, the complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then naturally, we can consider the following diagram for the complex quadric Q^m as follows:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\tilde{i}} & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \big\downarrow & & \pi \big\downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \tfrac{1}{2} \text{ and } g(x, y) = 0\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2})$ and $y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$, respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^\perp$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$, and F_z and $F_z(Q)$ are fibres which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} | \quad g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z | \quad g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)} \mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)} Q$, respectively. This gives that at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]} Q^m$ becomes a complex subspace of $T_{[z]} \mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [15]).

Now let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $-\bar{z}$. Then, by virtue of the Weingarten equation, it is defined by $A_{\bar{z}} w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]} Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}} w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda\bar{w} \\ &= \lambda A_{\bar{z}} \lambda\bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically, this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under the action of the centre SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way, the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that the complex structure J and each complex conjugation A are anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X \in V(A)$, $t = 0$, there exist many kinds of maximal 2-flats $\mathbb{R}X + \mathbb{R}Z$ for $Z \in V(A)$ orthogonal to $X \in V(A)$. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \pi/4$, it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by $\mathbb{R}(X + JY) + \mathbb{R}Z$ for any $Z \in V(A)$ orthogonal to $X \in V(A)$ or $\mathbb{R}(X + JY) + \mathbb{R}JZ$ for any $JZ \in JV(A)$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \quad \text{for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main theorem in the introduction.

Lemma 3.1 [20]. *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies the following

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurface M with the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transform JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the equation of Codazzi

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &\quad + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

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Putting $Z=\xi$ in (3.1) we get

$$\begin{aligned} &g((\nabla_X S)Y-(\nabla_Y S)X,\xi)\\ &\quad=-2g(\phi X,Y)+g(X,AN)g(Y,A\xi)-g(Y,AN)g(X,A\xi)\\ &\qquad-g(X,A\xi)g(JY,A\xi)+g(Y,A\xi)g(JX,A\xi). \end{aligned}$$

On the contrary, we have

$$\begin{aligned} &g((\nabla_X S)Y-(\nabla_Y S)X,\xi)\\ &\quad=g((\nabla_X S)\xi,Y)-g((\nabla_Y S)\xi,X)\\ &\quad=(X\alpha)\eta(Y)-(Y\alpha)\eta(X)+\alpha g((S\phi+\phi S)X,Y)-2g(S\phi SX,Y). \end{aligned}$$

Comparing the previous two equations and putting $X=\xi$ yields

$$Y\alpha=(\xi\alpha)\eta(Y)-2g(\xi,AN)g(Y,A\xi)+2g(Y,AN)g(\xi,A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} &g((\nabla_X S)Y-(\nabla_Y S)X,\xi)\\ &\quad=-2g(\xi,AN)g(X,A\xi)\eta(Y)+2g(X,AN)g(\xi,A\xi)\eta(Y)\\ &\qquad+2g(\xi,AN)g(Y,A\xi)\eta(X)-2g(Y,AN)g(\xi,A\xi)\eta(X)\\ &\qquad+\alpha g((\phi S+S\phi)X,Y)-2g(S\phi SX,Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0&=2g(S\phi SX,Y)-\alpha g((\phi S+S\phi)X,Y)-2g(\phi X,Y)\\ &\quad+g(X,AN)g(Y,A\xi)-g(Y,AN)g(X,A\xi)\\ &\quad-g(X,A\xi)g(JY,A\xi)+g(Y,A\xi)g(JX,A\xi)\\ &\quad+2g(\xi,AN)g(X,A\xi)\eta(Y)-2g(X,AN)g(\xi,A\xi)\eta(Y)\\ &\quad-2g(\xi,AN)g(Y,A\xi)\eta(X)+2g(Y,AN)g(\xi,A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z\in M$ we can choose $A\in\mathfrak{A}_z$ such that

$$N=\cos(t)Z_1+\sin(t)JZ_2$$

for some orthonormal vectors $Z_1,Z_2\in V(A)$ and $0\leqslant t\leqslant\pi/4$ (see proposition 3 in [**15**]). Note that t is a function on M. First of all, since $\xi=-JN$, we have

$$\begin{aligned} AN&=\cos(t)Z_1-\sin(t)JZ_2,\\ \xi&=\sin(t)Z_2-\cos(t)JZ_1,\\ A\xi&=\sin(t)Z_2+\cos(t)JZ_1. \end{aligned} \tag{3.3}$$

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This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &\quad - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

4. Killing normal Jacobi operator and a key lemma

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY \\ &\quad + g(SY, Z)SX - g(SX, Z)SY \end{aligned}$$

for any $X, Y, Z \in T_zM$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m$, $z \in M$, $\rho(X) = g(AX, N)$, where BX and $\rho(X)N$, respectively, denote the tangential and normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned} AN &= AJ\xi = -JA\xi = -JB\xi \\ &= -(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

By the equation of Gauss, the normal Jacobi operator $\bar{R}_N$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows:

$$\bar{R}_N(Y) = Y + 3\eta(Y)\xi + g(AN, N)AY - g(AY, N)AN - g(AY, \xi)A\xi.$$

for any $Y \in T_xM$, $x \in M$. Now the derivative of $\bar{R}_N$ is given by

$$(\nabla_X \bar{R}_N)Y = \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y). \tag{4.1}$$

Here we note that the connection ∇ on M in Q^m gives

$$\begin{aligned} (\nabla_X A)Y &= \bar{\nabla}_X(AY) - A\nabla_X Y \\ &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - A\nabla_X Y \\ &= q(X)JAY + A\sigma(X, Y) \\ &= q(X)JAY + g(SX, Y)AN. \end{aligned}$$

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So naturally, it follows that

$$\begin{aligned}(\nabla_X A)\xi &= \bar{\nabla}_X(A\xi) - A\nabla_X\xi \\ &= (\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi - A\nabla_X\xi \\ &= q(X)JA\xi + g(SX,\xi)AN.\end{aligned}$$

From this, together with (4.1) and Killing normal Jacobi operator, it follows that

$$\begin{aligned}0 &= (\nabla_X \bar{R}_N)Y + (\nabla_Y \bar{R}_N)X \\ &= 3\{g(\phi SX,Y) + g(\phi SY,X)\}\xi + 3\{\eta(Y)\phi SX + \eta(X)\phi SY\} \\ &\quad + \{q(X)g(JAN,N) - g(ASX,N) - g(AN,SX)\}AY \\ &\quad + \{q(Y)g(JAN,N) - g(ASY,N) - g(AN,SY)\}AX \\ &\quad + g(AN,N)\{q(X)JAY + q(Y)JAX + 2g(SX,Y)AN\} \\ &\quad - \{q(X)g(JAY,N) + q(Y)g(JAX,N) + 2g(SX,Y)g(AN,N)\}AN \\ &\quad + g(AY,SX)AN + g(AX,SY)AN \\ &\quad - g(AY,N)\{(\bar{\nabla}_X A)N + A\bar{\nabla}_X N\} - g(AX,N)\{(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N\} \\ &\quad - \{g((\bar{\nabla}_X A)Y,\xi) + g((\bar{\nabla}_Y A)X,\xi)\}A\xi \\ &\quad - \{g(AY,\phi SX + \sigma(X,\xi)) + g(AX,\phi SY + \sigma(Y,\xi))\}A\xi \\ &\quad - g(AY,\xi)\{(\nabla_X A)\xi + A\nabla_X\xi\} - g(AX,\xi)\{(\nabla_Y A)\xi + A\nabla_Y\xi\}.\end{aligned} \tag{4.2}$$

Here we have used the equation of Gauss $\bar{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi)$, where $\sigma(X,\xi)$ denotes the normal bundle $T^{\perp}M$ valued second fundament tensor on M in Q^m. From this, putting $Y=\xi$ and using $g(A\xi,N)=0$, $(\bar{\nabla}_X A)Y = q(X)JAY$, and $\bar{\nabla}_X N = -SX$ we have

$$\begin{aligned}0 &= 3\phi SX + g(AN,N)\{q(X)JA\xi + q(\xi)JAX + 2\alpha\eta(X)AN\} \\ &\quad - \{q(X)g(A\xi,\xi) + q(\xi)g(AX,\xi) + 2\alpha\eta(X)g(AN,N)\}AN \\ &\quad + g(A\xi,SX)AN + \alpha\eta(AX)AN - g(AX,N)\{(\bar{\nabla}_\xi A)N + A\bar{\nabla}_\xi N\} \\ &\quad - \{g((\bar{\nabla}_X A)\xi,\xi) + g((\bar{\nabla}_\xi A)X,\xi)\}A\xi \\ &\quad - \{g(A\xi,\phi SX + \sigma(X,\xi)) + g(AX,\sigma(\xi,\xi))\}A\xi \\ &\quad - g(A\xi,\xi)\{(\nabla_X A)\xi + A\nabla_X\xi\} - g(AX,\xi)\{(\nabla_\xi A)\xi + A\nabla_\xi\xi\}.\end{aligned} \tag{4.3}$$

On the contrary, we know the following

$$(\bar{\nabla}_\xi A)N = q(\xi)JAN, \quad \sigma(\xi,\xi) = g(S\xi,\xi)N = \alpha N, \quad \bar{\nabla}_\xi\xi = \alpha N.$$

Substituting these formulas into (4.3) gives the following

$$\begin{aligned} 0 = {} & 3\phi SX + g(AN,N)\{q(X)JA\xi + q(\xi)JAX + 2\alpha\eta(X)AN\} \\ & - \{q(X)g(A\xi,\xi) + q(\xi)g(AX,\xi) + 2\alpha\eta(X)g(AN,N)\}AN \\ & + g(A\xi,SX)AN + \alpha\eta(AX)AN \\ & - \{q(X)g(JA\xi,\xi) + q(\xi)g(JAX,\xi)\}A\xi \\ & - \{g(A\xi,\phi SX) + g(A\xi,\alpha\eta(X)N) + g(AX,\alpha N)\}A\xi \\ & - g(A\xi,\xi)\{q(X)JA\xi + A(\phi SX + \alpha\eta(X)N)\} \\ & - g(AX,\xi)\{q(\xi)JA\xi + \alpha AN\}. \end{aligned} \tag{4.4}$$

From this, by putting the Reeb vector field $X = \xi$ and using $JA\xi = -AN$, we have

$$\begin{aligned} 0 = {} & g(AN,N)\{-2q(\xi)AN + 2\alpha AN\} - \{q(\xi)g(A\xi,\xi) + 2\alpha g(AN,N)\}AN \\ & + 2\alpha\eta(A\xi)AN - q(\xi)g(JA\xi,\xi)A\xi - g(A\xi,\xi)\{q(\xi)JA\xi + \alpha AN\} \\ & - g(A\xi,\xi)\{q(\xi)JA\xi + \alpha AN\}. \end{aligned} \tag{4.5}$$

This gives that $q(\xi)g(AN,N)AN = 0$, which implies that $q(\xi) = 0$ or $g(AN,N) = 0$. The latter case means that the unit normal vector field N is $\mathfrak{A}$-isotropic.

Summing up the above discussions, we can assert an important lemma as follows:

LEMMA 4.1. *Let M be a Hopf real hypersurface in Q^m, $m \geqslant 3$, with Killing normal Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. In the above discussion, when $q(\xi) \neq 0$, we have proved that the unit normal N is $\mathfrak{A}$-isotropic. Now let us consider the case that $q(\xi) = 0$. Then taking the inner product of (4.4) with the unit normal N gives

$$\begin{aligned} 0 & = g(AN,N)q(X)g(JA\xi,N) + 2\alpha g(AN,N)^2\eta(X) \\ & \quad - \{q(X)g(A\xi,\xi) + 2\alpha\eta(X)g(AN,N)\}g(AN,N) \\ & \quad + g(A\xi,SX)g(AN,N) + \alpha\eta(AX)g(AN,N) + \alpha g(AX,N)g(AN,N) \\ & \quad - g(A\xi,\xi)\{q(X)g(JA\xi,N) + g(A\phi SX,N) + \alpha\eta(X)g(AN,N)\} \\ & \quad - \alpha g(AX,\xi)g(AN,N) \\ & = -q(X)g(AN,N)^2 + g(A\xi,SX)g(AN,N) + \alpha g(AX,N)g(AN,N) \\ & \quad - g(A\xi,\xi)g(A\phi SX,N). \end{aligned} \tag{4.6}$$

From this, putting $X = \xi$ and using $q(\xi) = 0$, it follows that

$$g(A\xi,S\xi)g(AN,N) = \alpha g(A\xi,\xi)g(AN,N) = 0.$$

Here, if the Reeb function $\alpha \neq 0$, then $g(AN,N) = 0$ gives that the unit normal vector field N is $\mathfrak{A}$-isotropic.

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When the Reeb function α is vanishing, by the formula in § 3, that is,

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi),$$

it follows that

$$g(Y, (AN)^T)g(\xi, A\xi) = 0.$$

Since in the second case we have assumed that N is not $\mathfrak{A}$-isotropic, we know $g(\xi, A\xi) \neq 0$. So it follows that $(AN)^T = 0$. This means that

$$AN = (AN)^T + g(AN, N)N = g(AN, N)N.$$

Then it implies that

$$N = A^2N = g(AN, N)AN = g^2(AN, N)N.$$

This gives that $g(AN, N) = \pm 1$, that is, we can take the unit normal N such that $AN = N$. So the unit normal N is $\mathfrak{A}$-principal, that is, $AN = N$. □

5. Proof of the main theorem with $\mathfrak{A}$-isotropic normal vector field

In this section, let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be put

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a (+1)-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (3.3) and the anti-commuting property $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \; g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0.$$

By virtue of these formulas for an $\mathfrak{A}$-isotropic unit normal, the normal Jacobi operator $\bar{R}_N$ is given by

$$\bar{R}_N(Y) = Y + 3\eta(Y)\xi - g(AY, N)AN - g(AY, \xi)A\xi.$$

Now let us assume that the normal Jacobi operator $\bar{R}_N$ on M is Killing. Then it gives that

$$\begin{aligned} 0 &= (\nabla_X \bar{R}_N)Y + (\nabla_Y \bar{R}_N)X \\ &= 3(\nabla_X\eta)(Y)\xi + 3\eta(Y)\nabla_X\xi + 3(\nabla_Y\eta)(X)\xi + 3\eta(X)\nabla_Y\xi \\ &\quad - g(\nabla_X(AN), Y)AN - g(\nabla_Y(AN), X)AN \\ &\quad - g(AN, Y)\nabla_X(AN) - g(AN, X)\nabla_Y(AN) \\ &\quad - g(Y, \nabla_X(A\xi))A\xi - g(X, \nabla_Y(A\xi))A\xi \\ &\quad - g(A\xi, Y)\nabla_X(A\xi) - g(A\xi, X)\nabla_Y(A\xi). \end{aligned} \tag{5.1}$$

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On the contrary, by using the equation of Gauss we know that

$$\begin{aligned}\nabla_X(AN) &= \bar{\nabla}_X(AN) - \sigma(X, AN)\\ &= (\bar{\nabla}_X A)N + A\bar{\nabla}_X N - \sigma(X, AN)\\ &= q(X)JAN - ASX - \sigma(X, AN),\\ &= q(X)A\xi - ASX - \sigma(X, AN),\end{aligned}$$

and

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi)\\ &= (\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi - \sigma(X, A\xi)\\ &= q(X)JA\xi + A\{\phi SX + \eta(SX)N\} - \sigma(X, A\xi)\\ &= -q(X)AN + A\phi SX + \eta(SX)AN - \sigma(X, A\xi).\end{aligned}$$

Now we use the facts that $\sigma(\xi, AN) = g(S\xi, AN)N = \alpha g(\xi, AN)N = 0$ and $\sigma(\xi, A\xi) = \alpha g(\xi, A\xi)N = 0$ for an $\mathfrak{A}$-isotropic unit normal N in the above equations. Then the two equations become the following respectively,

$$\nabla_\xi(AN) = q(\xi)A\xi - AS\xi - \sigma(\xi, AN) = \{q(\xi) - \alpha\}A\xi,$$

and

$$\begin{aligned}\nabla_\xi(AN) &= -q(\xi)AN + A\phi S\xi + \eta(S\xi)AN - \sigma(\xi, A\xi)\\ &= -\{q(\xi) - \alpha\}AN.\end{aligned}$$

By putting $Y = \xi$ and substituting these formulas into (5.1), we have

$$\begin{aligned}0 &= 3\phi SX - g(\{q(X)A\xi - ASX - \sigma(X, AN)\}, \xi)AN\\ &\quad - g(\{q(\xi) - \alpha\}A\xi, X)AN - g(AN, X)\{q(\xi) - \alpha\}A\xi\\ &\quad - g(\xi, q(X)AN + A\phi SX + \eta(SX)AN - \sigma(X, A\xi))A\xi\\ &\quad + g(X, \{q(\xi) - \alpha\}AN)A\xi + g(A\xi, X)\{q(\xi) - \alpha\}AN\\ &= 3\phi SX + g(ASX, \xi)AN - g(\xi, A\phi SX)A\xi\\ &= 3\phi SX + g(A\xi, SX)AN - g(AN, SX)A\xi.\end{aligned} \tag{5.2}$$

The formula (5.2) means that the vector field $\phi SX \in \mathrm{Span}\{A\xi, AN\}$. From this fact, together with the formulas $A\xi = \phi AN$ and $AN = -\phi A\xi$ into (5.2), it follows that

$$\begin{aligned}0 &= 3\phi SX + g(\phi AN, SX)AN + g(\phi A\xi, SX)A\xi\\ &= 3\phi SX - g(AN, \phi SX)AN - g(A\xi, \phi SX)A\xi\\ &= 3\phi SX - \phi SX.\end{aligned}$$

This gives that $\phi SX = 0$, which implies $SX = \alpha\eta(X)\xi$, because $\phi SX \in \mathrm{Span}\{A\xi, AN\} = \mathcal{Q}^\perp$. Then the hypersurface M is totally η-umbilical, that is, the shape operator S commutes with the structure tensor ϕ. Then by theorem B in the

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introduction, M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. But the tube is not η-umbilical. Accordingly, we assert that there do not exist any hypersurfaces with Killing normal Jacobi operator.

6. Proof of the main theorem with $\mathfrak{A}$-principal normal vector field

In this section, let us consider a real hypersurface M in Q^m with Killing normal Jacobi operator for the case that the unit normal N is $\mathfrak{A}$-principal. In this case, the normal Jacobi operator $\bar{R}_N$ is given by

$$\bar{R}_N(X) = X + 2\eta(X)\xi + AX,$$

where $AX = BX = (AX)^T$ denotes the tangential part of the $AX = BX + \rho(X)N$. In this case, we must have $\rho(X) = 0$ for an $\mathfrak{A}$-principal normal N. Then differentiating the above ones gives

$$\begin{aligned}(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= 2(\nabla_X\eta)(Y)\xi + 2\eta(Y)\nabla_X\xi + (\nabla_X B)Y.\end{aligned} \tag{6.1}$$

Now let us consider that the normal Jacobi operator $\bar{R}_N$ is Killing. Then it follows that

$$\begin{aligned}0 &= (\nabla_X \bar{R}_N)Y + (\nabla_Y \bar{R}_N)X \\ &= 2\{(\nabla_X\eta)(Y)\xi + (\nabla_Y\eta)(X)\xi\} + 2\eta(Y)\nabla_X\xi + 2\eta(X)\nabla_Y\xi \\ &\quad + (\nabla_X B)Y + (\nabla_Y B)X.\end{aligned} \tag{6.2}$$

From this, by putting $Y = \xi$, it follows that

$$0 = 2\nabla_X\xi + (\nabla_X B)\xi + (\nabla_\xi B)X. \tag{6.3}$$

On the contrary, for an $\mathfrak{A}$-principal unit normal N the derivative of the complex conjugation can be given as follows:

$$\begin{aligned}(\nabla_X B)Y &= (\nabla_X A)Y \\ &= \nabla_X(AY) - A\nabla_X Y \\ &= \bar{\nabla}_X(AY) - \sigma(X, AY) - A\nabla_X Y \\ &= (\bar{\nabla}_X A)Y + A(\bar{\nabla}_X Y) - \sigma(X, AY) - A\nabla_X Y \\ &= q(X)JAY + A\{\nabla_X Y + \sigma(X, Y)\} - \sigma(X, AY) - A\nabla_X Y \\ &= q(X)JAY + g(SX, Y)N - g(SX, AY)N.\end{aligned} \tag{6.4}$$

From this, by puttig $Y = \xi$ we have

$$\begin{aligned}(\nabla_X B)\xi &= (\nabla_X A)\xi \\ &= q(X)JA\xi + g(SX, \xi)N - g(SX, A\xi)N \\ &= -q(X)J\xi + 2\alpha\eta(X)N \\ &= -q(X)N + 2\alpha\eta(X)N\end{aligned}$$

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and

$$\begin{aligned}(\nabla_\xi B)X &= (\nabla_\xi A)X \\ &= (\nabla_\xi B)X \\ &= q(\xi)JAX + g(S\xi, X)N - g(SX, A\xi)N \\ &= q(\xi)JAX + 2\alpha\eta(X)N.\end{aligned}$$

Then substituting these formulas into (6.4) and using $A\xi = -\xi$, we have

$$\begin{aligned}0 = 2\phi SX - q(X)N + 2\alpha\eta(X)N \\ + q(\xi)\{\phi AX - \eta(X)N\} + 2\alpha\eta(X)N.\end{aligned}$$

From this, taking the tangential and normal part, respectively, we have

$$\begin{aligned}&0 = 2\phi SX + q(\xi)\phi AX, \quad \text{and} \\ &0 = -q(X) + 4\alpha\eta(X) - q(\xi)\eta(X).\end{aligned} \tag{6.5}$$

From the second equation of (6.5) we know that

$$q(X) = \{4\alpha - q(\xi)\}\eta(X). \tag{6.6}$$

Then $q(\xi) = 2\alpha$. Here we note that the 1-form q on M vanishes on $\mathcal{C} = \xi^\perp$, that is, (6.6) gives $q(X) = 0$ on any $X \in \mathcal{C}$, where $\xi^\perp$ denotes the orthogonal complement of the Reeb vector field ξ in T_zM, $z \in M$.

On the contrary, by applying the structure tensor ϕ to (6.5), and using $q(\xi) = 2\alpha$, we have

$$\begin{aligned}0 &= -2SX + 2\alpha\eta(X)\xi - q(\xi)AX - q(\xi)\eta(X)\xi \\ &= -2SX - q(\xi)AX.\end{aligned}$$

That is, we have

$$2SX = -q(\xi)AX. \tag{6.7}$$

From this, if we apply the complex conjugation A again, it follows that

$$2ASX = -q(\xi)X. \tag{6.8}$$

Since we have assumed that M is Hopf, we may consider an eigenvector $X \in \mathcal{C}$ such that $SX = \lambda X$. Then (6.8) implies that

$$2\lambda AX = -q(\xi)X = -2\alpha X. \tag{6.9}$$

Then from (6.9) we can consider two cases as follows:

First, we consider that at least one of the principal curvature λ vanishes. Then $q(\xi) = 2\alpha = 0$. From this, together with the Reeb function α vanishing and $q(X) = 0$ on $\mathcal{C}$ in (6.6), the 1-form q identically vanishes on M. But this gives a contradiction for a complex hypersurface Q^m in $\mathbb{C}P^{m+1}$, because $\tilde{\nabla}_X\bar{z} = -A_{\bar{z}}X + q(X)J\bar{z}$, where $\{\bar{z}, J\bar{z}\}$ denotes two unit normals of Q^m in $\mathbb{C}P^{m+1}$, and $\tilde{\nabla}$ a connection defined on the complex projective space $\mathbb{C}P^{m+1}$ (see Smyth [**16**]).

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Next, let us consider the case that any principal curvatures in (6.9) are non-vanishing, that is, $\lambda \neq 0$. Then (6.9) implies that

$$2\lambda X = 2\lambda A^2 X = -q(\xi)AX = \frac{q(\xi)^2}{2\lambda}X.$$

From this $q(\xi)^2 = 4\lambda^2$, so it follows that $q(\xi) = \pm 2\lambda$.

Now let us check two subcases as follows:

Subcase 2.1. $q(\xi) = 2\lambda$.

In this case, (6.9) gives that $AX = -X$ for any $X \in \mathcal{C}$. From this, together with $AN = N$ and $A\xi = -\xi$, the expression of the complex conjugation A on the decomposition $T_z Q^m = [N] \oplus [\xi] \oplus [\mathcal{C}]$ becomes the following

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -1 \end{bmatrix}$$

Then $TrA = -2(m-1)$. But it is known that TrA should vanish, by virtue of $T_z Q^m = V(A) \oplus JV(A)$, where $V(A) = \{X \in T_z Q^m | AX = X\}$ and $JV(A) = \{X \in T_z Q^m | AX = -X\}$. This gives a contradiction.

Subcase 2.2. $q(\xi) = -2\lambda$.

The formula (6.9) gives that $AX = X$ for any $X \in \mathcal{C}$. From this, also together with $AN = N$ and $A\xi = -\xi$, the expression of the complex conjugation A on the decomposition $T_z Q^m = [N] \oplus [\xi] \oplus [\mathcal{C}]$ becomes the following

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then $\mathrm{Tr}A = 2(m-1)$. But as mentioned above, the trace of the complex conjugation TrA should vanish. Even in this case we have a contradiction.

Summing up the above discussions, we conclude that there do not exist any real hypersurfaces in Q^m with Killing normal Jacobi operator for an $\mathfrak{A}$-principal unit normal N.

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