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Balanced viscosity solutions to a rate-independent system for damage†

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This article is the third one in a series of papers by the authors on vanishing-viscosity solutions to rate-independent damage systems. While in the first two papers (Knees, D. *et al.* 2013 *Math. Models Methods Appl. Sci.* **23**(4), 565–616; Knees, D. *et al.* 2015 *Nonlinear Anal. Real World Appl.* **24**, 126–162) the assumptions on the spatial domain Ω were kept as general as possible (i.e., non-smooth domain with mixed boundary conditions), we assume here that $\partial\Omega$ is smooth and that the type of boundary conditions does not change. This smoother setting allows us to derive *enhanced regularity spatial properties* both for the displacement and damage fields. Thus, we are in a position to work with a stronger solution notion at the level of the viscous approximating system. The vanishing-viscosity analysis then leads us to obtain the existence of a stronger solution concept for the rate-independent limit system. Furthermore, in comparison to [18, 19], in our vanishing-viscosity analysis we do not switch to an artificial arc-length parameterization of the trajectories but we stay with the true physical time. The resulting concept of Balanced Viscosity solution to the rate-independent damage system thus encodes a more explicit characterization of the system behaviour at time discontinuities of the solution.

Key words: Rate-independent damage system, vanishing-viscosity approximation, Balanced Viscosity solutions

1 Introduction

We consider in a three-dimensional spatial domain Ω the rate-independent system for damage evolution

$$-\operatorname{div}\big(g(z)\mathbb{C}\varepsilon(u+u_D)\big) = \ell \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$\partial\mathcal{R}_1(z_t) + A_q z + f'(z) + \tfrac{1}{2}g'(z)\mathbb{C}\varepsilon(u+u_D) : \varepsilon(u+u_D) \ni 0 \qquad \text{in } \Omega \times (0,T), \tag{1.1b}$$

with $q > 3$,

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A_q the q-Laplacian type operator

$$A_q z = -\mathrm{div}((1 + |\nabla z|^2)^{(q/2)-1}\nabla z),$$

and the one-homogeneous dissipation potential

$$\mathcal{R}_1(v) = \int_\Omega \mathrm{R}_1(v)\,\mathrm{d}x \qquad \text{with } \mathrm{R}_1(v) = \begin{cases} |v| & \text{if } v \leqslant 0, \\ \infty & \text{otherwise.} \end{cases}$$

Here, $u : [0,T] \times \Omega \to \mathbb{R}^3$ denotes the displacement field and $z : [0,T] \times \Omega \to \mathbb{R}$ characterizes the time and space-dependent damage state in the body $\Omega \subset \mathbb{R}^3$. The natural state spaces for u and z are $\mathcal{U} = H^1_0(\Omega;\mathbb{R}^3)$ and $\mathcal{Z} = W^{1,q}(\Omega)$. The energy potential is of the form

$$\mathcal{E}(t,u,z) = \int_\Omega g(z)\frac{1}{2}\mathbb{C}(x)\varepsilon(u + u_D(t)) : \varepsilon(u + u_D(t)) + f(z) + \frac{1}{q}(1 + |\nabla z|^2)^{\frac{q}{2}}\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T)$ ($w \in \mathcal{U}$) is the strain tensor and u_D denotes the Dirichlet datum. Since the underlying energy $\mathcal{E}(t,\cdot,\cdot)$ in general is non-convex and since $\mathcal{R}_1$ is of linear growth, solutions to (1.1) might be discontinuous in time. In order to select reasonable jump discontinuities, we adopt here the vanishing-viscosity approach to the weak solvability of the rate-independent systems. This approach was pioneered in [11] and developed both for abstract rate-independent systems, cf. e.g., [20, 22, 25], and for applied problems in fracture and plasticity, see for instance [5, 9, 10, 16]. In the context of damage, in addition to the previously mentioned [18,19], we quote the recent [8,27]. Let us stress that in all of these papers the vanishing-viscosity analysis is performed by suitably adapting the original reparameterization technique of [11]. In [17], a time-incremental alternate minimization scheme for a damage model of Ambrosio–Tortorelli type without viscous regularization was investigated. It turned out that in the time-continuous limit this procedure results in a class of solutions that is closely related but not identical to those obtained by vanishing viscosity limits. Also here, the reparameterization technique of [11] was applied.

Hence, we approximate the rate-independent flow rule for the damage parameter by its viscous regularization, and thus address the *rate-dependent* system

$$-\,\mathrm{div}(g(z)\mathbb{C}\varepsilon(u{+}u_D)) = \ell \qquad \text{in } \Omega \times (0,T), \qquad (1.2a)$$

$$\partial\mathcal{R}_1(z_t) + \epsilon z_t + A_q z + f'(z) + \tfrac{1}{2}g'(z)\mathbb{C}\varepsilon(u + u_D) : \varepsilon(u + u_D) \ni 0 \text{ in } \Omega \times (0,T), \qquad (1.2b)$$

where the underlying regularized dissipation potential is given by

$$\mathcal{R}_\epsilon : L^2(\Omega) \to [0,+\infty] \text{ given by } \mathcal{R}_\epsilon(v) := \mathcal{R}_1(v) + \frac{\epsilon}{2}\|v\|^2_{L^2(\Omega)}, \qquad (1.3)$$

and $\epsilon > 0$ is the viscosity parameter. The goal is to perform the limit passage as $\epsilon \downarrow 0$ from (1.2) to (1.1), without switching to an artificial arc-length reparameterization of the trajectories, but *staying with the true physical time*. The basics for this approach to the construction of the resulting concept of *Balanced Viscosity* (BV) solutions to the limit rate-independent system were set in [22,25] for abstract rate-independent systems in finite-dimensional and infinite-dimensional Banach spaces, respectively. A notable feature of this vanishing-viscosity technique is that it allows for a *direct* limit passage from the *time-discrete* version of (1.2) to (1.1), as the viscosity parameter ϵ and the time discretization step τ *simultaneously* tend to zero with $\frac{\epsilon}{\tau} \to \infty$. This provides a *constructive approach* to BV solutions of system (1.1), which could also be further advanced from a numerical viewpoint.

While the techniques applied here have been developed in an abstract context in [25], let us emphasize that the existence and convergence results therein, (in particular [25, Theorems. 3.11 and 3.12]), are not directly applicable to the present damage system. The main point is that, in contrast to [25] in our setting, the dissipation potential $\mathcal{R}_1$ may take the value $+\infty$ to enforce the unidirectionality of the damaging process. This causes additional technical difficulties for the derivation of uniform *a priori* bounds. Moreover, the definition of BV solution has to be carefully tailored to accommodate this irreversibility constraint. Further, analytical difficulties occur due to the presence of the quadratic term on the left-hand side of the differential inclusion (1.1*b*), which at a first glance belongs to $L^1(\Omega)$, only. This necessitates a careful study of the spatial regularity properties of the displacement and the damage fields, which was already initiated in [18, 19].

The main results of this paper are the following:

Regularity: Thanks to the assumed smoothness of $\partial\Omega$ (made precise in Section 2.1) and the assumption $q > 3$ on the q-Laplacian regularization in (1.1*b*), solutions $u = u(t,z)$ of (1.1*a*) belong to $H^2(\Omega;\mathbb{R}^d) \cap W^{1,p}(\Omega;\mathbb{R}^d)$, for every $p \geqslant 1$ if the external data ℓ, u_D are smooth enough. Here, we exploit that $W^{1,q}(\Omega)$ embeds into the space of Hölder continuous functions, which in turn ensures enough spatial regularity for the coefficient $g(z)$ of the elasticity operator in (1.1*a*). We derive explicit bounds for the corresponding norms of u in terms of z by adapting arguments from [6] to our situation. These results improve the integrability properties of the quadratic term in (1.1*b*) and in (1.2*b*) and allow us to test a regularized version of (1.2*b*) by $\partial_t A_q z$, which ultimately guarantees that $\mathrm{D}_z\mathcal{E}(t,u(t,z),z) \in L^2(\Omega)$, again with uniform bounds, see Section 3.1. Let us mention that, in the case of the standard Laplacian regularization (i.e., $q = 2$), this regularity estimate was first proposed in [4] for doubly non-linear differential inclusions in phase transition modelling.

Based on the improved integrability property of $\mathrm{D}_z\mathcal{E}(t,u(t,z),z)$, we may consider sub-differentials and convex conjugate functions of the dissipation potentials with respect to the $L^2(\Omega)$ duality, instead of the $\mathcal{Z} - \mathcal{Z}^*$ duality. Furthermore, based on these results, we derive a generalized λ-convexity property of the energy functional, cf. Corollary 2.14, and a chain rule identity, cf. Lemma 2.17. The latter is essential for the existence proof of BV solutions for the damage system.

This chain rule identity was not available in the earlier [19], which still addressed the case of a q-Laplacian regularization in the damage flow rule, whereas in [18] some technical difficulties were smeared out by taking as regularizing operator a fractional Laplacian. Hence, in [19], we had to deal with a weaker notion of vanishing-viscosity solution compared to the present paper. In particular, in [19], it could be shown that the vanishing-viscosity limits satisfied an energy-dissipation

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(hereafter, we will use the abbreviation ED) inequality but, due to the lack of an appropriate chain rule, this could not be improved to an ED identity.

Existence and approximation of* BV *solutions: The concept of BV solution to the rate-independent system (1.1) consists of a local stability condition and of an ED balance that encodes the possible onset of viscous behaviour in the jump regime. More precisely, let $u(t,z) \in \mathcal{U}$ be the unique solution of (1.1a) and $\mathcal{I}(t,z) := \mathcal{E}(t,u(t,z),z)$ the reduced energy. We call a curve $z \in L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$ with $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$ a *BV* solution to (1.1) if z satisfies the local stability $(\mathrm{S_{loc}})$ and the ED balance

$$-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0) \quad \text{for all } t \in [0,T]\backslash \mathrm{J}_z, \quad (\mathrm{S_{loc}})$$

$$Var_{\mathfrak{f}}(z;[0,t]) + \mathcal{I}(t,z(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r \quad \text{for all } t \in [0,T], \quad (\mathrm{ED})$$

where J_z denotes the countable jump set of z. The quantity $Var_{\mathfrak{f}}(\cdot;[0,t])$ is a total variation functional that encompasses both the dissipation with respect to the one-homogeneous potential $\mathcal{R}_1$ in continuous parts of the solution, as well as the dissipation at jump discontinuities. At jump discontinuities, it reflects the viscous regularization term from (1.2b). While referring to Section 5.1 for its precise definition and to [25] for more comments on it, we may mention here its structure at a jump from z_- to z_+ for $t \in \mathrm{J}_z$. Indeed, the *jump contribution* $\Delta_{\mathfrak{f}}(t;z_-,z_+)$ to $Var_{\mathfrak{f}}(z;[0,t])$ is given by

$$\Delta_{\mathfrak{f}}(t;z_-,z_+) := \inf_{\vartheta\in\mathcal{T}_t^{\varrho}(z_-,z_+)} \int_0^1 \mathfrak{f}_t(\vartheta(r),\vartheta'(r))\,\mathrm{d}r\,, \tag{1.4}$$

$$\mathfrak{f}_t(\vartheta,\vartheta') = \mathcal{R}_1(\vartheta') + \|\vartheta'\|_{L^2(\Omega)} \inf_{\xi\in\partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t,\vartheta)-\xi\|_{L^2(\Omega)}\,, \tag{1.5}$$

where $\mathcal{T}_t^{\varrho}(z_-,z_+)$ denotes the set of admissible transition curves connecting z_- with z_+ and satisfying certain properties.

The appearance of the term from (1.4) in the vanishing-viscosity limit of (1.2) can be motivated by a comparison with the ED balance that is valid for solutions of the viscous system (1.2). In fact, we will show in Theorem 4.1 that solutions to (1.2) exist and that they satisfy for all $t \in [0,T]$ the relation

$$\int_0^t \mathcal{R}_\epsilon(\dot z_\epsilon) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z_\epsilon(r)))\,\mathrm{d}r + \mathcal{I}(t,z_\epsilon(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(r,z_\epsilon(r))\,\mathrm{d}r \tag{1.6}$$

with $\mathcal{R}_\epsilon^*(\eta) = \frac{1}{2\epsilon}\inf_{\xi\in\partial\mathcal{R}_1(0)}\|\eta-\xi\|_{L^2(\Omega)}^2$ provided that $\eta \in L^2(\Omega)$. It turns out that

$$\mathfrak{f}_t(t,z,v) = \inf_{\varepsilon>0}\left(\mathcal{R}_\epsilon(v) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(t,z))\right).$$

The challenge here is to perform a sharp limit analysis for $\epsilon \to 0$ in order to show that the dissipation integral in (1.6) tends to $Var_{\mathfrak{f}}(z;[0,t])$ as $\epsilon \to 0$.

The *main result of this paper*, Theorem 5.7, states the existence of BV solutions to the damage system (1.1) under suitable assumptions on the data z_0, u_D and ℓ. They are obtained from a vanishing-viscosity analysis of the time discretized version of the viscous system (1.2) as the time step size τ, the viscosity parameter ϵ and the ratio τ/ϵ tend to zero. The convergence of discrete solutions of corresponding numerical schemes to BV solutions is an immediate consequence. Let us stress that with the techniques from [25], we could prove the existence of BV solutions also by taking the vanishing-viscosity analysis of the *time-continuous* system in (1.2), as standardly done in works on the vanishing-viscosity approach to rate-independent systems. Here, we have opted for this simultaneous limit passage to highlight the constructive character of this approach.

The paper is organized as follows: In Section 2, we collect and prove the basic regularity and differentiability properties of the reduced energy $\mathcal{I}$ and prove the chain rule identity. Some of the arguments are taken from the earlier paper [19] but are adapted to the enhanced smoothness assumptions on the boundary $\partial\Omega$. In Section 3, we study a time-discrete version of the viscous damage system (1.2), derive the necessary *a priori* estimates and provide an ED inequality for suitable interpolants of the time incremental solutions. The main part of Section 3 is devoted to proving that $A_q z_k \in L^2(\Omega)$ for time incremental solutions z_k. In Section 4, we shortly address the existence of viscous solutions to the system (1.2). The main focus of the paper lies on the analysis of the vanishing-viscosity limit as both the viscosity parameter and the time step size tend to zero simultaneously (Sections 5 and 6). The notion of BV solutions is introduced and explained at length in Section 5, where also the main existence theorem is formulated and where further properties of BV solutions are discussed. The corresponding proofs are collected in Section 6. A short Appendix collects some elliptic regularity results that are key for our analysis.

We conclude by fixing some notation that will be used throughout the paper.

Notation 1.1 Throughout the paper, for a given Banach space X, we will by $\|\cdot\|_X$ denote its norm. In the case of product spaces $X \times \cdots \times X$, we will mostly write $\|\cdot\|_X$ in place of $\|\cdot\|_{X\times\cdots\times X}$, still allowing for some exceptions: for instance, we will keep both notations $\|\varepsilon(u)\|_{L^p(\Omega)}$ and $\|\varepsilon(u)\|_{L^p(\Omega;\mathbb{R}^{3\times3})}$. We will denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X, using the symbol $(\cdot,\cdot)_X$ for the scalar product in X, if X is a Hilbert space.

We will denote most of the positive constants occurring in the calculations, and depending on known quantities, by the symbols c, c', C, C', ..., whose meaning may vary even within the same line. Furthermore, the symbols I_i, $i = 0, 1, \ldots$, will be used as abbreviations for several integral terms appearing in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, I_1 will appear several times with different meanings.

2 Preliminaries and properties of the reduced energy

We start by collecting our standing assumptions on the reference domain Ω and on the energy functional $\mathcal{E}$ in Section 2.1. Combining these requirements, in Section 2.2, we will obtain two regularity results for the Euler–Lagrange equation associated with the minimization of the elastic energy. In Section 2.3, such results will have a pivotal role in deriving a series of properties of the reduced energy $\mathcal{I}$, at the core of our subsequent analysis.

2.1 Setup

Throughout the paper, we shall suppose that

Assumption 2.1 (Regularity of the domain) *$\Omega \subset \mathbb{R}^3$ is a bounded $C^{1,1}$-domain with Dirichlet boundary $\Gamma_D = \partial\Omega$.*

From now on, we shall denote the state spaces for the variables u and z by

$$\mathcal{U} := H_0^1(\Omega;\mathbb{R}^3), \qquad \mathcal{Z} := W^{1,q}(\Omega) \quad \text{with } q > 3.$$

We will denote by

$$W^{-1,p}(\Omega) \text{ the dual space of } W_0^{1,p'}(\Omega) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.$$

For later use, we recall here two crucial properties of the elliptic operator A_q holding for all z_1, z_2, $w \in \mathcal{Z}$:

$$\langle A_q z_1 - A_q z_2, z_1 - z_2 \rangle_{\mathcal{Z}} \geqslant c_q \int_\Omega (1 + |\nabla z_1|^2 + |\nabla z_2|^2)^{\frac{q-2}{2}} |\nabla(z_1 - z_2)|^2 \, dx, \tag{2.1}$$

$$|\langle A_q z_1 - A_q z_2, w \rangle_{\mathcal{Z}}| \leqslant c_q' \int_\Omega (1 + |\nabla z_1|^2 + |\nabla z_2|^2)^{(q-2)/2} |\nabla(z_1 - z_2)||\nabla w| \, dx. \tag{2.2}$$

These inequalities rely on the corresponding estimates for the function $G_q : \mathbb{R}^3 \to \mathbb{R}$ defined by $G_q(A) := \frac{1}{q}(1+|A|^2)^{q/2}$ and its gradient. In particular the following monotonicity estimate is valid

$$(\nabla G_q(A) - \nabla G_q(B)) \cdot (A - B) \geqslant c_q (1+|A|^2+|B|^2)^{(q-2)/2} |A-B|^2 \qquad \text{for all } A, B \in \mathbb{R}^3 \tag{2.3}$$

with the constant $c_q > 0$ as in (2.1). This is a consequence of the estimates provided in [12, Lemma 8.3].

The energy functional $\mathcal{E} : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ consists of two contributions. The first one, $\mathcal{I}_1$, only depends on the damage variable. The second one, $\mathcal{E}_2 = \mathcal{E}_2(t,u,z)$, is given by the sum of an elastic energy of the type $\int_\Omega g(z) W(\varepsilon(x, u + u_D(t)))\, dx$ with u_D a Dirichlet datum, and of the external loading term.

Assumption 2.2 (The energy functional) *We consider*

$$\mathcal{I}_1 : \mathcal{Z} \to \mathbb{R} \;\; \textit{defined by} \;\; \mathcal{I}_1(z) := \mathcal{I}_q(z) + \int_\Omega f(z)\, dx \;\; \textit{with}$$

$$\mathcal{I}_q(z) := \frac{1}{q} \int_\Omega (1 + |\nabla z|^2)^{\frac{q}{2}} \, dx, \; q > 3,$$

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and f fulfilling

$$f \in C^2(\mathbb{R}) \quad \textit{and} \quad \exists K_1, K_2 > 0 \quad \forall x \in \mathbb{R} : \quad f(x) \geqslant K_1|x| - K_2. \tag{2.4}$$

As for $\mathcal{E}_2$, linearly elastic materials are considered with an elastic energy density

$$W(x,\eta) = \frac{1}{2}\mathbb{C}(x)\eta : \eta \quad \textit{for } \eta \in \mathbb{R}^{3\times 3}_{sym} \textit{ and almost every } x \in \Omega.$$

Hereafter, we shall suppose for the elasticity tensor that

$$\mathbb{C} \in C^0_{\mathrm{lip}}(\overline{\Omega}; \mathrm{Lin}(\mathbb{R}^{3\times 3}_{sym}, \mathbb{R}^{3\times 3}_{sym})) \textit{ with } \mathbb{C}(x)\xi_1 : \xi_2 = \mathbb{C}(x)\xi_2 : \xi_1 \textit{ for all } x \in \Omega, \xi_i \in \mathbb{R}^{3\times 3}_{sym}, \tag{2.5a}$$

$$\exists \gamma_0 > 0 \quad \textit{for all } \xi \in \mathbb{R}^{3\times 3}_{sym} \textit{ and almost all } x \in \Omega : \quad \mathbb{C}(x)\xi : \xi \geqslant \gamma_0|\xi|^2. \tag{2.5b}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a further constitutive function such that

$$g \in C^2(\mathbb{R}) \textit{ with } g', g'' \in L^\infty(\mathbb{R}), \quad \textit{and } \exists \gamma_1, \gamma_2 > 0 \, \forall z \in \mathbb{R} \ : \ \gamma_1 \leqslant g(z) \leqslant \gamma_2. \tag{2.6}$$

Then, we take the elastic energy

$$\begin{aligned} &\mathcal{E}_2 : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \ \text{ defined by} \\ &\mathcal{E}_2(t,u,z) := \int_\Omega g(z) W(x, \varepsilon(u + u_D(t)))\,\mathrm{d}x - \ \langle \ell(t), u \rangle_{\mathcal{U}}, \end{aligned}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized strain tensor and $\ell \in C^0([0,T],\mathcal{U}^*)$ an external loading. Further requirements on ℓ and u_D will be specified in Assumption 2.9 ahead. For $u \in \mathcal{U}$ and $z \in \mathcal{Z}$, the stored energy is then defined by

$$\mathcal{E}(t,u,z) := \mathcal{I}_1(z) + \mathcal{E}_2(t,u,z). \tag{2.7}$$

Minimizing the functional $\mathcal{E}$ with respect to the displacements we obtain the *reduced energy*

$$\begin{aligned} &\mathcal{I} : [0,T] \times \mathcal{Z} \to \mathbb{R} \ \text{ given by } \ \mathcal{I}(t,z) := \mathcal{I}_1(z) + \mathcal{I}_2(t,z) \text{ with} \\ &\mathcal{I}_2(t,z) := \inf\{\mathcal{E}_2(t,v,z) \ : \ v \in \mathcal{U}\}. \end{aligned} \tag{2.8}$$

Remark 2.3 With the choice $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) = z^2 + \eta$ for $z \in [-1,1]$, with $\eta > 0$ a fixed parameter, and by a suitable smooth extension to $\mathbb{R} \setminus [-1,1]$ in such a way that (2.6) holds, and with $f(z) = \mu(z-1)^2$, and $q = 2$, one obtains an energy of Ambrosio-Tortorelli-type [3].

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2.2 Preliminary regularity results

We focus on the regularity properties of the operator $L_{g(z)}: H^1_0(\Omega;\mathbb{R}^3) \to W^{-1,2}(\Omega;\mathbb{R}^3)$ associated with the following bilinear form describing linear elasticity, i.e.,

$$\langle L_{g(z)}u, v\rangle := \int_\Omega g(z)\mathbb{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x \quad \text{for all } u, v \in H^1_0(\Omega;\mathbb{R}^3), \tag{2.9}$$

where $\mathbb{C}$ is from (2.5), g from (2.6) and z is a fixed element in $\mathcal{Z} = W^{1,q}(\Omega)$, with $q > 3$. Our first result extends [19, Lemma 2.3] to a wider range of exponents, cf. Remark 2.5 below.

Lemma 2.4 *Under Assumption 2.1, let $\mathbb{C}$ and g comply with* (2.5) *and* (2.6)*, respectively. Then, there holds*

(a) *For every $p > 1$ and $z \in W^{1,q}(\Omega)$ the operator $L_{g(z)} : W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega)$ is a topological isomorphism.*

(b) *Uniform estimate: For every $p_* > 2$ there exists a constant $c_{q,p_*} > 0$ such that for all $z \in W^{1,q}(\Omega)$ and $p \in [p'_*, p_*]$ it holds*

$$\|L^{-1}_{g(z)}\|_{W^{-1,p}(\Omega;\mathbb{R}^3)\to W^{1,p}_0(\Omega;\mathbb{R}^3)} \leqslant c_{q,p_*}(1 + \|\nabla z\|_{L^q(\Omega)})^{\hat{k}_* \frac{p_*|p-2|}{p(p_*-2)}}, \tag{2.10}$$

where $\hat{k}_* \in \mathbb{N}$ is the smallest integer with $\hat{k}_* > \frac{3q}{2(q-3)}$.

Proof For every $z \in W^{1,q}(\Omega)$ the coefficients of the elliptic operator $L_{g(z)}$ are continuous and bounded. Since by Assumption 2.1 the boundary of Ω is continuous as well, we may apply [31, Theorem 3], see also [21, Theorem 7.1], to obtain claim (a). The uniform estimate follows along the same lines as in the proof of [19, Lemma 2.3], relying on a recursion argument originally developed in [6]. □

Remark 2.5 Lemma 2.4 enhances [19, Lemma 2.3] thanks to the stronger regularity condition on the reference domain Ω, which in [19] was only required to fulfill these properties:

(i) The spaces $W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) = \{u \in W^{1,p}(\Omega;\mathbb{R}^d) : u|_{\Gamma_D} = 0\}$, $p \in (1,\infty)$ (and Γ_D with positive Hausdorff measure, but possibly $\Gamma_D \subsetneq \partial\Omega$, was allowed in [19]), form an interpolation scale.

(ii) There exists $p_* > 3$ such that for all $p \in [2, p_*]$ the operator $L : W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ is an isomorphism.

It was for such $p_* > 3$, in fact, that the isomorphism property (a) and the uniform estimate (2.10) were obtained in [19, Lemma 2.3]. Let us highlight that, instead, in Lemma 2.4 property (a) is guaranteed for all $p > 1$, and (2.10) is shown for *every* $p_* > 2$.

The most relevant consequence of Assumption 2.1 for our analysis, though, is given by the following enhanced elliptic regularity result. Lemma 2.6 extends [6, Lemma A.1] to our situation.

Lemma 2.6 *Under Assumption 2.1, let* $\mathbb{C}$ *and* g *comply with* (2.5) *and* (2.6), *respectively. Then, for all* $z \in W^{1,q}(\Omega)$ *the operator* $L_{g(z)} : \mathcal{U} \to \mathcal{U}^*$ *fulfills*

$$L_{g(z)}^{-1}(h) \in H^2(\Omega;\mathbb{R}^3) \quad \text{for all } h \in L^2(\Omega;\mathbb{R}^3),$$

and there exists $c_0 > 0$ *such that for all* $z \in W^{1,q}(\Omega)$ *and all* $h \in L^2(\Omega;\mathbb{R}^3)$

$$\|u\|_{H^2(\Omega)} \leqslant c_0(1 + \|\nabla z\|_{L^q(\Omega)})^\alpha(\|h\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \tag{2.11}$$

where $u = L_{g(z)}^{-1}(h)$ *and* $\alpha \geqslant 2$ *is the smallest integer bigger than or equal to* $q/(q-3)$.

Proof The proof of [6, Lemma A.1] can be directly transferred to our situation having in mind that for every $p \in (1,\infty)$ the operator

$$L_{\mathbb{C}} = L_{g(1)} : W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \to L^p(\Omega), \quad u \mapsto -\text{div}\mathbb{C}\varepsilon(u)$$

is a continuous isomorphism, cf. Theorem A.3. □

Remark 2.7 Observe that $\sup_{p\in[p'_*,p_*]} \frac{p_*|p-2|}{p(p_*-2)} \leqslant 1$. Hence, we can estimate from above the right-hand side of (2.10) by $(1 + \|\nabla z\|_{L^q(\Omega)})^{\hat{k}_*}$. Therefore, whenever applying estimates (2.10) and (2.11), possibly with two different elements z_1, $z_2 \in \mathcal{Z}$, we will simply use the quantity

$$P(z_1, z_2) := (1 + \|\nabla z_1\|_{L^q(\Omega)} + \|\nabla z_2\|_{L^q(\Omega)})^{k_*}\,, \tag{2.12}$$

where $k_* := \max\{\hat{k}_*, \alpha\} + 1$ with $\hat{k}_*$ from Lemma 2.4 and α from (2.11). With this, (2.11) can be rewritten in terms of the quantity P as

$$\|u\|_{H^2(\Omega)} \leqslant c_0 P(z,0)(\|h\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

In the sequel, we will frequently use the following regularity result from [29, Theorem 2 & Remark 3.5] for solutions of the q-Laplace equation:

Proposition 2.8 *For every* $q > 2$, *there exists a constant* $C_q > 0$ *such that for all* $f \in L^{q'}(\Omega)$ *it holds: If* $z \in W^{1,q}(\Omega)$ *satisfies* $\langle A_q z, \widetilde{z}\rangle = \langle f, \widetilde{z}\rangle$ *for all* $\widetilde{z} \in W^{1,q}(\Omega)$, *then for all* $\sigma \in (0, \frac{1}{q})$ *the function* z *belongs to* $W^{1+\sigma,q}(\Omega)$ *and*

$$\|z\|_{W^{1+\sigma,q}(\Omega)} \leqslant C_q(\|f\|_{L^{q'}(\Omega)} + \|z\|_{L^q(\Omega)}). \tag{2.13}$$

Note that on the right-hand side of (2.13) the L^q-norm of z appears since A_q is not bijective on $W^{1,q}(\Omega)$.

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2.3 Properties of the reduced energy

Relying on Lemmas 2.4 and 2.6, we will show that the reduced energy functional $\mathcal{I}$ enjoys a series of differentiability properties, which in fact improve those obtained in [19, Section 2.3], under the additional

Assumption 2.9 (The external loadings) *From now on, we will suppose that ℓ and u_D comply with the following requirements*

$$\begin{aligned} &\ell \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap \mathrm{C}^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3)),\\ &u_D \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^3)) \cap \mathrm{C}^{1,1}([0,T];W^{1,3}(\Omega;\mathbb{R}^3)). \end{aligned} \tag{2.14}$$

The starting point is the following result, which improves [19, Lemmas 2.6, 2.7].

Lemma 2.10 (Existence of minimizers for $\mathcal{E}(t,\cdot,z)$ & their continuous dependence on the data) *Under Assumptions 2.1, 2.2 and 2.9, for every $(t,z) \in [0,T] \times \mathcal{Z}$ there exists a unique minimizer $u_{\min}(t,z) \in \mathcal{U}$ for the stored energy $\mathcal{E}(t,\cdot,z)$ (2.7). In fact, $u_{\min}(t,z) \in H^2(\Omega;\mathbb{R}^3)$. Moreover, there exist positive constants c_1 and c_2 such that for all (t,z), (t_1,z_1), $(t_2,z_2) \in [0,T] \times \mathcal{Z}$ and for all $p_* > 2$*

$$\|u_{\min}(t,z)\|_{H^2(\Omega)} \leqslant c_1 P(z,0)\left(\|\ell(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^2(\Omega)}\right); \tag{2.15}$$

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,p}(\Omega)}\\ &\leqslant c_2 P(z_1,z_2)^2\left(|t_1-t_2| + \|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,p}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,p}(\Omega))}\right) \end{aligned} \tag{2.16}$$

for all $p \in [p_*', \min\{p_*, 3\}]$, with $P(\cdot,\cdot)$ defined by (2.12). In particular, there holds

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,3}(\Omega)}\\ &\leqslant c_2 P(z_1,z_2)^2\left(|t_1-t_2| + \|z_1-z_2\|_{L^6(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,3}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,3}(\Omega))}\right), \end{aligned} \tag{2.17}$$

Finally, the reduced energy $\mathcal{I}$ from (2.8) is bounded from below and in particular satisfies the following coercivity estimate:

$$\begin{aligned} &\exists\, c_3, c_4 > 0 \quad \forall (t,z) \in [0,T] \times \mathcal{Z} \ :\\ &\qquad \mathcal{I}(t,z) \geqslant c_3\left(\|\nabla z\|_{L^q(\Omega)}^q + \|z\|_{L^1(\Omega)} + \|u_{\min}(t,z)\|_{H^1(\Omega;\mathbb{R}^3)}^2\right) - c_4. \end{aligned} \tag{2.18}$$

Proof We refer to [18, Lemma 2.1] for the proof of the existence and uniqueness of $u_{\min}(t,z)$, as well as for estimate (2.18). Clearly, $u_{\min}(t,z)$ satisfies $L_{g(z)}u_{\min}(t,z) = -L_{g(z)}u_D(t) - \ell(t)$. Observe that $L_{g(z)}u_D(t) \in L^2(\Omega)$. Indeed, by the assumptions on g, $\mathbb{C}$ and since $u_D(t) \in H^2(\Omega)$, we have $g(z)\mathrm{div}(\mathbb{C}\varepsilon(u_D(t)) \in L^2(\Omega)$. On the other hand, $\mathbb{C}\varepsilon(u_D(t))\nabla_x g(z) = g'(z)\mathbb{C}\varepsilon(u_D(t))\nabla z \in L^2(\Omega)$, which follows by Hölder's inequality taking into account that $H^1(\Omega) \subset L^6(\Omega)$ and that $q > 3$. Moreover, it holds $\|L_{g(z)}u_D(t)\|_{L^2(\Omega)} \leqslant c(1 + \|\nabla z\|_{L^q(\Omega)})\|u_D(t)\|_{H^2(\Omega)}$. Hence, it follows from (2.11), Remark

2.7 and (2.10) with $p = 2$ that

$$\begin{aligned}\|u_{\min}(t,z)\|_{H^2(\Omega)} &\leqslant c_0(1+\|\nabla z\|_{L^q(\Omega)})^\alpha(\|\ell(t)\|_{L^2(\Omega)} + \|\mathrm{div}(g(z)\mathbb{C}\varepsilon(u_D(t))\|_{L^2(\Omega)} \\ &\qquad + \|u_{\min}(t,z)\|_{H^1(\Omega)}) \\ &\leqslant c(1+\|\nabla z\|_{L^q(\Omega)})^\alpha\big(\|\ell(t)\|_{L^2(\Omega)} + (1+\|\nabla z\|_{L^q(\Omega)})\|u_D(t)\|_{H^2(\Omega)}\big) \\ &\leqslant c_1 P(z,0)\big(\|\ell(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^2(\Omega)}\big).\end{aligned}$$

All in all, we conclude (2.15).

Finally, in order to show (2.16), we mimic the argument from the proofs of [18, Lemma 2.2] and [19, Lemma 2.7]. Namely, for $i = 1, 2$, let $u_i := u_{\min}(t_i, z_i) \in H^2(\Omega;\mathbb{R}^3)$. From the corresponding Euler–Lagrange equations, we obtain that $u_1 - u_2$ satisfies for all $v \in \mathcal{U}$

$$\begin{aligned}\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2) : \varepsilon(v)\,\mathrm{d}x &= \int_\Omega \big(g(z_2)-g(z_1)\big)\mathbb{C}\varepsilon(u_2) : \varepsilon(v)\,\mathrm{d}x \\ &- \int_\Omega \big(g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\big) : \varepsilon(v)\,\mathrm{d}x + \int_\Omega (\ell(t_1)-\ell(t_2))\,v\,\mathrm{d}x.\end{aligned} \tag{2.19}$$

Observe that (2.19) extends to test functions $v \in W_0^{1,6/5}(\Omega;\mathbb{R}^3)$. This can be seen as follows: Since g is bounded, cf. (2.6), and since $u_j \in H^2(\Omega;\mathbb{R}^3)$, for $i, j \in \{1, 2\}$ the product $g(z_i)\varepsilon(u_j)$ belongs to $L^6(\Omega;\mathbb{R}^{3\times3})$. Thanks to Assumption 2.9, the same is true for the terms involving the data u_D and ℓ. Since $6' = 6/5$, we conclude *via* a density argument. Hence, $u_1 - u_2$ fulfills for all $v \in W_0^{1,6/5}(\Omega;\mathbb{R}^3)$ the relation

$$\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2) : \varepsilon(v)\,\mathrm{d}x = \langle\tilde{\ell}_{1,2}, v\rangle_{W_0^{1,6/5}(\Omega;\mathbb{R}^3)},$$

where $\tilde{\ell}_{1,2} \in W^{-1,6}(\Omega;\mathbb{R}^3)$ subsumes the terms on the right-hand side of (2.19). We now fix an arbitrary $p_* > 2$ and apply estimate (2.10) with $p \in [p_*', \min\{p_*, 3\}]$. Note that the restriction $p \leqslant 3$ is in view of conditions (2.14) on ℓ and u_D. We thus obtain $\|u_1-u_2\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leqslant c_{q,p_*}P(z_1,0)\|\tilde{\ell}_{1,2}\|_{W^{-1,p}(\Omega;\mathbb{R}^3)}$. Hence,

$$\begin{aligned}\|u_1-u_2\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leqslant c_{p_*,q}P(z_1,0)\,\big(&\|\ell(t_1)-\ell(t_2)\|_{W^{-1,p}(\Omega;\mathbb{R}^3)} \\ &+\|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^p(\Omega;\mathbb{R}^{3\times3})} \\ &+\|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\big).\end{aligned} \tag{2.20}$$

Now, the Lipschitz continuity of g (with the Lipschitz constant C_g) and Hölder's inequality imply that

$$\begin{aligned}\|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^p(\Omega;\mathbb{R}^{3\times3})} &\leqslant C_g\|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)}\|\varepsilon(u_2)\|_{L^6(\Omega;\mathbb{R}^{3\times3})} \\ &\leqslant CP(z_2,0)\,\big(\|\ell(t)\|_{L^2(\Omega)}+\|u_D(t)\|_{H^2(\Omega)}\big)\,\|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)},\end{aligned} \tag{2.21}$$

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where the second estimate follows from (2.15) and from the fact that $\|\varepsilon(u_2)\|_{L^6(\Omega;\mathbb{R}^{3\times 3})} \leqslant C\|u_2\|_{H^2(\Omega;\mathbb{R}^3)}$ by the Sobolev embeddings. Moreover,

$$\begin{aligned}
&\|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \\
&\quad\leqslant \|g(z_1)(\mathbb{C}\varepsilon(u_D(t_1)) - \mathbb{C}\varepsilon(u_D(t_2)))\|_{L^p(\Omega)} + \|(g(z_1) - g(z_2))\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \\
&\quad\leqslant \gamma_2|t_1 - t_2|\|u_D(t)\|_{C^1([0,T];W^{1,p}(\Omega))} + C\|u_D\|_{L^\infty(0,T;H^2(\Omega))}\|z_1 - z_2\|_{L^{6p/(6-p)}(\Omega)},
\end{aligned}$$

where the last estimate follows from the fact that $\|g(z_1)\|_{L^\infty(\Omega)} \leqslant \gamma_2$ by (2.6), and the fact that for $p \leqslant 6$, we have $\|\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \leqslant C\|u_D\|_{L^\infty(0,T;H^2(\Omega))}$. All in all, we conclude (2.16), whence (2.17) observing that, for $p = 3$ one has $\frac{6p}{6-p} = 6$. □

Concerning the differentiability in time, we have the following analogue of [19, Lemma 2.9], [18, Lemma 2.3]:

Lemma 2.11 *Under Assumptions 2.1, 2.2 and 2.9, for every $z \in \mathcal{Z}$ the map $t \mapsto \mathcal{I}(t,z)$ is in* $C^1([0,T];\mathbb{R})$ *with*

$$\partial_t\mathcal{I}(t,z) = \int_\Omega g(z)\mathbb{C}\varepsilon(u_{\min}(t,z) + u_D(t)) : \varepsilon(\dot{u}_D(t))\,\mathrm{d}x - \langle\dot{\ell}(t), u_{\min}(t,z)\rangle_{H_0^1(\Omega;\mathbb{R}^3)}. \tag{2.22}$$

Moreover, there exists a constant $c_5 > 0$ such that for all $t \in [0,T]$, $z \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t,z)| \leqslant c_5\big(\|u_D\|^2_{C^1([0,T];H^1(\Omega;\mathbb{R}^3))} + \|\ell\|^2_{C^1([0,T];W^{-1,2}(\Omega;\mathbb{R}^3))}\big). \tag{2.23}$$

Finally, there exists a constant $c_6 > 0$ depending on $\|\ell\|_{C^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3))}$ and $\|u_D\|_{C^{1,1}([0,T];W^{1,3}(\Omega))}$ such that for all $t_i \in [0,T]$ and $z_i \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t_1,z_1) - \partial_t\mathcal{I}(t_2,z_2)| \leqslant c_6P(z_1,z_2)^2\big(|t_1 - t_2| + \|z_1 - z_2\|_{L^2(\Omega)}\big). \tag{2.24}$$

Let us stress that the quantity on the right-hand side of estimate (2.23) is *independent* of $z \in \mathcal{Z}$.

Proof Relation (2.22) follows from direct calculations, while estimate (2.23) is a direct consequence of Hölder's inequality in combination with the uniform estimate (2.10) for $p = 2$. For the proof of (2.24), we start from

$$\begin{aligned}
&\partial_t\mathcal{I}(t_1,z_1) - \partial_t\mathcal{I}(t_2,z_2) \\
&= \int_\Omega (g(z_1) - g(z_2))\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1) + u_D(t_1))) : \varepsilon(\dot{u}_D(t_1))\,\mathrm{d}x \\
&\quad + \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1) + u_D(t_1)) - \varepsilon(u_{\min}(t_2,z_2) + u_D(t_2))) : \varepsilon(\dot{u}_D(t_1))\,\mathrm{d}x \\
&\quad + \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_2,z_2) + u_D(t_2))) : (\varepsilon(\dot{u}_D(t_1)) - \varepsilon(\dot{u}_D(t_2)))\,\mathrm{d}x \\
&\quad - \langle\dot{\ell}(t_1) - \dot{\ell}(t_2), u_{\min}(t_1,z_1)\rangle + \langle\dot{\ell}(t_2), u_{\min}(t_2,z_2) - u_{\min}(t_1,z_1)\rangle \doteq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

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To estimate I_1 and I_3, we apply Hölder's inequality and exploit the regularity of the data stated in (2.14), that g is bounded and Lipschitz-continuous and that $u_{\min}$ belongs to $H^2(\Omega)$, cf. (2.15). To estimate I_2, we additionally apply estimate (2.16) with $p = 3/2$ (which yields $6p/(6-p) = 2$) and obtain

$$\begin{aligned} I_2 &\leqslant c\|\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1)) - \varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))\|_{L^{3/2}(\Omega)}\|\varepsilon(\dot u_D(t_1))\|_{L^3(\Omega)} \\ &\leqslant cP(z_1,z_2)^2\left(|t_1-t_2| + \|z_1-z_2\|_{L^2(\Omega)}\right)\left(\|\ell\|_{C^1([0,T];W^{-1,3/2}(\Omega))} + \|u_D(t)\|_{C^1([0,T];W^{1,3/2}(\Omega))}\right). \end{aligned}$$

By (2.14) and (2.16), we also estimate I_4 and I_5. □

We now discuss the differentiability of $\mathcal{I}$ with respect to z. Let $\mathrm{D}_z\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathcal{Z}^*$ denote the Gâteaux-differential of the functional $\mathcal{I}(t,\cdot)$. For the proof of the following result, we refer to [19, Lemma 2.10], [18, Lemma 2.4].

Lemma 2.12 *Under Assumptions 2.1, 2.2 and 2.9, for all $t \in [0,T]$ the functional $\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is Gâteaux-differentiable at all $z \in \mathcal{Z}$, and for all $\eta \in \mathcal{Z}$ we have*

$$\langle \mathrm{D}_z\mathcal{I}(t,z),\eta\rangle_{\mathcal{Z}} = \langle A_q z,\eta\rangle_{\mathcal{Z}} + \int_\Omega f'(z)\eta\,\mathrm{d}x + \int_\Omega g'(z)\widetilde{W}(t,\nabla u_{\min}(t,z))\eta\,\mathrm{d}x, \tag{2.25}$$

where we use the abbreviation $\widetilde{W}(t,\nabla v) = W(x,\varepsilon(v+u_D(t))) = \frac{1}{2}\mathbb{C}\varepsilon(v+u_D(t)):\varepsilon(v+u_D(t))$. In particular, the following estimate holds with a constant c_7 that depends on the data ℓ, u_D, but is independent of t and z:

$$\forall\,(t,z) \in [0,T]\times\mathcal{Z} \;:\; \|\mathrm{D}_z\mathcal{I}(t,z)\|_{\mathcal{Z}^*} \leqslant c_7\left(\|z\|_{\mathcal{Z}}^{q-1} + \|f'(z)\|_{L^\infty(\Omega)} + 1\right). \tag{2.26}$$

Hereafter, we will use the short-hand notation

$$\widetilde{\mathcal{I}}(t,z) := \mathcal{I}_2(t,z) + \int_\Omega f(z)\,\mathrm{d}x \quad \text{for all } (t,z) \in [0,T]\times\mathcal{Z} \tag{2.27}$$

with $\mathcal{I}_2$ from (2.8) as the part of the reduced energy collecting all lower order terms. Accordingly, $\mathrm{D}_z\mathcal{I}$ from (2.25) decomposes as

$$\mathrm{D}_z\mathcal{I}(t,z) = A_q z + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \quad \text{for all } (t,z) \in [0,T]\times\mathcal{Z}. \tag{2.28}$$

In view of (2.25) and taking into account the $H^2(\Omega;\mathbb{R}^3)$-regularity of $u_{\min}$ from Lemma 2.6, the term $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ can be identified with an element of $L^2(\Omega)$. In Lemma 2.13, below we will even show that the map $(t,z) \mapsto \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ is Lipschitz continuous w.r.t. a suitable *Lebesgue* norm. Therefore, with the symbol $\mathrm{D}_z\widetilde{\mathcal{I}}$, we shall denote both the derivative of $\widetilde{\mathcal{I}}$ as an operator, and the corresponding density in $L^2(\Omega)$. Accordingly, we shall write

$$\text{for a given } v \in L^2(\Omega) \quad \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)v\,\mathrm{d}x \quad \text{in place of} \quad \langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,z),v\rangle_{L^2(\Omega)}. \tag{2.29}$$

For $h \in \mathrm{C}^0(\mathbb{R})$ and $z_1, z_2 \in \mathcal{Z}$ let

$$C_h(z_1,z_2) = \max\{\,|h(s)| \;:\; |s| \leqslant \|z_1\|_{L^\infty(\Omega)} + \|z_2\|_{L^\infty(\Omega)}\,\}. \tag{2.30}$$

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This notation will be used along the proof of the following lemma.

Lemma 2.13 *Under Assumptions 2.1, 2.2 and 2.9, there exists a constant* $c_8 > 0$ *that depends on the norms* $\|\ell\|_{C^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3))}$ *and* $\|u_D\|_{C^1([0,T];W^{1,3}(\Omega;\mathbb{R}^3))}$ *such that for all* $t_i \in [0,T]$ *and all* $z_i \in \mathcal{Z}$ *it holds*

$$\left|\widetilde{\mathcal{I}}(t_1,z_1) - \widetilde{\mathcal{I}}(t_2,z_2)\right| \leqslant c_8(1 + C_{f'}(z_1,z_2) + P(z_1,z_2)^3)\left(|t_1-t_2| + \|z_1-z_2\|_{L^3(\Omega)}\right), \quad (2.31)$$

with $C_{f'}(z_1,z_2)$ *as in* (2.30), *corresponding to* $h = f'$. *Further, now with* $h = f''$ *in* (2.30),

$$\begin{aligned}&\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^2(\Omega)} \\ &\quad \leqslant c_8\left(1 + C_{f''}(z_1,z_2) + P(z_1,z_2)^3\right)\left(|t_1-t_2| + \|z_1-z_2\|_{L^6(\Omega)}\right),\end{aligned} \quad (2.32)$$

$$\begin{aligned}&\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^{4/3}(\Omega)} \\ &\quad \leqslant c_8\left(1 + C_{f''}(z_1,z_2) + P(z_1,z_2)^3\right)\left(|t_1-t_2| + \|z_1-z_2\|_{L^4(\Omega)}\right),\end{aligned} \quad (2.33)$$

and

$$\|\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)\|_{L^2(\Omega)} \leqslant c_8(1 + \|f'(z)\|_{L^\infty(\Omega)} + P(z,0)^2) \qquad \text{for all } (t,z) \in [0,T]\times\mathcal{Z}. \quad (2.34)$$

Proof Although the proof follows the same lines as that of [19, Lemma 2.12], let us briefly see how the improved estimates (2.15) and (2.17) lead to (2.31), (2.32) and (2.33), while we will omit the calculations for (2.34). As for (2.31), we observe that

$$\begin{aligned}\left|\widetilde{\mathcal{I}}(t_1,z_1) - \widetilde{\mathcal{I}}(t_2,z_2)\right| \leqslant &\int_\Omega |f(z_1) - f(z_2)|\,\mathrm{d}x + \int_\Omega |g(z_1) - g(z_2)||\widetilde{W}(t_1,\nabla u_1)|\,\mathrm{d}x \\ &+ \int_\Omega |g(z_2)||\widetilde{W}(t_1,\nabla u_1) - \widetilde{W}(t_2,\nabla u_2)|\,\mathrm{d}x + |\langle \ell(t_1) - \ell(t_2), u_1\rangle_{\mathcal{U}}| \\ &+ |\langle \ell(t_2), u_1 - u_2\rangle_{\mathcal{U}}| \doteq I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where $u_i := u_{\min}(t_i,z_i) \in H^2(\Omega;\mathbb{R}^3)$ and, as above, $\widetilde{W}(t_i,\nabla u_i) = \frac{1}{2}\mathbb{C}\varepsilon(u_i{+}u_D(t_i)){:}\varepsilon(u_i{+}u_D(t_i))$ for $i = 1,2$. We observe that (cf. Notation (2.30))

$$\begin{aligned}I_1 &\leqslant C_{f'}(z_1,z_2)\|z_1 - z_2\|_{L^1(\Omega)}, \\ I_2 &\leqslant C\|z_1 - z_2\|_{L^2(\Omega)}\|\varepsilon(u_1 + u_D(t_1))\|_{L^3(\Omega)}\|\varepsilon(u_1 + u_D(t_1))\|_{L^6(\Omega)} \\ &\leqslant C'P(z_1,0)^2\|z_1 - z_2\|_{L^2(\Omega)}, \\ I_3 &\leqslant C\|\varepsilon(u_1 + u_D(t_1)) + \varepsilon(u_2 + u_D(t_2))\|_{L^2(\Omega)}\|\varepsilon(u_1 + u_D(t_1)) - \varepsilon(u_2 + u_D(t_2))\|_{L^2(\Omega)} \\ &\leqslant CP(z_1,z_2)P(z_1,z_2)^2(|t_1 - t_2| + \|z_1 - z_2\|_{L^3(\Omega)}), \\ I_4 &\leqslant C|t_1 - t_2|\|u_1\|_{H^1(\Omega)} \leqslant C'|t_1 - t_2|, \\ I_5 &\leqslant C\|u_1 - u_2\|_{H^1(\Omega)} \leqslant CP(z_1,z_2)^2(|t_1 - t_2| + \|z_1 - z_2\|_{L^3(\Omega)}),\end{aligned}$$

where, in the estimate for I_2 we have exploited (2.15), while in the estimates for I_3 and I_5 we have also resorted to (2.16) with $p = 2$. The estimate for I_4 follows from (2.14). All in all, we conclude (2.31).

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As for (2.32), we have that

$$\begin{aligned}\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1)-\mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^2(\Omega)} &\leqslant \|f'(z_1)-f'(z_2)\|_{L^2(\Omega)}+\|(g'(z_1)-g'(z_2))\widetilde{W}(t_1,\nabla u_1)\|_{L^2(\Omega)}\\ &\quad+\|g'(z_2)(\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2))\|_{L^2(\Omega)}\doteq I_6+I_7+I_8\,.\end{aligned}$$

We observe that $I_6 \leqslant C_{f''}(z_1,z_2)\|z_1-z_2\|_{L^2(\Omega)}$, while

$$\begin{aligned}I_7 &\leqslant C\|z_1-z_2\|_{L^3(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|_{L^6(\Omega)} \leqslant C'\|z_1-z_2\|_{L^3(\Omega)}P(z_1,0),\\ I_8 &\leqslant C\|\varepsilon(u_1+u_D(t_1))+\varepsilon(u_2+u_D(t_2))\|_{L^6(\Omega)}\|\varepsilon(u_1+u_D(t_1))-\varepsilon(u_2+u_D(t_2))\|_{L^3(\Omega)}\\ &\leqslant C'P(z_1,z_2)^3(|t_1-t_2|+\|z_1-z_2\|_{L^6(\Omega)})\,.\end{aligned}$$

thanks to estimates (2.15) and (2.17) and the fact that $g', g'' \in L^\infty(\mathbb{R})$. The proof of (2.33) follows the very same lines: we estimate $\|f'(z_1)-f'(z_2)\|_{L^{4/3}(\Omega)}$ by means of $C_{f''}(z_1,z_2)\|z_1-z_2\|_{L^{4/3}(\Omega)}$, while we have with Hölder's inequality

$$\begin{aligned}\|(g'(z_1)-g'(z_2))\widetilde{W}(t_1,\nabla u_1)\|_{L^{4/3}(\Omega)} &\leqslant C\|z_1-z_2\|_{L^4(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|^2_{L^4(\Omega)}\\ &\leqslant C'\|z_1-z_2\|_{L^4(\Omega)},\end{aligned}$$

where in the last estimate we applied (2.10) with $p=4$ to the term $\|\varepsilon(u_1)\|_{L^4(\Omega)}$. Finally,

$$\begin{aligned}&\|g'(z_2)(\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2))\|_{L^{4/3}(\Omega)}\\ &\quad\leqslant C\|\varepsilon(u_1+u_D(t_1))+\varepsilon(u_2+u_D(t_2))\|_{L^4(\Omega)}\|\varepsilon(u_1+u_D(t_1))-\varepsilon(u_2+u_D(t_2))\|_{L^2(\Omega)}\\ &\quad\leqslant C'P(z_1,z_2)^3(|t_1-t_2|+\|z_1-z_2\|_{L^3(\Omega)}).\end{aligned}$$

This concludes the proof. □

From all of the above results, and in particular from Lemma 2.13, we now draw a series of consequences on which our subsequent analysis will rely. First of all, we observe the Fréchet differentiability of the functional $z \in \mathcal{Z} \mapsto \mathcal{I}(t,z)$. This is due to the continuity of the mapping $z \in \mathcal{Z} \mapsto \mathrm{D}_z\mathcal{I}(t,z) \in \mathcal{Z}^*$, which relies on the continuity of $z \mapsto A_q z$ and of $z \mapsto \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$. If restricted to bounded sets in $\mathcal{Z}$, the latter mapping is even continuous with values in $L^2(\Omega)$ w.r.t. to $L^6(\Omega)$-convergence for z, cf. (2.32). The restriction of the power functional $\partial_t\mathcal{I}$ is continuous w.r.t. $L^2(\Omega)$-convergence for z. Taking into account that $\mathcal{Z} \Subset L^6(\Omega)$, we may then claim the continuity of $\mathrm{D}_z\widetilde{\mathcal{I}}$ and $\partial_t\mathcal{I}$ w.r.t. weak convergence in $\mathcal{Z}$.

Corollary 2.14 (Fréchet differentiability of $\mathcal{I}$) *Under Assumptions 2.1, 2.2 and 2.9, the functional $\mathcal{I}$ is Fréchet differentiable on $[0,T]\times\mathcal{Z}$ and*

$$t_n \to t \text{ and } z_n \to z \text{ strongly in } \mathcal{Z} \text{ implies } \mathrm{D}_z\mathcal{I}(t_n,z_n) \to \mathrm{D}_z\mathcal{I}(t,z) \text{ strongly in } \mathcal{Z}^*. \tag{2.35}$$

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Furthermore,

$$\begin{aligned}&t_n \to t \text{ and } z_n \rightharpoonup z \text{ in } \mathcal{Z} \text{ implies}\\ &\liminf_{n\to\infty} \mathcal{I}(t_n,z_n) \geqslant \mathcal{I}(t,z), \quad \widetilde{\mathcal{I}}(t_n,z_n) \to \widetilde{\mathcal{I}}(t,z), \quad \partial_t\mathcal{I}(t_n,z_n) \to \partial_t\mathcal{I}(t,z),\\ &\mathrm{D}_z\widetilde{\mathcal{I}}(t_n,z_n) \to \mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \text{ strongly in } L^2(\Omega).\end{aligned} \tag{2.36}$$

We now derive a generalized λ-convexity property for $\mathcal{I}(t,\cdot)$ involving the $H^1(\Omega)$ and the $L^1(\Omega)$-norm, see (2.38) below. This estimate is valid on bounded sets in $\mathcal{Z}$. Indeed, note that the constant modulating the $L^1(\Omega)$-norm in (2.38) depends on the radius of a $\mathcal{Z}$-ball.

Corollary 2.15 (λ-convexity of $\mathcal{I}$) *Under Assumptions 2.1, 2.2 and 2.9, there exists a constant $\beta > 0$ and for every $M > 0$ there exists $\Lambda_M > 0$ such that for every $t \in [0,T]$, z_1, $z_2 \in \mathcal{Z}$ with $\|z_1\|_{\mathcal{Z}} + \|z_2\|_{\mathcal{Z}} \leqslant M$ and for every $\theta \in [0,1]$ the functional $\mathcal{L}$ with*

$$\mathcal{L}(t,z) := \mathcal{I}(t,z) + \frac{1}{2}\|z\|^2_{L^2(\Omega)} \tag{2.37}$$

complies with

$$\begin{aligned}\mathcal{L}(t,(1-\theta)z_1 + \theta z_2) \leqslant &(1-\theta)\mathcal{L}(t,z_1) + \theta\mathcal{L}(t,z_2)\\ &- \theta(1-\theta)(\beta\|z_1-z_2\|^2_{H^1(\Omega)} - \Lambda_M\|z_1-z_2\|^2_{L^1(\Omega)}).\end{aligned} \tag{2.38}$$

Proof From (2.3) it follows that the mapping $A \in \mathbb{R}^3 \mapsto G_q(A) - \frac{c_q}{2}|A|^2$ is convex. This entails that $A \mapsto G_q(A)$ is c_q-convex, i.e., there holds $G_q((1-\theta)A_1 + \theta A_2) \leqslant (1-\theta)G_q(A_1) + \theta G_q(A_2) - \frac{c_q}{2}\theta(1-\theta)|A_1-A_2|^2$ for every A_1, $A_2 \in \mathbb{R}^3$ and $\theta \in [0,1]$. As a consequence, we have that

$$\mathcal{I}_q((1-\theta)z_1 + \theta z_2) \leqslant (1-\theta)\mathcal{I}_q(z_1) + \theta\mathcal{I}_q(z_2) - \frac{c_q}{2}\theta(1-\theta)\int_\Omega |\nabla(z_1-z_2)|^2\,\mathrm{d}x\,. \tag{2.39}$$

As for $\widetilde{\mathcal{I}}$, with trivial calculations, we have that

$$\begin{aligned}&\widetilde{\mathcal{I}}(t,(1-\theta)z_1 + \theta z_2) - (1-\theta)\widetilde{\mathcal{I}}(t,z_1) - \theta\widetilde{\mathcal{I}}(t,z_2)\\ &\quad = (1-\theta)\left(\widetilde{\mathcal{I}}(t,(1-\theta)z_1 + \theta z_2) - \widetilde{\mathcal{I}}(t,z_1)\right) + \theta\left(\widetilde{\mathcal{I}}(t,(1-\theta)z_1 + \theta z_2) - \widetilde{\mathcal{I}}(t,z_2)\right) \doteq I_1 + I_2.\end{aligned}$$

There holds

$$\begin{aligned}I_1 &= (1-\theta)\int_0^1\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-s)z_1 + s((1-\theta)z_1 + \theta z_2))\theta(z_2-z_1)\,\mathrm{d}x\,\mathrm{d}s\\ &= (1-\theta)\theta\int_0^1\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-s)z_1 + s((1-\theta)z_1 + \theta z_2)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1)\right)(z_2-z_1)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad - (1-\theta)\theta\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1)(z_1-z_2)\,\mathrm{d}x \doteq I_{1,1} + I_{1,2}\,.\end{aligned}$$

We now estimate $I_{1,1}$ by using Hölder's inequality and inequality (2.33), taking into account that $(1-s)z_1 + s((1-\theta)z_1 + \theta z_2) - z_1 = s\theta(z_2-z_1)$. Therefore,

$$|I_{1,1}| \leqslant c_8\theta(1-\theta)\int_0^1 (1 + C_{f'}(z_1,\zeta_{1,2}) + P(z_1,\zeta_{1,2})^3)\|s\theta(z_2-z_1)\|_{L^4(\Omega)}\|z_2-z_1\|_{L^4(\Omega)}\,\mathrm{d}s$$
$$\leqslant \widetilde{C}_1(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)},$$

where we have used the place-holder $\zeta_{1,2} := (1-s)z_1 + s((1-\theta)z_1 + \theta z_2)$, and where $\widetilde{C}_1(M) > 0$ depends on the constant M that bounds $\|z_1\|_{\mathcal{Z}}$ and $\|z_1\|_{\mathcal{Z}}$. With analogous calculations one has that

$$I_2 \leqslant \widetilde{C}_1(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)} + \underbrace{(1-\theta)\theta\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_2)(z_1-z_2)\,\mathrm{d}x}_{I_{2,2}}.$$

Therefore, estimating $I_{1,2} + I_{2,2} \leqslant \widetilde{C}_2(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)}$ with the same arguments as above, we conclude that

$$\widetilde{\mathcal{I}}(t,(1-\theta)z_1 + \theta z_2) \leqslant (1-\theta)\widetilde{\mathcal{I}}(t,z_1) + \theta\widetilde{\mathcal{I}}(t,z_2) + \frac{\widetilde{C}(M)}{2}(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)} \tag{2.40}$$

for some $\widetilde{C}(M) > 0$. We now combine (2.39) with (2.40). Adding to this the trivial identity

$$\frac{1}{2}\|(1-\theta)z_1 + \theta z_2\|^2_{L^2(\Omega)} = \frac{(1-\theta)}{2}\|z_1\|^2_{L^2(\Omega)} + \frac{\theta}{2}\|z_2\|^2_{L^2(\Omega)} - \frac{(1-\theta)\theta}{2}\|z_1-z_2\|^2_{L^2(\Omega)},$$

and using Ehrling's Lemma, cf. e.g., [28, Theorem 7.30], to estimate $\|\eta\|^2_{L^4(\Omega)} \leqslant \delta\|\eta\|^2_{H^1(\Omega)} + C(\delta)\|\eta\|^2_{L^1(\Omega)}$ for arbitrary $\delta > 0$, finally results in (2.38). □

A slight generalization of property (2.38) was proposed in [25, Section 3.4, (3.63)] as a sufficient condition for a sort of "uniform differentiability" condition for $\mathcal{I}(t,\cdot)$, cf. (2.41) ahead, which was in turn introduced in [25, Section 2.1, (E.3)]. As we will see, (2.41) is at the core of key chain rule properties for viscous solutions to (1.2) and for BV solutions to (1.1), cf. Lemma 2.17 and Theorem 5.8 ahead. As a trivial consequence of (2.41), we have a monotonicity property for the Fréchet subdifferential $\mathcal{D}_z\mathcal{I}$, which will allow us to prove the uniqueness of solutions for the time-incremental problems giving rise to discrete solutions. The uniqueness is crucial for our analysis.

Corollary 2.16 *Under Assumptions 2.1, 2.2 and 2.9, for every $M > 0$ there exist constants c_9, $c_{10}(M) > 0$ such that for all $t \in [0,T]$, $z_i \in \mathcal{Z}$, $i = 1,2$, with $\|z_1\|_{\mathcal{Z}} + \|z_2\|_{\mathcal{Z}} \leqslant M$, we have for $\mathcal{L}$ from* (2.37)

$$\mathcal{L}(t,z_2) - \mathcal{L}(t,z_1) \geqslant \langle \mathrm{D}_z\mathcal{L}(t,z_1), z_2-z_1\rangle_{\mathcal{Z}} + \beta\|z_1-z_2\|^2_{H^1(\Omega)} - \Lambda_M\|z_1-z_2\|^2_{L^1(\Omega)}. \tag{2.41}$$

As a consequence, there holds

$$\|z_1-z_2\|^2_{L^2(\Omega)} + \langle \mathrm{D}_z\mathcal{I}(t,z_1)-\mathrm{D}_z\mathcal{I}(t,z_2), z_1-z_2\rangle_{\mathcal{Z}} \geqslant c_9\|z_1-z_2\|^2_{H^1(\Omega)} - c_{10}(M)\|z_1-z_2\|^2_{L^2(\Omega)}. \tag{2.42}$$

Note that, in accordance with (2.38) and (2.41), only the constant c_{10} depends on M.

Proof Estimate (2.41) can be deduced from (2.38) by the very same calculations as in the proof of [25, Lemma 3.26], while (2.42) can be obtained by adding (2.41) with the estimate obtained exchanging z_1 with z_2, and observing that $-\|z_1 - z_2\|^2_{L^1(\Omega)} \geqslant -C\|z_1 - z_2\|^2_{L^2(\Omega)}$. □

The validity of the chain rule identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t)) - \partial_t\mathcal{I}(t,z(t)) = \langle \mathrm{D}_z\mathcal{I}(t,z(t)), z'(t)\rangle_{L^2(\Omega)} \qquad \text{for a.a. } t \in (0,T), \tag{2.43}$$

along solution curves $z : [0,T] \to \mathcal{Z}$ with $\mathrm{D}_z\mathcal{I}(t,z(t)) \in L^2(\Omega)$ is a key ingredient for the proof of energy identities in the context of solutions to the *viscous* damage system (1.2) (cf. Section 4), and of BV solutions to the rate-independent (1.1) (cf. Section 5). In fact, a chain rule *inequality* would suffice. Since $\mathcal{I} \in \mathrm{C}^1([0,T]\times\mathcal{Z})$, the validity of (2.43) with the duality pairing $\langle\cdot,\cdot\rangle_{\mathcal{Z}}$ is guaranteed along any curve $z \in \mathrm{AC}([0,T];\mathcal{Z})$. The following result extends (2.43) to curves z with weaker regularity and summability properties. We will use the chain rule under assumption (2.44*a*) in the analysis of the viscous system (1.2) (cf. the proof of Theorem 4.1), and under assumption (2.44*b*) in the analysis of BV solutions to the rate-independent system (1.1) (cf. the proof of Proposition 5.8).

Lemma 2.17 (Chain rule for $\mathcal{I}$ in $L^2(\Omega)$) *Under Assumptions 2.1, 2.2 and 2.9, let a curve z fulfill*

$$\text{either} \quad z \in L^\infty(0,T;\mathcal{Z}) \cap H^1(0,T;L^2(\Omega)), \quad \text{with } A_q z \in L^2(0,T;L^2(\Omega)), \tag{2.44a}$$

$$\text{or} \quad z \in L^\infty(0,T;\mathcal{Z}) \cap W^{1,1}(0,T;L^2(\Omega)), \quad \text{with } A_q z \in L^\infty(0,T;L^2(\Omega)). \tag{2.44b}$$

Then, the map $t \mapsto \mathcal{I}(t,z(t))$ is absolutely continuous on $[0,T]$, and (2.43) holds.

Proof We will prove this result assuming (2.44*a*), as the argument under the alternative condition is perfectly analogous. Preliminarily, let us observe that, due to estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$ with $\widetilde{\mathcal{I}}$ from (2.27), it follows from (2.44*a*) that the function $t \mapsto \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ belongs to $L^2(0,T;L^2(\Omega))$. Therefore, $\mathrm{D}_z\mathcal{I}(t,z(t)) = A_q(z(t)) + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ belongs to $L^2(0,T;L^2(\Omega))$, as well and the integral on the R.H.S. of (2.43) is well defined for almost all $t \in (0,T)$.

First of all, we show the absolute continuity of $t \mapsto \mathcal{I}(t,z(t))$. We will in fact show that $t \mapsto \mathcal{L}(t,z(t))$ is absolutely continuous, with $\mathcal{L}$ from (2.37). With this aim, for every $0 \leqslant s \leqslant t \leqslant T$, we estimate

$$\mathcal{L}(t,z(t)) - \mathcal{L}(s,z(s)) = \mathcal{L}(t,z(t)) - \mathcal{L}(s,z(t)) + \mathcal{L}(s,z(t)) - \mathcal{L}(s,z(s)) \doteq I_1 + I_2\,.$$

Since $\partial_t\mathcal{L} = \partial_t\mathcal{I}$, we have

$$|I_1| \leqslant \int_s^t \partial_t\mathcal{I}(r,z(t))\,\mathrm{d}r \overset{(1)}{\leqslant} C(t-s) \tag{2.45}$$

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with (1) due to (2.23). As for I_2, from the uniform differentiability property (2.41), we deduce that

$$I_2 \geqslant \int_\Omega \mathrm{D}_z\mathcal{L}(t,z(s))(z(t)-z(s))\,\mathrm{d}x + \alpha\|z(t)-z(s)\|^2_{H^1(\Omega)} - \Lambda_M\|z(t)-z(s)\|^2_{L^1(\Omega)}, \quad (2.46)$$

cf. notation (2.29). Here, we have used that by (2.44a) and estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$ the function $s \mapsto \mathrm{D}_z\mathcal{I}(s,z(s))$ belongs to $L^2(0,T;L^2(\Omega))$. This extends to $s \mapsto \mathrm{D}_z\mathcal{L}(t,z(s))$ due to (2.32). All in all we arrive at

$$\begin{aligned} |\mathcal{L}(s,z(s)) - \mathcal{L}(t,z(t))| &\leqslant 2\Lambda_M\|z(t)-z(s)\|^2_{L^1(\Omega)} + 2c|t-s| \\ &\quad + \left(\|\mathrm{D}_z\mathcal{L}(t,z(t))\|_{L^2(\Omega)} + \|\mathrm{D}_z\mathcal{L}(s,z(s))\|_{L^2(\Omega)}\right)\|z(t)-z(s)\|_{L^2(\Omega)}. \end{aligned} \quad (2.47)$$

Up to a suitable reparameterization, cf. [2, Lemma 1.1.4], we can suppose that $z \in W^{1,\infty}(0,\widetilde{T};L^2(\Omega))$ with Lipschitz constant 1. With [2, Lemma 1.2.6], we finally conclude from (2.47) the absolute continuity of $t \mapsto \mathcal{L}(t,z(t))$, which gives the same property for $t \mapsto \mathcal{I}(t,z(t))$.

For the proof of identity (2.43), we may repeat the very same argument as in the proof of [24, Proposition 2.4]. □

3 A priori estimates for the time-discrete solutions

We construct time-discrete solutions to the Cauchy problem for the viscous damage system (1.2) by solving the following time incremental minimization problems: for fixed $\epsilon > 0$, we consider a uniform partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of the time interval $[0,T]$ with fineness $\tau = t_{k+1}^\tau - t_k^\tau = T/N$. The elements $(z_k^\tau)_{0\leqslant k\leqslant N}$ are determined through $z_0^\tau := z_0 \in \mathcal{Z}$ and

$$z_{k+1}^\tau \in \mathrm{Argmin}\left\{ \mathcal{I}(t_{k+1}^\tau, z) + \tau\mathcal{R}_\epsilon\left(\frac{z-z_k^\tau}{\tau}\right) : z \in \mathcal{Z} \right\}, \qquad k \in \{0,\ldots,N-1\}. \quad (3.1)$$

Our first result, Proposition 3.1 below, states the existence of minimizers for problem (3.1), which is an immediate outcome of classical variational arguments, as well as the uniqueness of solutions to the associated Euler–Lagrange equation (3.2) below. This will be a key ingredient in the proof of the main result of this section, Proposition 3.2 ahead. Indeed, in order to obtain some of the *a priori* estimates stated therein, we shall have to perform calculations on an approximate version of (3.2). Then, the above mentioned uniqueness property will ensure that the *a priori* estimates also hold for the solutions to (3.2), i.e., for the minimizers from (3.1).

Proposition 3.1 *Under Assumptions 2.1, 2.2 and 2.9, for every $\epsilon, \tau > 0$ and for every $k \in \{1,\ldots,N-1\}$ the minimum problem* (3.1) *admits a solution z_{k+1}^τ satisfying the Euler–Lagrange equation*

$$\omega + \epsilon\frac{z-z_k^\tau}{\tau} + \mathrm{D}_z\mathcal{I}(t_{k+1}^\tau, z) = 0 \qquad \textit{in } \mathcal{Z}^*, \qquad \textit{with } \omega \in \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1\left(\frac{z-z_k^\tau}{\tau}\right), \quad (3.2)$$

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where $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ *is the convex analysis subdifferential of* $\mathcal{R}_1$*. Moreover, for every* $\epsilon > 0$ *and for every* $M > 0$*, there exists* $\tau(\epsilon, M) > 0$ *such that for all* $0 < \tau \leqslant \tau(\epsilon, M)$ *the Euler–Lagrange equation* (3.2) *admits at most one solution in the closed ball* $\overline{B}_M(0)$ *in* $\mathcal{Z}$*.*

Suppose in addition that f *and* g *comply with the following condition:*

$$f(0) \leqslant f(z), \quad g(0) \leqslant g(z) \quad \textit{for all } z \leqslant 0, \tag{3.3}$$

and that the initial datum z_0 *fulfills* $z_0(x) \in [0,1]$ *for all* $x \in \Omega$*. Then, the minimizer* z^τ_{k+1} *from* (3.1) *also fulfills* $z^\tau_{k+1}(x) \in [0,1]$ *for all* $x \in \Omega$ *and all* $k \in \{0, \dots, N-1\}$*.*

Proof The existence of minimizers can be checked *via* the direct method in the calculus of variations. Observe that every minimizer fulfills (3.2), where we have used that the convex analysis subdifferential $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_\epsilon : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is given by $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_\epsilon(\eta) = \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\eta) + \epsilon\eta$ for every $\eta \in \mathcal{Z}$. Here and in what follows, for notational simplicity we write η in place of $J(\eta)$, with $J : \mathcal{Z} \to \mathcal{Z}^*$ the Riesz isomorphism.

In order to check that the Euler–Lagrange equation (3.2) has a unique solution, let $M > 0$ and z_1, $z_2 \in \mathcal{Z}$ be solutions to (3.2) such that $\|z_1\|_{\mathcal{Z}} + \|z_2\|_{\mathcal{Z}} \leqslant M$. Subtracting the equation for z_2 from that for z_1 and testing the obtained relation by $z_1 - z_2$, we find

$$\begin{aligned} 0 = {} & \langle \omega_1 - \omega_2, z_1 - z_2\rangle_{\mathcal{Z}} + \frac{\epsilon}{\tau}\|z_1 - z_2\|^2_{L^2(\Omega)} + \langle \mathrm{D}_z\mathcal{I}(t^\tau_{k+1}, z_1) - \mathrm{D}_z\mathcal{I}(t^\tau_{k+1}, z_2), z_1 - z_2\rangle_{\mathcal{Z}} \\ \geqslant {} & \left(\frac{\epsilon}{\tau} - c_{10}(M) - 1\right)\|z_1 - z_2\|^2_{L^2(\Omega)} + c_9\|z_1 - z_2\|^2_{H^1(\Omega)}, \end{aligned}$$

where $\omega_i \in \partial\mathcal{R}_1\left(\frac{z_i - z^\tau_k}{\tau}\right)$ for $i = 1, 2$, and the second inequality follows from the monotonicity estimate (2.42). Hence, for $\tau \leqslant \tau(\epsilon, M) := \frac{\epsilon}{(c_{10}(M)+1)}$, we conclude that $\|z_1 - z_2\|^2_{H^1(\Omega)} \leqslant 0$, whence $z_1 = z_2$.

For the proof of the property $z^\tau_k \in [0,1]$ in Ω under (3.3), we refer to [18, Proposition 4.5]. □

The following piecewise constant and piecewise linear interpolation functions will be used:

$$\begin{aligned} &\overline{z}_\tau(t) = z^\tau_{k+1} \text{ for } t \in (t^\tau_k, t^\tau_{k+1}], \ \ \underline{z}_\tau(t) = z^\tau_k \text{ for } t \in [t^\tau_k, t^\tau_{k+1}), \\ &\widehat{z}_\tau(t) = z^\tau_k + \frac{t - t^\tau_k}{\tau}(z^\tau_{k+1} - z^\tau_k) \text{ for } t \in [t^\tau_k, t^\tau_{k+1}]. \end{aligned}$$

Furthermore, we shall use the notation

$$\begin{array}{lll} \tau(r) & = \tau & \text{for } r \in (t^\tau_k, t^\tau_{k+1}), \\ \overline{t}_\tau(r) & = t^\tau_{k+1} & \text{for } r \in (t^\tau_k, t^\tau_{k+1}], \\ \underline{t}_\tau(r) & = t^\tau_k & \text{for } r \in [t^\tau_k, t^\tau_{k+1}), \\ \overline{u}_\tau(r) & = u_{\min}(\overline{t}_\tau(r), \overline{z}_\tau(r)) & \text{for } r \in (t^\tau_k, t^\tau_{k+1}], \\ \underline{u}_\tau(r) & = u_{\min}(\underline{t}_\tau(r), \underline{z}_\tau(r)) & \text{for } r \in [t^\tau_k, t^\tau_{k+1}), \\ \widehat{u}_\tau(r) & = \underline{u}_\tau(r) + \frac{r - \underline{t}_\tau(r)}{\tau}(\overline{u}_\tau(r) - \underline{u}_\tau(r)) & \text{for } r \in [t^\tau_k, t^\tau_{k+1}]. \end{array}$$

Clearly,

$$\overline{t}_\tau(t),\ \underline{t}_\tau(t) \to t \qquad \text{as } \tau \to 0 \text{ for all } t \in (0,T), \text{ and } \underline{t}_\tau(0) = 0,\ \overline{t}_\tau(T) = T. \tag{3.4}$$

We will also denote by $\overline{\ell}_\tau$ and $\overline{u}_{D,\tau}$ the (left-continuous) piecewise constant interpolants of the values $(\ell_k^\tau := \ell(t_k^\tau))_{k=0}^N$, $(u_{D,k}^\tau := u_D(t_k^\tau))_{k=0}^N$ and, for a given N-uple $\{v_\tau^k\}_{k=0}^N$, use the short-hand notation

$$\triangle_k^\tau(v) := v_{k+1}^\tau - v_k^\tau.$$

In view of (3.2) and of formula (2.28) for $\mathrm{D}_z\mathcal{I}$, the above interpolants fulfill for almost all $t \in (0,T)$

$$\overline{\omega}_\tau(t) + \epsilon\widehat{z}_\tau'(t) + A_q\overline{z}_\tau(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_\tau(t)) = 0 \quad \text{in } \mathcal{Z}^*, \quad \text{with } \overline{\omega}_\tau(t) \in \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1\left(\widehat{z}_\tau'(t)\right). \tag{3.5}$$

The following result collects all the *a priori* estimates on the functions $(\overline{z}_\tau, \widehat{z}_\tau, \overline{u}_\tau, \widehat{u}_\tau)_\tau$, uniform w.r.t. the parameters ϵ, $\tau > 0$. These estimates are at the core of the existence of solutions of the viscous system, cf. Theorem 4.1 ahead, and of its vanishing-viscosity analysis developed in Section 5. Let us mention that the estimates for $(\overline{u}_\tau, \widehat{u}_\tau)_\tau$ have to be understood as side results, while the really relevant bounds for the limit passage are those for $(\overline{z}_\tau, \widehat{z}_\tau)$. We also prove that the Euler–Lagrange equation (3.5) holds in $L^2(\Omega)$, with $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1$ replaced by the subdifferential operator $\partial_{L^2(\Omega)}\mathcal{R}_1 : L^2(\Omega) \rightrightarrows L^2(\Omega)$. From now on, we will denote the latter operator by $\partial\mathcal{R}_1$.

Proposition 3.2 *Under Assumptions 2.1, 2.2 and 2.9, suppose that the initial datum $z_0 \in \mathcal{Z}$ fulfills in addition*

$$A_q z_0 \in L^2(\Omega). \tag{3.6}$$

Then, for every $\epsilon > 0$ there exists $\overline{\tau}_\epsilon > 0$, only depending on ϵ and on the problem data (cf. (3.14) *ahead), such that for every $\tau \in (0, \overline{\tau}_\epsilon)$ there holds*

$$A_q\overline{z}_\tau \in L^\infty(0,T;L^2(\Omega)) \text{ and } \overline{\omega}_\tau \in L^\infty(0,T;L^2(\Omega)), \tag{3.7}$$

with $\overline{\omega}_\tau$ a selection in $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}_\tau')$ which fulfills (3.5)*. Therefore, the functions $(\overline{t}_\tau, \overline{z}_\tau, \widehat{z}_\tau)$ satisfy*

$$\partial\mathcal{R}_1(\widehat{z}_\tau'(t)) + \epsilon\widehat{z}_\tau'(t) + \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)) \ni 0 \qquad \text{in } L^2(\Omega) \text{ for a.a. } t \in (0,T). \tag{3.8}$$

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Furthermore, there exist constants C, $C(\epsilon)$, $C(\sigma) > 0$, *with* $C(\epsilon) \uparrow +\infty$ *as* $\epsilon \downarrow 0$, *such that for all* $\epsilon > 0$ *and* $\tau \in (0, \overline{\tau}_\epsilon)$ *the following estimates hold:*

$$\sup_{t\in[0,T]} |\mathcal{I}(\bar{t}_\tau(t), \overline{z}_\tau(t))| \leqslant C, \tag{3.9a}$$

$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))} + \|\widehat{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leqslant C, \tag{3.9b}$$

$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1+\sigma,q}(\Omega))} \leqslant C(\sigma) \text{ for all } 0 < \sigma < \tfrac{1}{q}, \tag{3.9c}$$

$$\|\widehat{z}_\tau'\|_{L^2(0,T;H^1(\Omega))} + \|\widehat{z}_\tau'\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(\epsilon), \tag{3.9d}$$

$$\|\widehat{z}_\tau\|_{W^{1,1}(0,T;H^1(\Omega))} \leqslant C, \tag{3.9e}$$

$$\|A_q(\overline{z}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{3.9f}$$

$$\|\overline{\varpi}_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C \tag{3.9g}$$

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^2(\Omega))} \leqslant C, \tag{3.9h}$$

$$\|\widehat{u}_\tau'\|_{L^2(0,T;W^{1,3}(\Omega))} \leqslant C(\epsilon), \tag{3.9i}$$

$$\|\widehat{u}_\tau\|_{W^{1,1}(0,T;W^{1,3}(\Omega))} \leqslant C, \tag{3.9j}$$

$$\|\mathrm{D}_z\mathcal{I}(\bar{t}_\tau, \overline{z}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C. \tag{3.9k}$$

Based on Proposition 3.2, we derive a discrete energy inequality, cf. (3.11) below, involving the Fenchel–Moreau conjugate of the functional $\mathcal{R}_\epsilon$ w.r.t. the scalar product in $L^2(\Omega)$, namely the functional

$$\mathcal{R}_\epsilon^* : L^2(\Omega) \to [0, +\infty) \qquad \text{defined by } \mathcal{R}_\epsilon^*(\xi) := \frac{1}{2\epsilon} \min_{\eta\in\partial\mathcal{R}_1(0)} \|\xi - \eta\|^2_{L^2(\Omega)}. \tag{3.10}$$

Observe that we are in a position to work with this Legendre transform of $\mathcal{R}_\epsilon$, and not with the one w.r.t. the $(\mathcal{Z}, \mathcal{Z}^*)$-duality, relying on the fact that $\mathrm{D}_z\mathcal{I}(\bar{t}_\tau(t), \overline{z}_\tau(t)) \in L^2(\Omega)$ for almost all $t \in (0, T)$, thanks to (3.7).

Corollary 3.3 *Under Assumptions 2.1, 2.2 and 2.9, suppose that the initial datum* z_0 *fulfills* (3.6).

Then, there exists $C > 0$ *such that for every* $\epsilon > 0$ *and* $\tau \in (0, \overline{\tau}_\epsilon)$ *the functions* $\overline{z}_\tau$, $\widehat{z}_\tau$ *comply with the* discrete ED inequality *for every* $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned} &\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_\tau(r), \overline{z}_\tau(r)))\right) \mathrm{d}r + \mathcal{I}(t, \widehat{z}_\tau(t)) \\ &\leqslant \mathcal{I}(s, \widehat{z}_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)} \partial_t\mathcal{I}(r, \widehat{z}_\tau(r))\,\mathrm{d}r \\ &+ C \sup_{t\in[0,T]} \|\overline{z}_\tau(t) - \widehat{z}_\tau(t)\|_{L^2(\Omega)} \int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)} (|\bar{t}_\tau(r) - r| + \|\overline{z}_\tau(r) - \widehat{z}_\tau(r)\|_{L^6(\Omega)})\,\mathrm{d}r. \end{aligned} \tag{3.11}$$

Therefore, there exists a constant $C > 0$ *such that for every* $\epsilon > 0$ *and* $\tau \in (0, \overline{\tau}_\epsilon)$

$$\sup_{t\in[0,T]} |\mathcal{I}(t, \widehat{z}_\tau(t))| \leqslant C, \tag{3.12a}$$

$$\int_0^T \left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_\tau(r), \overline{z}_\tau(r)))\right) \mathrm{d}r \leqslant C. \tag{3.12b}$$

Let us mention that (3.11) will be the starting point of the vanishing-viscosity analysis developed in Section 6. We postpone the proof of Corollary 3.3 to the end of this section.

Let us now comment on the proof of Proposition 3.2: The enhanced spatial regularity (3.7) leads to (3.8) as a subdifferential inclusion in $L^2(\Omega)$. Estimates (3.9) and (3.7) will be proved by performing on equation (3.5) the following *a priori* estimates (the last of which can be carried out only formally):

Energy estimate based on the ED inequality

$$\mathcal{I}(\bar{t}_\tau(t), \overline{z}_\tau(t)) + \int_0^{\bar{t}_\tau(t)} \mathcal{R}_\epsilon(\widehat{z}'_\tau(s))\,\mathrm{d}s \leqslant \mathcal{I}(0, z_0) + \int_0^{\bar{t}_\tau(t)} \partial_t \mathcal{I}(s, \underline{z}_\tau(s))\,\mathrm{d}s \tag{3.13}$$

for every $t \in [0, T]$, it leads to the uniform bounds (3.9*a*)–(3.9*b*). Observe that the proof of (3.13) works for *every* $\tau > 0$.

We then choose

$$\overline{\tau}_\epsilon := \tau(\epsilon, M) \text{ according to Proposition 3.1, with } M \geqslant \sup_{\tau>0} \|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))}. \tag{3.14}$$

First regularity estimate: In view of estimate (2.15), from the estimate for $\overline{z}_\tau$ in $L^\infty(0, T; W^{1,q}(\Omega))$, we deduce (3.9*h*).

Enhanced energy estimate: It consists in differentiating (3.5) w.r.t. time (on the time-discrete level), and testing it by $\widehat{z}'_\tau$. In view of the coercivity property (2.1) of the elliptic operator A_q, this gives estimates (3.9*d*) & (3.9*e*) for $\widehat{z}'_\tau$.

Second regularity estimate: Estimates (3.9*i*) and (3.9*j*) for $\widehat{u}'_\tau$ derive from (3.9*d*) and (3.9*e*), respectively, *via* the continuous dependence estimate (2.17).

Third regularity estimate: It consists in testing (3.5) by (the formally written term) $\partial_t A_q \overline{z}_\tau$. This gives rise to estimate (3.9*f*), which induces the spatial regularity (3.9*c*) by applying Proposition 2.8, and it induces (3.9*g*) by a comparison argument in (3.5).

We will carry out the proof of the above mentioned *a priori* estimates in Lemmas 3.4 and 3.5 ahead. More precisely,

(1) The energy and the enhanced energy estimates (3.9*a*)–(3.9*b*), (3.9*d*)–(3.9*e*) and (3.9*h*)–(3.9*j*) will be proved in Lemma 3.4.

These estimates can be rendered rigorously on the discrete equation (3.5). In their proof, we shall revisit the calculations developed in [19, Section 5], relying on the novel estimates provided by Lemmas 2.11 and 2.13.

(2) The third regularity estimate (3.9*f*), along with its consequences (3.9*c*), (3.9*g*) and (3.9*k*), will be proved in Lemma 3.5.

Observe that it cannot be performed directly on (3.5). In fact, it would involve testing the subdifferential inclusion (3.5), set in $\mathcal{Z}^*$, by the difference $\frac{1}{\tau}(A_q\overline{z}_\tau(t) - A_q\underline{z}_\tau(t))$, which then should belong to $\mathcal{Z}$. This cannot be rigorous, since $A_q\overline{z}_\tau(t)$ is in general an element of $\mathcal{Z}^*$, only. Therefore, in the proof of Lemma 3.5, we shall perform all the calculations on an approximate version of (3.5), featuring a regularized version of the dissipation potential $\mathcal{R}_1$, cf. (3.28) ahead.

Lemma 3.4 *Under Assumptions 2.1, 2.2 and 2.9, and the condition that the initial datum* $z_0 \in \mathcal{Z}$ *fulfills* (3.6), *estimates* (3.9*a*)–(3.9*b*), (3.9*d*)–(3.9*e*) *and* (3.9*h*)–(3.9*j*) *hold true for every* $\tau > 0$.

Proof The discrete ED inequality (3.13) can be derived by choosing the competitor $z = z_k^\tau$ in the minimum problem (3.1), which leads to

$$\mathcal{I}(t_{k+1}^\tau, z_{k+1}^\tau) + \tau \mathcal{R}_\epsilon\left(\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right) \leqslant \mathcal{I}(t_{k+1}^\tau, z_k^\tau) = \mathcal{I}(t_k^\tau, z_k^\tau) + \int_{t_k^\tau}^{t_{k+1}^\tau} \partial_t \mathcal{I}(s, z_k^\tau)\, \mathrm{d}s.$$

Then, (3.13) follows upon summing over the index k. In view of estimate (2.23), on the power functional $\partial_t \mathcal{I}$, and Assumption 2.9, the right-hand side of (3.13) is uniformly bounded. Since the second term on its L.H.S. is non-negative, we immediately conclude estimate (3.9*a*). Then, the coercivity property (2.18), combined with Poincaré's inequality, gives (3.9*b*) for $\overline{z}_\tau$. The bound for $\widehat{z}_\tau$ then trivially follows. From the bound for $\int_0^T \mathcal{R}_\epsilon(\widehat{z}_\tau'(t))\,\mathrm{d}t$, we also infer that $\epsilon^{1/2}\|\widehat{z}_\tau'\|_{L^2(0,T;L^2(\Omega))} \leqslant C$.

Thanks to (2.15), we have that

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^2(\Omega))} \leqslant c_1 \sup_{t\in(0,T)} P(0, \overline{z}_\tau(t)) \left(\|\overline{\ell}_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}_{D,\tau}\|_{L^\infty(0,T;H^2(\Omega))}\right) \leqslant C',$$

where we have used estimate (3.9*b*), as well as Assumption 2.9. Then, (3.9*h*) follows.

In order to derive estimates (3.9*d*) and (3.9*e*), we follow the proof of [19, Lemma 5.3] and observe that, by the one-homogeneity of $\mathcal{R}_1$, (3.5) rewrites as

$$\begin{cases} \langle \overline{h}_\tau(\rho), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} = \mathcal{R}_1(\widehat{z}_\tau'(\rho)) & \text{for all } \rho \in (t_k^\tau, t_{k+1}^\tau) \\ \langle \overline{h}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} \leqslant \mathcal{R}_1(\widehat{z}_\tau'(\rho)) & \text{for all } r \in [0,T] \setminus \{t_0^\tau, \dots, t_N^\tau\}, \end{cases}$$

where we have used the place-holder $\overline{h}_\tau(\rho) := -(\epsilon \widehat{z}_\tau'(\rho) + A_q \overline{z}_\tau(\rho) + \mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)))$. Subtracting the second relation from the first one gives $\tau^{-1}\langle \overline{h}_\tau(\rho) - \overline{h}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} \geqslant 0$ for $\rho \in (t_k^\tau, t_{k+1}^\tau)$ and $r \in (t_{k-1}^\tau, t_k^\tau)$. Hence, we get

$$\begin{aligned} \epsilon\tau^{-1} \underbrace{\int_\Omega (\widehat{z}_\tau'(\rho) - \widehat{z}_\tau'(r))\widehat{z}_\tau'(\rho)\,\mathrm{d}x}_{=I_1} + \underbrace{\tau^{-1}\langle A_q \overline{z}_\tau(\rho) - A_q \overline{z}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}}}_{=I_2} \\ \leqslant \underbrace{-\tau^{-1} \int_\Omega (\mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)) - \mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_\tau(r)))\widehat{z}_\tau'(\rho)\,\mathrm{d}x}_{=I_3}. \end{aligned} \tag{3.15}$$

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Observe that $I_1 \geqslant \frac{1}{2}\int_\Omega(|\widehat{z}'_\tau(\rho)|^2 - |\widehat{z}'_\tau(r)|^2)\,\mathrm{d}x$, whereas it follows from estimate (2.1) that

$$
\begin{aligned}
I_2 &\geqslant c_q \int_\Omega \left(1+|\nabla \overline{z}_\tau(\rho)|^2+|\nabla \overline{z}_\tau(r)|^2\right)^{(q-2)/2} |\nabla \widehat{z}'_\tau(\rho)|^2\,\mathrm{d}x \\
&\geqslant C_q \int_\Omega \left(1+|\nabla \widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2} |\nabla \widehat{z}'_\tau(\rho)|^2\,\mathrm{d}x
\end{aligned} \tag{3.16}
$$

for some positive constant C_q, where we have used that $|\nabla\widehat{z}_\tau(\rho)|^2 \leqslant 2(|\nabla\overline{z}_\tau(\rho)|^2+|\nabla\overline{z}_\tau(r)|^2)$. As for I_3, by the Hölder inequality

$$
|I_3| \;\leqslant C\tau^{-1}\|\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho),\overline{z}_\tau(\rho)) - \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(r),\overline{z}_\tau(r))\|_{L^2(\Omega)}\|\widehat{z}'_\tau(\rho)\|_{L^2(\Omega)}\,.
$$

Relying on (2.32), we then find

$$
|I_3| \leqslant C(1+\|\widehat{z}'_\tau(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_\tau(\rho)\|_{L^2(\Omega)}\,, \tag{3.17}
$$

where we have also used that $\sup_{t\in[0,T]} C_{f''}(\overline{z}_\tau(\rho),\overline{z}_\tau(r)) + P(\overline{z}_\tau(\rho),\overline{z}_\tau(r))) \leqslant C$ thanks to Assumption 2.2 and the previously proved estimate (3.9*b*). Hence, multiplying (3.15) by τ, we infer

$$
\begin{aligned}
&\frac{\epsilon}{2}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)} + C_q\tau\int_\Omega \left(1+|\nabla\widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_\tau(\rho)|^2\,\mathrm{d}x \\
&\quad\leqslant \frac{\epsilon}{2}\|\widehat{z}'_\tau(r)\|^2_{L^2(\Omega)} + \tau C(1+\|\widehat{z}'_\tau(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_\tau(\rho)\|_{L^2(\Omega)}\,,
\end{aligned} \tag{3.18}
$$

which leads, upon summation, to the following estimate on the time interval (t_0,t), with $t_0\in(0,t_1^\tau)$ and $t\in(t_k^\tau,t_{k+1}^\tau)$:

$$
\begin{aligned}
&\frac{\epsilon}{2}\|\widehat{z}'_\tau(t)\|^2_{L^2(\Omega)} + C_q\int_{t_1^\tau}^{\overline{t}_\tau(t)}\int_\Omega \left(1+|\nabla\widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_\tau(\rho)|^2\;\mathrm{d}x\;\mathrm{d}\rho \\
&\qquad\leqslant \frac{\epsilon}{2}\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + \frac{C_q}{4}\int_{t_1^\tau}^{\overline{t}_\tau(t)}(1+\|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)})\;\mathrm{d}\rho + C\int_{t_1^\tau}^{\overline{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)}\;\mathrm{d}\rho,
\end{aligned} \tag{3.19}
$$

where we have used Young's inequality, and the continuous embedding $H^1(\Omega)\subset L^6(\Omega)$, to handle the last term on the R.H.S. of (3.18). For the first time step with $t_0\in(0,t_1^\tau)$, following the very same calculations as in the proof of [19, Lemma 5.3], we obtain

$$
\begin{aligned}
&\epsilon\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + C_q\tau\int_\Omega(1+|\nabla\widehat{z}_\tau(t_0)|^2)^{(q-2)/2}|\nabla\widehat{z}'_\tau(t_0)|^2\,\mathrm{d}x \\
&\quad\leqslant \frac{\epsilon}{2}\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|^2_{L^2(\Omega)} + \frac{C_q\tau}{4}(1+\|\widehat{z}'_\tau(t_0)\|^2_{H^1(\Omega)}) + C\tau\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{3.20}
$$

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Summing (3.19) with (3.20), and adding the term $\frac{C_q\tau}{2}\int_0^{\bar t_\tau(t)}\|\widehat z_\tau'(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho$ to both sides, we thus end up with the following estimate:

$$\frac{\epsilon}{2}\|\widehat z_\tau'(t)\|^2_{L^2(\Omega)} + C_q\int_0^{\bar t_\tau(t)}\|\widehat z_\tau'(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho \\ \leqslant C + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|^2_{L^2(\Omega)} + \frac{C_q}{4}\int_0^{\bar t_\tau(t)}\|\widehat z_\tau'(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho + C\int_0^{\bar t_\tau(t)}\|\widehat z_\tau'(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho\,. \tag{3.21}$$

Applying the discrete Gronwall Lemma (cf., e.g., [14, Chap. 2.2] or even [7, Lemme A.5]), we get estimate (3.9*d*).

In order to prove (3.9*e*), which is uniform w.r.t. ϵ, we follow the proof of [19, Lemma 5.5]. Since $\widehat z_\tau'$ is not defined in the points t_k^τ, we write (3.15) for $\rho = m_k$ and $r = m_{k-1}$, with $m_k := \frac{1}{2}(t_{k-1}^\tau + t_k^\tau)$, $k \in \{2,\ldots,N\}$, and set $\widehat z_\tau'(m_0) := 0$. Following the lines in the proof of [19, Lemma 5.5], where also the first time step is discussed in detail, we arrive at the following estimate (cf. [19, Formula (5.26)]): For all $k \in \{1,\ldots,N\}$

$$\frac{\epsilon}{\tau}\int_\Omega \left(\widehat z_\tau'(m_k) - \widehat z_\tau'(m_{k-1})\right)\widehat z_\tau'(m_k)\,\mathrm{d}x + \tau^{-1}\langle A_q\overline z_\tau(m_k) - A_q\underline z_\tau(m_k), \widehat z_\tau'(m_k)\rangle_{\mathcal{Z}} + \|\widehat z_\tau'(m_k)\|^2_{L^2(\Omega)} \\ \leqslant -\frac{1}{\tau}\int_\Omega\left(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline z_\tau(m_k)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\underline z_\tau(m_k))\right)\widehat z_\tau'(m_k)\,\mathrm{d}x + \|\widehat z_\tau'(m_k)\|^2_{L^2(\Omega)} \\ + \frac{\delta_{1,k}}{\tau}\left|\int_\Omega \mathrm{D}_z\mathcal{I}(0,z_0)\widehat z_\tau'(m_1)\,\mathrm{d}x\right|, \tag{3.22}$$

with the Kronecker symbol $\delta_{i,j}$. Arguing as in the proof of [19, Lemma 5.5], by estimate (2.1) and the fact that $|\nabla\widehat z_\tau(m_k)|^2 \leqslant 2|\nabla\overline z_\tau(m_k)|^2 + 2|\nabla\overline z_\tau(m_{k-1})|^2$, it follows that the left-hand side of (3.22) can be bounded from below by

$$\text{L.H.S.} \geqslant \frac{\epsilon}{2\tau}\|\widehat z_\tau'(m_k)\|_{L^2(\Omega)}\left(\|\widehat z_\tau'(m_k)\|_{L^2(\Omega)} - \|\widehat z_\tau'(m_{k-1})\|_{L^2(\Omega)}\right) + \mathrm{M}_k^2, \tag{3.23}$$

with the abbreviation

$$\mathrm{M}_k^2 := C_q\int_\Omega (1 + |\nabla\widehat z_\tau(m_k)|^2)^{\frac{q-2}{2}}|\nabla\widehat z_\tau'(m_k)|^2\,\mathrm{d}x + \|\widehat z_\tau'(m_k)\|^2_{L^2(\Omega)}$$

and C_q from (3.16). As for the first term of the right-hand side of (3.22), in place of estimate (3.17), we shall use that

$$\left|\frac{1}{\tau}\int_\Omega\left(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline z_\tau(m_k)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline z_\tau(m_{k-1}))\right)\widehat z_\tau'(m_k)\,\mathrm{d}x\right| \leqslant C(1 + \|\widehat z_\tau'(m_k)\|_{L^4(\Omega)})\|\widehat z_\tau'(m_k)\|_{L^4(\Omega)}, \tag{3.24}$$

which derives from estimate (2.33) for $\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline z_\tau(m_k)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline z_\tau(m_{k-1}))\|_{L^{4/3}(\Omega)}$. We then continue (3.24) by using the trivial estimate $C(1 + \|\widehat z_\tau'(m_k)\|_{L^4(\Omega)})\|\widehat z_\tau'(m_k)\|_{L^4(\Omega)} \leqslant C\|\widehat z_\tau'(m_k)\|^2_{L^4(\Omega)} + C$, and then applying the Gagliardo–Nirenberg estimate $\|\zeta\|^2_{L^4(\Omega)} \leqslant$

$c\|\zeta\|_{L^1(\Omega)}^{2(1-\theta)}\|\zeta\|_{H^1(\Omega)}^{2\theta}$, with $\theta = 9/10$, and Young's inequality, so that

$$\left|\frac{1}{\tau}\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline{z}_\tau(m_k)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline{z}_\tau(m_{k-1}))\right)\widehat{z}'_\tau(m_k)\,\mathrm{d}x\right| \\ \leqslant \frac{1}{2}\min\{C_q,1\}\|\widehat{z}'_\tau(m_k)\|_{H^1(\Omega)}^2 + C\|\widehat{z}'_\tau(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}'_\tau(m_k)) + C\,,$$

where we have also used that $\|\widehat{z}'_\tau(m_k)\|_{L^1(\Omega)}^2 \leqslant \|\widehat{z}'_\tau(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}'_\tau(m_k))$. Therefore, the right-hand side of (3.22) can be bounded as follows:

$$\text{R.H.S.} \leqslant \frac{1}{2}\mathrm{M}_k^2 + C\left(1 + \|\widehat{z}'_\tau(m_k)\|_{L^2(\Omega)}\mathcal{R}_1(\widehat{z}'_\tau(m_k)) + \delta_{1,k}\tau^{-1}|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\widehat{z}'_\tau(m_1)\rangle_{\mathcal{Z}}|\right). \quad (3.25)$$

From (3.23) and (3.25), after some algebra it results that (cf. [19, (5.28)])

$$2\|\widehat{z}'_\tau(m_k)\|_{L^2(\Omega)}\left(\|\widehat{z}'_\tau(m_k)\|_{L^2(\Omega)} - \|\widehat{z}'_\tau(m_{k-1})\|_{L^2(\Omega)}\right) + \frac{\tau}{\epsilon}\|z'_\tau(m_k)\|_{L^2(\Omega)}^2 + \frac{\tau}{\epsilon}\mathrm{M}_k^2 \\ \leqslant \frac{4C\tau}{\epsilon} + \frac{4C\tau}{\epsilon}\|\widehat{z}'_\tau(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}'_\tau(m_k)) + 4C\frac{\delta_{1,k}}{\epsilon\tau}\left|\int_\Omega \mathrm{D}_z\mathcal{I}(0,z_0)\widehat{z}'_\tau(m_1)\,\mathrm{d}x\right|. \quad (3.26)$$

At this point, we apply a new version of Gronwall Lemma, exactly tailored to estimate (3.26), cf. [19, Lemma B.1]. We then perform the very same calculations as in the proof of [19, Lemma 5.5]. Thus, we obtain (3.9*e*).

Finally, we use (2.17) and deduce that for almost all $t \in (0,T)$ there holds

$$\|\widehat{u}'_\tau(t)\|_{W^{1,3}(\Omega)} = \frac{1}{\tau}\|u_{k+1}^\tau - u_k^\tau\|_{W^{1,3}(\Omega)} \\ \leqslant \frac{c_2}{\tau}P(z_\tau^k,z_\tau^{k+1})^2\left(\tau + \|z_{k+1}^\tau - z_k^\tau\|_{L^6(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,3}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,3}(\Omega))}\right) \\ \leqslant C(1 + \|\widehat{z}'_\tau(t)\|_{L^6(\Omega)}),$$

where the second inequality follows from (3.9*b*) and Assumption 2.9. Hence, estimates (3.9*d*) and (3.9*e*) imply (3.9*i*) and (3.9*j*), respectively. □

We postpone to Section 3.1 the proof of the forthcoming Lemma 3.5.

Lemma 3.5 *Under Assumptions 2.1, 2.2 and 2.9, and, in addition,* (3.6) *on the initial datum* z_0*, for every* $\tau \in (0,\overline{\tau}_\epsilon)$ *the enhanced regularity* (3.7) *and estimates* (3.9*f*)–(3.9*g*) *hold true, whence* (3.9*c*) (3.9*k*). *Furthermore, the subdifferential inclusion* (3.8) *is satisfied in* $L^2(\Omega)$.

The *proof of Proposition 3.2* now follows from combining Lemmas 3.4 and 3.5. ■

Let us finally give the *proof of Corollary 3.3:* the very same calculations as in the proof of [19, Lemma 6.1] (cf. also the proof of Theorem 4.1 ahead), show that the interpolants

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$\overline{z}_\tau$, $\widehat{z}_\tau$ fulfill at every $0 \leqslant s \leqslant t \leqslant T$

$$\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau')(r) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_\tau(r),\overline{z}_\tau(r)))\right)\,\mathrm{d}r + \mathcal{I}(t,\widehat{z}_\tau(t))$$
$$= \mathcal{I}(s,\widehat{z}_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)} \partial_t\mathcal{I}(r,\widehat{z}_\tau(r))\,\mathrm{d}r$$
$$\underbrace{-\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_\Omega \left(A_q\overline{z}_\tau(r) - A_q\widehat{z}_\tau(r)\right)\widehat{z}_\tau'(r)\,\mathrm{d}r}_{F_1}$$
$$\underbrace{-\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_\tau(r),\overline{z}_\tau(r)) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,\widehat{z}_\tau(r))\right)\widehat{z}_\tau'(r)\,\mathrm{d}r}_{F_2}\,.$$

Observe that the terms F_1 and F_2 feature integrals, instead of duality pairings between $\mathcal{Z}^*$ and $\mathcal{Z}$, thanks to (2.34) and (3.7). By monotonicity, we have $F_1 \leqslant 0$, whereas, the very same argument leading to (3.17) yields

$$\begin{aligned}|F_2| &\leqslant C \textstyle\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)}(|(\bar{t}_\tau(r)-r| + \|\overline{z}_\tau(r)-\widehat{z}_\tau(r)\|_{L^6(\Omega)})\|\overline{z}_\tau(r)-\widehat{z}_\tau(r)\|_{L^2(\Omega)}\,\mathrm{d}r\\ &\leqslant C\sup_{t\in[0,T]}\|\overline{z}_\tau(t)-\widehat{z}_\tau(t)\|_{L^2(\Omega)}\textstyle\int_{\underline{t}_\tau(s)}^{\bar{t}_\tau(t)}(|(\bar{t}_\tau(r)-r| + \|\overline{z}_\tau(r)-\widehat{z}_\tau(r)\|_{L^6(\Omega)})\,\mathrm{d}r,\end{aligned}$$

whence (3.11).

It follows from (3.11) and (2.23) that

$$\begin{aligned}&\int_0^{\bar{t}_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_\tau(r),\overline{z}_\tau(r)))\right)\,\mathrm{d}r + \mathcal{I}(t,\widehat{z}_\tau(t)) \leqslant \mathcal{I}(0,z_0) + C\\ &\quad + C\left(\|\overline{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\widehat{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))}\right)\left(1 + \int_0^{\bar{t}_\tau(t)}\|\overline{z}_\tau(r)-\widehat{z}_\tau(r)\|_{L^6(\Omega)}\,\mathrm{d}r\right) \leqslant C\,,\end{aligned}$$

where the very last estimate ensues from (3.9*b*) and (3.9*e*). Recalling that $\mathcal{I}$ is bounded from below (cf. (2.18)), we thus infer that $\sup_{t\in[0,T]}|\mathcal{I}(t,\widehat{z}_\tau(t))| \leqslant C$, i.e., (3.12*a*), as well as (3.12*b*). ∎

3.1 Proof of Lemma 3.5

Observe that, once estimate (3.9*f*) is proved, (3.9*k*) then follows by observing that $\mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_\tau,\overline{z}_\tau)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ in view of estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$, combined with the previously obtained (3.9*b*).

Hence, let us now turn to the proof of (3.9*f*), which is a consequence of the *Third regularity estimate*. In order to render it on the time-discrete level, we need to work on an approximate version of the discrete equation (3.5), where the dissipation metric R_1 inducing $\mathcal{R}_1$ is replaced, for technical reasons that will be apparent in the proof of Lemma 3.6 below, by a *twice-differentiable* function. Observe that the standard Yosida

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approximation of R_1, namely the function

$$\mathrm{R}_{1,\nu} : \mathbb{R} \to \mathbb{R} \ \text{ defined by } \ \mathrm{R}_{1,\nu}(r) := \min_{y\in\mathbb{R}} \left(\frac{|y-r|^2}{2\nu} + \mathrm{R}_1(y) \right) = \begin{cases} \frac{1}{2\nu} r^2 & \text{if } r > -\nu\kappa \\ -\kappa r - \frac{\nu\kappa^2}{2} & \text{if } r \leqslant -\nu\kappa \end{cases} \tag{3.27}$$

with $\nu > 0$ fixed, does not enjoy this regularity, as it is only differentiable on $\mathbb{R}$, cf. [7].

We will thus resort to a regularization of R_1 devised in [13] and defined in this way. Let $\varrho \in \mathrm{C}^\infty(\mathbb{R})$ satisfy $\mathrm{supp}(\varrho) \subset [-1, 1]$ and $\|\varrho\|_{L^1(\mathbb{R})} = 1$. We then define

$$\overline{\mathrm{R}}_{1,\nu}(r) := \int_0^r \int_{-\nu^2}^{\nu^2} \mathrm{R}'_{1,\nu}(\sigma - s)\varrho_\nu(s)\,\mathrm{d}s\,\mathrm{d}\sigma \tag{3.28}$$

where $\varrho_\nu(s) = \nu^{-2}\varrho(s/\nu^2)$. In [13], it has been proved that

$$\overline{\mathrm{R}}_{1,\nu} \in \mathrm{C}^\infty(\mathbb{R}) \text{ is convex and satisfies } -\nu|r| \leqslant \overline{\mathrm{R}}_{1,\nu}(r) \leqslant \mathrm{R}_1(r) + \nu|r| \quad \text{for all } r \in \mathbb{R}. \tag{3.29a}$$

Of course, for $r > 0$ the latter estimate is trivially satisfied, since in that case, $\mathrm{R}_1(r) = \infty$. Inequality (3.29*a*) in fact derives from the estimate

$$|\overline{\mathrm{R}}'_{1,\nu}(r) - \mathrm{R}'_{1,\nu}(r)| \leqslant \nu \quad \text{for all } r \in \mathbb{R}. \tag{3.29b}$$

Since $\mathrm{R}'_{1,\nu}$ is Lipschitz, from (3.29*b*), we in fact deduce that $\overline{\mathrm{R}}_{1,\nu}$ grows at most quadratically on $\mathbb{R}$. The function $\overline{\mathrm{R}}_{1,\nu}$ induces an integral functional

$$\overline{\mathcal{R}}_{1,\nu} : L^2(\Omega) \to \mathbb{R} \quad \text{defined by} \quad \overline{\mathcal{R}}_{1,\nu}(\eta) := \int_\Omega \overline{\mathrm{R}}_{1,\nu}(\eta(x))\,\mathrm{d}x \qquad \text{for all } \eta \in L^2(\Omega). \tag{3.29c}$$

Observe that $\overline{\mathcal{R}}_{1,\nu}$ is Gâteaux-differentiable on $L^2(\Omega)$, with derivative $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)$ defined by $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)(x) := \overline{\mathrm{R}}'_{1,\nu}(\eta(x))$ for almost all $x \in \Omega$. In fact, $\overline{\mathrm{R}}'_{1,\nu}(\eta) \in L^2(\Omega)$ by the linear growth of $\overline{\mathrm{R}}'_{1,\nu}$. Indeed, as soon as $\eta \in \mathcal{Z}$, $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)$ coincides with the Gâteaux derivative $\mathrm{D}_{\mathcal{Z},\mathcal{Z}^*}\overline{\mathcal{R}}_{1,\nu}(\eta)$. For our purposes, the following closedness property relating $\mathrm{D}\overline{\mathcal{R}}_{1,\nu} : L^2(\Omega) \to L^2(\Omega)$ to the convex subdifferential $\partial\mathcal{R}_1 : L^2(\Omega) \rightrightarrows L^2(\Omega)$ will have a prominent role: for any $(t_0, t_1) \subset (0, T)$ and all sequences $(\eta_\nu)_\nu$, η, $\xi \in L^2(t_0, t_1; L^2(\Omega))$ there holds

$$\begin{cases} \eta_\nu \rightharpoonup \eta & \text{as } \nu \downarrow 0 \quad \text{in } L^2(t_0, t_1; L^2(\Omega)), \\ \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta_\nu) \rightharpoonup \xi & \text{as } \nu \downarrow 0 \quad \text{in } L^2(t_0, t_1; L^2(\Omega)), \\ \limsup_{\nu\downarrow 0} \int_{t_0}^{t_1}\int_\Omega \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta_\nu)\eta_\nu\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{t_0}^{t_1}\int_\Omega \xi\eta\,\mathrm{d}x\,\mathrm{d}t \end{cases}$$
$$\Rightarrow \ \xi(t) \in \partial\mathcal{R}_1(\eta(t)) \qquad \text{for almost all } t \in (t_0, t_1). \tag{3.29d}$$

We refer to [13, Proposition 3.1] for the proof of (3.29*d*).

For a fixed time step $\tau > 0$, given a partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of $[0, T]$, we now incrementally solve the minimum problems featuring the regularized functionals $\overline{\mathcal{R}}_{1,\nu}$.

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Namely, starting from $z_0^{\tau,\nu} := z_0$, we set

$$z_{k+1}^{\tau,\nu} \in \mathrm{Argmin}\Big\{ \mathcal{I}(t_{k+1}^{\tau}, z) + \tau\overline{\mathcal{R}}_{1,\nu}\left(\frac{z - z_k^{\tau}}{\tau}\right) + \frac{\epsilon}{\tau}\left\|\frac{z - z_k^{\tau}}{\tau}\right\|_{L^2(\Omega)}^2 \ : z \in \mathcal{Z} \Big\},$$
$$k \in \{1, \dots, N-1\}. \tag{3.30}$$

The analogue of Proposition 3.1 holds. In particular, the (left- and right-continuous) piecewise constant and linear interpolants $\overline{z}_{\tau,\nu}$, $\underline{z}_{\tau,\nu}$ and $\widehat{z}_{\tau,\nu}$ of the elements $(z_k^{\tau,\nu})_{k=0}^N$ satisfy the following approximate version of (3.8):

$$\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(t)) + \epsilon\widehat{z}'_{\tau,\nu}(t) + A_q\overline{z}_{\tau,\nu}(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t)) = 0 \quad \text{in } L^2(\Omega) \quad \text{for a.a. } t \in (0,T), \tag{3.31}$$

where we have in fact used that $\mathrm{D}_{\mathcal{Z},\mathcal{Z}^*}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}) = \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$. In particular, observe that, by comparison in (3.31), there holds

$$A_q\overline{z}_{\tau,\nu}(t) \in L^2(\Omega) \qquad \text{for almost all } t \in (0,T). \tag{3.32}$$

For the functions $(\overline{z}_{\tau,\nu}, \widehat{z}_{\tau,\nu}, \overline{u}_{\tau,\nu}, \widehat{u}_{\tau,\nu})_{\tau,\nu}$, where $\overline{u}_{\tau,\nu}$, $\widehat{u}_{\tau,\nu}$ are the interpolants of the elements $u_{\min}(t_\tau^k, z_k^{\tau,\nu})$, we are now able to derive *only some* of estimates (3.9) (in fact, (3.9a, 3.9b, 3.9h)) *uniformly w.r.t. all* parameters ε, τ and ν. The remaining estimates will be proved only with a constant blowing up with $\epsilon > 0$, cf. also Remark 3.7 later on. However, this will be sufficient for our purposes. On the one hand, passing to the limit with $\nu \downarrow 0$ we will derive from estimates (3.33*f*) and (3.33*g*) that the discrete solutions of (3.5) enjoy the additional properties $A_q(\overline{z}_\tau) \in L^\infty(0,T;L^2(\Omega))$, $\overline{\omega}_\tau \in L^\infty(0,T;L^2(\Omega))$ for a selection $\overline{\omega}_\tau \in \partial_{\mathcal{Z},\mathcal{Z}^*}(\mathcal{R}_1(\widehat{z}'_\tau))$, and thus we will conclude that (3.5) in fact holds as subdifferential inclusion in $L^2(\Omega)$. On the other hand, estimate (3.34) below will be the starting point for deriving estimate (3.9*f*) for $A_q(\overline{z}_\tau)$ with constant *uniform* w.r.t. ϵ, relying on the previously proved estimate (3.9*e*) for $\widehat{z}_\tau$.

Lemma 3.6 *Under Assumptions 2.1, 2.2 and 2.9, and under condition* (3.6) *on the initial datum* z_0*, there exist constants* C', $C'(\epsilon)$, $C'(\varepsilon,\sigma) > 0$*, with* $C'(\epsilon)$, $C'(\varepsilon,\sigma) \uparrow +\infty$ *as* $\epsilon \downarrow 0$*, such that for all* $\epsilon > 0$, $\tau \in (0, \overline{\tau}_\epsilon)$*, and* $\nu > 0$ *the following estimates hold:*

$$\sup_{t\in[0,T]} |\mathcal{I}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t))| \leqslant C, \tag{3.33a}$$
$$\|\overline{z}_{\tau,\nu}\|_{L^\infty(0,T;W^{1,q}(\Omega))} + \|\widehat{z}_{\tau,\nu}\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leqslant C, \tag{3.33b}$$
$$\|\overline{z}_{\tau,\nu}\|_{L^\infty(0,T;W^{1+\sigma,q}(\Omega))} \leqslant C(\varepsilon,\sigma) \text{ for all } 0 < \sigma < \tfrac{1}{q}, \tag{3.33c}$$
$$\|\widehat{z}_{\tau,\nu}\|_{W^{1,2}(0,T;H^1(\Omega))} + \|\widehat{z}'_{\tau,\nu}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(\epsilon), \tag{3.33d}$$
$$\|\widehat{z}_{\tau,\nu}\|_{W^{1,1}(0,T;H^1(\Omega))} \leqslant C(\epsilon), \tag{3.33e}$$
$$\|A_q(\overline{z}_{\tau,\nu})\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(\epsilon), \tag{3.33f}$$
$$\|\overline{\omega}_{\tau,\nu}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(\epsilon), \tag{3.33g}$$
$$\|\overline{u}_{\tau,\nu}\|_{L^\infty(0,T;H^2(\Omega))} \leqslant C, \tag{3.33h}$$
$$\|\widehat{u}_{\tau,\nu}\|_{W^{1,2}(0,T;W^{1,3}(\Omega))} \leqslant C(\epsilon), \tag{3.33i}$$

with $\overline{\varpi}_{\tau,\nu} := \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$. *Furthermore, there exists a constant* $\overline{C} > 0$ *such that for all* ε, $\tau \in (0, \bar{\tau}_\epsilon)$, *and* $\nu > 0$ *there holds*

$$\|A_q \overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)} \leqslant \overline{C}\left(1 + \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\,\mathrm{d}\rho\right) \qquad \textit{for all } t \in [0,T]. \tag{3.34}$$

Proof Estimates (3.33*a*)–(3.33*b*) (and, consequently, (3.33*h*) for $\overline{u}_{\tau,\nu}$) can be derived by the very same arguments as in the proof of Lemma 3.4. Let us point out that we may suppose that $\sup_{\tau,\nu} \|\overline{z}_{\tau,\nu}\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leqslant M$, with M the same constant as in (3.14).

Instead, the calculations for (3.33*d*) have to be slightly modified in comparison to the proof of Lemma 3.4. Indeed, the calculations therein rely on the one-homogeneity of $\mathcal{R}_1$, whereas $\overline{\mathcal{R}}_{1,\nu}$ no longer has this property. Therefore, we argue in this way: keeping the short-hand notation $\widehat{h}_{\tau,\nu}(t) := -(\epsilon\widehat{z}'_{\tau,\nu}(t) + A_q\overline{z}_{\tau,\nu}(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_\tau(t), \overline{z}_{\tau,\nu}(t)))$, and writing $\overline{\varpi}_{\tau,\nu}(t)$ in place of $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(t))$, (3.31) rephrases as $\overline{\varpi}_{\tau,\nu}(t) = \widehat{h}_{\tau,\nu}(t)$. We subtract (3.31) at time $r \in (t^\tau_{k-1}, t^\tau_k)$ from (3.31) at time $t \in (t^\tau_k, t^\tau_{k+1})$ and test the resulting relation by $\widehat{z}'_{\tau,\nu}(t)$. Therefore, we obtain

$$\begin{aligned}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t)) - \overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(r)) &\leqslant \int_\Omega \left(\overline{\varpi}_{\tau,\nu}(t) - \overline{\varpi}_{\tau,\nu}(r)\right)\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x \\ &= \int_\Omega \left(\widehat{h}_{\tau,\nu}(t) - \widehat{h}_{\tau,\nu}(r)\right)\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x,\end{aligned} \tag{3.35}$$

where $\overline{\mathcal{R}}^*_{1,\nu}$ denotes the Fenchel–Moreau convex conjugate of $\overline{\mathcal{R}}_{1,\nu}$, and we have used that

$$\widehat{z}'_{\tau,\nu}(t) \in \partial\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t)) \quad \text{for all } t \in (t^\tau_k, t^\tau_{k+1}) \text{ and for all } k = 0, \ldots, N-1. \tag{3.36}$$

From (3.35), we then obtain the analogue of (3.15), namely

$$\begin{aligned}&\tfrac{1}{\tau}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t)) + \tfrac{\epsilon}{\tau}\textstyle\int_\Omega(\widehat{z}'_{\tau,\nu}(t) - \widehat{z}'_{\tau,\nu}(r))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x + \tfrac{1}{\tau}\int_\Omega(A_q\overline{z}_{\tau,\nu}(t) - A_q\overline{z}_{\tau,\nu}(r))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x \\ &\quad\leqslant \tfrac{1}{\tau}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(r)) - \tfrac{1}{\tau}\textstyle\int_\Omega(\mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_\tau(t), \overline{z}_{\tau,\nu}(t)) - \mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_\tau(r), \overline{z}_{\tau,\nu}(r)))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x.\end{aligned} \tag{3.37}$$

Observe that (3.37) contains the same terms as in (3.15), but with the additional contribution coming from $\overline{\mathcal{R}}^*_{1,\nu}$. Following the lines of the proof of Lemma 3.4 (see also [19, Lemma 5.3]) we 'integrate' over the time interval (t_0, t) with $t_0 \in (0, t^\tau_1)$ and $t \in (t^\tau_k, t^\tau_{k+1})$ and get

$$\begin{aligned}&\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t)) + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t)\|^2_{L^2(\Omega)} + C_q\int_{t^\tau_1}^{\bar{t}_\tau(t)}\int_\Omega \left(1 + |\nabla\widehat{z}_{\tau,\nu}(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_{\tau,\nu}(\rho)|^2\,\mathrm{d}x\,\mathrm{d}\rho \\ &\leqslant \overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t_0)) + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} + C\int_{t^\tau_1}^{\bar{t}_\tau(t)}(1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho,\end{aligned} \tag{3.38}$$

with C_q from (3.16). We observe that $\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t)) \geqslant 0$, and therefore on the left-hand side we get the exact analogue of the left-hand side of (3.19). For the right-hand side, we have to deal with the 'extra'-term $\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t_0))$. For this, we observe that

$$\begin{aligned}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t_0)) = \overline{\mathcal{R}}^*_{1,\nu}(\overline{\varpi}_{\tau,\nu}(t_0)) - \overline{\mathcal{R}}^*_{1,\nu}(0) &\leqslant \int_\Omega \left(\frac{z_1^{\tau,\nu} - z_0}{\tau}\right) \overline{\varpi}_{\tau,\nu}(t_0)\,\mathrm{d}x \\ &= \int_\Omega \frac{(z_1^{\tau,\nu} - z_0)}{\tau}\left(-\epsilon\frac{z_1^{\tau,\nu} - z_0}{\tau} - \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^{\tau,\nu})\right)\mathrm{d}x \\ &= -\epsilon\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} - \int_\Omega \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^{\tau,\nu})\widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x\end{aligned} \tag{3.39}$$

and therefore, the right-hand side of (3.38) can be bounded as follows:

$$\begin{aligned}\text{R.H.S.} \leqslant &- \int_\Omega \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^{\tau,\nu})\widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x - \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} \\ &+ C\int_{t_1^\tau}^{\bar{t}_\tau(t)} (1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho.\end{aligned} \tag{3.40}$$

Writing $\mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^{\tau,\nu}) = A_q(z_1^{\tau,\nu}) - A_q(z_0) + \mathrm{D}_z\widetilde{\mathcal{I}}(t_1^\tau, z_1^{\tau,\nu}) - \mathrm{D}_z\widetilde{\mathcal{I}}(0, z_0) + \mathrm{D}_z\mathcal{I}(0, z_0)$ and performing calculations analogous to those developed in the proof of Lemma 3.4, we obtain

$$\begin{aligned}-\int_\Omega \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^{\tau,\nu})\widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x \leqslant &-C_q\tau\int_\Omega (1+|\nabla\widehat{z}_{\tau,\nu}(t_0)|^2)^{(q-2)/2}|\nabla\widehat{z}'_{\tau,\nu}(t_0)|^2\,\mathrm{d}x + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} \\ &+ \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + c\tau(1 + \|\widehat{z}'_{\tau,\nu}(t_0)\|_{L^6(\Omega)})\|\widehat{z}'_{\tau,\nu}(t_0)\|_{L^2(\Omega)}.\end{aligned}$$

Combining this with (3.40), summing the resulting inequality with (3.38), and adding $C_q\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho$ to both terms of the resulting estimate, we obtain

$$\begin{aligned}&\frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t)\|^2_{L^2(\Omega)} + C_q\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho \\ &\qquad + C_q\int_0^{\bar{t}_\tau(t)}\int_\Omega \left(1 + |\nabla\widehat{z}_{\tau,\nu}(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_{\tau,\nu}(\rho)|^2\,\mathrm{d}x\,\mathrm{d}\rho \\ &\quad\leqslant \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + C_q\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho \\ &\qquad + C\int_0^{\bar{t}_\tau(t)}(1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho \\ &\quad\leqslant C + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + C\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho + \frac{C_q}{4}\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho,\end{aligned} \tag{3.41}$$

where in the last inequality we have used Young's inequality, and the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$, for the last term in the R.H.S. of (3.40) exactly as in the proof of Lemma 3.4. Absorbing $\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho$, into the left-hand side, we conclude estimate (3.33*d*) for $\widehat{z}'_{\tau,\nu}$, uniformly with respect to τ and ν (but not w.r.t. ϵ), and therefore also the bound (3.33*i*) for $\widehat{u}'_{\tau,\nu}$.

We are now in a position to carry out the time-discrete analogue of the *Third regularity estimate*. We multiply (3.31), written at time $\rho \in (t_k^\tau, t_{k+1}^\tau)$, by the difference

$(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))$, with $r \in (t^\tau_{k-1}, t^\tau_k)$, and integrate in space. Observe that this is now a legal test, in view of (3.32). We thus obtain

$$\underbrace{\int_\Omega \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_1} + \epsilon \underbrace{\int_\Omega \widehat{z}'_{\tau,\nu}(\rho)(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_2}$$
$$+ \underbrace{\int_\Omega A_q\overline{z}_{\tau,\nu}(\rho)(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_3} = \underbrace{-\int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho))(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_4}\,. \tag{3.42}$$

Now, we have that

$$\begin{aligned} I_1 =& \int_\Omega \nabla\left(\mathrm{R}'_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))\right)\cdot\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x \\ =& \int_\Omega \mathrm{R}''_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))\nabla\widehat{z}'_{\tau,\nu}(\rho) \\ & \times\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x \overset{(1)}{\geqslant} 0, \end{aligned}$$

where for the first equality we have used that $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho)) = \mathrm{R}'_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))$ is an element in $W^{1,q}(\Omega)$: indeed, $\widehat{z}'_{\tau,\nu}(\rho) \in W^{1,q}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$, so that there exists a constant $M > 0$ with $|\widehat{z}'_{\tau,\nu}(\rho)| \leqslant M$ a.e. in Ω; on the other hand $\mathrm{R}'_{1,\nu} \in \mathrm{C}^\infty(\mathbb{R})$, hence its restriction to the ball $\overline{B}_M(0)$ is Lipschitz, and the composition of a Lipschitz function with an element in $W^{1,q}(\Omega)$ belongs to $W^{1,q}(\Omega)$. Estimate (1) follows from the fact that $\mathrm{R}''_{1,\nu} \geqslant 0$ on $\mathbb{R}$, and from the convexity inequality

$$(A-B)\cdot\Big((1+|A|^2)^{q/2-1}A-(1+|B)|^2)^{q/2-1}B\Big) \geqslant 0 \quad \text{for all } A,\, B \in \mathbb{R}^3,$$

applied with $A = \nabla\overline{z}_{\tau,\nu}(\rho)$ and $B = \nabla\overline{z}_{\tau,\nu}(r)$. Analogously, we have

$$I_2 = \int_\Omega \nabla\widehat{z}'_{\tau,\nu}(\rho)\cdot\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x \geqslant 0.$$

We have

$$I_3 \geqslant \frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)} - \frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(r)\|^2_{L^2(\Omega)}\,.$$

Finally,

$$\begin{aligned} I_4 =& \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho))A_q\overline{z}_{\tau,\nu}(\rho)\,\mathrm{d}x - \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_{\tau,\nu}(r))A_q\overline{z}_{\tau,\nu}(r)\,\mathrm{d}x \\ & - \int_\Omega \Big(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho))-\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_{\tau,\nu}(r))\Big)\,A_q\overline{z}_{\tau,\nu}(r)\,\mathrm{d}x\,. \end{aligned}$$

Summing with respect to the index k, we thus obtain for any $t \in (t_1^\tau, T)$ and for $\sigma \in (0, t_1^\tau)$ (remember that $\overline{z}_{\tau,\nu}(r) = \underline{z}_{\tau,\nu}(\rho)$ and $\overline{t}_\tau(r) = \underline{t}_\tau(\rho)$ for $r \in (t_{k-1}^\tau, t_k^\tau]$ and $\rho \in [t_k^\tau, t_{k+1}^\tau)$)

$$\begin{aligned}\frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2 \leqslant & \frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2 + \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\sigma), \overline{z}_{\tau,\nu}(\sigma))A_q\overline{z}_{\tau,\nu}(\sigma)\,\mathrm{d}x \\ & - \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t))A_q\overline{z}_{\tau,\nu}(t)\,\mathrm{d}x \\ & + \int_{t_1^\tau}^{\overline{t}_\tau(t)} \int_\Omega \frac{1}{\tau}\Big(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho)) - \mathrm{D}\widetilde{\mathcal{I}}(\underline{t}_\tau(\rho), \underline{z}_{\tau,\nu}(\rho))\Big) A_q\underline{z}_{\tau,\nu}(\rho)\,\mathrm{d}x\,\mathrm{d}\rho \\ \doteq & I_5 + I_6 + I_7 + I_8.\end{aligned}$$

We estimate *via* Hölder's and Young's inequalities

$$\begin{aligned}|I_6| &\leqslant \|\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\sigma), \overline{z}_{\tau,\nu}(\sigma))\|_{L^2(\Omega)}^2 + \frac{1}{4}\|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2 \overset{(2)}{\leqslant} C + \frac{1}{4}\|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2, \\ |I_7| &\leqslant \|\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t))\|_{L^2(\Omega)}^2 + \frac{1}{4}\|A_q\overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2 \overset{(1)}{\leqslant} C + \frac{1}{4}\|A_q\overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2, \\ |I_8| &\leqslant \int_0^{\overline{t}_\tau(t)} \frac{1}{\tau}\|\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho)) - \mathrm{D}\widetilde{\mathcal{I}}(\underline{t}_\tau(\rho), \underline{z}_{\tau,\nu}(\rho))\|_{L^2(\Omega)}\|A_q\underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho \\ &\overset{(3)}{\leqslant} C\int_0^{\overline{t}_\tau(t)} \frac{1}{\tau}\|\overline{z}_{\tau,\nu}(\rho) - \underline{z}_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\|A_q\underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho.\end{aligned}$$

Here, (1) and (2) follow from (2.34) and from the bound $\|f'(\overline{z}_{\tau,\nu}(t))\|_{L^\infty(\Omega)} + P(\overline{z}_{\tau,\nu}(t), 0) \leqslant C$, for a constant uniform w.r.t. $t \in [0, T]$, thanks to estimate (3.33*b*) for $(\overline{z}_{\tau,\nu})_{\tau,\nu}$; instead, (3) is due to (2.32), again taking into account that $\sup_{\rho\in[0,T]}(C_{f''}(\overline{z}_{\tau,\nu}(\rho), \underline{z}_{\tau,\nu}(\rho)) + P(\overline{z}_{\tau,\nu}(\rho), \underline{z}_{\tau,\nu}(\rho))^3) \leqslant C$ due to the bound (3.33*b*). All in all, we conclude

$$\frac{1}{4}\|A_q\overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2 \leqslant \frac{3}{4}\|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2 + C\left(1 + \int_0^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\|A_q\underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho\right),$$

and, with the discrete version of Gronwall's Lemma from [14], we conclude that

$$\|A_q\overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)} \leqslant C\left(1 + \|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)} + \int_0^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\,\mathrm{d}\rho\right). \tag{3.43}$$

It now remains to estimate $\|A_q\overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)} = \|A_q z_1^{\tau,\nu}\|_{L^2(\Omega)}$. For this, we use the Euler–Lagrange equation

$$\mathrm{D}\overline{\mathcal{R}}_{1,\nu}\left(\frac{z_1^{\tau,\nu} - z_0}{\tau}\right) + \epsilon\frac{z_1^{\tau,\nu} - z_0}{\tau} + A_q z_1^{\tau,\nu} + \mathrm{D}_z\widetilde{\mathcal{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu}) = 0$$

and test it by $A_q z_1^{\tau,\nu} - A_q z_0$. We repeat the same calculations as above and arrive at

$$\begin{aligned}\frac{1}{2}\|A_q z_1^{\tau,\nu}\|_{L^2(\Omega)}^2 \leqslant & \frac{1}{2}\|A_q z_0\|_{L^2(\Omega)}^2 + \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(0, z_0)A_q z_0\,\mathrm{d}x - \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu})A_q z_1^{\tau,\nu}\,\mathrm{d}x \\ & + \int_\Omega \Big(\mathrm{D}\widetilde{\mathcal{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu}) - \mathrm{D}\widetilde{\mathcal{I}}(0, z_0)\Big) A_q z_0\,\mathrm{d}x,\end{aligned}$$

whence

$$\|A_q z_1^{\tau,\nu}\|^2_{L^2(\Omega)} \leqslant C\left(1 + \|A_q z_0\|^2_{L^2(\Omega)} + \|z_1^{\tau,\nu} - z_0\|^2_{L^6(\Omega)}\right) \leqslant C, \tag{3.44}$$

the last inequality due to (3.6) and bound (3.33*b*). Observe that the constant C in (3.44) is also uniform w.r.t. ϵ. Combining (3.44) with (3.43), we infer (3.34). Then, estimate (3.33*f*) follows in view of the previously proved bound (3.33*d*) for $\widehat{z}'_{\tau,\nu}$.

Estimate (3.33*g*) for $\overline{\varpi}_{\tau,\nu} = \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$ follows from a comparison argument in (3.31), in view of estimate (3.33*d*), and the previously used bound for $\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\cdot), \overline{z}_{\tau,\nu}(\cdot))$ in $L^\infty(0,T;L^2(\Omega))$ due to (2.34) and (3.33*b*). Finally, we obtain (3.33*c*) from Proposition 2.8 combined with (3.33*b*), (3.33*f*) and (3.33*g*), while (3.33*e*) is a consequence of (3.33*d*). □

Remark 3.7 It remains an open problem to prove the analogue of the BV estimate (3.9*e*) for the functions $\widehat{z}_{\tau,\nu}$ (and, consequently, (3.9*j*) for $\widehat{u}_{\tau,\nu}$), with a constant *uniform* w.r.t. ϵ. The reason is that, mimicking the calculations from the proof of Lemma 3.4 one obtains the analogue of (3.26) featuring additional terms with $\overline{\mathcal{R}}^*_{1,\nu}$, just like these terms appeared in (3.38). Such terms did not pop up in the calculations for Lemma 3.4 as $\mathcal{R}_1^*$ is an indicator function. In fact, they prevent us from applying the Gronwall-type result from [19, Lemma B.1] and to conclude (3.9*e*), uniformly w.r.t. ϵ, for $\widehat{z}_{\tau,\nu}$.

Thanks to Lemma 3.6 we are now in a position to conclude the *proof of Lemma 3.5:* For *fixed* positive τ and ϵ, let $(\overline{z}_{\tau,\nu}, \widehat{z}_{\tau,\nu})_\nu$ be a family of solutions to (3.31). The bound (3.33*c*) implies that also the sequence $(\widehat{z}_{\tau,\nu})_\nu$ is bounded in that space. Therefore, applying the Aubin–Lions type compactness results from [30] to $(\widehat{z}_{\tau,\nu})_\nu$, we infer that there exists a function $\widehat{z}$ such that, along a (not relabelled) subsequence, as $\nu \downarrow 0$ the following convergences hold:

$$\begin{aligned} \widehat{z}_{\tau,\nu} &\rightharpoonup^* \widehat{z} \quad \text{in } L^\infty(0,T;W^{1+\sigma,q}(\Omega)) \cap H^1(0,T;H^1(\Omega)) \\ &\qquad \cap W^{1,\infty}(0,T;L^2(\Omega)) \quad \text{for all } 0 < \sigma < \frac{1}{q}, \\ \widehat{z}_{\tau,\nu} &\to \widehat{z} \quad \text{in } \mathrm{C}^0([0,T];\mathcal{Z}), \end{aligned} \tag{3.45a}$$

where the last convergence follows from the compact embedding $W^{1+\sigma,q}(\Omega) \Subset \mathcal{Z}$ for all $\sigma \in (0, \frac{1}{q})$. From the estimate for $(\widehat{z}'_{\tau,\nu})_\nu$ in $L^1(0,T;H^1(\Omega))$, we gather that

$$\|\overline{z}_{\tau,\nu}\|_{\mathrm{BV}([0,T];H^1(\Omega))} \leqslant C$$

for a constant independent of ν (and τ). Therefore, thanks to an infinite-dimensional version of Helly's Theorem, see e.g., [26, Theorem 6.1], we conclude that there exists $\overline{z} \in \mathrm{BV}([0,T];H^1(\Omega))$ such that, up to the further extraction of a subsequence, $\overline{z}_{\tau,\nu}(t) \rightharpoonup \overline{z}(t)$ in $H^1(\Omega)$, as $\nu \downarrow 0$ for every $t \in [0,T]$. Since $(\overline{z}_{\tau,\nu})_\nu$ is bounded in $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$, we ultimately conclude that $\overline{z}_{\tau,\nu}(t) \rightharpoonup \overline{z}(t)$ in $W^{1+\sigma,q}(\Omega)$ for every $t \in [0,T]$. Thus, we infer

$$\overline{z}_{\tau,\nu}(t) \to \overline{z}(t) \qquad \text{in } \mathcal{Z} \quad \text{for every } t \in [0,T], \tag{3.45b}$$

and thus

$$\underline{z}_{\tau,\nu}(t) = z(t-\tau) \to \overline{z}(t-\tau) =: \underline{z}(t) \qquad \text{in } \mathcal{Z} \quad \text{for every } t \in [\tau, T], \tag{3.45c}$$

(observe that $\underline{z}_{\tau,\nu}(t) = z_0$ and thus we may define $\underline{z}(t) \equiv z_0$ for $t \in [0, \tau)$). Then, *a fortiori* one has that

$$\overline{z}_{\tau,\nu} \rightharpoonup^* \overline{z} \text{ in } L^\infty(0,T;\mathcal{Z}), \qquad \overline{z}_{\tau,\nu} \to \overline{z} \text{ in } L^p(0,T;\mathcal{Z}) \text{ for every } 1 \leqslant p < \infty, \tag{3.45d}$$

and we have the analogous convergences of $(\underline{z}_{\tau,\nu})_\nu$ to $\underline{z}$. Finally, there exists $\overline{\varpi} \in L^\infty(0,T;L^2(\Omega))$ such that, up to a further extraction,

$$\overline{\varpi}_{\tau,\nu} \rightharpoonup^* \overline{\varpi} \qquad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{3.45e}$$

It follows from (3.45*b*), combined with the bound (3.33*e*), that

$$A_q\overline{z}_{\tau,\nu}(t) \rightharpoonup A_q\overline{z}(t) \qquad \text{in } L^2(\Omega) \quad \text{for every } t \in [0,T]. \tag{3.46}$$

Also in view of (3.45*b*), it is not difficult to deduce that

$$A_q\overline{z}_{\tau,\nu} \rightharpoonup^* A_q\overline{z} \qquad \text{in } L^\infty(0,T;L^2(\Omega)).$$

Furthermore, combining estimate (2.32) with (3.33*b*) and convergence (3.45*b*), we find that for every $t \in [0,T]$

$$\begin{aligned}\|\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}_{\tau,\nu}(t)) - \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t))\|_{L^2(\Omega)} &\leqslant C\left(C_f'(\overline{z}_{\tau,\nu}(t),\overline{z}(t)) + P(\overline{z}_{\tau,\nu}(t),\overline{z}(t))^3\right)\|\overline{z}_{\tau,\nu}(t)-\overline{z}(t)\|_{L^6(\Omega)}\\ &\leqslant C\|\overline{z}_{\tau,\nu}(t)-\overline{z}(t)\|_{L^6(\Omega)} \to 0\end{aligned}$$

as $\nu \downarrow 0$. Since $(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}))_\nu$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by (2.34) and (3.9*b*), we also have

$$\begin{aligned}&\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}) \rightharpoonup^* \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}) \text{ in } L^\infty(0,T;L^2(\Omega)),\\ &\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}) \to \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}) \text{ in } L^p(0,T;L^2(\Omega)) \quad \text{for every } 1 \leqslant p < \infty.\end{aligned}$$

Therefore, also on account of convergences (3.45*a*) and (3.45*e*), we can pass to the limit as $\nu \downarrow 0$ in an integrated-in-time version of (3.31) and, with a standard argument, conclude that the triple $(\overline{z},\widehat{z},\overline{\varpi})$ satisfies

$$\overline{\varpi}(t) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t)) = 0 \quad \text{ in } L^2(\Omega) \quad \text{for a.a. } t \in (t_k^\tau, t_{k+1}^\tau) \tag{3.47}$$

and for every $k \in \{0,\ldots,N-1\}$. We can also prove that

$$\limsup_{\nu\downarrow 0} \int_{t_k^\tau}^{t_{k+1}^\tau} \int_\Omega \overline{\varpi}_{\tau,\nu}\widehat{z}_{\tau,\nu}' \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{t_k^\tau}^{t_{k+1}^\tau} \int_\Omega \overline{\varpi}\widehat{z}' \,\mathrm{d}x\,\mathrm{d}t\,.$$

This follows from multiplying (3.31) by $\widehat{z}_{\tau,\nu}'$ and taking the limit in each of the terms, on account of the convergences proved so far.

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Therefore, thanks to (3.29d), we infer that $\overline{\varpi}(t) \in \partial\mathcal{R}_1(\widehat{z}'(t))$ for almost all $t \in (t_k^\tau, t_{k+1}^\tau)$. All in all, the pair $(\overline{z}, \widehat{z})$ fulfills the differential inclusion

$$\begin{aligned}&\partial\mathcal{R}_1(\widehat{z}'(t)) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}(t)) \ni 0 \text{ in } L^2(\Omega)\\ &\text{for a.a. } t \in (t_k^\tau, t_{k+1}^\tau)\ \forall k \in \{0, \ldots, N-1\}.\end{aligned} \tag{3.48}$$

A fortiori, since $\partial\mathcal{R}_1(\widehat{z}'(t)) \subset \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}'(t))$, we conclude that $(\overline{z}, \widehat{z})$ fulfill

$$\begin{aligned}&\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}'(t)) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}(t)) \ni 0 \quad \text{in } \mathcal{Z}^*\\ &\text{for a.a. } t \in (t_k^\tau, t_{k+1}^\tau) \quad \forall k \in \{0, \ldots, N-1\}.\end{aligned}$$

Since the latter has a unique solution in the closed ball $\overline{B}_M(0)$ of $\mathcal{Z}$ for $\tau < \overline{\tau}_\epsilon$ (cf. Proposition 3.1), and since $\overline{z}$ and $\overline{z}_\tau$ take value in that ball, we obtain that

$$\overline{z}(t) = \overline{z}_\tau(t), \quad \underline{z}(t) = \underline{z}_\tau(t), \quad \widehat{z}'(t) = \widehat{z}_\tau'(t) \quad \text{for a.a. } t \in (t_k^\tau, t_{k+1}^\tau) \quad \forall k \in \{0, \ldots, N-1\},$$

and, therefore, a.e. in $(0, T)$. Here, we have also used the fact that $\hat{z}$ is piecewise affine, $\overline{z}$ is piecewise constant on (t_k^τ, t_{k+1}^τ) and that $\hat{z}(t_k^\tau) = \overline{z}(t_k^\tau)$ for all k. In particular, we find that $A_q\overline{z}_\tau \in L^\infty(0, T; L^2(\Omega))$. In view of convergences (3.45d) and recalling that $\widehat{z}_{\tau,\nu}' = \frac{\overline{z}_{\tau,\nu} - \underline{z}_{\tau,\nu}}{\tau}$ a.e. in $(0, T)$, we ultimately have

$$\widehat{z}_{\tau,\nu}' \to \widehat{z}_\tau' \quad \text{in } L^p(0, T; \mathcal{Z}) \text{ for every } 1 \leqslant p < \infty.$$

Therefore, also on account of the pointwise convergence (3.46), we are in a position to pass to the limit in estimate (3.34) and deduce that

$$\|A_q\overline{z}_\tau(t)\|_{L^2(\Omega)} \leqslant \overline{C}\left(1 + \int_0^{\overline{t}_\tau(t)} \|\widehat{z}_\tau'(\rho)\|_{L^6(\Omega)}\,\mathrm{d}\rho\right) \qquad \text{for all } t \in [0, T]. \tag{3.49}$$

Combining (3.49) with the, uniform w.r.t. ϵ, estimate (3.9e) for $\widehat{z}_\tau'$, we ultimately conclude

$$\|A_q\overline{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C,$$

for a constant independent of ε and $\tau < \overline{\tau}_\epsilon$. A comparison in (3.47) also yields

$$\|\overline{\varpi}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C.$$

Therefore, we set $\overline{\varpi}_\tau := \overline{\varpi}$ and ultimately conclude (3.7) as well as (3.9f) and (3.9g). Finally, from (3.48) we gather the validity of (3.8). This concludes the proof of Lemma 3.5. ∎

4 Existence of viscous solutions

In this section, we briefly comment on the existence of solutions to the viscous system (1.2).

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By passing to the limit with $\epsilon > 0$ fixed in the time-discrete scheme (3.5), we are able to prove the existence of a solution to (1.2), formulated as a subdifferential inclusion in $L^2(\Omega)$, namely

$$\omega(t) + \epsilon z'(t) + A_q(z(t)) + \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } t \in (0, T), \tag{4.1}$$

with $\omega(\cdot)$ a selection in the subdifferential $\partial\mathcal{R}_1(z'(\cdot)) \subset L^2(\Omega)$. Furthermore, along the footsteps of [24], we obtain an ED balance featuring the conjugate $\mathcal{R}_\epsilon^*$ of $\mathcal{R}_\epsilon$, cf. (3.10).

Theorem 4.1 *Let $\epsilon > 0$ be fixed. Under Assumptions 2.1, 2.2 and 2.9, and under condition* (3.6) *on the initial datum z_0, there exist*

$$z \in L^\infty(0, T; W^{1+\sigma,q}(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \text{ for every } \sigma \in (0, \tfrac{1}{q}), \\ \text{with } A_q z \in L^\infty(0, T; L^2(\Omega)) \tag{4.2}$$

and $\omega \in L^\infty(0, T; L^2(\Omega))$ fulfilling the subdifferential inclusion (4.1) *and the Cauchy condition $z(0) = z_0$.*

Furthermore, z complies with the ED balance

$$\begin{aligned}\int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-A_q(z(r)) - \mathrm{D}_z\widetilde{\mathcal{I}}(r, z(r)))\,\mathrm{d}r + \mathcal{I}(t, z(t)) \\ = \mathcal{I}(s, z(s)) + \int_s^t \partial_t\mathcal{I}(r, z(r))\,\mathrm{d}r\end{aligned} \tag{4.3}$$

for every $0 \leqslant s \leqslant t \leqslant T$.

Proof Let $(\tau_j)_j$ be a null sequence of time steps, and let $(\overline{z}_{\tau_j})_j$, $(\widehat{z}_{\tau_j})_j$ be the approximate solutions to the viscous subdifferential inclusion (1.2) constructed in Section 3. For them, estimates (3.9) hold with a constant uniform w.r.t. $j \in \mathbb{N}$ (recall that $\epsilon > 0$ is fixed).

Adapting the arguments from the proof of [19, Proposition 6.2], combining (3.9) with Aubin–Lions type compactness results (cf., e.g., [30, Theorem 5, Corollary 4]) and arguing in the same way as in the proof of Lemma 3.5, cf. also Lemma 6.2 ahead, we may show that there exist a (not relabelled) subsequence and a curve z as in (4.2) such that the following convergences hold

$$\begin{aligned}
&\overline{z}_{\tau_j}, \widehat{z}_{\tau_j} \to z && \text{in } L^\infty(0, T; \mathcal{Z}),\\
&\widehat{z}_{\tau_j} \rightharpoonup^* z && \text{in } H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)),\\
&\mathcal{I}(\overline{t}_{\tau_j}(t), \overline{z}_{\tau_j}(t)),\ \mathcal{I}(t, \widehat{z}_{\tau_j}(t)) \to \mathcal{I}(t, z(t)) && \text{for all } t \in [0, T],\\
&\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_j}(t), \overline{z}_{\tau_j}(t)) \rightharpoonup^* \mathrm{D}_z\mathcal{I}(t, z(t)) && \text{in } L^\infty(0, T; L^2(\Omega)),\\
&\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_j}(t), \overline{z}_{\tau_j}(t)) \to \mathrm{D}_z\mathcal{I}(t, z(t)) && \text{in } L^\infty(0, T; \mathcal{Z}^*).
\end{aligned}$$

With the limit passage arguments from [19, Theorem 3.5] we deduce that z complies with the variational inequality

$$\mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geqslant \langle -A_q z(t), w\rangle_{\mathcal{Z}} + \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x \\ - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(t, z(t))(w - z'(t))\,\mathrm{d}x \quad \text{for all } w\in\mathcal{Z} \quad \text{for a.a. } t\in(0,T), \tag{4.4}$$

which in fact defined the concept of *weak solution* to the viscous system considered in [19].

We now enhance (4.4) by relying on the information that $A_q z \in L^\infty(0,T;L^2(\Omega))$. Due to this, $\int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x = \int_\Omega A_q(z(t))z'(t)\,\mathrm{d}x$, so that (4.4) reads for almost all $t\in(0,T)$

$$\mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geqslant \; - \int_\Omega A_q z(t)(w - z'(t))\,\mathrm{d}x - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(t, z(t))(w - z'(t))\,\mathrm{d}x \\ \text{for all } w\in\mathcal{Z}.$$

This extends to all $w\in L^2(\Omega)$ by a density argument, and therefore we conclude that

$$-A_q z(t) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t)) \in \partial\mathcal{R}_\epsilon(z'(t)) \qquad \text{in } L^2(\Omega) \tag{4.5}$$

for almost all $t\in(0,T)$, namely the validity of (4.1).

The ED balance (4.3) ensues from integrating on the generic interval $(s,t)\subset(0,T)$ the following chain of identities

$$\mathcal{R}_\epsilon(z'(r)) + \mathcal{R}_\epsilon^*(-A_q z(r) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r))) \overset{(1)}{=} \int_\Omega \Big(-A_q z(r) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r))\Big)\, z'(t)\,\mathrm{d}x \\ \overset{(2)}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(r,z(r)) + \partial_t\mathcal{I}(r,z(r)) \quad \text{for a.a. } r\in(0,T),$$

where (1) is a reformulation of (4.5), while (2) follows from the chain rule (2.43). □

5 Balanced viscosity solutions to the rate-independent damage system

The main result of this section, Theorem 5.7 ahead, states the convergence of the sequences

$$(\overline{z}_{\tau,\epsilon})_{\tau,\epsilon}, \; (\widehat{z}_{\tau,\epsilon})_{\tau,\epsilon} \tag{5.1}$$

of discrete solutions constructed in Section 3 to a BV solution of the rate-independent damage system (1.1), as ϵ and τ *simultaneously* tend to zero. That is why we stress the dependence on the parameter ϵ in the notation (5.1). The proof of Theorem 5.7 will be carried out in Section 6.

In Section 5.1, we provide a precise definition of this solution concept, after revisiting and suitably modifying all the preliminary definitions and notions given in [25, Section 3.1]. Indeed, the latter paper addressed the case of a *non-smooth* energy functional driving the abstract gradient system under consideration, and developed the vanishing-viscosity analysis solely relying on the *basic energy* estimates for viscous solutions, cf. the discussion prior to Lemma 3.4. In the present context, on one hand, we will work with simpler definitions that are tailored to the smoothness properties of $\mathcal{I}$ and to the enhanced estimates holding for our own damage system. On the other hand, our definitions shall reflect the fact that the dissipation potential $\mathcal{R}_1$ takes the value $+\infty$, whereas the analysis in [25] is confined to the case of a *continuous* potential $\mathcal{R}_1$.

In Section 5.2 , we gain further insight into to the properties of BV solutions and again revisit and adapt a series of results given in [25, Sections 3.2, 3.3, 3.4].

5.1 The notion of Balanced Viscosity solution

In order to define the notion of BV solution for the damage system (1.1),

we start by introducing the *vanishing-viscosity contact potential* $\mathfrak{p}$ induced by the viscous dissipation potentials $\mathcal{R}_\epsilon$ from (1.3). This functional will enter into the Finsler cost describing the energy dissipated at jumps. We define $\mathfrak{p} : L^2(\Omega) \times L^2(\Omega) \to [0, +\infty]$ *via*

$$\begin{aligned}\mathfrak{p}(v, \xi) &:= \inf_{\epsilon>0} \left(\mathcal{R}_\epsilon(v) + \mathcal{R}^*_\epsilon(\xi)\right) \\ &= \mathcal{R}_1(v) + \|v\|_{L^2(\Omega)} \inf_{z\in\partial\mathcal{R}_1(0)} \|\xi - z\|_{L^2(\Omega)} .\end{aligned}$$

From this, one defines the *dissipation functional* $\mathfrak{f} : [0, T] \times \mathcal{Z} \times L^2(\Omega) \to [0, +\infty]$ *via*

$$\mathfrak{f}_t(z, v) := \mathfrak{p}(v, -\mathrm{D}_z\mathcal{I}(t, z)) = \mathcal{R}_1(v) + \|v\|_{L^2(\Omega)} \min_{\zeta\in\partial\mathcal{R}_1(0)} \| - \mathrm{D}_z\mathcal{I}(t, z) - \zeta\|_{L^2(\Omega)} ,$$

where v plays the role of z'. Observe that for all $z \in \mathcal{Z}, v \in L^2(\Omega)$, we have

$$\mathfrak{f}_t(z, v) \geqslant \langle -\mathrm{D}_z\mathcal{I}(t, z), v\rangle_{L^2(\Omega)} ,$$

provided that $\mathrm{D}_z\mathcal{I}(t, z) \in L^2(\Omega)$. We are now in a position to define the Finsler cost associated with $\mathfrak{f}$, obtained by minimizing suitable integral quantities along *admissible curves*. Let us mention in advance that our definition of the class of admissible curves reflects the enhanced estimates available in the present setting for the discrete viscous solutions, cf. Remark 5.2 below for more details.

Definition 5.1 *Let $t \in [0, T]$ and z_0, $z_1 \in \mathcal{Z}$ be fixed.*

(1) *We call a curve $\vartheta : [r_0, r_1] \to \mathcal{Z}$, for some $r_0 < r_1$, an* admissible transition curve *between z_0 and z_1, at the time $t \in [0, T]$, if*

(a) *$\vartheta \in L^\infty(r_0, r_1; \mathcal{Z}) \cap \mathrm{AC}([r_0, r_1]; L^2(\Omega))$;*

(b) *$\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$.*

We denote by $\mathcal{T}_t(z_0, z_1)$ the set of admissible curves connecting z_0 and z_1.

(2) *The (possibly asymmetric) Finsler cost induced by $\mathfrak{f}_t$ at the time t is given by*

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) := \inf_{\vartheta\in\mathcal{T}_t(z_0,z_1)} \int_{r_0}^{r_1} \mathfrak{f}_t(\vartheta(r), \vartheta'(r))\,\mathrm{d}r \tag{5.2}$$

with the usual convention of setting $\Delta_{\mathfrak{f}}(t; u_0, u_1) = +\infty$ if the set $\mathcal{T}_t(z_0, z_1)$ of admissible curves connecting z_0 and z_1 is empty.

As in the proof of Lemma 2.17, we observe that, since $\vartheta \in L^\infty(r_0, r_1; \mathcal{Z})$, requiring $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$ is equivalent to asking for $A_q(\vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$.

We trivially have

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) \geqslant \mathcal{R}_1(z_1 - z_0) \quad \text{for every } t \in [0, T] \text{ and } z_0,\, z_1 \in \mathcal{Z}. \tag{5.3}$$

Furthermore, observe that, for a given admissible transition curve, $\int_{r_0}^{r_1} \mathfrak{f}_t(\vartheta(r), \vartheta'(r))\,\mathrm{d}r < \infty$ implies that $\vartheta' \leqslant 0$ a.e. in (r_0, r_1). Therefore,

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) < \infty \text{ implies that } z_1(x) \leqslant z_0(x) \text{ for all } x \in \Omega\,.$$

Hereafter, upon writing $\Delta_{\mathfrak{f}}(t; z_0, z_1)$ and $\mathcal{T}_t(z_0, z_1)$ we will, most often implicitly, assume that $z_1 \leqslant z_0$ in Ω.

Up to a reparameterization, due to the positive homogeneity of the Finsler metric $\mathfrak{f}_t(z, \cdot)$, we can suppose that the admissible transition curves are defined on $[0, 1]$. For later use, we also introduce, for a fixed $\varrho > 0$ and z_0, $z_1 \in \mathcal{Z}$ with $z_1 \leqslant z_0$ in Ω, the set of admissible transition curves lying in a suitable ball of radius ϱ, i.e.,

$$\mathcal{T}_t^\varrho(z_0, z_1) := \{\vartheta \in \mathcal{T}_t(z_0, z_1) : \|\vartheta\|_{L^\infty(0,1;\mathcal{Z})} + \|\vartheta'\|_{L^1(0,1;L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot))\|_{L^\infty(0,1;L^2(\Omega))} \leqslant \varrho\} \tag{5.4a}$$

and, accordingly,

$$\Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) := \inf_{\vartheta \in \mathcal{T}_t^\varrho(z_0, z_1)} \int_{r_0}^{r_1} \mathfrak{f}_t(\vartheta(r), \vartheta'(r))\,\mathrm{d}r\,. \tag{5.4b}$$

Since for every $\varrho > 0$ there holds $\mathcal{T}_t^\varrho(z_0, z_1) \subset \mathcal{T}_t(z_0, z_1)$, one has $\Delta_{\mathfrak{f}}(t; z_0, z_1) \leqslant \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1)$. Indeed,

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) = \inf_{\varrho > 0} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) \qquad \text{for every } t \in [0, T] \text{ and } z_0,\, z_1 \in \mathcal{Z}. \tag{5.5}$$

For later use, we also record the following monotonicity property

$$\Delta_{\mathfrak{f}}^{\bar\varrho}(t; z_0, z_1) = \inf_{0 < \varrho < \bar\varrho} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) = \sup_{\varrho > \bar\varrho} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) \text{ for every } t \in [0, T],\ z_0,\, z_1 \in \mathcal{Z} \text{ and } \bar\varrho > 0, \tag{5.6}$$

since $\mathcal{T}_t^\varrho(z_0, z_1) \subset \mathcal{T}_t^{\bar\varrho}(z_0, z_1)$ for every $0 < \varrho < \bar\varrho$. Observe that, for every fixed $\varrho > 0$, the inf in definition (5.4*b*) is attained, cf. Proposition 6.1 ahead, whereas it need not be attained in the definition of $\Delta_{\mathfrak{f}}$, cf. (5.2). In fact, the dissipation functional $\mathfrak{f}$ does not control the norms of the spaces where we look for admissible transition curves.

Remark 5.2 The most striking difference between the present definition of admissible curve and the one given in [25, Definition 3.4] resides in the fact that, in contrast with conditions (a) & (b) from Definition 5.1, in [25] it was only required

$$\begin{aligned} &\vartheta|_{G_t[\vartheta]} \in \mathrm{AC}(G_t[\vartheta]; L^2(\Omega)) \quad \text{with the open set} \\ &G_t[\vartheta] := \{r \in [r_0, r_1] \;:\; \min_{\zeta \in \partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t, z) - \zeta\|_{L^2(\Omega)} > 0\}\,. \end{aligned} \tag{5.7}$$

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The stronger condition $\vartheta \in \mathrm{AC}([r_0.r_1];L^2(\Omega))$ reflects the fact that the discrete viscous solutions $(\overline{z}_\tau)_\tau$ enjoy an estimate in $\mathrm{BV}([0,T];L^2(\Omega))$ uniformly w.r.t. both parameters ϵ and τ. In fact this estimate is even valid in $\mathrm{BV}([0,T];H^1(\Omega))$, cf. (3.9$e$). Instead, in the general framework considered in [25] only the *basic* energy estimate

$$\int_0^T \mathfrak{p}(\widehat{z}'_\tau(t), -\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t),\overline{z}_\tau(t)))\,\mathrm{d}t \leqslant \int_0^T \left(\mathcal{R}_\epsilon(\widehat{z}'_\tau(t)) + \mathcal{R}^*_\epsilon(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t),\overline{z}_\tau(t)))\right)\,\mathrm{d}t \leqslant C$$

was available. In accordance with that, only (5.7) was required on admissible curves.

Condition (b) in Definition 5.1 reflects the enhanced estimate (3.9k). It is also a peculiarity of the present framework, and in particular it is motivated by the fact that we impose unidirectionality of damage evolution, thus allowing $\mathcal{R}_1$ to take the value $+\infty$. In order to explain this, let us observe that, in the setting considered in [25], it was not necessary to specify the summability properties of $\mathrm{D}_z\mathcal{I}(t,\vartheta(\cdot))$ within the definition of admissible curve. Indeed, outside the set $G_t[\vartheta]$ one had $\mathrm{D}_z\mathcal{I}(t,\vartheta(\cdot)) \in \partial\mathcal{R}_1(0)$, a bounded subset of $L^2(\Omega)$ since the dissipation potential $\mathcal{R}_1$ was everywhere continuous. Instead, on the set $G_t[\vartheta]$ an estimate for the quantity $\min_{\zeta\in\partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t,z)-\zeta\|_{L^2(\Omega)}$ would morally provide a bound for $-\mathrm{D}_z\mathcal{I}(t,z)$, as well, by comparison arguments, again thanks to the boundedness $\partial\mathcal{R}_1(0)$. Instead, in the present setting, since the set $\partial\mathcal{R}_1(0)$ is unbounded, it is necessary to encompass a suitable summability condition on $\mathrm{D}_z\mathcal{I}(t,\vartheta(\cdot))$ in the definition of admissible curve.

We are now ready to introduce the jump variation induced by $\mathfrak{f}$, accounting for the energy dissipated at the jumps of a given curve $z \in \mathrm{BV}([0,T];L^1(\Omega))$, with countable jump set

$$\mathrm{J}_z := \{t \in [0,T] \;:\; z(t_-) \neq z(t) \text{ or } z(t_+) \neq z(t)\}$$

and $z(t_\pm)$ the right/left limits of z at $t \in [0,T]$. Based on the jump variation associated with $\mathfrak{f}$ in (5.11) ahead, we introduce a novel notion of total variation for the curve z, alternative to the total variation induced by the dissipation potential $\mathcal{R}_1$. Then, for a given $[a,b] \subset [0,T]$, the $\mathcal{R}_1$-variation of z on $[a,b]$ is defined by

$$\mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) := \sup\{\sum_{m=1}^M \mathcal{R}_1(z(t_m)-z(t_{m-1})) \;:\; a = t_0 < t_1 < \ldots < t_{M-1} < t_M = b\}. \tag{5.8}$$

In particular, the contribution at the jumps induced by $\mathcal{R}_1$ is

$$\begin{aligned}\mathrm{Jump}_{\mathcal{R}_1}(z;[a,b]) &:= \mathcal{R}_1(z(a_+)-z(a)) + \mathcal{R}_1(z(b)-z(b_-)) \\ &\quad + \sum_{t\in \mathrm{J}_z\cap(a,b)} \mathcal{R}_1(z(t_+)-z(t)) + \mathcal{R}_1(z(t)-z(t_-)).\end{aligned}$$

Clearly, $\mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) < \infty$ guarantees that $t \mapsto z(t)$ is decreasing on $[a,b]$, i.e.,

$$\text{for all } a \leqslant t_1 \leqslant t_2 \leqslant b, \text{ for almost all } x \in \Omega \qquad z(t_2,x) \leqslant z(t_1,x). \tag{5.9}$$

For later convenience, we also introduce the scalar function

$$V(t) := \begin{cases} 0 & \text{if } t \leqslant 0, \\ \mathrm{Var}_{\mathcal{R}_1}(z;[0,t]) & \text{if } t \in (0,T), \\ \mathrm{Var}_{\mathcal{R}_1}(z;[0,T]) & \text{if } t \geqslant T \end{cases} \qquad \text{with distributional derivative } \mu = \frac{\mathrm{d}}{\mathrm{d}t}V\,. \tag{5.10}$$

Recall that μ is a finite Borel measure supported on $[0,T]$, and it can be decomposed as $\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{J}}$, with μ_{J} the jump part, concentrated on the countable jump set J_z, and μ_{d} the diffuse part, given by the sum of the absolutely continuous and of the Cantor parts, so that $\mu_{\mathrm{d}}(\{t\}) = 0$ for every $t \in \mathbb{R}$.

We are now in a position to give the notion of total variation induced by $\mathfrak{f}$. Let us mention in advance that it is obtained by replacing the $\mathrm{Jump}_{\mathcal{R}_1}$-contribution to the total variation $\mathrm{Var}_{\mathcal{R}_1}$, with the $\mathfrak{f}$-jump variation, cf. (5.12) below.

Definition 5.3 (Jump and total variation induced by $\mathfrak{f}$) *Let z in $\mathrm{BV}([0,T];L^1(\Omega))$, with $z(t) \in \mathcal{Z}$ for all $t \in [0,T]$, be a given curve with jump set J_z. Let $[a,b] \subset [0,T]$:*

(1) *The* jump variation *of z on $[a,b]$ induced by $\mathfrak{f}$ is*

$$\begin{aligned}\mathrm{Jump}_{\mathfrak{f}}(z;[a,b]) := {} & \Delta_{\mathfrak{f}}(a;z(a),z(a_+)) + \Delta_{\mathfrak{f}}(b;z(b_-),z(b)) \\ & + \sum_{t\in \mathrm{J}_z\cap(a,b)} \big(\Delta_{\mathfrak{f}}(t;z(t_-),z(t)) + \Delta_{\mathfrak{f}}(t;z(t),z(t_+))\big).\end{aligned} \tag{5.11}$$

(2) *The total variation of z on $[a,b]$ induced by $\mathfrak{f}$ is*

$$\mathit{Var}_{\mathfrak{f}}(z;[a,b]) := \mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) - \mathrm{Jump}_{\mathcal{R}_1}(z;[a,b]) + \mathrm{Jump}_{\mathfrak{f}}(z;[a,b]) \tag{5.12}$$

$$= \mu_{\mathrm{d}}([a,b]) + \mathrm{Jump}_{\mathfrak{f}}(z;[a,b])\,. \tag{5.13}$$

For a given $\varrho > 0$, we use the symbols $\mathrm{Jump}^{\varrho}_{\mathfrak{f}}(z;[a,b])$ and $\mathit{Var}^{\varrho}_{\mathfrak{f}}$ for the total variation induced by the cost $\Delta^{\varrho}_{\mathfrak{f}}$.

Again, $\mathit{Var}_{\mathfrak{f}}(z;[a,b]) < \infty$ ensures that the curve z fulfills the monotonicity property (5.9). As already pointed out in [23, Rmk. 3.5], $\mathit{Var}_{\mathfrak{f}}$ is not a *standard* total variational functional: it is neither induced by any distance on $L^1(\Omega)$, nor it is lower semi-continuous w.r.t. pointwise convergence in $L^1(\Omega)$. Yet, it enjoys the additivity property.

We are finally in a position to give our definition of BV solution to the rate-independent damage system. Again, we will consider a slightly stronger version than that given in [25, Definition 3.10], where $z \in \mathrm{BV}([0,T];L^1(\Omega))$ was only required. Instead, here we will consider curves z in $\mathrm{BV}([0,T];L^2(\Omega))$ and, for technical reasons that will be apparent in the proof of the BV-chain rule from Proposition 5.8 ahead, we will also restrict to curves z such that $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$. Furthermore, unlike what was done in [25], we will claim an energy balance involving a total variation $\mathit{Var}^{\varrho}_{\mathfrak{f}}(z;[0,t])$ with a threshold $\varrho > 0$ such that

$$\varrho \geqslant \|z\|_{L^\infty(0,T;\mathcal{Z})\cap \mathrm{BV}([0,T];L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot))\|_{L^\infty(0,T;L^2(\Omega))}\,. \tag{5.14}$$

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Definition 5.4 *A curve z in $L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$, monotonically decreasing in the sense of* (5.9), *with*

$$z(t) \in \mathcal{Z} \quad \text{and} \quad \mathrm{D}_z\mathcal{I}(t,z(t)) \in L^2(\Omega) \qquad \text{for all } t \in [0,T] \tag{5.15}$$

and $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$, is a BV *solution of the rate-independent damage system* (1.1) *if the* local stability ($\mathrm{S}_{\mathrm{loc}}$) *and the* ($\mathrm{E}_{\mathfrak{f}}$)-energy balance *hold:*

$$-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0) \quad \text{for all} \quad t \in [0,T] \setminus \mathrm{J}_z, \tag{$\mathrm{S}_{\mathrm{loc}}$}$$

$$Var_{\mathfrak{f}}^{\varrho}(z;[0,t]) + \mathcal{I}(t,z(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(s,z(s))\,\mathrm{d}s \qquad \text{for all } t \in (0,T]. \tag{$\mathrm{E}_{\mathfrak{f}}$}$$

with $\varrho > 0$ fulfilling (5.14).

Remark 5.5 The requirement $z \in L^\infty(0,T;\mathcal{Z})$ in Definition 5.4 is redundant and has been added only for the sake of clarity. Indeed, since $\mathcal{I}(0,z(0)) \leqslant C$ as $z(0) \in \mathcal{Z}$ (cf. (2.15)), and taking into account that $t \mapsto \partial_t\mathcal{I}(t,z(t))$ is in $L^\infty(0,T)$ thanks to (2.23), from ($\mathrm{E}_{\mathfrak{f}}$), we deduce that $|\mathcal{I}(t,z(t))| \leqslant C$. Recall that $\mathcal{I}$ is bounded from below thanks to (2.18). In turn, this gives $z \in L^\infty(0,T;\mathcal{Z})$.

On the other hand, combining the information $z \in L^\infty(0,T;\mathcal{Z})$ with estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$, we conclude that $\mathrm{D}_z\widetilde{\mathcal{I}}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$. Therefore, what we are really requiring in Definition 5.4 is that $A_q z \in L^\infty(0,T;L^2(\Omega))$, which enhances the regularity of z to the space $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$ for every $0 < \sigma < \frac{1}{q}$ by Proposition 2.8.

Prior to stating the *main result of the paper*, Theorem 5.7 below, we need to give the following definition, where z_- and z_+ are place-holders for the left and right limits of a curve z at a jump point.

Definition 5.6 *Let $\varrho > 0$, $t \in [0,T]$, and z_-, $z_+ \in \mathcal{Z}$ with $z_+ \leqslant z_-$ in Ω be such that*

$$-\mathrm{D}_z\mathcal{I}(t,z_-) \in \partial\mathcal{R}_1(0) \text{ and } -\mathrm{D}_z\mathcal{I}(t,z_+) \in \partial\mathcal{R}_1(0)\,. \tag{5.16}$$

We say that an admissible transition curve $\vartheta \in \mathcal{T}_t^{\varrho}(z_-,z_+)$ is an optimal transition *between z_- and z_+ if*

$$\mathcal{I}(t,z_-) - \mathcal{I}(t,z_+) = \varDelta_{\mathfrak{f}}^{\varrho}(t;z_-,z_+) = \int_0^1 \mathfrak{f}_t(\vartheta(r),\vartheta'(r))\,\mathrm{d}r = \mathfrak{f}_t(\vartheta(r),\vartheta'(r)) \qquad \text{for a.a. } r \in (0,1). \tag{5.17}$$

We will denote by $\mathcal{O}_t^{\varrho}(z_-,z_+)$ the collection of such transitions.

A few comments are in order. First of all, with (5.16) we are imposing that the points z_- and z_+ to be connected and to fulfill the local stability condition. It is not difficult to check that this is satisfied whenever z_- and z_+ are the left and right limits at a jump point of a BV solution. Second, let us gain further insight into (5.17): with the second equality, we are asking that ϑ (which we may always supposed to be defined on $[0,1]$) is a minimizer in the definition of $\Delta_{\mathfrak{f}}^{\varrho}(t;z_-,z_+)$; with the third one, we ask that ϑ has constant '$\mathfrak{f}_t$-velocity', which can be obtained by a rescaling argument. The first equality relates to the jump conditions verified along any BV solution, cf. (5.27) ahead.

We are now in a position to give Theorem 5.7 , stating the convergence of the discrete solutions of the viscous damage system to a BV solution of the rate-independent damage system, as the parameters ϵ and τ tend to zero *simultaneously*, with $\frac{\varepsilon}{\tau}\uparrow\infty$. In fact, we will retrieve a BV solution z with enhanced properties:

(i) we have that $z\in \mathrm{BV}([0,T];H^1(\Omega))$, which reflects the enhanced discrete BV-estimate (3.9*e*);

(ii) at all jump points t of z, the left and right limits $z(t_-)$ and $z(t_+)$ can be connected by an optimal jump transition in the sense of Definition 5.6, so that the set $\mathcal{O}_t^{\varrho}(z(t_-),z(t_+))$ is non-empty. Additionally, such transition has finite $H^1(\Omega)$-length. Furthermore, the total $H^1(\Omega)$-length of the connecting paths is finite.

Observe that property (ii) is not encoded in Definition 5.4, which gives $Var_{\mathfrak{f}}(z;[0,T])<\infty$, since $Var_{\mathfrak{f}}(z;[0,T])$ only controls the '$\mathfrak{f}$-length' of the optimal jump paths.

This enhanced concept of BV solution was already introduced in the general setting of [25], cf. Section 3.4 therein. Along the footsteps of [25], we will refer to these solutions as $H^1(\Omega)$-*parameterizable BV solutions.*

Theorem 5.7 *Under Assumptions 2.1, 2.2 and 2.9, let $z_0\in\mathcal{Z}$, fulfilling* (3.6), *be approximated by discrete initial data $(z^0_{\tau,\epsilon})_{\tau,\epsilon}$ such that*

$$z^0_{\tau,\epsilon}\to z_0 \quad \text{in } \mathcal{Z},\qquad \mathcal{I}(0,z^0_{\tau,\epsilon})\to\mathcal{I}(0,z_0),\quad \mathrm{D}_z\mathcal{I}(0,z^0_{\tau,\epsilon})\rightharpoonup \mathrm{D}_z\mathcal{I}(0,z_0)\quad \text{in } L^2(\Omega), \tag{5.18}$$

and let $(\overline{z}_{\tau,\epsilon})_{\tau,\epsilon}$, $(\widehat{z}_{\tau,\epsilon})_{\tau,\epsilon}$ be the discrete solutions to the viscous damage system (1.2) *starting from the data $(z^0_{\tau,\epsilon})_{\tau,\epsilon}$.*

Then, there exists $\bar\varrho>0$, only depending on the problem data (cf. (6.2) *below) such that for all sequences $(\tau_k,\epsilon_k)_k$ satisfying*

$$\lim_{k\to\infty}\epsilon_k=0 \quad \text{and}\quad \lim_{k\to\infty}\frac{\tau_k}{\epsilon_k}=0, \tag{5.19}$$

there exist a (not relabelled) subsequence, and a BV solution z to the rate-independent damage system (1.1), *fulfilling the initial condition $z(0)=z_0$, condition* (5.14), *the energy balance* ($\mathrm{E}_{\mathfrak{f}}$) *with*

$$Var_{\mathfrak{f}}^{\bar\varrho}(z;[0,t])=\sup_{\varrho\geqslant\bar\varrho}Var_{\mathfrak{f}}^{\varrho}(z;[0,t])=\inf_{\varrho\geqslant\bar\varrho}Var_{\mathfrak{f}}^{\varrho}(z;[0,t])\quad \text{for every } t\in[0,T] \tag{5.20}$$

and such that the following convergences hold as $k\to\infty$, at every $t\in[0,T]$:

$$\overline{z}_{\tau_k,\epsilon_k}(t),\ \widehat{z}_{\tau_k,\epsilon_k}(t)\to z(t)\quad \text{in } \mathcal{Z}, \tag{5.21a}$$

$$\mathcal{I}(t,\overline{z}_{\tau_k,\epsilon_k}(t)),\ \mathcal{I}(t,\widehat{z}_{\tau_k,\epsilon_k}(t))\to\mathcal{I}(t,z(t)), \tag{5.21b}$$

$$\int_0^{\bar t_\tau(t)}\left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r))+\mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar t_\tau(r),\overline{z}_\tau(r)))\right)\mathrm{d}r\to Var_{\mathfrak{f}}^{\bar\varrho}(z;[0,t]). \tag{5.21c}$$

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Furthermore, z is a $H^1(\Omega)$-parameterizable BV solution, namely $z \in \mathrm{BV}([0,T];H^1(\Omega))$, and

$$\text{(1)} \qquad \forall\, t \in \mathrm{J}_z \ \exists\, \vartheta_t \in \mathscr{O}_t^{\bar{\varrho}}(z(t_-), z(t_+)) \ \text{s.t.}\ \vartheta_t \in \mathrm{AC}([0,1];H^1(\Omega)); \tag{5.22a}$$

$$\text{(2)} \qquad \sum_{t\in \mathrm{J}_z} \int_0^1 \|\vartheta_t'(r)\|_{H^1(\Omega)}\,\mathrm{d}r < \infty. \tag{5.22b}$$

Observe that (5.20) is an additional property, cf. (5.6). This, as well as the constant $\bar{\varrho}$ will be specified along the proof of Theorem 5.7, postponed to Section 6. Instead, in the forthcoming Section 5.2, we gain further insight into the notion of BV solution for our damage system, in particular focusing on the description of the behaviour of the system at jumps.

5.2 Properties of balanced viscosity solutions

One of the cornerstones of the proof of Theorem 5.7 is a characterization of BV solutions in terms of the local stability condition ($\mathrm{S}_{\mathrm{loc}}$), combined with the *upper energy estimate* in ($\mathrm{E}_{\mathfrak{f}}$). The proof of this characterization relies on a *chain-rule inequality* for $\mathcal{E}$, evaluated along a *locally stable* curve with the regularity and summability properties specified in Definition 5.4. This inequality involves the non-standard total variation functional $\mathit{Var}_{\mathfrak{f}}$.

Proposition 5.8 (BV-chain rule inequality) *Under Assumptions 2.1, 2.2 and 2.9, let $z \in L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$, with $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$, also fulfill* (5.15). *Let ϱ fulfill* (5.14). *Suppose that z satisfies the local stability condition* ($\mathrm{S}_{\mathrm{loc}}$)*, with $\mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]) < \infty$. Then, the map $t \mapsto \mathcal{I}(t,u(t))$ belongs to* $\mathrm{BV}([0,T])$ *and satisfies the chain rule inequality*

$$\left|\mathcal{I}(t_1,u(t_1)) - \mathcal{I}(t_0,u(t_0)) - \int_{t_0}^{t_1} \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t\right| \leqslant \mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[t_0,t_1]) \qquad \text{for all } 0 \leqslant t_0 \leqslant t_1 \leqslant T\,. \tag{5.23}$$

We postpone its *proof* to Section 6. We now characterize BV solutions in terms of the local stability ($\mathrm{S}_{\mathrm{loc}}$), joint with the upper energy estimate in ($\mathrm{E}_{\mathfrak{f}}$), which is sufficient to be given on the whole time interval $[0,T]$. Namely we have

Corollary 5.9 *Under Assumptions 2.1, 2.2 and 2.9, a curve $z \in \mathrm{BV}([0,T];L^2(\Omega))$ is a BV solution of the rate-independent damage system* (1.1) *(in the sense of Definition 5.4) if and only if it satisfies* ($\mathrm{S}_{\mathrm{loc}}$) *and*

$$\mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]) + \mathcal{I}(T,z(T)) \leqslant \mathcal{I}(0,z(0)) + \int_0^T \partial_t\mathcal{I}(s,z(s))\,\mathrm{d}s \tag{5.24}$$

for some ϱ fulfilling (5.14).

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For the *proof*, we refer the reader to the argument for [25, Corollary 3.14]. Corollary 5.9 will play a crucial role in the proof of Theorem 5.7, for it will allow us to focus on the proof of (S_{loc}) and of the energy inequality (5.24), only, in place of the balance ($E_{\mathfrak{f}}$). In turn, (5.24) will be achieved by means of careful lower semi-continuity arguments. The second outcome of the characterization provided by Corollary 5.9 is the following Proposition 5.10, which was proved in the abstract setting in [25, Theorem 3.15]. It shows that a locally stable curve is a BV solution of the rate-independent system if and only if it fulfills

(i) an ED inequality only featuring the $\mathcal{R}_1$-total variation functional from (5.8), cf. (5.26) below, and

(ii) at each jump point, the jump conditions (5.27) featuring the Finsler cost $\Delta_{\mathfrak{f}}$ induced by $\mathfrak{f}$.

Concerning (i), let us also mention that it is possible to show (cf. [25, Theorem 3.16]) that any BV solution also satisfies the subdifferential inclusion

$$\partial\mathcal{R}_1(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } L^2(\Omega) \tag{5.25}$$

at every $t \in (0, T)$ that is not a jump point, hence for almost all $t \in (0, T)$. The system behaviour at jump points is instead described by the jump conditions (5.27) below. This further characterization of the BV concept in terms of (i) and (ii) highlights how it differs in comparison to the standard Global Energetic notion. The latter can be characterized in terms of the *global stability* condition, the ED inequality (5.26), and the analogues of the jump conditions (5.27), with the cost $\Delta_{\mathfrak{f}}(t; \cdot, \cdot)$ replaced by $\mathcal{R}_1$. Conditions (5.27) highlight that the viscous approximation, from which BV solutions originate, enters into play in the description of the energetic behaviour of the system at jumps.

Proposition 5.10 *A curve* $z \in \mathrm{BV}([0, T]; L^2(\Omega))$ *is a BV solution of the rate-independent damage system* (1.1) *if and only if it satisfies* (S_{loc})*, the* ($\mathcal{R}_1$)*-ED inequality*

$$Var_{\mathcal{R}_1}(z; [s, t]) + \mathcal{I}(t, z(t)) \leqslant \mathcal{I}(s, z(s)) + \int_s^t \partial_t\mathcal{I}(s, z(s))\,\mathrm{d}s \quad \textit{for all } 0 \leqslant s \leqslant t \leqslant T, \tag{5.26}$$

and the jump conditions

$$\begin{aligned}
\mathcal{I}(t, z(t)) - \mathcal{I}(t, z(t_-)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t; z(t_-), z(t)),\\
\mathcal{I}(t, z(t_+)) - \mathcal{I}(t, z(t)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t; z(t), z(t_+)),\\
\mathcal{I}(t, z(t_+)) - \mathcal{I}(t, z(t_-)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t; z(t_-), z(t_+))\\
&= -\left(\Delta_{\mathfrak{f}}^{\varrho}(t; z(t_-), z(t)) + \Delta_{\mathfrak{f}}^{\varrho}(t; z(t), z(t_+))\right)
\end{aligned} \tag{5.27}$$

at every $t \in \mathrm{J}_z$*, for some* ϱ *fulfilling* (5.14).

The *proof* follows the very same lines as the argument for [25, Theorem 3.15].

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We conclude this section by shedding further light into the fine properties of optimal jump transitions. Following [25, Section 3.4], we say that an optimal transition $\vartheta \in \mathcal{O}_t^{\varrho}(z_-, z_+)$ is of

- *sliding* type if $-\mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \in \mathcal{R}_1(0)$ for every $r \in [0,1]$;
- *viscous* type if $-\mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \notin \mathcal{R}_1(0)$ for every $r \in [0,1]$.

The forthcoming result on sliding and viscous optimal transitions follows from the very same argument as in the proof of [25, Proposition 3.19].

Proposition 5.11 *Let $\varrho > 0$, $t \in [0,T]$, and z_-, $z_+ \in \mathcal{Z}$ fulfilling* (5.16) *be given. Let $\vartheta \in \mathcal{O}_t^{\varrho}(z_-, z_+)$. Then,*

(1) *ϑ is of sliding type if and only if it satisfies*

$$\partial\mathcal{R}_1(\vartheta'(r)) + \mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } r \in (0,1);$$

(2) *ϑ is of viscous type if and only if there exists a map $\epsilon : (0,1) \to (0,+\infty)$ such that ϑ and ϵ satisfy*

$$\partial\mathcal{R}_1(\vartheta'(r)) + \epsilon(r)\vartheta'(r) + \mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } r \in (0,1);$$

(3) *Every optimal transition ϑ can be decomposed in a canonical way into an (at most) countable collection of optimal* sliding *and* viscous *transitions.*

6 Proofs

The main focus of this Section is on the proof of Theorem 5.7. Prior to carrying it out, we

(1) collect the main properties of the Finsler cost $\Delta_{\mathfrak{f}}$ in Proposition 6.1;
(2) prove the chain rule from Proposition 5.8, which is an essential ingredient for Theorem 5.7.

On the other hand,

(3) the proof of Theorem 5.7 is itself split in various steps, in which we prove intermediate results.

Proposition 6.1, which is the counterpart to [25, Theorem 3.7]. Nonetheless, a comparison between the latter result and Proposition 6.1 below reflects the major differences between the present context and that of [25]: The transition curves by means of which the Finsler cost $\Delta_{\mathfrak{f}}$ from (5.2) is defined have better properties than their analogues in [25], cf. also Remark 5.2. This is also apparent from item (3) of the ensuing statement, yielding the existence of a transition path ϑ in the space $W^{1,\infty}(0,1;H^1(\Omega))$, even, in accordance with the uniform bound (3.9*e*) for the discrete solutions.

Proposition 6.1 *Let $t \in [0,T]$ and $z_0, z_1 \in \mathcal{Z}$ be fixed with $z_1 \leqslant z_0$ in Ω. Then,*

(1) *For every $\varrho > 0$ such that $\max_{i=0,1}(\|z_i\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t,z_i)\|_{L^2(\Omega)}) \leqslant \varrho$ and $\Delta_{\mathfrak{f}}^{\varrho}(t;z_0,z_1) < +\infty$, there exists an optimal transition path $\vartheta \in \mathcal{T}_t^{\varrho}(z_0,z_1)$ attaining the* inf *in the definition of $\Delta_{\mathfrak{f}}^{\varrho}(t;z_0,z_1)$, cf. (5.4);*

(2) *Let $(z_0^n)_n, (z_1^n)_n \subset \mathcal{Z}$ fulfill*

$$z_0^n \to z_0, \quad z_1^n \to z_1 \quad \text{in } \mathcal{Z}.$$

Then,

$$\liminf_{n\to\infty} \Delta_{\mathfrak{f}}^{\varrho}(t;z_0^n,z_1^n) \geqslant \Delta_{\mathfrak{f}}^{\varrho}(t;z_0,z_1) \tag{6.1}$$

for every $\varrho \geqslant \sup_{i=1,2,n\in\mathbb{N}}(\|z_i\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t,z_i)\|_{L^2(\Omega)})$.

(3) *Let the sequences $(\alpha_k)_k, (\beta_k)_k \subset [0,T]$, $(\widehat{z}_k)_k \subset L^\infty(\alpha_k,\beta_k;\mathcal{Z}) \cap \mathrm{AC}([\alpha_k,\beta_k];H^1(\Omega))$, $(\overline{z}_k)_k \subset L^\infty(\alpha_k,\beta_k;\mathcal{Z})$, fulfill*

$$\lim_{k\to\infty} \alpha_k = t = \lim_{k\to\infty} \beta_k, \quad \overline{z}_k(\alpha_k) \to z_0 \text{ in } \mathcal{Z}, \quad \overline{z}_k(\beta_k) \to z_1 \text{ in } \mathcal{Z},$$
$$\lim_{k\to\infty} \sup_{r\in[\alpha_k,\beta_k]} \|\overline{z}_k(r) - \widehat{z}_k(r)\|_{H^1(\Omega)} = 0,$$
$$\exists\, \bar{\varrho} > 0 \quad \forall k \in \mathbb{N} :$$
$$\|\widehat{z}_k\|_{L^\infty(\alpha_k,\beta_k;\mathcal{Z})\cap W^{1,1}(\alpha_k,\beta_k;H^1(\Omega))} + \|\overline{z}_k\|_{L^\infty(\alpha_k,\beta_k;\mathcal{Z})} + \|\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k},\overline{z}_k)\|_{L^\infty(\alpha_k,\beta_k;L^2(\Omega))} \leqslant \bar{\varrho}\,. \tag{6.2}$$

Then, there exists a (not relabelled) increasing subsequence of (k), increasing and surjective time rescalings $\mathsf{t}_k \in \mathrm{AC}([0,1];[\alpha_k,\beta_k])$ and an admissible transition $\vartheta \in \mathcal{T}_t^{\bar{\varrho}}(z_0,z_1)$ such that

$$\lim_{k\to\infty} \sup_{s\in[0,1]} \|\overline{z}_k \circ \mathsf{t}_k(s) - \vartheta(s)\|_{H^1(\Omega)} = \lim_{k\to\infty} \sup_{s\in[0,1]} \|\widehat{z}_k \circ \mathsf{t}_k(s) - \vartheta(s)\|_{H^1(\Omega)} = 0, \tag{6.3a}$$

in addition, ϑ is in $W^{1,\infty}(0,1;H^1(\Omega))$, and (6.3b)

$$\Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z_0,z_1) \leqslant \int_0^1 \mathfrak{f}_t[\vartheta(s),\vartheta'(s)]\,\mathrm{d}s \leqslant \liminf_{k\to\infty} \int_{\alpha_k}^{\beta_k} \left(\mathcal{R}_{\epsilon_k}(\widehat{z}_k'(r)) + \mathcal{R}_{\epsilon_k}^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\overline{z}_k(r)))\right) \mathrm{d}r\,. \tag{6.3c}$$

Proof We start by addressing the proof of *(2)*: Along the footsteps of the proof of [25, Theorem 3.7], we consider a sequence of admissible transitions $\vartheta_n \in \mathcal{T}_t^{\varrho}(z_0^n,z_1^n)$ such that

$$\int_0^1 \mathfrak{f}_t(\vartheta_n(r),\vartheta_n'(r))\,\mathrm{d}r \leqslant \Delta_{\mathfrak{f}}^{\varrho}(t;z_0^n,z_1^n) + \eta_n \qquad \text{with } \eta_n \geqslant 0 \text{ and } \lim_{n\to\infty} \eta_n = \eta \geqslant 0\,.$$

We perform the change of variable

$$\mathsf{s}_n(r) := c_n \left(r + \int_0^r \|\vartheta_n'(\sigma)\|_{L^2(\Omega)}\,\mathrm{d}\sigma \right), \;\; \mathsf{r}_n := \mathsf{s}_n^{-1} : [0,\mathsf{S}] \to [0,1], \;\; \theta_n := \vartheta_n \circ \mathsf{r}_n : [0,\mathsf{S}] \to \mathcal{Z}, \tag{6.4}$$

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with c_n a normalization constant such that $\mathsf{S} = \mathsf{s}_n(1)$ is independent of $n \in \mathbb{N}$. In view of the estimate $\|\vartheta_n'\|_{L^1(0,1;L^2(\Omega))} \leqslant \varrho$ encoded in the definition of $\varDelta_{\mathfrak{f}}^{\varrho}$, we have that $c_n \geqslant \bar{c} > 0$ for all $n \in \mathbb{N}$. The curves $(\mathsf{r}_n, \theta_n)_n$ fulfill the normalization condition

$$\mathsf{r}_n'(s) + \|\theta_n'(s)\|_{L^2(\Omega)} = \frac{1}{c_n} \leqslant \frac{1}{\bar{c}} \qquad \text{for a.a. } s \in (0, \mathsf{S}) \tag{6.5a}$$

and, moreover,

$$\|\theta_n\|_{L^\infty(0,\mathsf{S};\mathcal{Z})} + \|\theta_n'\|_{L^1(0,\mathsf{S};L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(t, \theta_n(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))} \leqslant \varrho. \tag{6.5b}$$

It follows from the first bound in (6.5*b*) and from (2.34) that $\|\mathrm{D}_z\widetilde{\mathcal{I}}(t, \theta_n(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))} \leqslant C$. Therefore, we deduce that $\|A_q(\theta_n)\|_{L^\infty(0,\mathsf{S};L^2(\Omega))} \leqslant C$, which yields, in view of the aforementioned regularity results from Proposition 2.8, a bound for $(\theta_n)_n$ in $L^\infty(0, \mathsf{S}; W^{1+\sigma,q}(\Omega)))$ for all $0 < \sigma < \frac{1}{q}$. In view of (6.5*a*), there exists $\mathsf{r} \in W^{1,\infty}(0, \mathsf{S})$ such that, up to a not relabelled subsequence, $\mathsf{r}_n \to r$ uniformly in $[0, \mathsf{S}]$ and weakly* in $W^{1,\infty}(0, \mathsf{S})$. Furthermore, by Aubin–Lions type compactness results (cf., e.g., [30, Theorem 5, Cor. 4]), there exists a curve $\theta \in L^\infty(0, \mathsf{S}; W^{1+\sigma,q}(\Omega)) \cap \mathrm{C}^0([0, \mathsf{S}]; \mathcal{Z}) \cap W^{1,\infty}(0, \mathsf{S}; L^2(\Omega))$ for all $0 < \sigma < \frac{1}{q}$, with $\mathrm{D}_z\mathcal{I}(t, \theta(\cdot)) \in L^\infty(0, \mathsf{S}; L^2(\Omega))$, such that

$$\begin{aligned}
&\theta_n \rightharpoonup^* \theta && \text{in } L^\infty(0, \mathsf{S}; W^{1+\sigma,q}(\Omega)) \cap W^{1,\infty}(0, \mathsf{S}; L^2(\Omega)) \quad \text{for all } 0 < \sigma < \frac{1}{q},\\
&\theta_n \to \theta && \text{in } \mathrm{C}^0([0, \mathsf{S}]; \mathcal{Z}),\\
&\mathrm{D}_z\mathcal{I}(t, \theta_n) \rightharpoonup^* \mathrm{D}_z\mathcal{I}(t, \theta) && \text{in } L^\infty(0, \mathsf{S}; L^2(\Omega)).
\end{aligned} \tag{6.6}$$

The latter convergence property follows from the fact that $\mathrm{D}_z\mathcal{I}(t, \theta_n) = A_q(\theta_n) + \mathrm{D}_z\widetilde{\mathcal{I}}(t, \theta_n)$ converges strongly to $\mathrm{D}_z\mathcal{I}(t, \theta)$ in $L^\infty(0, \mathsf{S}; \mathcal{Z}^*)$ in view of the second of (6.6), combined with (2.36). Therefore,

$$\|\theta\|_{L^\infty(0,\mathsf{S};\mathcal{Z})} + \|\theta'\|_{L^1(0,\mathsf{S};L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(t, \theta(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))} \leqslant \varrho.$$

We thus conclude that $\theta \in \mathcal{T}_t^{\varrho}(z_0, z_1)$; up to a reparameterization, we may suppose θ to be defined on $[0, 1]$. Arguing in the very same way as in the proof of [18, Theorem 5.1], [19, Theorem 7.4], we see that

$$\begin{aligned}
\eta + \liminf_{n\to\infty} \varDelta_{\mathfrak{f}}^{\varrho}(t; z_0^n, z_1^n) &\geqslant \liminf_{n\to\infty} \int_0^1 \mathfrak{f}_t(\vartheta_n(r), \vartheta_n'(r))\,\mathrm{d}r = \liminf_{n\to\infty} \int_0^{\mathsf{S}} \mathfrak{f}_t(\theta_n(s), \theta_n'(s))\,\mathrm{d}s\\
&\geqslant \int_0^{\mathsf{S}} \mathfrak{f}_t(\theta(s), \theta'(s))\,\mathrm{d}s \geqslant \varDelta_{\mathfrak{f}}^{\varrho}(t; z_0, z_1).
\end{aligned}$$

Observe that the last inequality follows from the fact that θ is an admissible curve between z_0 and z_1. Since $\eta \geqslant 0$ is arbitrary, this concludes the proof of *(2)*; a slight modification of this argument yields *(1)*, as well.

In order to prove *(3)*, we can confine the discussion to the case $z_0 \neq z_1$. Up to the extraction of a (not relabelled) subsequence, we may suppose that the $\liminf$ in (6.3*c*) is in fact a limit, so that

$$\lim_{k\to\infty}\int_{\alpha_k}^{\beta_k}\left(\mathcal{R}_{\epsilon_k}(\widehat{z}_k'(r))+\mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\overline{z}_k(r)))\right)\,\mathrm{d}r =: L \geqslant \mathcal{R}_1(z_1-z_0)>0\,.$$

In analogy with (6.4), but taking now into account that $(\widehat{z}_k)_k$ is bounded in $W^{1,1}(\alpha_k,\beta_k;H^1(\Omega))$ by (6.2), we define

$$\mathsf{s}_k(r) := c_k\left(r+\int_0^r \|\widehat{z}_k'(\sigma)\|_{H^1(\Omega)}\,\mathrm{d}\sigma\right)\quad \text{for all } r\in[\alpha_k,\beta_k]\,,$$

where the normalization constant c_k is now chosen in such a way as to have $\mathsf{s}_k(\beta_k-\alpha_k)=1$. Thus, we set

$$\mathsf{t}_k := \mathsf{s}_k^{-1} : [0,1]\to[\alpha_k,\beta_k],\qquad \overline{\mathsf{z}}_k := \overline{z}_k\circ\mathsf{t}_k,\ \widehat{\mathsf{z}}_k := \widehat{z}_k\circ\mathsf{t}_k : [0,1]\to\mathcal{Z},$$

and observe that the following estimates hold

$$\|\mathsf{t}_k\|_{W^{1,\infty}(0,1)}+\|\widehat{\mathsf{z}}_k\|_{W^{1,\infty}(0,1;H^1(\Omega))}\leqslant C, \tag{6.7a}$$
$$\|\overline{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})}+\|\widehat{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})}+\|\widehat{\mathsf{z}}_k'\|_{L^1(0,1;H^1(\Omega))}+\|\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))}\leqslant\bar{\varrho}\,, \tag{6.7b}$$

where (6.7*a*) is due to the analogue of the normalization condition (6.5*a*), while (6.7*b*) derives from (6.2). From the bound for $\|\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))}$, taking into account that $\|\mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))}\leqslant C$ in view of (2.34) and the estimate $\|\overline{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})}\leqslant C$, we also deduce

$$\|A_q(\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))}\leqslant C\,. \tag{6.7c}$$

Combining estimates (6.7) with the compactness results [30, Theorem 5, Corollary 4], and taking into account that $(\overline{\mathsf{z}}_k)$ and $(\widehat{\mathsf{z}}_k)_k$ converge to the same limit in view of the second of (6.2), with the very same arguments as in the proof of *(2)*, we conclude that there exists ϑ such that

$$\widehat{\mathsf{z}}_k \rightharpoonup^* \vartheta \qquad \text{in } L^\infty(0,1;\mathcal{Z})\cap W^{1,\infty}(0,1;H^1(\Omega)), \tag{6.8a}$$
$$\overline{\mathsf{z}}_k \rightharpoonup^* \vartheta \qquad \text{in } L^\infty(0,1;W^{1+\sigma,q}(\Omega))\quad \text{for all } 0<\sigma<\frac{1}{q}, \tag{6.8b}$$
$$\overline{\mathsf{z}}_k \to \vartheta \qquad \text{in } L^\infty(0,1;\mathcal{Z}), \tag{6.8c}$$
$$\widehat{\mathsf{z}}_k \to \vartheta \qquad \text{in } \mathrm{C}^0([0,1],H^1(\Omega))\,, \tag{6.8d}$$

whence (6.3*a*) and (6.3*b*). Furthermore, observe that $A_q(\overline{\mathsf{z}}_k)\rightharpoonup^* A_q(\vartheta)$ in $L^\infty(0,1;L^2(\Omega))$ and that, as $k\to\infty$,

$$\begin{aligned}&\|\mathrm{D}_z\widetilde{\mathcal{I}}(\bar{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)-\mathrm{D}_z\widetilde{\mathcal{I}}(t,\vartheta)\|_{L^\infty(0,1;L^2(\Omega))}\\ &\quad\overset{(1)}{\leqslant} C\sup_{s\in[0,1]}\left(|\bar{t}_{\tau_k}(\mathsf{t}_k(s))-t|+\|\overline{\mathsf{z}}_k(s)-\vartheta(s)\|_{L^6(\Omega)}\right)\overset{(2)}{\to}0\end{aligned} \tag{6.8e}$$

with (1) due to (2.32). Convergence (2) is due to (6.8*c*) and the fact that $\sup_{s\in[0,1]}|\mathsf{t}_k(s)-t|\to 0$, since t_k takes values in the interval $[\alpha_k,\beta_k]$ that shrinks to $\{t\}$. All in all,

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$\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}\circ \mathsf{t}_k, \bar{\mathsf{z}}_k) \rightharpoonup^* \mathrm{D}_z\mathcal{I}(t,\vartheta)$ in $L^\infty(0,1;L^2(\Omega))$. It follows from estimates (6.7*b*) and convergences (6.8) that $\vartheta \in \mathcal{T}_t^{\bar{\varrho}}(z_0,z_1)$. It remains to conclude (6.3*c*). For this limit passage, we rely on convergences (6.8) and refer the reader to the proof of [25, Proposition 7.1], cf. also [18, Theorem 5.1], [19, Theorem 7.4].

This finishes the proof of Proposition 6.1. □

We continue this section by carrying out the *proof of Proposition 5.8*, by suitably adapting the argument for the chain-rule result [25, Theorem 3.13]. From now on, we will suppose that $t_0 = 0$ and $t_1 = T$ for the sake of simplicity. Let $\varrho > 0$ fulfill (5.14).

First of all, for any $z \in \mathrm{BV}([0,T];L^2(\Omega))$ fulfilling the conditions of the statement we construct a *parameterized* curve $(\mathsf{t},\mathsf{z}) : [0,\mathsf{S}] \to [0,T]\times\mathcal{Z}$ with the following properties:

$$z(t) \in \{\mathsf{z}(s) \ : \mathsf{t}(s) = t\}$$

and

- t is non-decreasing, surjective, Lipschitz,
- $\mathsf{z} \in L^\infty(0,\mathsf{S};\mathcal{Z}) \cap \mathrm{AC}([0,\mathsf{S}];L^2(\Omega))$ and $\mathrm{D}_z\mathcal{I}(\cdot,\mathsf{z}(\cdot)) \in L^\infty(0,\mathsf{S};L^2(\Omega))$.

The integrability and regularity requirements on z coincide with those on admissible transition curves, cf. Definition 5.1. Hence, we will call (t,z) *admissible parameterized curve*. We borrow the construction of (t,z), starting from the BV-curve z, from the proof of [25, Proposition 4.7]: first, we introduce the parameterization

$$\mathsf{s}(t) := t + Var_{L^2(\Omega)}(z;[0,t]), \quad \mathsf{S} := \mathsf{s}(T).$$

We define

$$\mathsf{t} := \mathsf{s}^{-1} : [0,\mathsf{S}]\setminus I \to [0,T], \qquad \mathsf{z} := z\circ\mathsf{t},$$

where the set I is given by $I = \cup_n I_n$, with $I_n = (\mathsf{s}(t_{n-}),\mathsf{s}(t_{n+}))$ and the points $(t_n)_n$ constitute the countable jump set of z, which in fact coincides with the jump set of s. We extend t and z to I by setting

$$\mathsf{t}(s) := t_n, \quad \mathsf{z}(s) := \vartheta_n(\mathsf{r}_n(s)) \quad \text{if } s\in I_n,$$

with $\mathsf{r}_n : \overline{I_n} \to [0,1]$ the unique affine and strictly increasing function from I_n to $[0,1]$ and $\vartheta_n \in \mathcal{T}_{t_n}^{\varrho}(z(t_{n-}),z(t_{n+}))$ an admissible transition curve satisfying $\vartheta_n(\mathsf{r}_n(\mathsf{s}(t_n))) = z(t_n)$ and the optimality condition

$$\int_0^1 \mathfrak{f}_{t_n}(\vartheta_n(r),\vartheta_n'(r))\,\mathrm{d}r = \Delta_{\mathfrak{f}}^{\varrho}(t_n;z(t_{n-}),z(t_n)) + \Delta_{\mathfrak{f}}^{\varrho}(t_n;z(t_n),z(t_{n+}))\,.$$

The existence of such an optimal transition follows from Proposition 6.1(1). Indeed, let $t_* \in \mathrm{J}_z$. Observe that in $(t_*,z(t_{*\pm}))$ the assumptions of the proposition are satisfied, which can be seen as follows. First of all, $\Delta_{\mathfrak{f}}^{\varrho}(t_*;z(t_{*-}),z(t_*)) < \infty$ and $\Delta_{\mathfrak{f}}^{\varrho}(t_*;z(t_*),z(t_{*+})) < \infty$ since $Var_{\mathfrak{f}}^{\varrho}(z;[0,T]) < +\infty$. Moreover, choose a sequence $s_k \to t_{*-}$ for $k\to\infty$ such that the assumptions of Proposition 6.1(1) are satisfied along this sequence and such that $z(s_k) \rightharpoonup z(t_{*-})$ in $\mathcal{Z}$. Consequently, by Corollary 2.14, $\mathrm{D}_z\widetilde{\mathcal{I}}(s_k,z(s_k)) \to \mathrm{D}_z\widetilde{\mathcal{I}}(t_*,z(t_{*-}))$ and $\|A_q(z(s_k))\|_{L^2(\Omega)} \leqslant C$, which translates into a uniform bound of the sequence

$(z(s_k))_k$ in $W^{1+\sigma,q}(\Omega)$ for $0 < \sigma < \frac{1}{q}$, cf. Proposition 2.8. Thus, we finally conclude that $\mathrm{D}_z\mathcal{I}(t_*, z(t_{*-})) \in L^2(\Omega)$ and that $\|z(t_{*-})\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t_*, z(t_{*-}))\|_{L^2(\Omega)} \leqslant \varrho$. A similar argument applies to t_{*+}.

By construction, $\mathsf{z} \in W^{1,\infty}(0,\mathsf{S}; L^2(\Omega))$. Indeed, let $s_1 < s_2 \in [0,\mathsf{S}]$ and $\sigma_i := \mathsf{t}(s_i)$. Hence, $s_i = \sigma_i + Var_{L^2(\Omega)}(z; [0,\sigma_i])$. This implies that

$$\|\mathsf{z}(s_1) - \mathsf{z}(s_2)\|_{L^2(\Omega)} \leqslant |\sigma_2 + Var_{L^2(\Omega)}(z; [0,\sigma_2]) - (\sigma_1 + Var_{L^2(\Omega)}(z; [0,\sigma_1]))| = |s_2 - s_1|\,.$$

Hence, altogether (t,z) is an admissible parameterized curve.

By repeating the very same calculations as in the proof of [25, Proposition 4.7], we may show that

$$Var_{\mathfrak{f}}^{\varrho}(z; [0,T]) = \int_0^{\mathsf{S}} \mathfrak{f}_{\mathsf{t}(s)}(\mathsf{z}(s), \mathsf{z}'(s))\,\mathrm{d}s\,. \tag{6.9}$$

Second, we observe that the chain rule from Lemma 2.17 extends to the admissible parameterized curve (t,z), yielding

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s)) - \partial_t\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{t}'(s) = \int_\Omega \mathrm{D}_z\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{z}'(s)\,\mathrm{d}x \qquad \text{for a.a.}\, s \in (0,\mathsf{S})\,.$$

Therefore, with a simple calculation (cf. also the proof of [25, Theorem 4.4]) we infer that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s)) - \partial_t\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{t}'(s)\right| \leqslant \mathfrak{f}_{\mathsf{t}(s)}(\mathsf{z}(s),\mathsf{z}'(s)) \qquad \text{for a.a.}\, s \in (0,\mathsf{S})\,. \tag{6.10}$$

Combining (6.9) and (6.10), we obtain the desired chain-rule inequality (5.23). ■

We are now in a position to give the *proof of Theorem 5.7*. We will split the proof in several steps and give some intermediate results. Let us mention in advance that, in their statements, we will always tacitly suppose that Assumptions 2.1, 2.2 and 2.9, as well as condition (5.18), from Theorem 5.7 hold. More precisely,

- we start by fixing the compactness properties of the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ in Lemma 6.2 below.
- Throughout Steps 1–3, we show that any limit curve z of $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ complies with the local stability ($\mathrm{S}_{\mathrm{loc}}$) and with the ED inequality (5.24), obtained by passing to the limit in its discrete counterpart (3.11). By virtue of Corollary 5.9, we thus conclude that z is a BV solution to the rate-independent system (1.1).
- Steps 4 and 5 are devoted to finalizing the proof of convergences (5.21), and to showing that z is a $H^1(\Omega)$-parameterizable solution, cf. (5.22).

Step* 0*: Compactness.
We prove the following

Lemma 6.2 *Let $(\tau_k,\epsilon_k)_k$ be null sequences. There holds*

$$\exists\, C > 0\ \forall k \in \mathbb{N}\ :\ \sup_{t\in[0,T]} \|\overline{z}_{\tau_k,\epsilon_k}(t) - \widehat{z}_{\tau_k,\epsilon_k}(t)\|_{H^1(\Omega)} \leqslant C\left(\frac{\tau_k}{\epsilon_k}\right)^{1/2}. \tag{6.11}$$

Suppose in addition (5.19). *Then, there exists a curve* $z \in L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];H^1(\Omega))$ *such that, up to a (not relabelled) subsequence, the following convergences hold:*

$$\overline{z}_{\tau_k,\epsilon_k},\ \widehat{z}_{\tau_k,\epsilon_k} \rightharpoonup^* z \qquad \text{in } L^\infty(0,T;\mathcal{Z}), \tag{6.12a}$$
$$\overline{z}_{\tau_k,\epsilon_k}(t),\ \widehat{z}_{\tau_k,\epsilon_k}(t) \to z(t) \qquad \text{in } \mathcal{Z} \quad \text{for all } t \in [0,T], \tag{6.12b}$$
$$\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(t), \overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup \mathrm{D}_z\mathcal{I}(t,z(t)) \qquad \text{in } L^2(\Omega) \quad \text{for all } t \in [0,T]. \tag{6.12c}$$

Proof The first estimate follows from observing that for every $t \in (0,T)$

$$\|\overline{z}_{\tau_k,\epsilon_k}(t) - \widehat{z}_{\tau_k,\epsilon_k}(t)\|_{H^1(\Omega)} \leqslant \int_{\underline{t}_\tau(t)}^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau_k,\epsilon_k}(r)\|_{H^1(\Omega)}\,\mathrm{d}r \leqslant \tau_k^{1/2} \|\widehat{z}'_{\tau_k,\epsilon_k}\|_{L^2(\underline{t}_\tau(t),\overline{t}_\tau(t);H^1(\Omega))},$$

and then (6.11) is a consequence of the a priori estimate (3.9*d*).

Convergences (6.12*a*) follow from estimate (3.9*b*): observe that the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ converge to the same limit, weakly star in $L^\infty(0,T;\mathcal{Z})$, in view of the fact that

$$\|\overline{z}_{\tau_k,\epsilon_k} - \widehat{z}_{\tau_k,\epsilon_k}\|_{L^\infty(0,T;H^1(\Omega))} \to 0 \tag{6.13}$$

as $k \to \infty$ by (6.11) combined with condition (5.19) on the sequences $(\tau_k, \epsilon_k)_k$.

It follows from estimate (3.9*e*) that the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ are bounded in $\mathrm{BV}([0,T];H^1(\Omega))$. Due to the previously mentioned [26, Theorem 6.1], up to a subsequence they pointwise converge on $[0,T]$ w.r.t. the weak $H^1(\Omega)$-topology to the same function $\tilde{z}$, c.f. (6.13). Now, by the additional estimate (3.9*f*), $(\overline{z}_{\tau_k,\epsilon_k})_k$ is bounded in $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$ for every $0 < \sigma < \frac{1}{q}$, cf. Proposition 2.8, and so is $(\widehat{z}_{\tau_k,\epsilon_k})_k$. Therefore, by compactness the above pointwise convergence to $\tilde{z}$ improves to a strong convergence in $\mathcal{Z}$. But then, $\overline{z}_{\tau_k,\epsilon_k}, \widehat{z}_{\tau_k,\epsilon_k} \to \tilde{z}$ in $L^p(0,T;\mathcal{Z})$ for every $1 \leqslant p < \infty$, which allows us to conclude that $\tilde{z} = z$. All in all, we have obtained convergence (6.12*b*).

Finally, we address (6.12*c*): Observe that $A_q(\overline{z}_{\tau_k,\epsilon_k}(t)) \to A_q(z(t))$ in $\mathcal{Z}^*$ as a consequence of the strong convergence (6.12*b*). *A fortiori*, by the $L^\infty(0,T;L^2(\Omega))$-bound on $(A_q(\overline{z}_{\tau_k,\epsilon_k}))_k$, we find that $A_q(\overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup A_q(z(t))$ in $L^2(\Omega)$. We combine this with (2.36), giving that $\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_{\tau_k}(t), \overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ in $L^2(\Omega)$, and arrive at (6.12*c*). □

***Step* 1*: the local stability* ($\mathrm{S_{loc}}$).**

On one hand, the very same argument leading to the proof of estimate (3.12*a*) in Corollary 3.3 also shows that

$$\sup_k \int_0^T \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r), \overline{z}_{\tau_k,\epsilon_k}(r)))\,\mathrm{d}r \leqslant C\,. \tag{6.14}$$

On the other hand, $\mathcal{R}^*_\epsilon$ Mosco-converges, w.r.t. the $L^2(\Omega)$-topology, to the indicator functional

$$\mathrm{I}_{\partial\mathcal{R}_1(0)} : L^2(\Omega) \to [0,+\infty] \quad \text{defined by} \quad \mathrm{I}_{\partial\mathcal{R}_1(0)}(v) := \begin{cases} 0 & \text{if } v \in \partial\mathcal{R}_1(0), \\ +\infty & \text{else.} \end{cases}$$

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Hence, we have in view of (6.12c) that

$$\liminf_{k\to\infty} \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(t),\bar{z}_{\tau_k,\epsilon_k}(t))) \geqslant \mathrm{I}_{\partial\mathcal{R}_1(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t))) \qquad \text{for every } t\in[0,T]. \quad (6.15)$$

Therefore, from (6.14) and (6.15) *via* the Fatou Lemma we infer that

$$\int_0^T \mathrm{I}_{\partial\mathcal{R}_1(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t)))\,\mathrm{d}t < +\infty\,, \text{ whence } \mathrm{I}_{\partial\mathcal{R}_1(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t))) = 0 \text{ for a.a. } t\in(0,T).$$

From this we conclude with an approximation argument $-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0)$ for every $t\in[0,T]\setminus \mathrm{J}_z$, and that $-\mathrm{D}_z\mathcal{I}(t,z(t_\pm)) \in \partial\mathcal{R}_1(0)$ for every $t\in\mathrm{J}_z$, i.e., ($\mathrm{S}_{\mathrm{loc}}$).

Step* 2*: the key lower semi-continuity inequality.
We aim to prove the following

Lemma 6.3 *For every* $0\leqslant s\leqslant t\leqslant T$ *there holds*

$$\liminf_{k\to\infty}\int_{\underline{t}_{\tau_k}(s)}^{\bar{t}_{\tau_k}(t)} \mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r))\,\mathrm{d}r + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\bar{z}_{\tau_k,\epsilon_k}(r)))\,\mathrm{d}r \geqslant \mathit{Var}^{\bar{\varrho}}_{\mathfrak{f}}(z;[s,t]) \qquad (6.16)$$

with $\bar{\varrho}$ *given by*

$$\bar{\varrho} := \sup_k \Big(\int_0^T \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_k(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\bar{z}_k(r)))\big)\,\mathrm{d}r + \|\widehat{z}_k\|_{L^\infty(0,T;\mathcal{Z})\cap W^{1,1}(0,T;H^1(\Omega))} + \|\bar{z}_k\|_{L^\infty(0,T;\mathcal{Z})} + \|\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k},\bar{z}_k))\|_{L^\infty(0,T;L^2(\Omega))}\Big)$$

Proof Along the footsteps of the [25, proof of Theorem 7.3], we introduce the non-negative Borel measures on $[0,T]$

$$\nu_k := \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k},\bar{z}_{\tau_k,\epsilon_k}))\big)\,\mathscr{L}^1,$$

with $\mathscr{L}^1$ the one-dimensional Lebesgue measure. It follows from estimate (3.12b) that the sequence $(\nu_k)_k$ is bounded in the space of Radon measures, hence there exists a positive measure ν such that $\nu_k \rightharpoonup^* \nu$ as $k\to\infty$. Like in the proof of [25, Theorem 7.3], we observe that for every interval $[a,b]\subset[0,T]$

$$\begin{aligned}\nu([a,b]) \geqslant \limsup_{k\to\infty}\nu_k([a,b]) &\geqslant \limsup_{k\to\infty}\int_a^b \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\bar{z}_{\tau_k,\epsilon_k}(r)))\big)\,\mathrm{d}r \\ &\geqslant \liminf_{k\to\infty}\int_a^b \mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r))\,\mathrm{d}r \\ &\geqslant \liminf_{k\to\infty}\mathrm{Var}_{\mathcal{R}_1}(\bar{z}_{\tau_k,\epsilon_k};[a,b]) \overset{(1)}{\geqslant} \mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) \overset{(2)}{\geqslant} \mu_{\mathrm{d}}([a,b]),\end{aligned}$$

where (1) follows from the pointwise convergence (6.12b) and the lower semi-continuity of the variation functional $\mathrm{Var}_{\mathcal{R}_1}$, and (2) from the definition (5.10) of the measure μ. We thus conclude that

$$\nu \geqslant \mu_{\mathrm{d}}. \tag{6.17}$$

We now check

$$\nu(\{t\}) \geqslant \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t_-),z(t)) + \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t),z(t_+)) \geqslant \mu_{\mathrm{J}}(\{t\}) \qquad \text{for every } t \in \mathrm{J}_z. \tag{6.18}$$

With this aim, for fixed $t \in \mathrm{J}_z$ let us fix two sequences $\alpha_k \uparrow t$ and $\beta_k \downarrow t$ such that

$$\begin{cases} \overline{z}_{\tau_k,\epsilon_k}(\alpha_k) \to z(t_-), \\ \overline{z}_{\tau_k,\epsilon_k}(\beta_k) \to z(t_+) \end{cases} \qquad \text{in } \mathcal{Z} \text{ as } k \to \infty.$$

Thus, we have

$$\begin{aligned} \limsup_{k\to\infty} \nu_k([\alpha_k,\beta_k]) &\geqslant \liminf_{k\to\infty} \int_{\alpha_k}^{\beta_k} \left(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\overline{z}_{\tau_k,\epsilon_k}(r)))\right) \mathrm{d}r \\ &\overset{(1)}{\geqslant} \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t_-),z(t_+)), \end{aligned}$$

where (1) ensues from Proposition 6.1 by applying (6.3) with the choices $\overline{z}_k := \overline{z}_{\tau_k,\epsilon_k}$, $\widehat{z}_k := \widehat{z}_{\tau_k,\epsilon_k}$. With analogous arguments we check that

$$\liminf_{k\to\infty} \nu_k([\alpha_k,t]) \geqslant \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t_-),z(t)), \qquad \liminf_{k\to\infty} \nu_k([t,\beta_k]) \geqslant \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t),z(t_+)). \tag{6.19}$$

All in all, we have

$$\begin{aligned} \nu(\{t\}) &\overset{(1)}{\geqslant} \limsup_{k\to\infty} \nu_k([\alpha_k,\beta_k]) \geqslant \liminf_{k\to\infty} \nu_k([\alpha_k,t]) + \liminf_{k\to\infty} \nu_k([t,\beta_k]) \\ &\geqslant \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t_-),z(t)) + \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t;z(t),z(t_+)) \overset{(2)}{\geqslant} \mu_{\mathrm{J}}(\{t\}), \end{aligned}$$

where (1) is a property of the weak*-convergence of measures and (2) ensues from (5.3). Hence, inequality (6.18) is proved.

Combining (6.17)–(6.19) and repeating the very same calculations as in the proof of [25, Theorem 7.3], we ultimately conclude (6.16). □

Step* 3*: the energy-dissipation inequality (5.24)**.**
We now pass to the limit in the discrete ED inequality (3.11), written for $s = 0$ and $t = T$. For the first term on the left-hand side, we resort to the lower semi-continuity inequality (6.16) from Step 2. It follows from the pointwise convergence (6.12*b*) and the lower semi-continuity (2.36) of $\mathcal{I}$ that

$$\liminf_{k\to\infty} \mathcal{I}(T,\widehat{z}_{\tau_k,\epsilon_k}(T)) \geqslant \mathcal{I}(T,z(T)),$$

whereas by hypothesis (5.18), we have that $\mathcal{I}(0,\widehat{z}_{\tau_k,\epsilon_k}(0)) \to \mathcal{I}(0,z_0)$. Furthermore, it follows from (2.23), (2.24) and the Lebesgue Theorem that

$$\lim_{k\to\infty} \int_0^T \partial_t\mathcal{I}(t,\widehat{z}_{\tau_k,\epsilon_k}(t))\,\mathrm{d}t = \int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.$$

Finally, observe that the very last term on the right-hand side of (3.11) converges to zero by virtue of estimates (3.9) and convergence (6.13).

Thus, (5.24) is proven with $\mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z;[0,T])$ and, by virtue of Corollary 5.9, we deduce that z is a BV solution to the rate-independent damage system (1.1).

Finally, (5.20) follows from the following chain of inequalities (which in fact holds for every $t \in [0,T]$)

$$\sup_{\varrho\geqslant\bar{\varrho}} \mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]) \stackrel{(1)}{=} \mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z;[0,T]) \stackrel{(2)}{=} \mathcal{I}(0,z(0)) - \mathcal{I}(T,z(T)) + \int_0^T \partial_t \mathcal{I}(s,z(s))\,\mathrm{d}s$$
$$\stackrel{(3)}{\leqslant} \inf_{\varrho\geqslant\bar{\varrho}} \mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]),$$

with (1) due to (5.6), (2) to ($\mathrm{E}_{\mathfrak{f}}$) involving the total variation functional $\mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z;[0,T])$, and (3) from the chain-rule inequality (5.23) (observe that ϱ therein is arbitrary, provided it fulfills (5.14)).

***Step* 4*: convergences* (5.21).**
The convergences of the energies $(\mathcal{I}(t,\bar{z}_{\tau_k,\epsilon_k}(t)))_k$ follows from the pointwise convergence (6.12*a*) of $(\bar{z}_{\tau_k,\epsilon_k}(t))_k$. In order to prove the convergence of $(\mathcal{I}(t,\widehat{z}_{\tau_k,\epsilon_k}(t)))_k$ and of the dissipation integrals in (5.21*c*), we repeat the very same arguments as in the proof of [25, Theorem 3.11].

***Step* 5*:* (5.22).**
We may repeat the proof of [25, Theorem 3.22], to which we refer the reader, relying on Proposition 6.1(3).

This concludes the proof of Theorem 5.7. ∎

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Appendix A Some references on elliptic regularity

For $d \geqslant 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$-domain with Dirichlet boundary $\partial\Omega$. Let further $\mathbb{C}$ satisfy (2.5).

Reference [31, Theorem 3], see also [21, Theorem 7.1], yields

Theorem A.1 *For every* $p \in (1,\infty)$ *the operator* $L_{\mathbb{C}} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ *is a continuous isomorphism.*

Moreover, Theorem 10.5 from [1] (there it is assumed that the domain has a C^2-boundary, but the coefficients need to be continuous, only, instead of Lipschitz continuous) provides the following *a priori* estimate:

Theorem A.2 *For every* $p \in (1,\infty)$ *there exist constants* $c_p, \widetilde{c}_p > 0$ *such that for every* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *it holds*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c_p \left(\|L_{\mathbb{C}} u\|_{L^p(\Omega)} + \widetilde{c}_p \|u\|_{L^p(\Omega)} \right) . \tag{A 1}$$

Thanks to Theorem A.1, for every $p \in (1,\infty)$ the operator

$$L_{\mathbb{C}} : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega), \quad u \mapsto -\mathrm{div}(\mathbb{C}\varepsilon(u)) \tag{A 2}$$

is injective, which implies that estimate (A 1) is valid with $\widetilde{c}_p = 0$ and that $L_{\mathbb{C}}$ has a closed range. By [15, Chapter 3.5.5], one finally concludes that the operator $L_{\mathbb{C}}$ from (A 2) is surjective for every $p \in (1,\infty)$. This finally results in

Theorem A.3 *For every* $p \in (1,\infty)$ *the operator in* (A 2) *is a continuous isomorphism.*