Proceedings of the Edinburgh Mathematical Society (2021) **64**, 590–593
doi:10.1017/S0013091521000328

ON CLASS 2 QUOTIENTS OF LINEAR GROUPS

YONG YANG

Department of Mathematics, Texas State University, San Marcos, TX 78666, USA
(yang@txstate.edu)

(Received 13 September 2020; first published online 9 June 2021)

Abstract In this paper, we study the relation of the size of the class two quotients of a linear group and the size of the vector space. We answer a question raised in Keller and Yang [Class 2 quotients of solvable linear groups, J. Algebra 509 (2018), 386-396].

Keywords: Finite linear group; class 2 group; vector space

2020 Mathematics subject classification: Primary 20C15; 20C20; 20D10; 20D15

1. Introduction

Glauberman proved that if n is a positive integer, p is a prime and $G \leq \mathrm{GL}(n, p)$ is a p'-group which is nilpotent of class 2, then $|G| < |V|$ in [4, Proposition 1].

The goal of this paper is to generalize this result to an arbitrary linear group. For a finite group G, we define $G^c = [G, G, G] = [G', G]$; i.e., G^c is the intersection of all normal subgroups of G whose quotient group is nilpotent of class 2, so that G/G^c is the (maximal) class 2 quotient of G. In [6], Keller and the author generalized the above-mentioned result of Glauberman to solvable linear groups. It was conjectured in the same paper that the main result of that paper remains true for arbitrary finite groups in place of solvable groups. In this short note, we settle this conjecture, and our main results are as follows.

Theorem 1.1. *Let a finite group G act faithfully on a finite group V. Then any one of the following conditions guarantees that*

$$|G/G^c| \leq |V|.$$

(1) *V is a p-group and $O_p(G) = 1$;*

(2) *V is a completely reducible G-module, possibly of mixed characteristic.*

Of course, when G is nilpotent of class 2, this is just Glauberman's result.

We note that Theorem 1.1 can also be viewed as a strengthening of a result by Aschbacher and Guralnick, see [1, Theorem 1]. They proved that the order of the abelian quotient $|G/G'|$ of G, i.e., the class 1 quotient of G, is strictly bounded above by $|V|$, where G is a finite faithful linear group on the finite module V such that $O_p(G) = 1$ for the characteristic p of V. Our result shows that even the class 2 quotient of G is strictly bounded above by $|V|$. We point out that the main result in [1] has been generalized in a different direction in [5, 7], where the upper bound $|V|$ is replaced by the largest orbit size. We shall also point out that our result also generalizes a result of Flavell [3, Theroem A (i)], since we do not need coprimeness here.

2. Preliminary results

In this section, we state a few lemmas and propositions.

The following lemma is elementary.

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$. Then*

$$|G/G^c| = |G/G^cN| \cdot |N : N \cap G^c|$$

and

$$|G : G^c| \text{ divides } |G/N : (G/N)^c||N : N^c|.$$

Lemma 2.2. *Let G be a finite group and let N be a normal subgroup of G such that G/N is nilpotent. Let p be a given prime and $N = S_1 \times \cdots \times S_t$ be a direct product of isomorphic non-abelian simple groups S_i. Assume that S_i are all normal in G and that $C_G(N) = 1$. Then there exists an abelian p'-subgroup A of G such that $2^t|G/N| \leq |A|$.*

Proof. This is [8, Lemma 2.2]. □

Lemma 2.3. *Let $N \trianglelefteq G$. Assume that $H/N \leq G/N$ and $|(G/N) : (G/N)^c| \leq |(H/N) : (H/N)^c|$. Then $|G : G^c| \leq |H : H^c|$.*

Proof. $|G : G^c| \leq |G : G^c||G^c \cap N|/|H^c \cap N| = |G : G^cN||N : H^c \cap N| \leq |H : H^cN||N : H^c \cap N| = |H : H^c|$. □

Lemma 2.4. *Let G be a finite solvable group. Then G has a nilpotent subgroup H such that $|G/G^c| \leq |H/H^c|$.*

Proof. This follows from the first half of the proof of [6, Theorem 2.3], where it reduces G to be nilpotent. In fact, the proof shows that for any finite solvable group G, we have $|G : G^c| \leq |F : F^c|$ where F is the Fitting subgroup of G. □

Proposition 2.5. *Let G be a finite group with $O_p(G) = 1$ for some prime p. Then G has a solvable subgroup H with $O_p(H) = 1$ such that $|G : G^c| \leq |H : H^c|$.*

Proof. We may assume G is non-solvable, and we work by induction on $|G|$. Let E be a minimal normal subgroup of G and write $D/E = O_p(G/E)$. Assume that G/D is

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non-solvable. By induction, G/D has a solvable subgroup B/D with $O_p(B/D) = 1$ such that $|(G/D) : (G/D)^c| \leq |(B/D) : (B/D)^c|$. By Lemma 2.3, we have

$$|G : G^c| \leq |B : B^c|.$$

Since $O_p(D) \leq O_p(G) = 1$ and $O_p(B/D) = 1$, we have $O_p(B) = 1$. Suppose that B is solvable. Then B meets the requirements. Suppose that B is non-solvable. By induction, B has a solvable subgroup H with $O_p(H) = 1$ such that $|B : B^c| \leq |H : H^c|$. Then

$$|G : G^c| \leq |B : B^c| \leq |H : H^c|,$$

so H meets the requirements. Hence, we may assume that G/D and G/E are solvable for all minimal normal subgroups E of G.

Since G is non-solvable, it forces that E is non-solvable and is the unique minimal normal subgroup of G. In particular, $C_G(E) = 1$. By Lemma 2.4, G/E has a nilpotent subgroup U/E such that

$$|G/G^c| = |G/G^cE| \leq |U/U^cE| = |U/U^c|.$$

Observe that $O_p(U) \leq C_G(E) = 1$. Suppose that $U < G$, applying the inductive hypothesis to U, we get the required result. Hence, we may assume that G/E is nilpotent.

Write $E = S_1 \times \cdots \times S_t$, where $S_1, \ldots, S_t$ are isomorphic non-abelian simple groups. Let us investigate $D = \bigcap_{i=1}^t N_G(S_i)$. Since all S_i are normal subgroups of D, by Lemma 2.2 there exists an abelian p'-subgroup H of D such that $2^t|D/E| \leq |H|$. Observe that G acts on $\{S_1, \ldots, S_t\}$ with the kernel D, it follows that G/D is a nilpotent permutation group of degree t. As is well known (see, for example, [2]), we have that $|G/D| \leq 2^{t-1}$. Now

$$|G : G^c| \leq 2|G/E| \leq |H| = |H : H^c|,$$

so H meets the requirements. □

3. The main result

We now prove the main result of this paper, Theorem 1.1.

Proof. (1) Set $K = G \ltimes V$. It is clear that $F(K)$ is nilpotent, so we can write $F(K) = P \times Q$, where P is a p-group and Q is a p'-group. Note that both P and Q are normal in K, which implies that PV/V is a normal p-subgroup of $K/V \cong G$ and $[Q, V] \leq Q \cap V = 1$. Now the assumptions that $O_p(G) = 1$ and $\mathbf{C}_G(V) = 1$ guarantee that $P \leq V$ and $Q = 1$, so $F(K) = P \leq V$. But the converse containment is clear, so $\Phi(K) \leq F(K) = V$. If $\Phi(K) = V$, then $K = GV = G\Phi(K)$, which forces $K = G$ and hence $V = 1$, a trivial case. Thus, we have $F(K) = V$ and $\Phi(K) < V$.

Set

$$\overline{G} = G\Phi(K)/\Phi(K), \overline{V} = V/\Phi(K), \overline{K} = K/\Phi(K) = \overline{G} \ltimes \overline{V}.$$

Observe that $\overline{G}$ also acts faithfully on $\overline{V}$. We note this assertion is equivalent to saying that G acts faithfully on $\overline{V}$. Let $C = \mathbf{C}_G(\overline{V})$, and take A to be a p'-subgroup of C. Then $[V, A] \leq \Phi(K)$. Since A acts coprimely on V, we have $V = [V, A]\mathbf{C}_V(A) = \Phi(K)\mathbf{C}_V(A)$

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and thus $K = GV = \Phi(K)G\mathbf{C}_V(A)$. This forces $GV = G\mathbf{C}_V(A)$ and hence $V = \mathbf{C}_V(A)$. Note that we are assuming that $\mathbf{C}_G(V) = 1$, so $A = 1$, which means that C is a p-group. But $O_p(G) = 1$ by hypotheses, so $C = 1$, as required.

Assume that $\Phi(K) > 1$. By induction,

$$|G : G^c| = |\overline{G} : \overline{G}^c| < |\overline{V}| < |V|,$$

and we are done. Therefore, we may assume that $\Phi(K) = 1$. By Proposition 2.5, we may assume that the group G is solvable. Since $F(K) = V$, V is a faithful and completely reducible G-module over a field of characteristic p by Gaschütz's theorem. Now the result follows by [6, Theorem 1.1].

(2) Suppose that V is a finite, faithful and completely reducible G-module. We work by induction on $|G| + |V|$. Assume that $V = V_1 \oplus V_2$, where V_1, V_2 are non-trivial G-submodules of V. Let $C = C_G(V_1)$. Observe that V_1 is a faithfully and completely reducible G/C-module, while V_2 is a faithfully and completely reducible C-module (note that C is normal in G). By induction,

$$|G/C : (G/C)^c| \leq |V_1|$$

and

$$|C : C^c| \leq |V_2|.$$

Thus, $|G : G^c| \leq |G/C : (G/C)^c| \cdot |C : C^c| < |V_1||V_2| = |V|$, and we are done.

Hence, we may assume that V is an irreducible G-module over a finite field of characteristic p. Clearly, $O_p(G) = 1$ because G acts faithfully on V. Now the required result follows by (1). □

Acknowledgements. The author's research was supported by a grant from the Simons Foundation (No 499532). The author is very grateful to the referee's helpful comments and suggestions which improve the manuscript.

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