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An application of the theorem on Sums to viscosity solutions of degenerate fully nonlinear equations

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We prove Hölder continuous regularity of bounded, uniformly continuous, viscosity solutions of degenerate fully nonlinear equations defined in all of $\mathbb{R}^n$ space. In particular, the result applies also to some operators in Carnot groups.

Keywords: Degenerate elliptic operators; nonlinear elliptic operators; carnot groups; viscosity solutions; theorem on sums; Hölder regularity

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1. Introduction

In this paper, we continue the research regarding the properties of the viscosity solutions of some nonlinear PDEs started in [**7**, **8**]. In those papers we studied the case of non-divergence nonlinear equations modelled on vector fields in the Heisenberg group. We proved there that bounded uniformly continuous functions that are also viscosity solutions of some nonlinear degenerate uniformly elliptic equations in all the Heisenberg group $\mathbb{H}^1$ are also Hölder continuous in the classical sense.

In the cited papers we did not need to prove Harnack inequality in advance, as it is customary to do in order to prove Hölder continuity.

Our main goal is now to deal with a larger class of operators, intrinsically uniformly elliptic with respect to square matrices, of order less or equal to the dimension of the Euclidean given space, obtained by considering some smooth vector fields. Even if these operators are not elliptic in the classical sense defined in [**3**], we obtain similar regularity results to the ones proved in [**7**, **8**] without making use of a Harnack inequality.

This research has been inspired by reading [**12**]. In that paper the author applied the theorem on sums, see [**4**], to elliptic linear operators having quite sufficiently smooth coefficients.

The key point that we explore in our approach is based on the existence of square root matrices, sufficiently smooth and of the symmetric matrix associated with the second order term of the equation, the so-called leading term.

Since in this paper we consider many different families of operators in non-divergence form, we prove that our approach works even in those cases in which, instead of considering the classical square root matrix, there exist rectangular square roots matrices σ such that $\sigma^T\sigma$ represents, possibly, the degenerate square matrix associated with the second order term of the equation. A typical application of this case appears in Carnot groups, but many other examples exist.

In order to better explain the result, we introduce the classes of operators that we deal with. In the sequel, we denote with S^m the set of $m \times m$ square symmetric matrices, $m \in \mathbb{N}$, $m \geqslant 1$.

Definition 1.1. Let $0 < \lambda \leqslant \Lambda$ be given real numbers and let $0 < m \leqslant n$ be two positive integers. Let σ be a $m \times n$ matrix, with Lipschitz continuous coefficients defined in $\Omega \subseteq \mathbb{R}^n$. Let $G : S^m \to \mathbb{R}$ be a given function such that for every $A, B \in S^m$, if $B \leqslant A$ then

$$\lambda \mathrm{Tr}(A - B) \leqslant G(A) - G(B) \leqslant \Lambda \mathrm{Tr}(A - B).$$

We define the function $F : S^n \times \Omega \to \mathbb{R}$ in such a way that, for every $M \in S^n$ and for every $x \in \Omega$

$$F(M, x) = G(\sigma(x) M \sigma(x)^T).$$

We sometime denote for every $x \in \Omega$, the $n \times n$ matrix $P(x) = \sigma(x)^T \sigma(x)$.

We postpone to § 2 some comments about the novelty of this family of operators and we state immediately our main result.

Theorem 1.2. *Let $f, c \in C(\mathbb{R}^n)$ be continuous functions and let L_c, L_f, β, β' be positive real numbers such that $\beta, \beta' \in (0, 1]$ and for every $x, y \in \mathbb{R}^n$, $|c(x) - c(y)| \leqslant L_c|x - y|^\beta$, $|f(x) - f(y)| \leqslant L_c|x - y|^{\beta'}$. Let us suppose $\inf_{x\in\mathbb{R}^n} c(x) := c_0 > 0$. Let F be an operator satisfying definition 1.1 where σ is Lipschitz continuous in $\mathbb{R}^n$ and $P = \sigma^T\sigma$. Assume that there exists a positive constant $\bar{c}$, $c_0 \geqslant \bar{c} > 0$. If $u \in C(\mathbb{R}^n)$ is a bounded uniformly continuous viscosity solution of the equation*

$$F(D^2u(x), x) - c(x)u(x) = f(x), \quad \mathbb{R}^n,$$

and

$$\limsup_{|x|\to\infty} \left(\frac{Tr(P(x))}{|x|^2} - \frac{c_0}{2\Lambda} \right) \leqslant 0, \tag{1.1}$$

then there exist $0 < \alpha := \alpha(c_0, \bar{c}, L_c, L_f, \Lambda) \in (0, 1]$, $\alpha \leqslant \min\{\beta, \beta'\}$, and $L := L(c_0, \bar{c}, L_c, L_f, \Lambda) > 0$ such that for every $x, y \in \mathbb{R}^n$

$$|u(x) - u(y)| \leqslant L|x - y|^\alpha.$$

We point out that in our presentation we do not distinguish the operators by considering their possible degeneracy since the approach that we introduce applies independently to the fact that the operator is degenerate elliptic or it is not.

In fact, it is well known that viscosity theory existence is independent to the lack of strict ellipticity. Namely, the construction of Perron solutions can be done independently to the possible degenerate nature of the equation.

As a consequence, even when we deal with PDEs in Carnot groups, we state our results always with respect to the classical notions of regularity. For instance, in our main result, we obtain a Hölder modulus of continuity in the classical sense of the viscosity solutions.

We point out this aspect since there exists also a wide literature in which the obtained results are introduced by using an intrinsic notion of regularity, see for example, [**16**]. In particular, those results are stated by exploiting the intrinsic notions of distance and differentiability associated with the geometry of the operator. From this point of view, we recall that the intrinsic distance associated with degenerate PDEs in non-commutative groups, usually, is not equivalent to the Euclidean one. For the reader's convenience, we shall come back at the end of § 2 to this remark.

After this introduction, the paper is organized as follows: in § 2, we list some cases to which our result applies and we introduce the main tools we need to; in § 3, we show the proof of theorem 1.2 and in § 4, we discuss some final remarks and conclusions. Concerning the recent literature about this subject, in addition to [**1**, **7**, **8**, **23**], we would like to cite also, [**18**, **19**].

2. Examples and preliminary tools

We begin this section by listing some examples of the operators belonging to the family introduced in definition 1.1. All the fully nonlinear operators F, that are uniformly elliptic, see [**3**], belong to our class when $P \equiv I$. In this case, $\sigma = I \in S^n$ and $m = n$.

In addition, in order to give an explicit nontrivial example belonging to the class of fully nonlinear operator studied in [**3**], we consider in $\mathbb{R}^3$ the matrix

$$P_{\mathbb{H}^1}(x) = \begin{bmatrix} 1, & 0, & 2x_2 \\ 0, & 1, & -2x_1 \\ 2x_2, & -2x_1, & 4(x_1^2 + x_2^2) \end{bmatrix}.$$

For every point $x \in \mathbb{H}^1$, the $\det(P(x)) = 0$, $\operatorname{rank}(P(x)) = 2$ and $P(0) \geqslant 0$.

Nevertheless,

$$\sqrt{P_{\mathbb{H}^1}(x)} = \begin{bmatrix} \frac{x_2^2 + \frac{x_1^2}{\sqrt{1+4(x_1^2+x_2^2)}}}{x_1^2+x_2^2}, & \frac{x_1 x_2\left(1 - \frac{1}{\sqrt{1+4(x_1^2+x_2^2)}}\right)}{x_1^2+x_2^2}, & \frac{2x_2}{\sqrt{1+4(x_1^2+x_2^2)}} \\ \frac{x_1 x_2\left(1 - \frac{1}{\sqrt{1+4(x_1^2+x_2^2)}}\right)}{x_1^2+x_2^2}, & \frac{x_1^2 + \frac{x_2^2}{\sqrt{1+4(x_1^2+x_2^2)}}}{x_1^2+x_2^2}, & -\frac{2x_1}{\sqrt{1+4(x_1^2+x_2^2)}} \\ \frac{2x_2}{\sqrt{1+4(x_1^2+x_2^2)}}, & -\frac{2x_1}{\sqrt{1+4(x_1^2+x_2^2)}}, & \frac{4(x_1^2+x_2^2)}{\sqrt{1+4(x_1^2+x_2^2)}} \end{bmatrix}.$$

In the class of our operators we find the following ones:

$$\mathcal{P}^+_{\mathbb{H}^1}(M, x) = \max_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{Tr}(A\sqrt{P(x)}M\sqrt{P(x)})$$

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and

$$\mathcal{P}^{-}_{\mathbb{H}^1}(M,x) = \min_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(A\sqrt{P(x)}M\sqrt{P(x)}),$$

where

$$\mathcal{A}_{\lambda,\Lambda} = \{A \in S^3 : \quad \lambda|\xi|^2 \leqslant \langle A\xi,\xi\rangle \leqslant \Lambda|\xi|^2\}.$$

They are the analogous ones of the Pucci's extremal operators belonging to the class of fully nonlinear uniformly elliptic operators, see [3].

In this framework, also the particular case given by the sublaplacian in the Heisenberg group

$$\Delta_{\mathbb{H}^1}u(x) \equiv \mathrm{Tr}(P_{\mathbb{H}^1}(x)D^2u(x)) = G(D^2u(x)) = F(D^2u(x),x),$$

where

$$G(M) = \mathrm{Tr}(\sqrt{P_{\mathbb{H}^1}(x)}M\sqrt{P_{\mathbb{H}^1}(x)}),$$

belongs to the same class.

Indeed

$$\lambda\Delta_{\mathbb{H}^1}u(x) \leqslant G(\sigma_{\mathbb{H}^1}(x)D^2u(x)\sigma(x)^T_{\mathbb{H}^1}) \leqslant \Lambda\Delta_{\mathbb{H}^1}u.$$

Thus, we conclude that these operators are not uniformly elliptic in the classical sense described in [3].

It is worth to say that we can also consider those operators F obtained coherently with our definition by remarking that if σ is not a squared matrix, but it is simply a rectangular matrix, we can construct, at least apparently, another family of operators.

For example, one more time considering for simplicity the Heisenberg group $\mathbb{H}^1$, that is the simplest case of a nontrivial Carnot group, we have:

$P_{\mathbb{H}^1}(x) = \sigma^T(x)\sigma(x)$ where:

$$\sigma_{\mathbb{H}^1}(x) = \begin{bmatrix} 1, & 0, 2x_2 \\ 0, & 1, -2x_1 \end{bmatrix}.$$

As a consequence for every $M \in S^{3\times 3}$

$$F(M,x) = G(\sigma_{\mathbb{H}^1}(x)M\sigma_{\mathbb{H}^1}(x)^T).$$

This approach can be extended to every Carnot group by considering the matrix $\sigma_{\mathbb{G}}$ whose rows are given by the coefficients that determine the vector fields of the first stratum of the Lie algebra in a Carnot group $\mathbb{G}$. Namely, we construct the matrix

$$\sigma_{\mathbb{G}}^T(x) = \begin{bmatrix} X_1(x) \\ X_2(x) \\ \vdots \\ X_m(x) \end{bmatrix},$$

where

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, \ldots, X_m\},$$

$$\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1], \quad \mathfrak{g}_{k+1} = [\mathfrak{g}_1, \mathfrak{g}_k], \;\; k \leqslant p-1, \quad \bigoplus_{j=1}^{p} \mathfrak{g}_j = \mathfrak{g},$$

$\mathfrak{g}$ is the Lie algebra of the group $\mathbb{G}$ and p is its step. We refer to [**2**] for further details.

It is important to point out that, by considering different Carnot groups to the Heisenberg one, our definition

$$\lambda \operatorname{Tr}(\sigma(x) M \sigma(x)^T) \leqslant F(M, x) = G(\sigma(x) M \sigma(x)^T) \leqslant \Lambda \operatorname{Tr}(\sigma(x) M \sigma(x)^T)$$

does not necessarily translate into the following equivalent condition

$$\lambda \Delta_{\mathbb{G}} u(x) \leqslant F(D^2 u(x), x) = G(\sigma(x) D^2 u(x) \sigma(x)^T) \leqslant \Lambda \Delta_{\mathbb{G}} u(x),$$

as in the Heisenberg group. Indeed, it is well known that there exist Carnot groups such that

$$\operatorname{Tr}(\sigma_{\mathbb{G}}(x) M \sigma(x)_{\mathbb{G}}^T) \neq \Delta_{\mathbb{G}} u(x),$$

assuming, by definition, $\Delta_{\mathbb{G}} u(x) := \sum_{j=1}^m X_j^2 u(x)$, and $\{X_1 \ldots, X_m\}$ endowed with the first stratum of the Lie algebra of the Carnot group.

For instance, let us consider the Engel group $\mathbb{E}^1 \equiv \mathbb{R}^4$, endowed by the non-commutative inner product law,

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 - y_1 x_2, x_4 + y_4 + \frac{1}{2} y_1^2 x_2 - y_1 x_3\right), \tag{2.1}$$

where the Jacobian basis, see [**2**], is

$$\begin{aligned} X_1 &= \partial_1 - x_2 \partial_3 - x_3 \partial_4 & X_2 &= \partial_2 \\ X_3 &= \partial_3 & X_4 &= \partial_4. \end{aligned}$$

The matrix $\sigma_{\mathbb{E}^1}$ becomes

$$\sigma_{\mathbb{E}^1}(x) = \begin{bmatrix} 1, & 0, & -x_2, & -x_3 \\ 0, & 1, & 0, & 0 \end{bmatrix}$$

and

$$\operatorname{Tr}(\sigma_{\mathbb{E}^1}(x) D^2 u(x) \sigma(x)_{\mathbb{E}^1}^T) = X_1^2 u + X_2^2 u - x_2 \frac{\partial u}{\partial x_4}.$$

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In this case, the class of operators that we have defined does not contain explicitly the intrinsic sublaplacian on the Engel group given by $\Delta_{\mathbb{E}^1} u = X_1^2 u + X_2^2 u$. Nevertheless, $\mathrm{Tr}(\sigma_{\mathbb{E}^1}(x) D^2 u(x) \sigma(x)_{\mathbb{E}^1}^T)$ is still a degenerate operator, having the smallest eigenvalue always 0 in all of $\mathbb{R}^4$, see lemma 2.2 in the next subsection.

We spend some words making a digression about the intrinsic notion of distance in Carnot groups. It is well known that in the framework of this non-commutative structure, it is defined as a natural distance associated with the geometry of the group. In literature it is called the Carnot-Charathéodoty distance. This distance can be constructed in many ways. For instance, we briefly describe the following approach. If

$$\mathfrak{g}_1(P) = \mathrm{span}\{X_1(P), \ldots, X_m(P)\},$$

for every $P \in \mathbb{G}$ and the set $\{X_1(P), \ldots, X_m(P)\}$ is braking generating all the space $\mathbb{R}^n \equiv \mathbb{G}$, then for every function $\phi : [0,1] \to \mathbb{G} \equiv \mathbb{R}^n$ parametrizing a path $\gamma \subset \mathbb{G}$ such that for every $t \in [0,1]$, $\phi'(t) \in \mathfrak{g}_1(\phi(t))$, we define the length of γ by

$$\mathrm{l}(\gamma) = \int_0^1 \sqrt{\sum_{k=1}^m \langle \phi'(t), X(\phi(t)) \rangle^2}\, \mathrm{d}t.$$

Then for every $P_0, P_1 \in \mathbb{G}$ we call:

$$\begin{aligned} &d_{CC}^{\mathbb{G}}(P_0, P_1) \\ &= \inf\{\mathrm{l}(\gamma_{P_0,P_1}) : \ \gamma_{P_0,P_1}, \text{ is horizontal path connecting } P_0, \text{ with } P_1\} \end{aligned} \tag{2.2}$$

the Carnot-Charathéodory distance between P_0, with P_1.

This distance is not equivalent to the Euclidean distance since it holds only that if $K \subset \mathbb{G}$ is bounded, then there exist $C_1, C_2 > 0$ such that, for every $P_1, P_2 \in K$

$$C_1 |P_1 - P_2|_E \leqslant d_{CC}(P_1, P_2) \leqslant C_2 |P_1 - P_2|_E^{1/p},$$

where p denotes the step of the Carnot group. For instance, in the Heisenberg group $p = 2$, in the Engel group $p = 4$. Further details and a complete list of references about these topics can be found in [**2**]. Thus, as we pointed out in the Introduction, we remark that in the statement of theorem 1.2 we make use only of the usual Euclidean distance and the classical Hölder modulus of continuity of the viscosity solutions.

Thus all the results that we state in this paper have to be understood in the classical usual sense since we do not use explicitly the Carnot-Charathéodory distance (2.2).

2.1. Preliminary tools

In this subsection, we list some useful key tools concerning the eigenvalues of matrices obtained as the product of rectangular matrices, and the statement of the theorem on sums, see [**4**]. For the notation, the definition of viscosity solution and other symbols like sub/super jets $J^{2,\pm}u(x)$, we refer, one more time, to [**4**] and [**6**].

LEMMA 2.1. *Let A be a symmetric $n \times n$ matrix such that for every $i = 1, \ldots, n$, $a_{ii} > 0$ then all the eigenvalues of A are strictly positive.*

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We omit the trivial proof. The following result is known in literature even if I do not know a specific citation of it in this form. Thus, coherently, we also show a direct proof.

LEMMA 2.2. *Let σ be a $m \times n$ matrix $m \leqslant n$ such that rank$(\sigma) = m$ then $\sigma\sigma^T$ is an $m \times m$ strictly positive matrix while if $m < n$, then $\sigma^T\sigma$ is a degenerate matrix whose eigenvalues different to 0 are the same of $\sigma\sigma^T$ and if $m = n$ then $\sigma^T\sigma$ is invertible and its eigenvalues are the same of $\sigma\sigma^T$.*

Proof. Let λ be an eigenvalue of $\sigma\sigma^T$ and v one of its eigenvectors. Then

$$\sigma\sigma^T v = \lambda v,$$

so that $\langle \sigma\sigma^T v, v\rangle = \lambda||v||^2$, so that $\langle \sigma^T v, \sigma^T v\rangle = \lambda||v||^2$ implies that $\lambda > 0$ whenever $v \notin \mathrm{Ker}\sigma^T$. Indeed $v \notin \mathrm{Ker}\sigma^T$ because by hypothesis $\mathrm{rank}(\sigma) = m$. Thus, we conclude that $\sigma\sigma^T$ is an $m \times m$ strictly positive, in particular also invertible, matrix. Consider now an eigenvalue λ of the matrix $\sigma\sigma^T$. If $\lambda \neq 0$ and $v \in \mathrm{Ker}(\sigma\sigma^T - \lambda I)$ then

$$\sigma\sigma^T v = \lambda v.$$

Thus $\sigma^T\sigma(\sigma^T v) = \lambda\sigma^T v$, that is λ is also an eigenvalue of $\sigma^T\sigma$. This proves that all the nonzero eigenvalues of $\sigma\sigma^T$ are eigenvalues of $\sigma^T\sigma$. On the other hand, if $\gamma > 0$ is an eigenvalue of $\sigma^T\sigma$ then

$$\sigma^T\sigma w = \gamma w,$$

$w \in \mathrm{Ker}(\sigma^T\sigma - \gamma I)$, then

$$(\sigma\sigma^T)\sigma w = \gamma(\sigma w),$$

then γ is also an eigenvector of $\sigma\sigma^T$, because $\sigma w \neq (0)$ since $\mathrm{rank}(\sigma) = m$. As a consequence, the nonzero eigenvalues of $\sigma\sigma^T$ are only the strictly positive eigenvalues of $\sigma^T\sigma$. The case $m = n$ is now trivial. □

The following result is an obvious consequence of the definition of trace of a matrix.

LEMMA 2.3. *Let $A, B \in S^n$ be given. For every $m \times n$ matrices σ_1, σ_2 then*

$$Tr(\sigma_1^T\sigma_1 A - \sigma_2^T\sigma_2 B) = Tr(\sigma_1 A\sigma_1^T - \sigma_2 B\sigma_2^T).$$

We recall now the maximum principle for semiconvex functions, sometimes also named theorem on the sum, see [4].

THEOREM 2.4 (Crandall-Ishii-Lions). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$. Let $\phi \in C^2(W)$ where W is open and $\Omega \times \Omega \subset W \subseteq \mathbb{R}^n \times \mathbb{R}^n$. If*

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there exists $(\hat{x},\hat{y}) \in \Omega$ *such that*

$$u(\hat{x}) - v(\hat{y}) - \phi(\hat{x},\hat{y}) = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} (u(x) - v(y) - \phi(x,y)), \tag{2.3}$$

then for each $\mu > 0$, *there exist* $A = A(\mu)$ *and* $B = B(\mu)$ *such that*

$$(D_x\phi(\hat{x},\hat{y}), A) \in \overline{J}^{2,+}u(\hat{x}), \quad (-D_y\phi(\hat{x},\hat{y}), B) \in \overline{J}^{2,-}u(\hat{y}), \quad and$$

$$-(\mu + ||D^2\phi(\hat{x},\hat{y})||)\begin{bmatrix} I, & 0 \\ 0, & I \end{bmatrix} \leqslant \begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leqslant D^2\phi(\hat{x},\hat{y}) + \frac{1}{\mu}(D^2\phi(\hat{x},\hat{y}))^2.$$

Where:

$$D^2\phi(\hat{x},\hat{y}) = \begin{bmatrix} D^2_{xx}\phi(\hat{x},\hat{y}), & D^2_{yx}\phi(\hat{x},\hat{y}) \\ D^2_{xy}\phi(\hat{x},\hat{y}), & D^2_{yy}\phi(\hat{x},\hat{y}) \end{bmatrix}$$

and $||M||$ *is the norm given by the maximum, in absolute value, of the eigenvalues of the symmetric matrix* $M \in S^{2n}$.

LEMMA 2.5. *Let* $\phi(x,y) = |x-y|^\alpha$. *If* $x \neq y$ *then*

$$D^2\phi(x,y) = \begin{bmatrix} M, & -M \\ -M, & M \end{bmatrix}, \tag{2.4}$$

where

$$M = L\alpha|x-y|^{\alpha-2}\left((\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I\right),$$

$$(D^2\phi(x,y))^2 = 2\begin{bmatrix} M^2, & -M^2 \\ -M^2, & M^2 \end{bmatrix}, \tag{2.5}$$

and

$$M^2 = \alpha^2L^2|x-y|^{2(\alpha-2)}\left(\alpha(\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I\right). \tag{2.6}$$

Proof. The proof follows by straightforward calculation. □

It is well known, at least since [**13**], that viscosity solutions of the equation

$$F(D^2u(x)) = f(x), \quad \Omega,$$

F uniformly elliptic, in the usual sense (see [**3**]), homogeneous of degree one, are $C^{0,\alpha}$ in every ball $B \subset 4B \subset\subset \Omega$, whenever $f \in C(\Omega)$.

We want to adapt the previous result to the case of degenerate elliptic operators that we are dealing with in this paper. Before doing this, we recall in the next subsection this approach.

2.2. C^α regularity for uniformly elliptic operators without Harnack inequality

It is possible to prove C^α regularity of viscosity solutions without proving first the Harnack inequality. Indeed it is sufficient to reduce the problem to a ball of radius 1 for a non-constant function $0 < u < 1$. The scheme of the proof, see for example, the idea in [17] or in [10] for a slightly different but equivalent approach, is the following one:

Let $w(x,y) = u(x) - u(y) - L|x-y|^\alpha - 2|x-z|^2$, for every $z \in B_{1/4}$ and denote $\phi(x,y) = L|x-y|^\alpha$ so that $w(x,y) = u(x) - u(y) - \phi(x,y) - 2|x-z|^2$, Let

$$\max_{B_1(0)\times B_1(0)} w(x,y) = w(\hat{x},\hat{y}) := \theta.$$

Assume by contradiction that $\theta > 0$. Then $\hat{x} \neq \hat{y}$. Thanks to the localization term $2|x-z|^2$, then $(\hat{x},\hat{y}) \in B_{1/4}(0)$.

By the theorem of the sums, for every $\mu > 0$, there exist $A = A(\mu)$ and $B = B(\mu)$ such that

$$(D_x\phi(\hat{x},\hat{y}), A) \in \overline{J}^{2,+}u(\hat{x}), \quad (-D_y\phi(\hat{x},\hat{y}), B) \in \overline{J}^{2,-}u(\hat{y}), \quad \text{and}$$
$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leqslant D^2\phi(\hat{x},\hat{y}) + \frac{1}{\mu}(D^2\phi(\hat{x},\hat{y}))^2.$$

In particular, this implies

$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leqslant \begin{bmatrix} M, & -M \\ -M, & M \end{bmatrix} + \frac{2}{\mu}\begin{bmatrix} M^2, & -M^2 \\ -M^2, & M^2 \end{bmatrix},$$

so that for every $\xi \in \mathbb{R}^n$

$$\langle (A-B)\xi, \xi\rangle \leqslant 0.$$

In addition, we conclude that for every $\xi \in \mathbb{R}^n$

$$\langle (A-B)\xi, \xi\rangle \leqslant 2\langle \left(M + \frac{2}{\mu}M^2\right)\xi, \xi\rangle.$$

Moreover, taking $\bar{\xi} = (x-y)/|x-y|$ and choosing μ in the right way, we conclude that:

$$\langle (A-B)\bar{\xi}, \bar{\xi}\rangle \leqslant L\alpha(\alpha-1)|\hat{x}-\hat{y}|^{\alpha-2} < 0.$$

In this way taking L sufficiently large we obtain a contradiction concluding that $\theta \leqslant 0$. Indeed

$$\begin{aligned} -2||f||_{L^\infty} &\leqslant f(\hat{x}) - f(\hat{y}) \leqslant F(A+2I) - F(B) \leqslant \Lambda \mathrm{Tr}(A-B) + n\Lambda \\ &\leqslant L\alpha(\alpha-1)|\hat{x}-\hat{y}|^{\alpha-2} + n\Lambda \to -\infty, \end{aligned}$$

as $L \to +\infty$.

So that by choosing $z = \hat{x} \in B_{1/4}(0)$ we get that for every $x \in B_{1/4}(0)$:

$$u(x) - u(y) \leqslant L|x-y|^{\alpha}.$$

This proof can be, in a sense, partially adapted to our operators. Nevertheless, see for instance, even the subelliptic Laplace operator in Heinseberg group, we did not manage to prove that $\theta < 0$ following the previous proof.

Nevertheless, in a paper by Ishii, [**12**], see also [**5**, **11**, **14**], there is a proof that in some sense works for some, possibly degenerate, linear operators. We remind in the subsection below the main result, from our point of view, contained in [**12**].

2.3. A result for linear elliptic operators

In the paper [**12**] it was proven the following result.

If

$$Lu(x) = \mathrm{Tr}(H(x)D^2u(x)) + \langle b(x), Du(x)\rangle - c(x)u(x),$$

where $H^T = H \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^{2n})$, $b, c, f \in C^{0,1}(\mathbb{R}^n)$, and there exist a matrix σ and a positive number $\Lambda > 0$ such that $H \geqslant 0$, $\sigma^T\sigma = H$, and

$$H \leqslant \Lambda. \tag{2.7}$$

Moreover, denoting by

$$\lambda_0 = \sup_{x\neq y}\left\{\frac{\mathrm{Tr}(\sigma(x)-\sigma(y))^2 + \langle (b(x)-b(y)), x-y\rangle}{|x-y|^2}\right\}$$

and

$$c_0 = \inf_{\mathbb{R}^n} c.$$

Then, see [**12**], we get the following result.

THEOREM 2.6 (Ishii). *Let $c_0 \geqslant 0$ and assume that $c, f \in C^{0,1}(\mathbb{R}^n)$. Let $u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution of $Lu = f$ that is also uniformly continuous in $\mathbb{R}^n$. If $c_0 > \lambda_0$ then $u \in C^{0,1}(\mathbb{R}^n)$ and*

$$|Du|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{1}{c_0 - \lambda_0}\left(|Df|_{L^\infty(\mathbb{R}^n)} + |Dc|_{L^\infty(\mathbb{R}^n)}|u|_{L^\infty(\mathbb{R}^n)}\right).$$

REMARK 2.7. If $H(x) = I$, then $\lambda_0 \leqslant L_b$, where L_b denotes the Lipschitz constant associated with b. Moreover, if $H(x) = P(x)$, and $b = 0$, that is in the case of the Heisenberg group, then $\mathrm{Tr}(P(x)D^2u(x)) = \Delta_{\mathbb{H}^1}u$. Nevertheless, condition (2.7) is not satisfied because $P(x) \leqslant 1 + 4(x_1^2 + x_2^2)$. Anyhow the approach seems useful to get the first result in the direction we desire as we shall prove in the next § 3.

We are now in a position to give the proof of our main result.

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3. Proof of theorem 1.2

Let

$$\Phi(x,y) = u(x) - u(y) - L|x-y|^{\alpha} - \delta|x|^2 - \epsilon.$$

We claim that there exists $L_0(c, ||u||_{L^\infty}, ||f||_{L^\infty})$ such that for every $\epsilon, \delta > 0$, if $L \geqslant L_0$ then

$$\sup_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x,y) \leqslant 0.$$

Indeed, arguing by contradiction, if there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for $\delta \leqslant \delta_0$, $\epsilon \leqslant \epsilon_0$

$$\sup_{\mathbb{R}^n \times \mathbb{R}^n} \{u(x) - u(y) - L|x-y|^{\alpha} - \delta|x|^2 - \epsilon\} = \theta > 0,$$

then invoking theorem of the sums, see theorem 2.4 in this paper, and denoting $\phi = L|x-y|^{\alpha}$, we get that there exist $A = A(\mu)$ and $B = B(\mu)$ such that

$$(D_x\phi(\hat{x},\hat{y}), A + 2\delta I) \in \overline{J}^{2,+}u(\hat{x}), \quad (-D_y\phi(\hat{x},\hat{y}), B) \in \overline{J}^{2,-}u(\hat{y}),$$

and the following estimate holds:

$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leqslant D^2\phi(\hat{x},\hat{y}) + \frac{1}{\mu}(D^2\phi(\hat{x},\hat{y}))^2.$$

We remark that denoting

$$M := \alpha L|x-y|^{\alpha-2}\left((\alpha-2)\frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} + I\right),$$

then, keeping in mind also lemma 2.5,

$$M \leqslant \alpha L|x-y|^{\alpha-2} I$$

and

$$M^2 \leqslant \alpha^2 L^2 |x-y|^{2(\alpha-2)} I.$$

Thus

$$\begin{aligned} D^2\phi(\hat{x},\hat{y}) &+ \frac{1}{\mu}(D^2\phi(\hat{x},\hat{y}))^2 \\ &= \begin{bmatrix} M, & -M \\ -M, & M \end{bmatrix} + \frac{2}{\mu}\begin{bmatrix} M^2, & -M^2 \\ -M^2, & M^2 \end{bmatrix} \\ &= \begin{bmatrix} I, & -I \\ -I, & I \end{bmatrix}\begin{bmatrix} M, & 0 \\ 0, & M \end{bmatrix} \\ &\quad + \frac{2}{\mu}\begin{bmatrix} I, & -I \\ -I, & I \end{bmatrix}\begin{bmatrix} M^2, & 0 \\ 0, & M^2 \end{bmatrix} \\ &\leqslant \alpha L|x-y|^{\alpha-2}\left(1 + \frac{\alpha L}{\mu}|x-y|^{\alpha-2}\right)\begin{bmatrix} I, & -I \\ -I, & I \end{bmatrix}, \end{aligned}$$

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that is

$$\equiv L\alpha|x-y|^{\alpha-2}\eta\begin{bmatrix} I, & -I \\ -I, & I\end{bmatrix}.$$

Here $\eta > 1$ and $\eta \to 1$ possibly taking μ larger and larger.

On the other hand, we have to adapt our inequality to the degenerate part of our operator encoded in the coefficients of the matrix in the second order operator. Thus from

$$\begin{bmatrix} A, & 0 \\ 0, & -B\end{bmatrix} \leqslant L\alpha|x-y|^{\alpha-2}\eta\begin{bmatrix} I, & -I \\ -I, & I\end{bmatrix},$$

it follows that

$$\begin{aligned}&\mathrm{Tr}\left([\sigma(\hat{x}),\sigma(\hat{y})]\begin{bmatrix} A, & 0 \\ 0, & -B\end{bmatrix}\begin{bmatrix}\sigma(\hat{x})^T \\ \sigma(\hat{y})^T\end{bmatrix}\right)\\ &\quad\leqslant L\alpha|x-y|^{\alpha-2}\eta\mathrm{Tr}\left([\sigma(\hat{x}),\sigma(\hat{y})]\begin{bmatrix} I, & -I \\ -I, & I\end{bmatrix}\begin{bmatrix}\sigma(\hat{x})^T \\ \sigma(\hat{y})^T\end{bmatrix}\right).\end{aligned}$$

Performing the computation for both sides of previous inequality we get

$$\begin{aligned}&\mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T-\sigma(\hat{y})B\sigma(\hat{x})^T) = \mathrm{Tr}(\sigma(\hat{x})^T\sigma(\hat{x})A)) - \mathrm{Tr}(\sigma(\hat{y})B\sigma(\hat{x})^T)\\ &\quad\leqslant L\alpha|x-y|^{\alpha-2}\eta\mathrm{Tr}(\sigma(\hat{x})\sigma(\hat{x})^T-\sigma(\hat{x})\sigma(\hat{y})^T-\sigma(\hat{y})\sigma(\hat{x})^T+\sigma(\hat{y})\sigma(\hat{y})^T)\\ &\quad= L\alpha|x-y|^{\alpha-2}\eta(\sigma(\hat{x})-\sigma(\hat{y}))(\sigma(\hat{x})-\sigma(\hat{y}))^T\\ &\quad= L\alpha|x-y|^{\alpha-2}\eta(\sigma(\hat{x})-\sigma(\hat{y}))^2.\end{aligned}\tag{3.1}$$

We can now exploit some information contained in the fact that u is a viscosity solution of the equation. Indeed recalling that $\theta > 0$ we get

$$L|\hat{x}-\hat{y}|^\alpha + \delta c_0|x|^2 \leqslant u(\hat{x})-u(\hat{y})$$

and

$$\begin{aligned}Lc_0|\hat{x}-\hat{y}|^\alpha + \delta c_0|x|^2 &\leqslant c_0(u(\hat{x})-u(\hat{y})) \leqslant c(\hat{x})(u(\hat{x})-u(\hat{y}))\\ &= c(\hat{x})u(\hat{x})-c(\hat{y})u(\hat{y})+u(\hat{y})(c(\hat{y})-c(\hat{x})).\end{aligned}$$

By the theorem of the sums and the definition of viscosity subsolution/supersolution we get

$$\begin{aligned}Lc_0|\hat{x}-\hat{y}|^\alpha + \delta c_0|x|^2 &\leqslant c_0(u(\hat{x})-u(\hat{y})) \leqslant c(\hat{x})(u(\hat{x})-u(\hat{y}))\\ &= c(\hat{x})u(\hat{x})-c(\hat{y})u(\hat{y})+u(\hat{y})(c(\hat{y})-c(\hat{x}))\\ &\leqslant F(A+2\delta I,\hat{x})-F(B,\hat{y})\\ &\quad + f(\hat{y})-f(\hat{x})+u(\hat{y})(c(\hat{y})-c(\hat{x}))\\ &= G(\sigma(\hat{x})^T(A+2\delta I)\sigma(\hat{x}))-G(\sigma(\hat{y})^TB\sigma(\hat{y}))\\ &\quad + f(\hat{y})-f(\hat{x})+u(\hat{y})(c(\hat{y})-c(\hat{x})).\end{aligned}$$

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Now, if $\sigma(\hat{x})(A+2\delta I)\sigma(\hat{x})^T \leqslant \sigma(\hat{y})B\sigma(\hat{y})^T$ we conclude by the elliptic degenerate property that

$$Lc_0|\hat{x}-\hat{y}|^\alpha + \delta c_0|x|^2 \leqslant f(\hat{y}) - f(\hat{x}) + u(\hat{y})(c(\hat{y}) - c(\hat{x}))$$

because $G(\sigma(\hat{x})(A+2\delta I)\sigma(\hat{x})^T) - G(\sigma(\hat{y})B\sigma(\hat{y})^T) \leqslant 0$.

On the contrary, if

$$\sigma(\hat{x})(A+2\delta I)\sigma(\hat{x})T > \sigma(\hat{y})B\sigma(\hat{y})^T$$

then

$$\begin{aligned} Lc_0|\hat{x}-\hat{y}|^\alpha + \delta c_0|\hat{x}|^2 &\leqslant \Lambda \mathrm{Tr}(\sigma(\hat{x})(A+2\delta I)\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) \\ &\quad + f(\hat{y}) - f(\hat{x}) + u(\hat{y})(c(\hat{y}) - c(\hat{x})) \\ &= \Lambda \mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) + 2\Lambda\delta \mathrm{Tr}(P(\hat{x})) \\ &\quad + f(\hat{y}) - f(\hat{x}) + u(\hat{y})(c(\hat{y}) - c(\hat{x})) \end{aligned}$$

Thus

$$\begin{aligned} Lc_0|\hat{x}-\hat{y}|^\alpha &\leqslant \Lambda \mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) \\ &\quad + 2\delta\Lambda|\hat{x}|^2\left(\frac{\mathrm{Tr}(P(\hat{x}))}{|\hat{x}|^2} - \frac{c_0}{2\Lambda}\right) + f(\hat{y}) - f(\hat{x}) + u(\hat{y})\,(c(\hat{y}) - c(\hat{x})) \\ &\leqslant \Lambda \mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) \\ &\quad + 2\delta\Lambda|\hat{x}|^2\left(\frac{\mathrm{Tr}(P(\hat{x}))}{|\hat{x}|^2} - \frac{c_0}{2\Lambda}\right) + L_f|\hat{y}-\hat{x}|^\beta + L_c|u|_{L^\infty}|\hat{y}-\hat{x}|^{\beta'} \end{aligned} \tag{3.2}$$

If $|\hat{x}|$ is bounded as $\delta \to 0$, then

$$2\delta\Lambda|\hat{x}|^2\left(\frac{\mathrm{Tr}(P(\hat{x}))}{|\hat{x}|^2} - \frac{c_0}{2\Lambda}\right) \to 0.$$

If $|\hat{x}|$ were unbounded as $\delta \to 0$, then $2\delta\Lambda|\hat{x}|^2(\mathrm{Tr}(P(\hat{x}))/|\hat{x}|^2 - c_0/2\Lambda) \leqslant 0$ whenever

$$\limsup_{|x|\to\infty} \frac{\mathrm{Tr}(P(x))}{|x|^2} < \frac{c_0}{2\Lambda}.$$

It remains to evaluate $\mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T)$. Indeed by recalling inequality (3.1) we get

$$\begin{aligned} &\mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) \\ &\quad \leqslant \eta\alpha L|\hat{x}-\hat{y}|^{\alpha-2}\mathrm{Tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 \leqslant \bar{C}\eta\alpha L|\hat{x}-\hat{y}|^{\alpha-2}|\hat{x}-\hat{y}|^2 \\ &\quad \leqslant C\alpha L|\hat{x}-\hat{y}|^\alpha \end{aligned} \tag{3.3}$$

thanks to our hypothesis on σ, where $\bar{C}$ and C are bounded and independent to $\hat{x}$ and $\hat{y}$.

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Summarizing, we have got that

$$Lc_0|\hat{x}-\hat{y}|^\alpha \leqslant C\alpha\Lambda L|\hat{x}-\hat{y}|^\alpha + L_f|\hat{y}-\hat{x}|^\beta + L_c|u|_{L^\infty}|\hat{y}-\hat{x}|^{\beta'},$$

that is

$$c_0 \leqslant C\alpha\Lambda + \frac{L_f}{L}|\hat{y}-\hat{x}|^{\beta-\alpha} + \frac{L_c}{L}|u|_{L^\infty}|\hat{y}-\hat{x}|^{\beta'-\alpha}.$$

So that by taking L sufficiently large and α sufficiently small ($\alpha < c_0/C\Lambda$), we get a contradiction. Indeed, since

$$\begin{aligned} L &\leqslant \frac{1}{c_0 - C\Lambda\alpha}\left(L_f|\hat{y}-\hat{x}|^{\beta-\alpha} + L_c|u|_{L^\infty}|\hat{y}-\hat{x}|^{\beta'-\alpha}\right) \\ &\leqslant \frac{1}{c_0 - C\Lambda\alpha}\left(L_f\left(\frac{|u|_{L^\infty}}{L}\right)^{\beta-\alpha} + L_c|u|_{L^\infty}\left(\frac{|u|_{L^\infty}}{L}\right)^{\beta'-\alpha}\right) \end{aligned}$$

so that keeping in mind that $|\hat{x}-\hat{y}| \leqslant |u|_{L^\infty}/L$, and for instance, if $\beta \leqslant \beta'$, then

$$L^{1+\beta'-\alpha} \leqslant \frac{1}{c_0 - C\Lambda\alpha}\left(L^{\beta-\beta'}[f]_{C^\beta}|u|_{L^\infty}^{\beta-\alpha} + [c]_{C^{\beta'}}|u|_{L^\infty}|u|_{L^\infty}^{\beta'-\alpha}\right)$$

getting a contradiction fixing

$$L > \left\{\frac{1}{c_0 - C\Lambda\alpha}\left([f]_{C^\beta}|u|_{L^\infty}^{\beta-\alpha} + [c]_{C^{\beta'}}|u|_{L^\infty}^{1+\beta'-\alpha}\right)\right\}^{1/(1+\beta'-\alpha)}.$$

Thus

$$u(x) - u(y) \leqslant L|x-y|^\alpha + \delta|x|^2 + \epsilon$$

and letting δ and ϵ go to 0 we conclude that

$$u(x) - u(y) \leqslant L|x-y|^\alpha.$$

4. Conclusions and remarks

4.1. Square root matrices and rectangular matrices

In case P was a sufficiently smooth square matrix, so that $\sigma = \sqrt{P}$, we have the required regularity of σ simply by invoking the result contained in [**15**] or [**12**] that reduces to [**22**]. In that case, we deduce that $\sqrt{P}$ is Lipschitz continuous whenever P is $C^{1,1}$. See also [**20**] for a different type of remark about the properties of the square root matrices.

In case P was obtained as the product of two rectangular matrices, the proof of the Lipschitz continuity it follows straightforwardly from the regularity of the coefficients of σ themselves. In that case, we have to assume that σ has to be at least Lipschitz continuous. In fact, in the case of the Heisenberg group, we start from analytic coefficients! See for instance, the Heisenberg case discussed in the introduction.

4.2. A little gain

Recalling the notation used in the proof of theorem 1.2, if we know that $\delta(\Lambda \mathrm{Tr}(P(\hat{x}) - c_0|\hat{x}|) \to 0$ as $\delta \to 0$, then we could improve the result simply requiring that

$$L > \frac{[f]_{C^\beta} + [c]_{C^{\beta'}}}{c_0 - C\Lambda\alpha}.$$

In the case of the Heisenberg group $\mathbb{H}^1$, for instance, concerning the sublaplacian, we have that the result is true if

$$4 \leqslant \frac{c_0}{2\Lambda},$$

because $\mathrm{Tr}(P(\hat{x})) = 2 + 4(x_1^2 + x_2^2)$.

4.3. The Carnot groups case

More in general, in Carnot groups, it results, in the nontrivial case, that $\sigma(x) = \sigma(x')$ where x' denotes the variables that do not contain the ones that are identified with the last stratum of the Lie algebra of the group, see for instance, remark 1.4.4, remark 1.4.5, remark 1.4.6 in [**2**]. Thus:

$$|\sigma(x) - \sigma(y)| \leqslant C|x' - y'|.$$

As a consequence, recalling the inequality (3.2) in the proof of theorem 1.2, or the quantity (1.1) entering in the statement of the theorem 1.2, we remark that:

$$\begin{aligned}
&2\delta\Lambda|\hat{x}|^2\left(\frac{\mathrm{Tr}(P(\hat{x}))}{|\hat{x}|^2} - \frac{c_0}{2\Lambda}\right) \\
&\quad = 2\delta\Lambda\left(\mathrm{Tr}(P(\hat{x})) - \frac{c_0}{2\Lambda}|\hat{x}|^2\right) \\
&\quad = 2\delta\Lambda\left(\mathrm{Tr}(P(\hat{x}')) - \frac{c_0}{2\Lambda}|\hat{x}|^2\right) = 2\delta\Lambda\left(\mathrm{Tr}(\sigma(\hat{x}')\sigma(\hat{x}')^T) - \frac{c_0}{2\Lambda}|\hat{x}|^2\right) \\
&\quad \leqslant 2\delta\Lambda\left((c + \phi(|\hat{x}'|)) - \frac{c_0}{2\Lambda}|\hat{x}|^2\right) \\
&\quad = 2\delta\Lambda c + 2\Lambda\delta|\hat{x}|^{2-\epsilon}\left(\frac{\phi(|\hat{x}'|))}{|\hat{x}|^{2-\epsilon}} - \frac{c_0}{2\Lambda}|\hat{x}|^\epsilon\right),
\end{aligned}$$

for a suitable positive number ϵ.

Here ϕ is a polynomial function depending only on $|x'|$ whose degree depends on the step of the group. In general, if the step of the group is p, then the degree is less or equal $2(p-1)$.

In this case, if ϕ does not grow up too much the result is true without restriction on the size of $c_0/2\Lambda$. For example, if $\phi(r) \sim r^{1+\nu}$ as $r \to \infty$ for some $\nu \in [0, 1)$.

4.4. Simple examples

It is easy to construct some examples. In very low dimension, $n = 2$, we are considering:

$$\sigma = \begin{bmatrix} 1, & 0 \end{bmatrix}.$$

Then

$$\sigma^T \sigma = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix},$$

so that for every operator like

$$F(D^2u(x), x) := G\left(\frac{\partial^2 u(x,y)}{\partial x^2}\right),$$

where $G : \mathbb{R} \to \mathbb{R}$ is monotone increasing and vanishing at 0, c, f Lipschitz continuous $\inf_{\mathbb{R}^2} c = c_0 > 0$, we deduce from theorem 1.2 that bounded uniformly continuous functions satisfying

$$F(D^2u(x,y), x, y) - cu = f, \quad \mathbb{R}^2$$

in a viscosity sense are Lipschitz continuous in $\mathbb{R}^2$.

Let

$$\sigma = \begin{bmatrix} \frac{x}{1+x^2}, & 0 \end{bmatrix}.$$

Then

$$\sigma^T \sigma = \begin{bmatrix} \frac{x^2}{(1+x^2)^2}, & 0 \\ 0, & 0 \end{bmatrix}.$$

so that for every operator like

$$F(D^2u(x), x) := G\left(\frac{x^2}{(1+x^2)^2}\frac{\partial^2 u(x,y)}{\partial x^2}\right),$$

where $G : \mathbb{R} \to \mathbb{R}$ is uniformly elliptic, c, f are Lipschitz continuous, $\inf_{\mathbb{R}^2} c = c_0 > 0$,we deduce from theorem 1.2 that bounded uniformly continuous functions satisfying

$$F(D^2u(x,y), x, y) - cu = f, \quad \mathbb{R}^2$$

in a viscosity sense are Lipschitz continuous in $\mathbb{R}^2$. Other examples can be easily constructed for degenerate structures without group structure. The first embrional approach in this direction can be found in [**21**] for a Grushin operator.

4.5. Limits to this approach

We are not able to improve our result assuming lower regularity on the coefficients. Indeed if in (3.3) we assume that $|\sigma(x) - \sigma(y)| \leqslant C|x - y|^\gamma$, $\gamma \in (0, 1]$ then we conclude that

$$\begin{aligned} &\mathrm{Tr}(\sigma(\hat{x})A\sigma(\hat{x})^T - \sigma(\hat{y})B\sigma(\hat{y})^T) \\ &\quad\leqslant \eta\alpha L|\hat{x} - \hat{y}|^{\alpha-2}\mathrm{Tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 \leqslant \bar{C}\eta\alpha L|\hat{x} - \hat{y}|^{\alpha-2}|\hat{x} - \hat{y}|^{2\gamma} \\ &\quad\leqslant C\alpha L|\hat{x} - \hat{y}|^{\alpha-2+2\gamma} \end{aligned} \tag{4.1}$$

but in order to get a contradiction we need to ask also that $\alpha - 2 + 2\gamma \geqslant \alpha$ and this happens only if $\gamma \geqslant 1$.

4.6. Conclusions

It is possible to deduce the Hölder regularity of viscosity solutions without knowing the Harnack inequality, under the hypotheses of theorem 1.2, even for degenerate nonlinear operators. Concerning the remark discussed in § 4.2, we cannot deduce that for linear operators like the sublaplacian in the Heisenberg group the result [**12**] applies, see also [**7**, **8**], since $P(\hat{x})$ might behave like $|\hat{x}|^2$ and $\delta|\hat{x}|^2$ is only bounded by $2|u|_{L^\infty}$. As a consequence, our result seems new, even in the linear case. It is worth to say, even it is well known in literature, that considering operators in divergence form, by recalling Hörmander approach, see [**9**], it is possible to prove, as it is well known, much more significative regularity results.

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