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NASH EQUILIBRIUM IN NONZERO-SUM GAMES OF OPTIMAL STOPPING FOR BROWNIAN MOTION

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Abstract

We present solutions to nonzero-sum games of optimal stopping for Brownian motion in [0, 1] absorbed at either 0 or 1. The approach used is based on the double partial superharmonic characterisation of the value functions derived in Attard (2015). In this setting the characterisation of the value functions has a transparent geometrical interpretation of 'pulling two ropes' above 'two obstacles' which must, however, be constrained to pass through certain regions. This is an extension of the analogous result derived by Peskir (2009), (2012) (semiharmonic characterisation) for the value function in zero-sum games of optimal stopping. To derive the value functions we transform the game into a free-boundary problem. The latter is then solved by making use of the double smooth fit principle which was also observed in Attard (2015). Martingale arguments based on the Itô–Tanaka formula will then be used to verify that the solution to the free-boundary problem coincides with the value functions of the game and this will establish the Nash equilibrium.

Keywords: Nonzero-sum optimal stopping game; Nash equilibrium; Brownian motion; double partial superharmonic characterisation; double smooth fit principle; Itô–Tanaka formula; optimal stopping; regular diffusion

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1. Introduction

The purpose of this work is to derive Nash equilibrium in two-player nonzero-sum games of optimal stopping for Brownian motion in [0, 1], absorbed at either 0 and 1. For this we shall use the results obtained in [1], in particular, the double partial superharmonic characterisation of the value functions of the two players and the double smooth fit principle.

This probabilistic approach for studying the value functions and the corresponding Nash equilibrium is in line with the results derived by Peskir in [14] and [15] for zero-sum games. In the case of absorbed Brownian motion in [0,1], the results of Peskir show that the value function in zero-sum games is equivalent to 'pulling a rope' between 'two obstacles' (semiharmonic characterisation) which, in turn, establishes the Nash equilibrium (by 'pulling a rope' between 'two obstacles', we mean finding the shortest path between the graphs of two functions). In nonzero-sum games, under certain assumptions on the payoff functions, we will show that the value functions are equivalent to 'pulling two ropes' above 'two obstacles' which must, however, be constrained to pass through certain regions. As in the case of zero-sum games this geometric explanation of the value function will establish the Nash equilibrium.

Literature on nonzero-sum games of optimal stopping are mainly concerned with existence of the Nash equilibrium. Initial studies in discrete time date back to Morimoto [10] wherein

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a fixed point theorem for monotone mappings was used to derive sufficient conditions for the existence of a Nash equilibrium point. Ohtsubo [12] derived equilibrium values via backward induction and in [13] the same author considered nonzero-sum games with the smaller gain processes having a monotone structure and gave sufficient conditions for a Nash equilibrium point to exist. Shmaya and Solan [18] proved that every two player nonzero-sum game in discrete time admits an ε-equilibrium in randomised stopping times. In continuous time, Bensoussan and Friedman [2] showed that, for diffusions, Nash equilibrium exists if there exists a solution to a system of quasivariational inequalities. However, the regularity and uniqueness of the solution remain open problems. Nagai [11] studied a nonzero-sum stopping game of symmetric Markov processes. A system of quasivariational inequalities was introduced in terms of Dirichlet forms and the existence of extremal solutions of a system of quasivariational inequalities was discussed. The Nash equilibrium is then established from these extremal solutions. Cattiaux and Lepeltier [3] studied right processes and they proved the existence of a quasi-Markov Nash equilibrium point. The authors follow Nagai's idea but use probabilistic tools rather than the theory of Dirichlet forms. Moreover, they completed Nagai's result (whose construction of the extremal solutions of the quasivariational inequalities is not complete) and extend it to nonsymmetric processes. Huang and Li in [8] proved the existence of a Nash equilibrium point for a class of nonzero-sum noncyclic stopping games using the martingale approach. Laraki and Solan [9] proved that every two-player nonzero-sum Dynkin game in continuous time admits an ε-equilibrium in randomised stopping times whereas Hamadène and Zhang in [7] proved the existence of Nash equilibrium points using the martingale approach, for processes with positive jumps.

The structure of this paper is as follows. In Section 2 we introduce the game and review the double partial superharmonic characterisation (DPSC) of the value functions (see [1]) when the underlying process is assumed to be absorbed Brownian motion in [0,1]. In Section 3 we formulate and solve an equivalent free-boundary problem for a certain class of payoff functions. Under additional assumptions on the payoff functions we then show that the solution is also unique. In Section 4 we use martingale arguments based on Itô–Tanaka formula to verify that the solution to the free-boundary problem coincides with the value functions of the game. Finally, in Section 5, we explain how these results can be extended to one-dimensional absorbed regular diffusions.

2. The DPSC of the value functions

Let X be Brownian motion in $[0, 1]$, started at $x \in [0, 1]$ and absorbed at either 0 or 1 and let $G_i, H_i \colon [0, 1] \to \mathbb{R}$ for $i = 1, 2$ be C^2 functions such that $G_i \le H_i$. Assume also that $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$. Consider the nonzero-sum game of optimal stopping in which player one wants to choose a stopping time τ_* and player two a stopping time σ_* such that their total average gains, which are respectively given by

$$M_x^1(\tau, \sigma) = \mathbb{E}_x[G_1(X_\tau)\mathbf{1}(\tau \le \sigma) + H_1(X_\sigma)\mathbf{1}(\sigma < \tau)],$$
$$M_x^2(\tau, \sigma) = \mathbb{E}_x[G_2(X_\sigma)\mathbf{1}(\sigma < \tau) + H_2(X_\tau)\mathbf{1}(\tau \le \sigma)]$$

are maximized. For a given strategy σ chosen by player two, we shall define the value function of player one by

$$V_\sigma^1(x) = \sup_\tau M_x^1(\tau, \sigma). \tag{1}$$

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Similarly, for a given strategy τ chosen by player one, we shall define the value function of player two by

$$V_\tau^2(x) = \sup_\sigma M_x^2(\tau, \sigma). \tag{2}$$

In this context, a saddle point of stopping times is characterized via Nash equilibrium. Formally, a pair of stopping times (τ_*, σ_*) is a saddle point if $M_x^1(\tau, \sigma_*) \leq M_x^1(\tau_*, \sigma_*)$ and $M_x^2(\tau_*, \sigma) \leq M_x^2(\tau_*, \sigma_*)$ for all stopping times τ and σ and for all $x \in [0, 1]$.

Under the mentioned assumptions on G_i and H_i, for $i = 1, 2$, the result on the DPSC of the value functions of player one and player two with the underlying process X introduced above becomes applicable (see [1]). It is well known that superharmonic/subharmonic functions of X are equivalent to concave/convex functions and that continuity in the fine topology is equivalent to continuity in the familiar Euclidean topology. Thus, in this setting, the DPSC of the value functions can be explained rigorously as finding two continuous functions u and v such that

$$u = \inf_{F \in \sup_v^1(G_1, K_1)} F \quad \text{and} \quad v = \inf_{F \in \sup_u^2(G_2, K_2)} F,$$

where

$$\begin{aligned}\sup_v^1(G_1, K_1) = \{F \colon [0, 1] \to [G_1, K_1] \colon & \ F \text{ is continuous}, F = H_1 \text{ in } D_2, \\ & \ F \text{ is concave in } D_2^c\}, \\ \sup_u^2(G_2, K_2) = \{F \colon [0, 1] \to [G_2, K_2] \colon & \ F \text{ is continuous}, F = H_2 \text{ in } D_1, \\ & \ F \text{ is concave in } D_1^c\}\end{aligned}$$

with $D_1 = \{u = G_1\}$, $D_2 = \{v = G_2\}$, and K_i, for $i = 1, 2$, is the smallest concave function majorizing H_i. Indeed, if the boundaries ∂D_1 and ∂D_2 of D_1 and D_2 are regular for their respective sets then the functions u and v solve the optimal stopping game, that is,

$$u(x) = V_{\sigma_*}^1(x) = \sup_\tau M_x^1(\tau, \sigma_*) \quad \text{and} \quad v(x) = V_{\tau_*}^2(x) = \sup_\sigma M_x^2(\tau_*, \sigma),$$

where $\tau_* = \inf\{t \geq 0 \colon X_t \in D_1\}$ and $\sigma_* = \inf\{t \geq 0 \colon X_t \in D_2\}$.

We initiate this study by showing that if $D_1 = [m, n] \cup \{0, 1\}$ and $D_2 = [r, l] \cup \{0, 1\}$, where $0 \leq m \leq n \leq 1$ and $0 \leq r \leq l \leq 1$, then the functions u and v are contained in the sets $\sup_v^1(G_1, K_1)$ and $\sup_u^2(G_2, K_2)$, respectively. We will prove this claim for u as the result for v follows by symmetry. Clearly, we have $u(x) = H_1(x)$ for all $x \in D_2$ and that u is bounded above by K_1. By the definition of the infimum, we also have $u(x) \geq G_1(x)$ for all $x \in [0, 1]$. Since the infimum of concave functions is concave, it follows that u is concave in D_2^c and so u is continuous in $\operatorname{int}(D_2^c)$, the interior of D_2^c (recall that concave functions defined on open sets are continuous). Continuity of u in D_2 follows from the continuity of H_1. So it remains to show that u is continuous at the boundary of D_2. To prove this we shall follow the line of thought of Ekström and Villeneuve in [5] and prove that u is lower semicontinuous at l (note that upper semicontinuity of u holds from the fact that u is the infimum of continuous functions). For this we shall assume, without loss of generality, that $r < l < 1$. So suppose for contradiction that u is not right-lower-semicontinuous at l (note that u is left-continuous at l by continuity of H_1). This means that there exists $\hat{\varepsilon} > 0$ such that $\lim_{x \downarrow l} u(x) < u(l) - \hat{\varepsilon}$. For given $\delta > 0$, let L be the line segment joining the points $(l, u(l) - \hat{\varepsilon})$ and $(l + \delta, u(l + \delta))$. By the continuity of L, it follows that there exists $y \in (l, l + \delta)$ such that $L(y) > u(y)$. By the definition of u, this means that there exists $F \in \sup_v^1(G_1, K_1)$ such that $F(y) < L(y)$. Since F is continuous

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in [0, 1] and concave in $(l, l+\delta)$, we have

$$\begin{aligned} F(l)\Big(\frac{l+\delta-y}{\delta}\Big) + L(l+\delta)\Big(\frac{y-l}{\delta}\Big) &= F(l)\Big(\frac{l+\delta-y}{\delta}\Big) + u(l+\delta)\Big(\frac{y-l}{\delta}\Big) \\ &\le F(l)\Big(\frac{l+\delta-y}{\delta}\Big) + F(l+\delta)\Big(\frac{y-l}{\delta}\Big) \\ &\le F(y) \\ &< L(y) \\ &= (u(l)-\hat{\varepsilon})\Big(\frac{l+\delta-y}{\delta}\Big) + L(l+\delta)\Big(\frac{y-l}{\delta}\Big). \end{aligned}$$

This implies that $F(l) < u(l) - \hat{\varepsilon}$, which contradicts the fact that $F \ge u$. Thus, u is right-lower-semicontinuous at l. Continuity of u at 0 can be proved as above by replacing l with 0. To show that u is continuous at r and 1, we can follow the steps above and prove, by contradiction, that u is left-lower-semicontinuous at these points. Note that (when $r < l$) u is right-continuous at r by the continuity of H_1.

3. Free-boundary problem

In this section we shall formulate a free-boundary problem by making use of the DPSC of the value functions. For this we will assume that there exist thresholds a, b with $0 \le a < b \le 1$, such that

$$G_1''(x) < 0 \quad \text{for } x \in [0, a), \qquad G_1''(x) = 0 \quad \text{for } x = a, \tag{3}$$
$$G_1''(x) > 0 \quad \text{for } x \in (a, 1], \qquad G_2''(x) > 0 \quad \text{for } x \in [0, b), \tag{4}$$
$$G_2''(x) = 0 \quad \text{for } x = b, \qquad G_2''(x) < 0 \quad \text{for } x \in (b, 1]. \tag{5}$$

In this setting, the DPSC of the value functions can be explained geometrically as follows. Suppose that two ropes are pulled above two obstacles G_1 and G_2 with their endpoints pulled to the ground. Let D_1' be the region where the first rope touches the first obstacle and let D_2' be the region where the second rope touches the second obstacle. Then on D_2' the first rope is constrained to pass through a certain region (this region corresponds to the value of H_1 on D_2') and so creates a (new) contact region with its obstacle G_1, say D_1''. Similarly, on D_1' the second rope is also constrained to pass through a certain region (as specified by the value of H_2 on D_1') and, thus, creates a (new) contact region with its obstacle G_2, say D_2''. All points of contact are then altered until both ropes touch their respective obstacles smoothly (it may also happen that the new regions coincide with the boundary points 0 and 1 in which case smoothness might break down). However, this must be done in such a way that the point of contact of the first rope with its obstacle G_1 must coincide with the point of contact of the second rope with H_2 and vice versa. With this intuitive explanation, we will search for a saddle point (τ_*, σ_*) of optimal stopping times of the form

$$\tau_* = \inf\{t \ge 0 \colon X_t \le A_*\} \wedge \rho_{0,1}, \qquad \sigma_* = \inf\{t \ge 0 \colon X_t \ge B_*\} \wedge \rho_{0,1},$$

where $0 \le A_* < B_* \le 1$ are optimal stopping boundaries that need to be determined and $\rho_{0,1} = \inf\{t \ge 0 \colon X_t \in \{0, 1\}\}$. Prior to formulating the free-boundary problem, we note that if there exists such optimal stopping boundaries then we must have $A_* \le a$ and $B_* \ge b$. This is a consequence of the DPSC of the value functions (which requires the value function of player one to be concave in $(0, B_*)$ and that of player two to be concave in $(A_*, 1)$).

We are now in a position to formulate the free-boundary problem for unknown points $0 \le A_* \le a < b \le B_* \le 1$ and unknown functions $u, v\colon [0, 1] \to \mathbb{R}$, that is

$$u''(x) = 0 \quad \text{and} \quad v''(x) = 0 \quad \text{for } x \in (A_*, B_*), \tag{6}$$
$$u(A_*) = G_1(A_*) \quad \text{and} \quad v(B_*) = G_2(B_*), \tag{7}$$
$$u(B_*) = H_1(B_*) \quad \text{and} \quad v(A_*) = H_2(A_*), \tag{8}$$
$$u(x) = G_1(x) \quad \text{for } x \in [0, A_*) \quad \text{and} \quad v(x) = G_2(x) \quad \text{for } x \in (B_*, 1], \tag{9}$$
$$u(x) > G_1(x) \quad \text{and} \quad v(x) > G_2(x) \quad \text{for } x \in (A_*, B_*), \tag{10}$$
$$u(x) = H_1(x) \quad \text{for } x \in (B_*, 1], \tag{11}$$
$$v(x) = H_2(x) \quad \text{for } x \in [0, A_*). \tag{12}$$

By means of straightforward calculations one can show that the solution of system (6)–(8) takes the form

$$u(x) = \frac{H_1(B_*) - G_1(A_*)}{B_* - A_*}x + G_1(A_*) - \frac{(H_1(B_*) - G_1(A_*))A_*}{B_* - A_*}, \tag{13}$$
$$v(x) = \frac{H_2(A_*) - G_2(B_*)}{A_* - B_*}x + G_2(B_*) - \frac{(H_2(A_*) - G_2(B_*))B_*}{A_* - B_*} \tag{14}$$

for all $x \in (A_*, B_*)$. In certain cases (which shall be specified below) the double smooth fit principle (see [1]) will also be satisfied, that is

$$u'(A_*) = G_1'(A_*) \quad \text{and} \quad v'(B_*) = G_2'(B_*). \tag{15}$$

If (13)–(15) hold, it follows that the optimal boundary points A_* and B_* must solve the system of nonlinear equations

$$G_1'(A_*)(B_* - A_*) - H_1(B_*) + G_1(A_*) = 0, \tag{16}$$
$$G_2'(B_*)(A_* - B_*) - H_2(A_*) + G_2(B_*) = 0. \tag{17}$$

For given $A, B \in [0, 1]$, let us denote the left-hand side expressions in (16) and (17) by $\Theta(A, B)$ and $\Gamma(A, B)$, respectively. Note that since G_i and H_i for $i = 1, 2$ are C^2 functions, Θ and Γ are C^1 functions on $[0, 1] \times [0, 1]$. Now since the process will be stopped when it reaches the absorption points 0 and 1, we will see that there can also be cases in which the double smooth fit principle breaks down because A_* and/or B_* coincide with 0 and 1, respectively. More precisely, we can obtain the following cases.

Case A. The double smooth fit principle breaks down at $A_* = 0$ and we will only have smoothness at B_*, that is $u'(A_*) > G_1'(A_*)$ and $v'(B_*) = G_2'(B_*)$. From (13) and (14), this implies that

$$\Theta(A_*, B_*) < 0 \quad \text{and} \quad \Gamma(A_*, B_*) = 0. \tag{18}$$

Case B. The double smooth fit principle breaks down at $B_* = 1$ and we will only have smoothness at A_*, that is $u'(A_*) = G_1'(A_*)$ and $v'(B_*) < G_2'(B_*)$. From (13) and (14), this implies that

$$\Theta(A_*, B_*) = 0 \quad \text{and} \quad \Gamma(A_*, B_*) < 0. \tag{19}$$

Case C. The smooth fit breaks down at $A_* = 0$ and $B_* = 1$ and $u'(A_*) > G_1'(A_*)$ and $v'(B_*) < G_2'(B_*)$. From (13) and (14), this implies that

$$\Theta(A_*, B_*) < 0 \quad \text{and} \quad \Gamma(A_*, B_*) < 0. \tag{20}$$

The link between the value functions of the game and the functions u and v in the free-boundary problem (6)–(12), where A_* and B_* are determined either from (16) and (17) or from one of the conditions in (18)–(20) will be provided in the verification results in Section 4, together with the corresponding optimal pair (τ_*, σ_*).

We are now in a position to show the existence of the free-boundaries $0 \le A_* < a < b < B_* \le 1$. For this we shall need the following elementary result from convex analysis (see, for example, [17]) and another preliminary result.

Proposition 1. *Let $I = [c, d]$ for some points $-\infty < c < d < \infty$ and suppose that $f\colon [c, d] \to \mathbb{R}$ is a differentiable strictly convex (respectively, strictly concave) function. Then*

$$f(x)(> [<]) f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \quad \text{for } c \le \bar{x} < x \le d.$$

Lemma 1. (i) *Let $B \in [b, 1]$ be given and fixed. Then $\Theta(A, B) < 0$ for all $A \in [a, 1]$ such that $A \ne B$. Similarly, if $A \in [0, a]$ is given and fixed then $\Gamma(A, B) < 0$ for all $B \in [0, b]$ such that $B \ne A$.*

(ii) *Let $B \in [b, 1]$ be given and fixed. If there exists $A_*^{\Theta,B} \in [0, a)$ such that $\Theta(A_*^{\Theta,B}, B) = 0$ then $A_*^{\Theta,B}$ is unique. Similarly, let $A \in [0, a]$ be given and fixed. If there exists $B_*^{\Gamma,A} \in (b, 1]$ such that $\Gamma(A, B_*^{\Gamma,A}) = 0$ then $B_*^{\Gamma,A}$ is unique.*

(iii) *Suppose that, for each $B \in [b_1, b_2]$, where $b \le b_1 < b_2 \le 1$, there exists a unique $A_*^{\Theta,B} \in [0, a)$ such that $\Theta(A_*^{\Theta,B}, B) = 0$. Then there exists a unique continuously differentiable function $\phi\colon [b_1, b_2] \to [0, a)$ such that $\Theta(\phi(B), B) = 0$ for all $B \in [b_1, b_2]$. Similarly, suppose that, for each $A \in [a_1, a_2]$, where $0 \le a_1 < a_2 \le a$, there exists a unique $B_*^{\Gamma,A} \in (b, 1]$ such that $\Gamma(B_*^{\Gamma,A}, A) = 0$. Then there exists a unique continuously differentiable function $\psi\colon [a_1, a_2] \to (b, 1]$ such that $\Gamma(A, \psi(A)) = 0$ for all $A \in [a_1, a_2]$.*

Proof. (i) Since G_1 is strictly convex in $[a, 1]$, we have

$$\Theta(A, B) \le G_1'(A)(B - A) - G_1(B) + G_1(A) < 0.$$

The first inequality follows from the fact that $G_1 \le H_1$, whereas the second inequality follows from Proposition 1. The result for Γ follows by symmetry.

(ii) This follows from the fact that for each $B \in [b, 1]$, $\Theta_A(A, B) = G_1''(A)(B - A) < 0$ for all $A \in [0, a)$ (recall that $G_1'' < 0$ in $[0, a)$) and so the mapping $A \mapsto \Theta(A, B)$ is strictly decreasing in $[0, a)$. The result for Γ follows by symmetry.

(iii) The existence and uniqueness of ϕ and ψ follows from the implicit function theorem upon noting that for a given $B \in [b_1, b_2]$, $\Theta_A(A, B) \ne 0$ for all $A \in [0, a)$ and similarly for a given $A \in [a_1, a_2]$, $\Gamma_B(A, B) \ne 0$ for all $B \in (b, 1]$. □

To determine A_* and B_* we shall first consider the case when there exists a unique (see Lemma 1(ii)) $A_*^{\Theta,1} \in [0, a)$ such that $\Theta(A_*^{\Theta,1}, 1) = 0$ and a $B_*^{\Theta,0} \in [b, 1]$ such that $\Theta(0, B_*^{\Theta,0}) = 0$. The proof will be divided in two steps (see 1° and 2° below). To this end let us introduce the following notation.

(I) If there exists at least one $A_*^{\Gamma,1} \in [0, a]$ such that $\Gamma(A_*^{\Gamma,1}, 1) = 0$, we will set

$$a_{\min}^{\Gamma,1} = \min\{A_*^{\Gamma,1} : \Gamma(A_*^{\Gamma,1}, 1) = 0\}. \tag{21}$$

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Moreover, we will assign

$$\tilde{a}^{\Gamma,1} = \max\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \leq A_*^{\Theta,1}\} \quad \text{and} \quad \hat{a}^{\Gamma,1} = \min\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \geq A_*^{\Theta,1}\}$$

whenever the sets $\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \leq A_*^{\Theta,1}\}$ and $\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \geq A_*^{\Theta,1}\}$ are nonempty. If, on the other hand, $\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \leq A_*^{\Theta,1}\} = \varnothing$ we will assign $\tilde{a}^{\Gamma,1} = 0$, whereas if $\{A_*^{\Gamma,1} : A_*^{\Gamma,1} \geq A_*^{\Theta,1}\} = \varnothing$ we shall set $\hat{a}^{\Gamma,1} = a$.

(II) We shall assign

$$b_{\max}^{\Theta,0} = \max\{B_*^{\Theta,0} : \Theta(0, B_*^{\Theta,0}) = 0\}. \tag{22}$$

If, in addition, there exists a unique (see Lemma 1(ii)) $B_*^{\Gamma,0} \in (b, 1]$ such that $\Gamma(0, B_*^{\Gamma,0}) = 0$, we set

$$\tilde{b}^{\Theta,0} = \max\{B_*^{\Theta,0} : B_*^{\Theta,0} \leq B_*^{\Gamma,0}\} \quad \text{and} \quad \hat{b}^{\Theta,0} = \min\{B_*^{\Theta,0} : B_*^{\Theta,0} \geq B_*^{\Gamma,0}\},$$

whenever the sets $\{B_*^{\Theta,0} : B_*^{\Theta,0} \leq B_*^{\Gamma,0}\}$ and $\{B_*^{\Theta,0} : B_*^{\Theta,0} \geq B_*^{\Gamma,0}\}$ are nonempty. In the case when $\{B_*^{\Theta,0} : B_*^{\Theta,0} \leq B_*^{\Gamma,0}\} = \varnothing$ we assign $\tilde{b}^{\Theta,0} = b$, whereas if $\{B_*^{\Theta,0} : B_*^{\Theta,0} \geq B_*^{\Gamma,0}\} = \varnothing$ we set $\hat{b}^{\Theta,0} = 1$.

Step 1°. Assume that there exists a unique $B_*^{\Gamma,0} \in (b, 1]$ such that $\Gamma(0, B_*^{\Gamma,0}) = 0$. If $\Theta(0, B_*^{\Gamma,0}) = 0$ then $A_* = 0$ and $B_* = B_*^{\Gamma,0}$ solve (16) and (17). If, on the other hand, $\Theta(0, B_*^{\Gamma,0}) < 0$ then $A_* = 0$ and $B_* = B_*^{\Gamma,0}$ satisfy (18). Finally, suppose that $\Theta(0, B_*^{\Gamma,0}) > 0$ and let $\tilde{b}^{\Theta,0}$ and $\hat{b}^{\Theta,0}$ be defined as in (II) above. Then, by the definition of $\tilde{b}^{\Theta,0}$ and $\hat{b}^{\Theta,0}$, we either have $B_*^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$ and $\Theta(0, B) > 0$ for all $B \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$ or $B_*^{\Gamma,0} = \hat{b}^{\Theta,0} = 1$ and $\Theta(0, B_*^{\Gamma,0}) > 0$ (note that $B_*^{\Gamma,0} \neq b$ since $\Gamma(0, b) < 0$). Suppose first that $B_*^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$. From Lemma 1(iii), we see that there exists a unique continuously differentiable function $\phi : [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}] \to [0, a)$ such that $\Theta(\phi(B), B) = 0$ for all $B \in [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}]$. Now recall, from the proof of Lemma 1(ii), that the mapping $B \mapsto \Gamma(0, B)$ is strictly increasing in $(b, 1]$. From this together with the fact that $\Gamma(0, B) < 0$ for all $B \in (0, b]$ (see Lemma 1(i)), we see that $\Gamma(0, B) > 0$ for all $B \in (B_*^{\Gamma,0}, 1]$.

(i) Suppose that $\Gamma(A, 1) > 0$ for all $A \in [0, a]$. Again from Lemma 1, we have the existence of a unique continuously differentiable function $\psi : [0, a] \to (b, 1]$ such that $\Gamma(A, \psi(A)) = 0$ for all $A \in [0, a]$. Since $B_*^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$, it follows that the sets $\{(\phi(B), B) : B \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\}$ and $\{(A, \psi(A)) : A \in (0, a)\}$ must intersect so there exists $(A_*, B_*) \in (0, a) \times (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$ solving (16) and (17).

(ii) Suppose that there exists at least one $A_*^{\Gamma,1} \in [0, a]$ such that $\Gamma(A_*^{\Gamma,1}, 1) = 0$. Let $a_{\min}^{\Gamma,1}$ be defined as in (21). Since $\Gamma(0, 1) > 0$ we have $a_{\min}^{\Gamma,1} > 0$ and $\Gamma(A, 1) > 0$ for all $A \in (0, a_{\min}^{\Gamma,1})$. Again, by using Lemma 1, we see that there exists a unique continuously differentiable function $\psi : [0, a_{\min}^{\Gamma,1}] \to (b, 1]$ such that $\Gamma(A, \psi(A)) = 0$ for all $A \in [0, a_{\min}^{\Gamma,1}]$. If either $\hat{b}^{\Theta,0} < 1$, or $\hat{b}^{\Theta,0} = 1$ and $\Theta(0, \hat{b}^{\Theta,0}) = 0$, the sets $\{(\phi(B), B) : B \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\}$ and $\{(A, \psi(A)) : A \in (0, a_{\min}^{\Gamma,1}]\}$ intersect and, hence, we conclude that there exists $(A_*, B_*) \in (0, a_{\min}^{\Gamma,1}] \times (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$ solving (16) and (17). Now suppose that $\hat{b}^{\Theta,0} = 1$ and $\Theta(0, \hat{b}^{\Theta,0}) > 0$ (note that $\Theta(0, \hat{b}^{\Theta,0})$ cannot be negative under the assumption that $\Theta(A_*^{\Theta,1}, 1) = 0$). If $a_{\min}^{\Gamma,1} \geq A_*^{\Theta,1}$ then the sets $\{(\phi(B), B) : B \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\}$ and $\{(A, \psi(A)) : A \in (0, a_{\min}^{\Gamma,1}]\}$ intersect and so there exists $(A_*, B_*) \in (0, a_{\min}^{\Gamma,1}] \times (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})$ solving (16) and (17). When $a_{\min}^{\Gamma,1} < A_*^{\Theta,1}$ we shall consider three cases. If $\Gamma(A_*^{\Theta,1}, 1) < 0$ then $A_* = A_*^{\Theta,1}$ and $B_* = 1$ satisfy (19). If, on the other

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hand, $\Gamma(A_*^{\Theta,1},1)=0$ then the $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (16) and (17). Finally, suppose that $\Gamma(A_*^{\Theta,1},1)>0$ and let $\tilde{a}^{\Gamma,1}$ and $\hat{a}^{\Gamma,1}$ be defined as in (II) above. Then, from Lemma 1, there exists a unique continuously differentiable function $\psi\colon[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]\to(b,1]$ such that $\Gamma(A,\psi(A))=0$ for all $A\in[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]$. Since $A_*^{\Theta,1}\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$, it follows that the sets $\{(\phi(B),B)\colon B\in(\tilde{b}^{\Theta,0},\hat{b}^{\Theta,0})\}$ and $\{(A,\psi(A))\colon A\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\}$ intersect and, hence, we conclude that there exists $(A_*,B_*)\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\times(\tilde{b}^{\Theta,0},\hat{b}^{\Theta,0})$ solving (16) and (17).

It remains to consider the case $B_*^{\Gamma,0}=\hat{b}^{\Theta,0}=1$ and $\Theta(0,B_*^{\Gamma,0})>0$. From Lemma 1 we obtain the existence and uniqueness of a continuously differentiable function $\phi\colon[\tilde{b}^{\Theta,0},1]\to[0,a)$ such that $\Theta(\phi(B),B)=0$ for all $B\in[\tilde{b}^{\Theta,0},1]$. Now suppose that $\Gamma(A_*^{\Theta,1},1)<0$. Then $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (19). If, on the other hand, $\Gamma(A_*^{\Theta,1},1)=0$ then $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (16) and (17). Finally, if $\Gamma(A_*^{\Theta,1},1)>0$ then by Lemma 1 there exists a unique continuously differentiable function $\psi\colon[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]\to(b,1]$ such that $\Gamma(A,\psi(A))=0$ for all $A\in[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]$ and the sets $\{(\phi(B),B)\colon B\in(\tilde{b}^{\Theta,0},1)\}$ and $\{(A,\psi(A))\colon A\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\}$ intersect. From this we conclude that there exists $(A_*,B_*)\in(\tilde{b}^{\Theta,1},1)\times(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ satisfying (16) and (17).

Step 2°. Let us now assume that $\Gamma(0,B)<0$ for all $B\in[b,1]$. If there exists no $A_*^{\Gamma,1}\in[0,a]$ such that $\Gamma(A_*^{\Gamma,1},1)=0$ then $\Gamma(A,B)<0$ in $[0,a]\times[b,1]$ (see Lemma 1(i)). In this case, $A_*=A_*^{\Theta,1}$ and $B_*=1$ satisfy (19). Suppose, on the other hand, that there exists such a $A_*^{\Gamma,1}$. If $\Gamma(A_*^{\Theta,1},1)=0$ then $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (16) and (17), whereas if $\Gamma(A_*^{\Theta,1},1)<0$ then $A_*=A_*^{\Theta,1}$ and $B_*=1$ satisfy (19). Finally, if $\Gamma(A_*^{\Theta,1},1)>0$ then there exists a unique continuously differentiable function $\psi\colon[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]\to(b,1]$ such that $\Gamma(A,\psi(A))=0$ for all $A\in[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]$. Let us define $b_{\max}^{\Theta,0}$ as in (22). Again, by using Lemma 1, we see that there exists a unique continuously differentiable function $\phi\colon[b_{\max}^{\Theta,0},1]\to[0,a)$ such that $\Theta(\phi(B),B)=0$ for all $B\in[b_{\max}^{\Theta,0},1]$. The fact that $A_*^{\Theta,1}\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ implies that the sets $\{(\phi(B),B)\colon B\in(b_{\max}^{\Theta,0},1)\}$ and $\{(A,\psi(A))\colon A\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\}$ intersect and, hence, we conclude that there exists $(A_*,B_*)\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\times(b_{\max}^{\Theta,0},1)$ solving (16) and (17).

We now consider the case $\Theta(A,1)<0$ for all $A\in[0,a)$ and when there exists $B_*^{\Theta,0}\in[b,1]$ such that $\Theta(0,B_*^{\Theta,0})=0$. The free boundaries A_* and B_* can be obtained by repeating steps 1° and 2° above. Note, however, that when considering the case $\Theta(0,B_*^{\Gamma,0})>0$, we must have $\hat{b}^{\Theta,0}<1$ (since $\Theta(0,1)<0$).

We next consider the case $\Theta(A,B)<0$ in $[0,a]\times[b,1]$. If there exists $B_*^{\Gamma,0}\in(b,1]$ such that $\Gamma(0,B_*^{\Gamma,0})=0$, we see that $A_*=0$ and $B_*=B_*^{\Gamma,0}$ satisfy (18). If, on the other hand, $\Gamma(0,B)<0$ for all $B\in[b,1]$ then $A_*=0$ and $B_*=1$ satisfy (20).

It remains to consider the case when there exists $A_*^{\Theta,1}\in[0,a)$ such that $\Theta(A_*^{\Theta,1},1)=0$ and when $\Theta(0,B)>0$ for all $B\in[b,1)$. By Lemma 1(iii) we see that there exists $\phi\colon[b,1]\to[0,a)$ such that $\Theta(\phi(B),B)=0$ for all $B\in[b,1]$. If $\Gamma(A_*^{\Theta,1},1)=0$ then $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (16) and (17). If, on the other hand, $\Gamma(A_*^{\Theta,1},1)<0$ then $A_*=A_*^{\Theta,1}$ and $B_*=1$ solve (19). Finally, suppose that $\Gamma(A_*^{\Theta,1},1)>0$. Then we must have either $A_*^{\Theta,1}\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ and $\Gamma(A,1)>0$ for all $A\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ (note that $A_*^{\Theta,1}\neq a$ since $\Theta(a,1)<0$) or $A_*^{\Theta,1}=\tilde{a}^{\Gamma,1}=0$ and $\Gamma(A_*^{\Theta,1},1)>0$. In the first case, by Lemma 1(iii), we see that there exists $\psi\colon[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]\to(b,1]$ such that $\Gamma(A,\psi(A))=0$ for all $A\in[\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1}]$. The fact that $A_*^{\Theta,1}\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ implies that the sets $\{(\phi(B),B)\colon B\in(b,1]\}$ and $\{(A,\psi(A))\colon A\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})\}$ intersect and so there exists $A_*\in(\tilde{a}^{\Gamma,1},\hat{a}^{\Gamma,1})$ and $B_*\in(b,1]$ which solve (16) and (17). In the second case, again by Lemma 1(iii), we see that there exists $\psi\colon[0,\hat{a}^{\Gamma,1}]\to(b,1]$ such that $\Gamma(A,\psi(A))=0$ for all $A\in[0,\hat{a}^{\Gamma,1}]$. This implies that the

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sets $\{(\phi(B), B)\colon B \in (b, 1)\}$ and $\{(A, \psi(A))\colon A \in (0, \hat{a}^{\Gamma,1})\}$ intersect and so there exists $A_* \in (0, \hat{a}^{\Gamma,1})$ and $B_* \in (b, 1)$, which solve (16) and (17).

3.1. Uniqueness of solution to the free-boundary problem

We now consider some special cases in which there exists a unique solution to the system of equations (16) and (17). In particular, we shall consider the case when there exist unique continuously differentiable functions $\phi\colon [b, 1] \to [0, a)$ and $\psi\colon [0, a] \to (b, 1]$ such that $\Theta(\phi(B), B) = 0$ for all $B \in [b, 1]$ and $\Gamma(A, \psi(A)) = 0$ for all $A \in [0, a]$, and for which the sets $\{(\phi(B), B)\colon B \in (b, 1]\}$ and $\{(A, \psi(A))\colon A \in [0, a)\}$ intersect. Let $\mathcal{A}_\Theta$ denote the range of the function ϕ. By continuity of ϕ, it follows that $\mathcal{A}_\Theta$ is a closed interval in $[0, a)$. Similarly, if we set $\mathcal{A}_\Gamma$ to be the range of the function ψ then by continuity of ψ it follows that $\mathcal{A}_\Gamma$ is a closed interval in $(b, 1]$.

Proposition 2. *Suppose that*

(i) $H_1'(B) > G_1'(A)$ *and* $H_2'(A) > G_2'(B)$ *for all* $(A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma$,

(ii) $H_1'(B) < G_1'(A)$ *and* $H_2'(A) < G_2'(B)$ *for all* $(A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma$.

Then the solution to (16) and (17) is unique.

Proof. We shall only prove (i) as the result for (ii) follows analogously. For this we note that, for any $A \in \mathcal{A}_\Theta$ given and fixed, $\Theta_B(A, B) < 0$ for all $B \in \mathcal{A}_\Gamma$ and so the mapping $B \mapsto \Theta(A, B)$ is decreasing in $\mathcal{A}_\Gamma$. Similarly, for $B \in \mathcal{A}_\Gamma$ given and fixed, $\Gamma_A(A, B) < 0$ for all $A \in \mathcal{A}_\Theta$ and so the mapping $A \mapsto \Gamma(A, B)$ is decreasing in $\mathcal{A}_\Theta$. Suppose, for contradiction, that there exist two pairs (A_*^1, B_*^1) and (A_*^2, B_*^2) in $\mathcal{A}_\Theta \times \mathcal{A}_\Gamma$, such that $(A_*^1, B_*^1) \neq (A_*^2, B_*^2)$, which solve (16) and (17). Suppose first that $A_*^1 < A_*^2$. If $B_*^1 \leq B_*^2$ we have $0 = \Theta(A_*^1, B_*^1) > \Theta(A_*^2, B_*^1) \geq \Theta(A_*^2, B_*^2) = 0$, where the first inequality follows from the fact that for $B \in [b, 1]$ the mapping $A \mapsto \Theta(A, B)$ is decreasing in $[0, a)$ (see Lemma 1(i)). So we must have $B_*^1 > B_*^2$ whenever $A_*^1 < A_*^2$. But if this is the case, we obtain $0 = \Gamma(A_*^1, B_*^1) > \Gamma(A_*^2, B_*^1) > \Gamma(A_*^2, B_*^2) = 0$. The second inequality follows from the fact that the mapping $B \mapsto \Gamma(A, B)$ is increasing for any given $A \in [0, a]$ (see Lemma 1(i)). From this it follows that $A_*^1 \geq A_*^2$. By symmetry one can see that this case is not possible either and so uniqueness of A_* and B_* follows. □

Proposition 3. *Suppose that* $H_1'(B) > G_1'(A)$ *and* $H_2'(A) < G_2'(B)$ *for all* $(A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma$. *Then, if* G_1'' *is increasing in* $\mathcal{A}_\Theta$, G_2'' *is decreasing in* $\mathcal{A}_\Gamma$, H_1 *is concave in* $\mathcal{A}_\Gamma$ *and* H_2 *is concave in* $\mathcal{A}_\Theta$, *the system of equations (16) and (17) is unique.*

Prior to proving Proposition 3 we need the following simple fact from convex analysis.

Lemma 2. *Let* f, g *be differentiable functions on some closed interval* $[l, m]$. *Suppose that there exists* $A \in [l, m)$ *such that* $f(A) = g(A)$. *If* f *is convex,* g *is concave, and* $f(m) < g(m)$, *then there exists no other point* $B \in [l, m)$ *such that* $f(B) = g(B)$.

Proof. We first show that $f(B) < g(B)$ for any $B \in (A, m)$. For this consider the lines $L_1(x)$ joining the points $(A, g(A))$ and $(m, g(m))$, and $L_2(x)$ joining the points $(A, f(A))$ and $(m, f(m))$. By concavity of g and convexity of f, we have $g(B) \geq L_1(B) > L_2(B) \geq f(B)$. We next show that $f(B) > g(B)$ for any $B \in [l, A)$. For this we note, by convexity of f and concavity of g (recall Proposition 1), that

$$f'(A) \leq \frac{f(m) - f(A)}{m - A} < \frac{g(m) - g(A)}{m - A} \leq g'(A). \tag{23}$$

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Again by convexity of f, concavity of g, and Proposition 1, we have

$$\begin{aligned} f(B) &\geq f(A) + f'(A)(B-A) \\ &= g(A) + f'(A)(B-A) \\ &\geq g(B) - g'(A)(B-A) + f'(A)(B-A) \\ &= g(B) + (f'(A) - g'(A))(B-A) \\ &> g(B) \quad \text{for all } B \in [l, A), \end{aligned}$$

where the last inequality follows from (23). □

Proof of Proposition 3. Since the functions ϕ and ψ are continuously differentiable we can take the partial derivatives on both sides of the equations $\Theta(\phi(B), B) = 0$ and $\Gamma(A, \psi(A)) = 0$ and rearranging terms to obtain

$$\begin{aligned} \phi'(B) &= -\frac{\Theta_B(\phi(B), B)}{\Theta_A(\phi(B), B)} = -\frac{G_1'(\phi(B)) - H_1'(B)}{G_1''(\phi(B))(B - \phi(B))} < 0, \\ \psi'(A) &= -\frac{\Gamma_A(A, \psi(A))}{\Gamma_B(A, \psi(A))} = -\frac{G_2'(\psi(A)) - H_2'(A)}{G_2''(\psi(A))(A - \psi(A))} < 0. \end{aligned}$$

The inequalities follow from the concavity properties of G_1 and G_2 and from the fact that $G_1'(\phi(B)) < H_1'(B)$ and $G_2'(\psi(A)) > H_2'(A)$. From this we conclude that ϕ and ψ are decreasing on $\mathcal{A}_\Gamma$ and $\mathcal{A}_\Theta$, respectively. Take any $B_1, B_2 \in \mathcal{A}_\Gamma$ such that $B_1 < B_2$. From the monotonicity property of ϕ together with the facts that $G_1'' < 0$ and is monotonic increasing on $\mathcal{A}_\Theta$, and that $B_1 - \phi(B_1) < B_2 - \phi(B_2)$, we have

$$-\frac{1}{G_1''(\phi(B_1))(B_1 - \phi(B_1))} > -\frac{1}{G_1''(\phi(B_2))(B_2 - \phi(B_2))} > 0. \tag{24}$$

Using again the concavity property of G_1 on $\mathcal{A}_\Theta$ and that of H_1 on $\mathcal{A}_\Gamma$, we obtain

$$G_1'(\phi(B_1)) - H_1'(B_1) < G_1'(\phi(B_2)) - H_1'(B_2) < 0, \tag{25}$$

where the last inequality follows by recalling that $G_1'(A) < H_1'(B)$ for all $(A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma$. Combining (24) and (25), we see that

$$\phi'(B_1) = -\frac{G_1'(\phi(B_1)) - H_1'(B_1)}{G_1''(\phi(B_1))(B_1 - \phi(B_1))} < -\frac{G_1'(\phi(B_2)) - H_1'(B_2)}{G_1''(\phi(B_2))(B_2 - \phi(B_2))} = \phi'(B_2),$$

from which the strict convexity property of ϕ on $\mathcal{A}_\Gamma$ follows. Analogously, one can show that ψ is strictly concave on $\mathcal{A}_\Gamma$. Since ϕ is continuously differentiable and $\phi' < 0$ in $[b, 1]$, it follows that the inverse function $\phi^{-1} \colon \mathcal{A}_\Theta \to [b, 1]$ is a decreasing continuously differentiable function. Moreover, using the fact that the inverse of convex decreasing functions is also convex we deduce that ϕ^{-1} is convex on $\mathcal{A}_\Theta$. Since $\mathcal{A}_\Theta$ is a closed interval, we can use Lemma 2 to deduce that the functions ϕ^{-1} and ψ intersect only once on $\mathcal{A}_\Theta$ and so we can conclude that there exists only one point $(A_*, B_*) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma$ which solves the system of equations (16) and (17). □

4. Verification theorem

We initiate this section by showing that if there exist $0 \leq A_* < a < b < B_* \leq 1$ solving (16) and (17) then the functions u and v in the free-boundary problem (6)–(12) coincide with the value functions of the nonzero-sum game (1) and (2).

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Theorem 1. *Let X be Brownian motion in $[0,1]$, started at $x \in [0,1]$ and absorbed at either 0 or 1. Suppose that G_i, H_i, for $i=1,2$, are C^2 functions on $[0,1]$ such that $G_i \le H_i$. Assume also that $G_i(0)=H_i(0)$, that $G_i(1)=H_i(1)$, and that G_i satisfy assumptions (3)–(5). If there exist $A_* \in [0,a)$ and $B_* \in (b,1]$, which solve (16) and (17) then the functions*

$$u(x)=\begin{cases} G_1(x) & \text{if } 0 \le x \le A_*, \\ u_*(x;A_*,B_*) & \text{if } A_* < x < B_*, \\ H_1(x) & \text{if } B_* \le x \le 1, \end{cases} \qquad v(x)=\begin{cases} H_2(x) & \text{if } 0 \le x \le A_*, \\ v_*(x;A_*,B_*) & \text{if } A_* < x < B_*, \\ G_2(x) & \text{if } B_* \le x \le 1, \end{cases}$$

where $u_*(x;A_*,B_*)$ takes the form (13) and $v_*(x;A_*,B_*)$ is given by (14), coincide with the value functions $V^1_{\sigma_*}(x)=\sup_\tau M^1_x(\tau,\sigma_*)$ and $V^2_{\tau_*}(x)=\sup_\sigma M^2_x(\tau_*,\sigma)$, respectively, where $\tau_*=\inf\{t\ge 0\colon X_t \le A_*\}\wedge\rho_{0,1}$ and $\sigma_*=\inf\{t\ge 0\colon X_t\ge B_*\}\wedge\rho_{0,1}$.

Proof. We first show that $V^1_{\sigma_*}(x) \le u(x)$ for all $x\in[0,1]$. Since G_1, H_1 and u_* are C^1 functions on $[0,1]$, it follows that u is absolutely continuous on $[0,1]$ and that u' (which exists almost everywhere) is of bounded variation. But this implies that u can be written as the difference of two convex functions. So we can apply the Itô–Tanaka formula (see [19]) to $u(X_t)$ to obtain

$$\begin{aligned} u(X_t) &= u(x)+\int_0^t u'_-(X_s)\,\mathrm{d}X_s+\frac12\int_0^1 l_t^x\,\mathrm{d}u'(x) \\ &= u(x)+\int_0^t u'_-(X_s)\,\mathrm{d}X_s+\frac12\int_0^1 l_t^x u''(x)\mathbf{1}(x\ne A_*,\ x\ne B_*)\,\mathrm{d}x \\ &\quad+\frac12\int_0^1 l_t^x\mathbf{1}(x=A_*)\,\mathrm{d}u'(x)+\frac12\int_0^1 l_t^x\mathbf{1}(x=B_*)\,\mathrm{d}u'(x) \\ &= u(x)+M_t+\frac12\int_0^t[G_1''(X_s)\mathbf{1}(0\le X_s<A_*)+0\mathbf{1}(A_*<X_s<B_*) \\ &\qquad\qquad +H_1''(X_s)\mathbf{1}(B_*<X_s\le 1)]\mathbf{1}(X_s\ne A_*,\ X_s\ne B_*)\,\mathrm{d}s \\ &\quad+\frac12 l_t^{B_*}(u'_+(B_*)-u'_-(B_*)) \\ &= u(x)+M_t+\frac12\int_0^t[G_1''(X_s)\mathbf{1}(0\le X_s<A_*) \\ &\qquad\qquad +0\mathbf{1}(A_*<X_s<B_*)+H_1''(X_s)\mathbf{1}(B_*<X_s\le 1)]\,\mathrm{d}s \\ &\quad+\frac12 l_t^{B_*}(H_1'(B_*)-G_1'(A_*)), \end{aligned}\tag{26}$$

where $(l_t^{B_*})_{t\ge0}$ is the local time of X at the point B_*, defined by

$$l_t^{B_*}=\mathbb{P}_x-\lim\frac{1}{\varepsilon}\int_0^t\mathbf{1}(B_*<X_s<B_*+\varepsilon)\,\mathrm{d}s$$

and $(M_t)_{t\ge0}$ is a local martingale, given by $\int_0^t u'_-(X_s)\,\mathrm{d}X_s$. The third equality follows from the occupation time space formula (see [6]) together with the definition of u and the fact that u

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is smooth at A_*. The last equality follows again from the definition of u. Since $\sigma_* = \inf\{t \geq 0 \colon X_t \geq B_*\} \wedge \rho_{0,1}$, we have

$$\begin{aligned} G_1(X_t)\mathbf{1}(t \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < t) &\leq u(X_t)\mathbf{1}(t \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < t) \\ &= u(X_t)\mathbf{1}(t \leq \sigma_*) + u(X_{\sigma_*})\mathbf{1}(\sigma_* < t) \\ &= u(X_{t\wedge\sigma_*}) \\ &\leq u(x) + M_{t\wedge\sigma_*} \end{aligned} \tag{27}$$

for any given $t \geq 0$. The first inequality can be seen by noting that since G_1 is concave in $[0, a]$ then the line $u_*(x; A_*, B_*)$ supports the hypograph of G_1 in $[A_*, a]$ and so $u \geq G_1$ in $[A_*, a]$. On the other hand, since $u(B_*) \geq G_1(B_*)$, it follows that u majorises the line joining the points $(a, G_1(a))$ and $(B_*, G_1(B_*))$ in the interval $[a, B_*]$, which, in turn, by convexity of G_1 in $(a, B_*]$, majorises G_1 in $[a, B_*]$. The first equality follows from the fact that $u(X_{\sigma_*}) \in \{0\} \cup [B_*, 1]$ and by the definition $u = H_1$ in $\{0\} \cup [B_*, 1]$. The second inequality follows from (26) upon noting that $G_1'' \leq 0$ in $[0, A_*)$ and that $l_t^{B_*}$ increases only when the process is at B_*. Now suppose that $(\tau_n)_{n=1}^{\infty}$ is a localizing sequence of stopping times for M. Then, from (27), we have

$$G_1(X_{\tau\wedge\tau_n})\mathbf{1}(\tau \wedge \tau_n \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < \tau \wedge \tau_n) \leq u(x) + M_{\tau\wedge\tau_n\wedge\sigma_*} \tag{28}$$

for every stopping time τ of X. Taking the $\mathbb{P}_x$-expectation in (28) we conclude, by the optional sampling theorem, that

$$\mathbb{E}_x[G_1(X_{\tau\wedge\tau_n})\mathbf{1}(\tau \wedge \tau_n \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < \tau \wedge \tau_n)] \leq u(x) \tag{29}$$

for all stopping times τ. Letting $n \to \infty$ in the left-hand side expression in (29), we obtain, by using the Lebesgue dominated convergence theorem and by noting that $\sigma_* < \infty$ $\mathbb{P}_x$-almost surely and G_1 and H_1 are bounded,

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_x[G_1(X_{\tau\wedge\tau_n})\mathbf{1}(\tau \wedge \tau_n \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < \tau \wedge \tau_n)] \\ &\quad = \mathbb{E}_x\Big[\lim_{n\to\infty}(G_1(X_{\tau\wedge\tau_n})\mathbf{1}(\tau \wedge \tau_n \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < \tau \wedge \tau_n))\Big] \\ &\quad = M_x^1(\tau, \sigma_*), \end{aligned}$$

and so we conclude that

$$M_x^1(\tau, \sigma_*) \leq u(x) \quad \text{for all } \tau. \tag{30}$$

Taking the supremum in (30) over all τ, it follows that $V_{\sigma_*}^1(x) \leq u(x)$. It remains to prove that (30) holds with equality if τ is replaced by τ_*. Indeed, from (26) and the structure of the stopping times τ_* and σ_*, we have

$$u(X_{\tau_*\wedge\tau_n\wedge\sigma_*}) = u(x) + M_{\tau_*\wedge\tau_n\wedge\sigma_*}. \tag{31}$$

Taking the $\mathbb{P}_x$-expectation on both sides of (31) and the limit as $n \to \infty$, we have, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}_x u(X_{\tau_*\wedge\sigma_*\wedge\tau_n}) = \mathbb{E}_x\Big[\lim_{n\to\infty} u(X_{\tau_*\wedge\sigma_*\wedge\tau_n})\Big] = \mathbb{E}_x u(X_{\tau_*\wedge\sigma_*}) = u(x).$$

Since $u(X_{\tau_*\wedge\sigma_*}) = G_1(X_{\tau_*})\mathbf{1}(\tau_* \leq \sigma_*) + H_1(X_{\sigma_*})\mathbf{1}(\sigma_* < \tau_*)$, we conclude that $M_x^1(\tau_*, \sigma_*) = u(x)$ and so $V_{\sigma_*}^1(x) \leq M_x^1(\tau_*, \sigma_*)$. By definition, $V_{\sigma_*}^1(x) \geq M_x^1(\tau_*, \sigma_*)$ and so the equality

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of $V^1_{\sigma_*}$ and u follows. To show that $V^2_{\tau_*}(x) = v(x)$ one can follow the same steps as for $V^1_{\sigma_*}(x)$. Note that the fact that the mapping $t \mapsto G_2(X_t)\mathbf{1}(t < \tau) + H_2(X_\tau)\mathbf{1}(\tau \le t)$ is right-continuous with left limits (in contrast with that of player one which is left-continuous with right limits) does not affect the proof since τ_* is finite, $G_2(0) = H_2(0)$ and $G_2(1) = H_2(1)$. □

We next provide three results to link the solution of the free-boundary problem with the value functions of the game in the case when A_* and B_* satisfy one of the conditions (18)–(20). The proofs can be carried out using similar arguments to the proof of Theorem 1 and, therefore, shall be omitted.

Theorem 2. *Consider the assumptions given in Theorem 1. Suppose that $A_* = 0$ and that there exists $B_* \in (b, 1]$ which satisfy (18). Then the functions*

$$u(x) = \begin{cases} u_*(x; 0, B_*) & \text{if } 0 \le x < B_*, \\ H_1(x) & \text{if } B_* \le x \le 1, \end{cases} \qquad v(x) = \begin{cases} v_*(x; 0, B_*) & \text{if } 0 \le x < B_*, \\ G_2(x) & \text{if } B_* \le x \le 1, \end{cases}$$

where $u_*(x; 0, B_*)$ takes the form (13) and $v_*(x; 0, B_*)$ is given by (14), coincide with the value functions $V^1_{\tau_*}(x) = \sup_\tau M^1_x(\tau, \sigma_*)$ and $V^2_{\sigma_*}(x) = \sup_\sigma M^2_x(\tau_*, \sigma)$, respectively, where $\tau_* = \rho_{0,1}$ and $\sigma_* = \inf\{t \ge 0 : X_t \ge B_*\} \wedge \rho_{0,1}$.

Theorem 3. *Consider the assumptions given in Theorem 1. Suppose that $B_* = 1$ and that there exists $A_* \in [0, a)$ which satisfy (19). Then the functions*

$$u(x) = \begin{cases} G_1(x) & \text{if } 0 \le x \le A_*, \\ u_*(x; A_*, 1) & \text{if } A_* < x \le 1, \end{cases} \qquad v(x) = \begin{cases} H_2(x) & \text{if } 0 \le x \le A_*, \\ v_*(x; A_*, 1) & \text{if } A_* < x \le 1, \end{cases}$$

where $u_*(x; A_*, 1)$ takes the form (13) and $v_*(x; A_*, 1)$ is given by (14), coincide with the value functions $V^1_{\tau_*}(x) = \sup_\tau M^1_x(\tau, \sigma_*)$ and $V^2_{\sigma_*}(x) = \sup_\sigma M^2_x(\tau_*, \sigma)$, respectively, where $\tau_* = \inf\{t \ge 0 : X_t \le A_*\} \wedge \rho_{0,1}$ and $\sigma_* = \rho_{0,1}$.

Theorem 4. *Consider the assumptions given in Theorem 1. Suppose that $A_* = 0$ and $B_* = 1$ satisfy (20). Then the functions*

$$u(x) = u_*(x; 0, 1) \quad \text{and} \quad v(x) = v_*(x; 0, 1),$$

where $u_*(x; 0, 1)$ takes the form (13) and $v_*(x; 0, 1)$ is given by (14), coincide with the value functions $V^1_{\tau_*}(x) = \sup_\tau M^1_x(\tau, \sigma_*)$ and $V^2_{\sigma_*}(x) = \sup_\sigma M^2_x(\tau_*, \sigma)$, respectively, where $\tau_* = \sigma_* = \rho_{0,1}$.

5. Regular diffusions

We shall now link nonzero-sum games of optimal stopping for one-dimensional regular diffusions with nonzero-sum games of optimal stopping for Brownian motion. In doing so one can then use the results in the previous sections to show that for a certain class of payoff functions, nonzero-sum optimal stopping games for one-dimensional regular diffusions admit a Nash equilibrium point. So let X be a one-dimensional regular diffusion in [0, 1], absorbed at either 0 or 1, and suppose that $\alpha \ge 0$ is a given constant. Let us assume that the fine topology coincides with the Euclidean topology and let $\mathbb{L}_X$ be the infinitesimal generator of X. It is well known that under regularity conditions (see, for example, [16]), $\mathbb{L}_X F = \frac{1}{2}\sigma^2(x)F_{xx} + \mu(x)F_x$ for $x \in (0, 1)$, where $\mu(x) \in \mathbb{R}$ is the drift and $\sigma^2(x)$ is the diffusion coefficient of X. Moreover, the second order $\mathbb{L}_X F = \alpha F$ admits two linearly independent solutions ψ and φ such that $\psi(0), \varphi(1) > 0$ and that ψ is increasing and φ is decreasing. These solutions are uniquely determined up to a multiplicative constant. In the case when $\alpha = 0$ we can take $\psi = S$ and $\varphi \equiv 1$, where S is the scale function of X.

Step 1°. Consider the nonzero-sum game of optimal stopping in which player one chooses a stopping time τ_* and player two a stopping time σ_* in order to maximize their expected payoffs, which are respectively given by

$$\mathbb{E}_x[\mathrm{e}^{-\alpha(\tau\wedge\sigma)}(G_1(X_\tau)\mathbf{1}(\tau \le \sigma) + H_1(X_\sigma)\mathbf{1}(\sigma < \tau))],$$
$$\mathbb{E}_x[\mathrm{e}^{-\alpha(\tau\wedge\sigma)}(G_2(X_\sigma)\mathbf{1}(\sigma < \tau) + H_2(X_\tau)\mathbf{1}(\tau \le \sigma))],$$

where $G_i, H_i\colon [0,1] \to \mathbb{R}$, for $i = 1, 2$, are continuous functions such that $G_i \le H_i$ with $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$. For a given stopping time σ chosen by player two, let

$$V_\sigma^{1,\alpha}(x) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-\alpha(\tau\wedge\sigma)}(G_1(X_\tau)\mathbf{1}(\tau \le \sigma) + H_1(X_\sigma)\mathbf{1}(\sigma < \tau))] \tag{32}$$

be the value function of player one and, for a given stopping time τ chosen by player one, let

$$V_\tau^{2,\alpha}(x) = \sup_\sigma \mathbb{E}_x[\mathrm{e}^{-\alpha(\tau\wedge\sigma)}(G_2(X_\sigma)\mathbf{1}(\sigma < \tau) + H_2(X_\tau)\mathbf{1}(\tau \le \sigma))] \tag{33}$$

be the value function of player two. Suppose that there exist continuous functions $u, v\colon [0,1] \to \mathbb{R}$ such that

$$u = \inf_{F\in\sup_v^1(G_1,K_1)} F, \qquad v = \inf_{F\in\sup_u^2(G_2,K_2)} F, \tag{34}$$

where

$$\begin{aligned}\sup\nolimits_v^1(G_1, K_1) = \{F\colon [0,1] \to [G_1, K_1]\colon F \text{ is continuous,}\\ F = H_1 \text{ in } D_2,\ F \text{ is } \alpha\text{-superharmonic in } D_2^c\},\end{aligned} \tag{35}$$

$$\begin{aligned}\sup\nolimits_u^2(G_2, K_2) = \{F\colon [0,1] \to [G_2, K_2]\colon F \text{ is continuous,}\\ F = H_2 \text{ in } D_1,\ F \text{ is } \alpha\text{-superharmonic in } D_1^c\},\end{aligned} \tag{36}$$

with K_i, for $i = 1, 2$, being the smallest α-superharmonic function (relative to X) majorizing H_i, $D_1 = \{u = G_1\}$ and $D_2 = \{v = G_2\}$ (recall that a measurable function $F\colon \mathbb{R} \to \mathbb{R}$ is α-superharmonic if $\mathbb{E}_x[\mathrm{e}^{-\alpha\tau} F(X_\tau)] \le F(x)$ for all stopping times τ of X and all $x \in [0,1]$). From the DPSC of the value functions $u(x) = V_{\sigma_{D_2}}^{1,\alpha}(x)$ and $v(x) = V_{\tau_{D_1}}^{2,\alpha}(x)$ for all $x \in [0,1]$, where $\tau_{D_1} = \inf\{t \ge 0\colon X_t \in D_1\}$ and $\sigma_{D_2} = \inf\{t \ge 0\colon X_t \in D_2\}$.

Step 2°. Let $I\colon [0,1] \to \mathbb{R}$ be a strictly increasing continuous function and $J\colon [0,1] \to \mathbb{R}$ a Borel measurable function. J is said to be I-concave if

$$J(x) \ge J(c)\bigg(\frac{I(d) - I(x)}{I(d) - I(c)}\bigg) + J(d)\bigg(\frac{I(x) - I(c)}{I(d) - I(c)}\bigg) \quad \text{for } 0 \le c < x < d \le 1.$$

It is known (see, for example, [4, Chapter 16] or [15, Proof of Theorem 3.2]) that a Borel measurable function J is α-superharmonic if and only if J/φ is I-concave or, equivalently, if and only if J/ψ is $\hat{I}$-concave, where I and $\hat{I}$ are strictly increasing continuous functions given by $I = \psi/\varphi$ and $\hat{I} = -1/I = -\varphi/\psi$. From this, it follows that the collections of the functions in (35) and (36) are equivalent to

$$\sup\nolimits_v^1(G_1, K_1) = \bigg\{F\colon [0,1] \to [G_1, K_1]\colon F \text{ is continuous,}\\ F = H_1 \text{ in } D_2,\ \frac{F}{\varphi} \text{ is } I\text{-concave in } D_2^c\bigg\} \tag{37}$$

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and

$$\sup_u^2(G_2, K_2) = \Big\{F\colon [0,1] \to [G_2, K_2]\colon F \text{ is continuous}, \\ F = H_2 \text{ in } D_1, \frac{F}{\varphi} \text{ is } I\text{-concave in } D_1^c\Big\}, \tag{38}$$

where K_i, for $i = 1, 2$, is the smallest function majorizing H_i such that K_i/φ is I-concave.

Step 3°. We show that the sets in (37) and (38) are equivalent to collections involving ordinary concave functions. For this let B be a Brownian motion in $[I(0), I(1)]$, absorbed at either $I(0)$ or $I(1)$ and consider the nonzero-sum game of optimal stopping in which player one chooses a stopping time γ_* and player two a stopping time β_* in order to maximize their expected payoffs, which are respectively given by

$$\mathbb{E}_y[\tilde{G}_1(B_\gamma)\mathbf{1}(\gamma \le \beta) + \tilde{H}_1(B_\beta)\mathbf{1}(\beta < \gamma)], \qquad \mathbb{E}_y[\tilde{G}_2(B_\beta)\mathbf{1}(\beta < \gamma) + \tilde{H}_2(B_\gamma)\mathbf{1}(\gamma \le \beta)]$$

for $y \in [I(0), I(1)]$, where $\tilde{G}_i := (G_i/\varphi) \circ I^{-1}$ and $\tilde{H}_i := (H_i/\varphi) \circ I^{-1}$ for $i = 1, 2$. Given stopping time β chosen by player two, let

$$W_\beta^{1,\alpha}(y) = \sup_\gamma \mathbb{E}_y[\tilde{G}_1(B_\gamma)\mathbf{1}(\gamma \le \beta) + \tilde{H}_1(B_\beta)\mathbf{1}(\beta < \gamma)] \tag{39}$$

be the value function of player one and similarly, given stopping time γ chosen by player one, let

$$W_\gamma^{2,\alpha}(y) = \sup_\beta \mathbb{E}_y[\tilde{G}_2(B_\beta)\mathbf{1}(\beta < \gamma) + \tilde{H}_2(B_\gamma)\mathbf{1}(\gamma \le \beta)] \tag{40}$$

be the value function of player two. Suppose that there exist continuous functions $\tilde{u}, \tilde{v}\colon [I(0), I(1)] \to \mathbb{R}$ such that

$$\tilde{u} = \inf_{F \in \widetilde{\sup}_{\tilde{v}}^1(\tilde{G}_1, \tilde{K}_1)} F \quad \text{and} \quad \tilde{v} = \inf_{F \in \widetilde{\sup}_{\tilde{u}}^2(\tilde{G}_2, \tilde{K}_2)} F,$$

where

$$\widetilde{\sup}_{\tilde{v}}^1(\tilde{G}_1, \tilde{K}_1) = \{F\colon [\mathbf{1}(0), \mathbf{1}(1)] \to [\tilde{G}_1, \tilde{K}_1]\colon F \text{ is continuous}, \\ F = \tilde{H}_1 \text{ in } \tilde{D}_2,\ F \text{ is concave in } \tilde{D}_2^c\} \tag{41}$$

and

$$\widetilde{\sup}_{\tilde{u}}^2(\tilde{G}_2, \tilde{K}_2) = \{F\colon [\mathbf{1}(0), \mathbf{1}(1)] \to [\tilde{G}_2, \tilde{K}_2]\colon F \text{ is continuous}, \\ F = \tilde{H}_2 \text{ in } \tilde{D}_1,\ F \text{ is concave in } \tilde{D}_1^c\}, \tag{42}$$

with $\tilde{K}_i$ for $i = 1, 2$, being the smallest concave function majorizing $\tilde{H}_i$, $\tilde{D}_1 = \{\tilde{u} = \tilde{G}_1\}$ and $\tilde{D}_2 = \{\tilde{v} = \tilde{G}_2\}$. Again from the DPSC of the value functions (note that $\tilde{G}_i(I(0)) = \tilde{H}_i(I(0))$ and $\tilde{G}_i(I(1)) = \tilde{H}_i(I(1))$ since $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$), we have

$$\tilde{u}(y) = W^{1,\alpha}_{\beta_{\tilde{D}_2}}(y) \quad \text{and} \quad \tilde{v}(y) = W^{2,\alpha}_{\gamma_{\tilde{D}_1}}(y) \quad \text{for all } y \in [I(0), I(1)], \tag{43}$$

where $\gamma_{\tilde{D}_1} = \inf\{t \ge 0\colon B_t \in \tilde{D}_1\}$ and $\beta_{\tilde{D}_2} = \inf\{t \ge 0\colon B_t \in \tilde{D}_2\}$.

Step 4°. We now link the value functions in (32) and (33) with those in (39) and (40) via the collections of functions in (37) and (38) and (41) and (42). It is easy to see that $\varphi(x)\tilde{u}(I(x)) \ge G_1(x)$ for all $x \in [0, 1]$ and that $\varphi(x)\tilde{u}(I(x)) = H_1(x)$ for all $x \in \{x \in [0, 1]\colon \varphi(x)(\tilde{v} \circ I)(x) = G_2(x)\}$. Since we know that $\tilde{u}$ is concave in $\{y \in [I(0), I(1)]\colon \tilde{v}(y) > \tilde{G}_2(y)\}$ then by writing

$\tilde{u} = (\tilde{u} \circ I) \circ I^{-1}$ and by making use of the fact that a Borel measurable function F on $D \subseteq [0, 1]$ is I-concave if and only if $F \circ I^{-1}$ is concave on $I(D) = \{I(x) \colon x \in [0, 1]\}$, it follows that $\tilde{u} \circ I$ is I-concave in $\{x \in [0, 1] \colon \varphi(x)(\tilde{v} \circ I)(x) > G_2(x)\}$. This fact can also be used to show that $\varphi(x)\tilde{K}_1(I(x)) = K_1(x)$ for all $x \in [0, 1]$. Repeating the above arguments for $\tilde{v}$ and comparing the functions $\varphi(\tilde{u} \circ I)$ and $\varphi(\tilde{v} \circ I)$ with the functions u and v in (34), it follows that

$$\varphi(x)\tilde{u}(I(x)) = V^{1,\alpha}_{\sigma_{D_2}}(x) \quad \text{and} \quad \varphi(x)\tilde{v}(I(x)) = V^{2,\alpha}_{\tau_{D_1}}(x) \quad \text{for } x \in [0, 1],$$

where $D_1 = \{x \in [0, 1] \colon \varphi(x)\tilde{u}(I(x)) = G_1(x)\}$ and $D_2 = \{x \in [0, 1] \colon \varphi(x)\tilde{v}(I(x)) = G_2(x)\}$. From (43), we can deduce that

$$V^{1,\alpha}_{\sigma_{D_2}}(x) = \varphi(x)W^{1,\alpha}_{\beta_{\tilde{D}_2}}(I(x)) \quad \text{and} \quad V^{2,\alpha}_{\tau_{D_1}}(x) = \varphi(x)W^{2,\alpha}_{\gamma_{\tilde{D}_1}}(I(x)) \quad \text{for } x \in [0, 1].$$

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