A NEW DERIVATION OF THE FUNDAMENTAL FORMULAE IN FOWLERIAN STATISTICAL MECHANICS

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Communicated by C. B. HASELGROVE

Received 23 June 1954

In this paper the notation and terminology of Fowler (1) are used, except when definitions of symbols are explicitly given.

Let P denote a property of an assembly of systems and let $P(n_{r0}, n_{r1}, \ldots, n_{rs}, \ldots)$ be the value of the property when n_{rs} are the occupation numbers of the states $\mathbf{s}$ for a complexion $\mathbf{r}$ of the assembly. We shall write $P(n_{r0}, \ldots, n_{rs}, \ldots)$ as $P(n_{rs})$ in the work which follows. The probability that the assembly is in a certain complexion $\mathbf{r}$ is proportional to the product of factors $w(n_{rs})$, one for each state $\mathbf{s}$. That is, the weight of a complexion $\mathbf{r}$ is $\prod_s w(n_{rs})$. The values of $w(n_{rs})$ for the three different types of statistics are:

classical $\quad w(n_{rs}) = (n_{rs}!)^{-1}$,

Einstein-Bose $\quad w(n_{rs}) = 1$,

Fermi-Dirac $\quad w(n_{rs}) = \begin{cases} 1 & \text{for} \quad n_{rs} = 0, 1, \\ 0 & \text{for} \quad n_{rs} > 1. \end{cases}$

The average value $\bar{P}$ of P is

$$\bar{P} = W/V, \tag{1}$$

where

$$W = \sum_r{}' P(n_{rs}) \prod_s w(n_{rs}) \tag{2a}$$

and

$$V = \sum_r{}' \prod_s w(n_{rs}). \tag{2b}$$

The summation over complexions, denoted by Σ', is limited (for instance) by constancy of number of particles and of total energy:

$$\sum_s n_{rs} = N, \tag{3a}$$

$$\sum_s n_{rs} \epsilon_s = E. \tag{3b}$$

These restrictions are normally imposed by constructing a suitable generating function. A more direct method is to define a δ-function (argument x) such that

$$\delta(0) = 1, \quad \delta(x) = 0 \quad \text{for} \quad x \neq 0. \tag{4}$$

Then, for instance,

$$V = \sum_r \delta(N - \sum_s n_{rs}) \, \delta(E - \sum_s n_{rs} \epsilon_s) \prod_s w(n_{rs}), \tag{5}$$

where the summation is now unrestricted. Since the n_{rs} are integers, we can use the integral representation

$$\delta(N - \sum_s n_{rs}) = \frac{1}{2\pi i} \oint \frac{du}{u^{N - \sum_s n_{rs} + 1}} \tag{6a}$$

with the contour of integration encircling the origin. It is not immediately clear that one can use a similar integral representation for the second δ-function occurring in (5), since the ratios of E and the various ϵ_s may not be rational. Nevertheless, each ϵ_s and E can be approximated as closely as we wish by energies ϵ'_s, E', such that the ratios of the ϵ'_s and E' are all rational. Then, choosing as energy unit the highest common factor of the ϵ'_s and E', we may write

$$\delta(E' - \sum_s n_{rs}\epsilon'_s) = \frac{1}{2\pi i}\oint \frac{dv}{v^{E' - \sum_s n_{rs}\epsilon'_s + 1}}. \tag{6b}$$

Assuming that the final results are continuous functions of the ϵ_s and E we can let (ϵ'_s, E') tend to (ϵ_s, E) finally. In effect, (6b) is then valid with (ϵ'_s, E') replaced by (ϵ_s, E).

From (5) and (6)

$$V = -\frac{1}{4\pi^2}\oint du \oint dv\, u^{-N-1} v^{-E-1} \sum_r \prod_s w(n_{rs})\, u^{n_{rs}} v^{n_{rs}\epsilon_s}$$

$$= -\frac{1}{4\pi^2}\oint du \oint dv \frac{\prod_s f(uv^{\epsilon_s})}{u^{N+1} v^{E+1}}$$

with

$$f(uv^{\epsilon_s}) = \sum_{n_{rs}=0}^{\infty} w(n_{rs})\,(uv^{\epsilon_s})^{n_{rs}}. \tag{7}$$

The forms of $f(x)$ for classical, Einstein-Bose and Fermi-Dirac statistics are $\exp(x)$, $(1-x)^{-1}$ and $(1+x)$. In terms of the grand partition function of Gibbs, defined by

$$Z(u,v) = \sum_s \log f(uv^{\epsilon_s}), \tag{8}$$

$$V = -\frac{1}{4\pi^2}\oint du \oint dv \frac{\exp Z(u,v)}{u^{N+1} v^{E+1}}. \tag{9}$$

It is assumed that $P(n_{rs})$ can be expressed as $P(n_{rs}) = \sum_s n_{rs} p_s$, so that p_s is the contribution to P of a single system in a state $\mathbf{s}$. Properties such that $P(n_{rs}) = \sum_s (n_{rs})^\delta p_s$, where δ is an integer, require only trivial modifications of the method. Then from (2a)

$$W = \sum_r (\sum_s n_{rs} p_s)\, \delta(N - \sum_s n_{rs})\, \delta(E - \sum_s n_{rs}\epsilon_s) \prod_s w(n_{rs})$$

$$= -\frac{1}{4\pi^2}\oint du \oint dv\, u^{-N-1} v^{-E-1} \sum_r (\sum_s n_{rs} p_s) \prod_t w(n_{rt})\, u^{n_{rt}} v^{n_{rt}\epsilon_t}.$$

Now

$$\frac{\partial}{\partial \epsilon_t} v^{n_{rt}\epsilon_t} = n_{rt} \log v\, v^{n_{rt}\epsilon_t}.$$

Hence

$$W = -\frac{1}{4\pi^2}\sum_s p_s \oint du \oint dv\, u^{-N-1} v^{-E-1} (\log v)^{-1} \frac{\partial}{\partial \epsilon_s} \prod_t f(uv^{\epsilon_t}),$$

or

$$W = -\frac{1}{4\pi^2}\sum_s p_s \oint du \oint dv \frac{\partial Z(u,v)}{\partial \epsilon_s} (\log v)^{-1} \exp Z(u,v)\, u^{-N-1} v^{-E-1}. \tag{10}$$

Expressions (9) and (10) are the integrals derived by Fowler; they are evaluated in most cases by direct application of steepest descent methods. The final result is obtained by putting $u = \theta, v = \psi$ (with θ, ψ the saddle-point coordinates) in the 'slowly varying' parts of the integrands of V and W. Then from (1),

$$\bar{P} = \sum_s p_s \frac{\partial Z(\theta, \psi)}{\partial \epsilon_s} (\log \psi)^{-1}. \tag{11}$$

Equation (11) applies only to assemblies with one type of particle present; the proof of the corresponding formulae in more general cases is identical, but the notation is more cumbersome.

The authors gratefully acknowledge the award of Nuffield Research Fellowships, during the tenure of which this work was done.

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THE STABILITY OF A COMBINED CURRENT AND VORTEX SHEET IN A PERFECTLY CONDUCTING FLUID

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Communicated by G. K. Batchelor

Received 26 August 1954

The role of a magnetic field in stabilizing hydrodynamical motions in conducting fluids has already been seen in several instances. For example, it has been shown for parallel flows by Stuart (4) and for rotating fluids by Chandrasekhar (2). A further instance which is discussed here is the effect of a parallel field on an inviscid vortex sheet. The instability of a vortex sheet in a fluid of uniform density was first noticed by Helmholtz (3). In the following note it is shown that a plane current sheet in a perfectly conducting inviscid fluid at rest is stable to small two-dimensional disturbances at the surface of discontinuity, and that when a current sheet coincides with a vortex sheet in the fluid the stability of the vortex sheet is improved by the presence of the current sheet.

A plane surface of discontinuity of tangential velocity and magnetic field exists in a perfectly conducting incompressible fluid of uniform density. The velocity com-