Proceedings of the Royal Society of Edinburgh, **147A**, 89–124, *2017*
DOI:10.1017/S0308210516000056

Optimal Hardy inequalities in cones

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(MS received 27 May 2015; accepted 2 October 2015)

Let Ω be an open connected cone in $\mathbb{R}^n$ with vertex at the origin. Assume that the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)}$$

is *subcritical* in Ω, where δ_Ω is the distance function to the boundary of Ω and $\mu \leqslant 1/4$. We show that under some smoothness assumption on Ω the improved Hardy-type inequality

$$\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \, \mathrm{d}x \geqslant \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2} \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega)$$

holds true, and the Hardy-weight $\lambda(\mu)|x|^{-2}$ is optimal in a certain definite sense. The constant $\lambda(\mu) > 0$ is given explicitly.

Keywords: Hardy inequality; minimal growth; positive solutions

2010 *Mathematics subject classification:* Primary 35A23
Secondary 35B09; 35J20; 35P05

1. Introduction

There is a huge literature devoted to the study of Hardy-type inequalities. The traditional point of view on this subject is to prove (using integration by parts, or the divergence theorem in one way or another) such an inequality, and then to find the *global* best constant. In [14] a natural notion of *optimal* Hardy-type inequality was introduced, and using a synthetic looking construction (the so-called *supersolution construction*), it has been proved that in many cases, this method provides an optimal Hardy-type inequality in this precise sense. The optimality of a Hardy-type inequality as introduced in [14] means in particular that the constant in the inequality is the best possible globally; however, let us emphasize that the optimality in the sense of [14] is significantly stronger than that. In this paper the main point is to provide another interesting and important class of examples for which the supersolution construction provides us with an optimal Hardy-type inequality. We now briefly explain what motivates this paper, and our results.

Let P be a symmetric second-order linear elliptic operator with real coefficients, defined in a domain Ω of $\mathbb{R}^n$, and denote by q its associated quadratic form. Suppose that $q(\varphi) \geqslant 0$ for all $\varphi \in C_0^\infty(\Omega)$, i.e. P is *non-negative* ($P \geqslant 0$) in Ω. Then P is called *subcritical* in Ω if there exists a non-trivial non-negative weight W such that the Hardy-type inequality

$$q(\varphi) \geqslant \lambda \int_\Omega W(x)|\varphi(x)|^2 \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega), \tag{1.1}$$

where $\lambda > 0$ is a constant, holds true. If $P \geqslant 0$ in Ω and (1.1) is not true for any $W \gneqq 0$, then P is called *critical* in Ω.

Given a subcritical operator P in Ω, there is a huge convex set of weights $W \gneqq 0$ satisfying (1.1). A natural question is to find a weight function W which is 'as large as possible' and satisfies (1.1) (see [1, p. 6]).

In [14] Devyver *et al.* constructed a Hardy weight W, for a subcritical operator P, which is *optimal* in a certain definite sense. For symmetric operators the main result of [14] reads as follows.

Theorem 1.1 (Devyver *et al.* [14, theorem 2.2]). *Assume that P is subcritical in Ω. Fix a reference point $x_0 \in \Omega$, and set $\Omega^\star := \Omega \setminus \{x_0\}$. There exists a non-zero non-negative weight W satisfying the following properties.*

(a) *Denote by $\lambda_0 = \lambda_0(P, W, \Omega^\star)$ the largest constant λ satisfying*

$$q(\varphi) \geqslant \lambda \int_{\Omega^\star} W(x)|\varphi(x)|^2 \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^\star). \tag{1.2}$$

Then $\lambda_0 > 0$ and the operator $P - \lambda_0 W$ is critical in $\Omega^\star$; that is, the inequality

$$q(\varphi) \geqslant \int_{\Omega^\star} V(x)|\varphi(x)|^2 \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid for any $V \gneqq \lambda_0 W$.

(b) *The constant λ_0 is also the best constant for (1.2) with test functions supported in $\Omega' \subset \Omega$, where Ω' is either the complement of any fixed compact set in Ω containing x_0 or any fixed punctured neighbourhood of x_0.*

(c) *The operator $P - \lambda_0 W$ is null-critical in $\Omega^\star$; that is, the corresponding Rayleigh–Ritz variational problem*

$$\inf_{\varphi \in \mathcal{D}_P^{1,2}(\Omega^\star)} \frac{q(\varphi)}{\int_{\Omega^\star} W(x)|\varphi(x)|^2 \,\mathrm{d}x} \tag{1.3}$$

admits no minimizer. Here $\mathcal{D}_P^{1,2}(\Omega^\star)$ is the completion of $C_0^\infty(\Omega^\star)$ with respect to the norm $u \mapsto \sqrt{q(u)}$.

(d) *If, furthermore, $W > 0$ in $\Omega^\star$, then the spectrum and the essential spectrum of the Friedrichs extension of the operator $W^{-1}P$ on $L^2(\Omega^\star, W\,\mathrm{d}x)$ are both equal to $[\lambda_0, \infty)$.*

Definition 1.2. A weight function that satisfies properties (a)–(d) is called an *optimal Hardy weight* for the operator P in Ω.

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For related spectral results concerning optimal Hardy inequalities see [13].

One may look at a punctured domain $\Omega^\star$ as a non-compact manifold with two ends $\bar{\infty}$ and x_0, where $\bar{\infty}$ denotes the ideal point in the one-point compactification of Ω. In fact, the results of theorem 1.1 are valid on such manifolds. In [14, theorem 11.6] Devyver *et al*. extended theorem 1.1 and got an optimal Hardy weight W in the *entire* domain Ω, in the case of *boundary singularities*, where the two singular points of the Hardy weight are located at $\partial\Omega \cup \{\bar{\infty}\}$ and not at $\bar{\infty}$ and at an isolated interior point of Ω as in theorem 1.1. The result reads as follows.

THEOREM 1.3 (Devyver *et al*. [14, theorem 11.6]). *Assume that P is subcritical in Ω. Suppose that the Martin boundary $\delta\Omega$ of the operator P in Ω is equal to the minimal Martin boundary and is equal to $\partial\Omega \cup \{\xi_0, \xi_1\}$, where $\partial\Omega \setminus \{\xi_0, \xi_1\}$ is assumed to be a regular manifold of dimension $(n-1)$ without boundary, and the coefficients of P are locally regular up to $\partial\Omega \setminus \{\xi_0, \xi_1\}$.*

Denote by $\hat{\Omega}$ the Martin compactification of Ω, and assume that there exists a bounded domain $D \subset \Omega$ such that ξ_0 and ξ_1 belong to two different connected components D_0 and D_1 of $\hat{\Omega} \setminus \bar{D}$ such that each D_j is a neighbourhood in $\hat{\Omega}$ of ξ_j, where $j = 0, 1$.

Let u_0 and u_1 be the minimal Martin functions at ξ_0 and ξ_1, respectively. Consider the supersolution $u_{1/2} := (u_0 u_1)^{1/2}$ of the equation $Pu = 0$ in Ω, and assume that

$$\lim_{\substack{x \to \zeta_0, \\ x \in \Omega}} \frac{u_1(x)}{u_0(x)} = \lim_{\substack{x \to \zeta_1, \\ x \in \Omega}} \frac{u_0(x)}{u_1(x)} = 0. \tag{1.4}$$

Then the weight $W := Pu_{1/2}/u_{1/2}$ is an optimal Hardy weight for P in Ω. In particular, if W does not vanish on $\hat{\Omega} \setminus \{\xi_0, \xi_1\}$, then the spectrum and the essential spectrum of the Friedrichs extension of the operator $W^{-1}P$ acting on $L^2(\Omega, W\,\mathrm{d}x)$ are $[1, \infty)$.

The following example illustrates theorem 1.3 and motivated the present study.

EXAMPLE 1.4 (Devyver *et al*. [14, example 11.1]). Let $P = P_0 := -\Delta$, and consider the cone Ω with vertex at the origin, given by

$$\Omega := \{x \in \mathbb{R}^n \mid r(x) > 0,\ \omega(x) \in \Sigma\}, \tag{1.5}$$

where Σ is a Lipschitz domain on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, $n \geqslant 2$, and (r, ω) denotes the spherical coordinates of x (i.e. $r = |x|$, and $\omega = x/|x|$). We assume that P is subcritical in Ω.

Let ϕ be the principal eigenfunction of the (Dirichlet) Laplace–Beltrami operator $-\Delta_S$ on Σ with principal eigenvalue $\sigma = \lambda_0(-\Delta_S, 1, \Sigma)$ (for the definition of λ_0 see (2.1)), and set

$$\gamma_\pm := \frac{2 - n \pm \sqrt{(2-n)^2 + 4\sigma}}{2}.$$

We denote by 1 the constant function taking the value 1 in Ω. Then the positive harmonic functions

$$u_\pm(r, \omega) := r^{\gamma_\pm}\phi(\omega)$$

are the Martin kernels at ∞ and 0 [32] (see also [5]).

The function

$$u_{1/2} := (u_+ u_-)^{1/2} = r^{(2-n)/2}\phi(\omega)$$

is a supersolution of the equation $Pu = 0$ in Ω (this is the so-called *supersolution construction for P in Ω with the pair* (u_+, u_-)).

Consequently, the associated Hardy weight is

$$W(x) := \frac{Pu_{1/2}}{u_{1/2}} = \frac{(n-2)^2 + 4\sigma}{4|x|^2},$$

and the corresponding Hardy-type inequality reads as

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \frac{(n-2)^2 + 4\sigma}{4} \int_\Omega \frac{|\varphi|^2}{|x|^2} \,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega). \tag{1.6}$$

It follows from theorem 1.3 that W is an *optimal* Hardy weight, in the sense of definition 1.2. Note that for $\Sigma = \mathbb{S}^{n-1}$ we obtain the classical Hardy inequality in the punctured space. We also remark that the Hardy-type inequality (1.6) and the *global* optimality of the constant $(n-2)^2/4+\sigma$ were previously known (see [26,30]), but again, such a result is significantly weaker than the optimality in the sense of definition 1.2.

Given the result of theorem 1.3, the following question is natural.

Question. What happens in theorem 1.3 if we drop the hypothesis that the coefficients of P are locally regular up to the boundary?

While it is probably not true that the result of theorem 1.3 holds in full generality without this regularity hypothesis, it could still hold in some interesting cases. However, we feel that this is a difficult problem. Therefore, in this paper, we focus on one natural class of operators that are not regular up to the boundary, for which we investigate the validity of theorem 1.3. More precisely, let

$$\delta(x) = \delta_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$$

be the distance function to the boundary of a domain Ω; from the point of view of Hardy inequalities, one of the most natural and interesting class of operators that do not satisfy the regularity assumption of theorem 1.3 are operators of the form

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)} \quad \text{in } \Omega,$$

where Ω is the cone defined by (1.5). Let us now explain our results, which deal with this class of operators. Define

$$\mu_0 = \sup\{\mu \in \mathbb{R}; P_\mu \geqslant 0\}.$$

It turns out that the $\mu < \mu_0$ case is easier, since in this case (as we will see in §§ 3 and 4) P_μ satisfies all the assumptions of theorem 1.3, except, of course, the up to the boundary regularity of the coefficients. The situation for $\mu = \mu_0$ is more delicate. First, it can happen that P_{μ_0} is critical in Ω (so the inequality $P_{\mu_0} \geqslant 0$ cannot be improved). Moreover, in the particular case in which $\mu_0 = 1/4$

https://doi.org/10.1017/S0308210516000056 Published online by Cambridge University Press

and $P_{1/4}$ is subcritical in Ω, we are not able to characterize the Martin boundary of $P_{1/4}$ in Ω. Nevertheless, in this case, under additional smoothness assumptions, we can still find an optimal Hardy weight by the supersolution construction. We note that if one assumes that Ω is (weakly) mean convex, then $\mu_0 = 1/4$ and $P_{1/4}$ is subcritical (proposition 5.8), so the situation is clearer for (weakly) mean convex cones. Our main results (theorems 5.4 and 5.6) assert that, in the subcritical case, the conclusion of theorem 1.3 remains true for the operators P_μ on Euclidean cones given by (1.5). Let us emphasize, however, that our results are not just an easy variation on theorem 1.3: the fact that the coefficients of P_μ are not regular up to the boundary introduce intrinsic difficulties that cannot be overcome by a simple modification of the proof of theorem 1.3; instead, we take advantage of the particular form of the operators P_μ and of the fact that Ω is a cone.

For some recent results (for example, estimates of the constants, existence of extremals, and applications to the study of related semilinear partial differential equations) on various Hardy inequalities with boundary singularities, see, for example, [10, 11, 17, 20, 21, 24] and references therein.

The outline of the paper is as follows. In § 2 we fix the setting and notation, and introduce some basic definitions. In § 3 we use an approximation argument to obtain two positive multiplicative solutions of the equation $P_\mu u = 0$ in Ω of the form $u_\pm(r, w) := r^{\gamma_\pm}\theta(\omega)$, while in § 4 we use the boundary Harnack principle of Ancona [4] and the methods in [25, 32] to obtain an explicit representation theorem for the positive solutions of the equation $P_\mu u = 0$ in Ω that vanish (in the potential theory sense) on $\partial\Omega\backslash\{0\}$. As a consequence of the results in §§ 3 and 4, the operators P_μ, $\mu < \mu_0$, satisfy all the hypotheses of theorem 1.3, except the up to the boundary regularity of the coefficients. The two linearly independent positive multiplicative solutions obtained are the building blocks of the supersolution construction that is used in § 5 to prove our main result, which extends theorem 1.3 to the class of operators P_μ on cones. In § 6 we consider a family of Hardy inequalities in the half-space $\mathbb{R}^n_+$ obtained by Filippas *et al*. [19], and we show (using similar methods as in the first five sections), for the appropriate case, the optimality of the corresponding weight.

We conclude the paper in § 7 by proving a closely related Hardy-type inequality with the best constant for the (non-negative) operator P_μ in Ω, where Ω is a domain in $\mathbb{R}^n$ such that $0 \in \partial\Omega$, and δ_Ω satisfies (in the weak sense) the linear differential inequality

$$-\Delta\delta_\Omega + \frac{n - 1 + \sqrt{1 - 4\mu}}{|x|^2}(x \cdot \nabla\delta_\Omega - \delta_\Omega) \geqslant 0 \quad \text{in } \Omega. \tag{1.7}$$

Finally, we note that parts of the results of this paper were announced in [15].

2. Preliminaries

In this section we fix our setting and notation, and introduce some basic definitions. We write $\mathbb{R}_+ := (0, \infty)$, and

$$\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}.$$

Throughout the paper Ω is a domain in $\mathbb{R}^n$, where $n \geqslant 2$. The distance function to the boundary of Ω is denoted by δ_Ω. We write $\Omega' \Subset \Omega$ whenever Ω is open and $\overline{\Omega'}$ is compact with $\overline{\Omega'} \subset \Omega$. By an *exhaustion* of Ω we mean a sequence $\{\Omega_k\}$ of smooth, relatively compact domains such that $x_0 \in \Omega_1$, $\Omega_k \Subset \Omega_{k+1}$, and $\bigcup_{N=1}^{\infty} \Omega_k = \Omega$.

Let $f, g \colon \Omega \to [0, \infty)$. We write $f \asymp g$ in Ω if there exists a positive constant C such that $C^{-1} g \leqslant f \leqslant C g$ in Ω. Also, we write $f \gneqq 0$ in Ω if $f \geqslant 0$ in Ω but $f \neq 0$ in Ω. $B_r(x)$ is the open ball of radius r centred at x. If Ω is a cone and $R > 0$, we denote by A_R the annulus

$$A_R := \{z \in \Omega \mid \tfrac{1}{2} R \leqslant |z| \leqslant 2R\}.$$

In this paper we consider a second-order linear elliptic operator P defined on a domain $\Omega \subset \mathbb{R}^n$, and let $W \gneqq 0$ be a given function. We write $P \geqslant 0$ in Ω if the equation $Pu = 0$ in Ω admits a positive (super)solution. Unless otherwise stated, it is assumed that $P \geqslant 0$ in Ω.

Throughout the paper it is assumed that the operator P is *symmetric* and locally uniformly elliptic. Moreover, we assume that coefficients of P and the function W are real valued and locally sufficiently regular in Ω (see [14]). For such an operator P, potential W, and $\lambda \in \mathbb{R}$, we write $P_\lambda := P - \lambda W$.

The following well-known Agmon–Allegretto–Piepenbrink (AAP) theorem holds (see, for example, [2] and references therein).

Theorem 2.1 (the AAP theorem). *Suppose that P is symmetric, and let q be the corresponding quadratic form. Then $P \geqslant 0$ in Ω if and only if $q(\varphi) \geqslant 0$ for every $\varphi \in C_0^\infty(\Omega)$.*

We recall the following definitions.

Definition 2.2. Let q be the quadratic form on $C_0^\infty(\Omega)$ associated with a symmetric non-negative operator P in Ω. We say that a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ of non-negative functions is a *null sequence* of the quadratic form q in Ω if there exists an open set $B \Subset \Omega$ such that

$$\lim_{k\to\infty} q(\varphi_k) = 0 \quad \text{and} \quad \int_B |\varphi_k|^2 \, \mathrm{d}x = 1.$$

We say that a positive function $\phi \in C^\alpha_{\mathrm{loc}}(\Omega)$ is an *(Agmon) ground state* of the functional q in Ω if ϕ is an $L^2_{\mathrm{loc}}(\Omega)$ limit of a null-sequence of q in Ω.

Definition 2.3. Let $K \Subset \Omega$, and let u be a positive solution of the equation $Pw = 0$ in $\Omega \setminus K$. We say that u is *a positive solution of minimal growth in a neighbourhood of $\bar{\infty}$ in Ω* if for any $K \Subset K' \Subset \Omega$ with smooth boundary and any (regular) positive supersolution $v \in C((\Omega \setminus K') \cup \partial K')$ of the equation $Pw = 0$ in $\Omega \setminus K'$ satisfying $u \leqslant v$ on $\partial K'$, we have $u \leqslant v$ in $\Omega \setminus K'$.

Theorem 2.4 (Pinchover and Tintarev [34]). *Suppose P is a non-negative symmetric operator in Ω, and let q be the corresponding quadratic form. Then the following assertions are equivalent.*

(i) *The operator P is critical in Ω.*

(ii) *The quadratic form admits a null-sequence and a ground state ϕ in Ω.*

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(iii) *The equation $Pu = 0$ admits a unique positive supersolution ϕ in Ω.*

(iv) *The equation $Pu = 0$ admits a positive solution in Ω of minimal growth in a neighbourhood of $\bar{\infty}$ in Ω.*

In particular, any ground state is the unique positive (super)solution of the equation $Pu = 0$ in Ω, and it has minimal growth in a neighbourhood of $\bar{\infty}$.

Let P and $W \gneqq 0$ be as above. The *generalized principal eigenvalue* is defined by

$$\lambda_0 := \lambda_0(P, W, \Omega) := \sup\{\lambda \in \mathbb{R} \mid P_\lambda = P - \lambda W \geqslant 0 \text{ in } \Omega\}. \tag{2.1}$$

We also define

$$\lambda_\infty = \lambda_\infty(P, W, \Omega) := \sup\{\lambda \in \mathbb{R} \mid \exists K \subset\subset \Omega \text{ such that } P_\lambda \geqslant 0 \text{ in } \Omega \setminus K\}.$$

Recall that if the operator P is symmetric in $L^2(\Omega, \mathrm{d}x)$, and $W > 0$, then λ_0 (respectively, λ_∞) is the infimum of the $L^2(\Omega, W\,\mathrm{d}x)$-spectrum (respectively, $L^2(\Omega, W\,\mathrm{d}x)$-essential spectrum) of the Friedrichs extension of the operator $\tilde{P} := W^{-1}P$ (see, for example, [2] and references therein). Note that $\tilde{P}$ is symmetric on $L^2(\Omega, W\,\mathrm{d}x)$, and has the same quadratic form as P.

Definition 2.5. Let $\Omega \subsetneqq \mathbb{R}^n$ be a domain. We say that Ω is *weakly mean convex* if δ_Ω is weakly superharmonic in Ω.

Recall that $\delta_\Omega \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Also, any convex domain is of course weakly mean convex, and if $\partial\Omega \in C^2$, then Ω is weakly mean convex if and only if the mean curvature at any point of $\partial\Omega$ is non-negative (see, for example, [36]).

Throughout the paper we fix a cone

$$\Omega := \{x \in \mathbb{R}^n \mid r(x) > 0,\ \omega(x) \in \Sigma\}, \tag{2.2}$$

where Σ is a Lipschitz domain in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, $n \geqslant 2$. For $x \in \Sigma$, we will denote by $d_\Sigma(x)$ the (spherical) distance from x to the boundary of Σ. Note that δ_Ω is clearly a homogeneous function of degree 1, that is,

$$\delta_\Omega(x) = |x|\delta_\Omega\left(\frac{x}{|x|}\right) = r\delta_\Omega(\omega). \tag{2.3}$$

Since the distance function to the boundary of any domain is Lipschitz continuous, Euler's homogeneous function theorem implies that

$$x \cdot \nabla\delta_\Omega(x) = \delta_\Omega(x) \quad \text{almost everywhere in } \Omega. \tag{2.4}$$

In fact, Euler's theorem characterizes all sufficiently smooth positive homogeneous functions. Hence, (2.4) characterizes the cones in $\mathbb{R}^n$. For spectral results and Hardy inequalities with homogeneous weights on $\mathbb{R}^n$, see [23].

We note that if Σ is C^2, then

$$\delta_\Omega(\omega) = \sin(d_\Sigma(\omega)) \quad \text{near the boundary of } \Sigma. \tag{2.5}$$

Indeed, for $\omega \in \Sigma$, let $z \in \partial\Omega$ such that $|z - \omega| = \delta_\Omega(\omega)$, and let $y \in \partial\Sigma$ realize $d_\Sigma(\omega)$. Since Σ is C^2, if ω is close enough to $\partial\Sigma$, then z is unique and not equal to 0, and the points 0, z, y are collinear. Moreover, the acute angle between the vectors $\overrightarrow{0y}$ and $\overrightarrow{0\omega}$ is equal to $d_\Sigma(\omega)$. Given that $\overrightarrow{0z}$ is orthogonal to $\overrightarrow{\omega z}$, by elementary trigonometry in the triangle 0, ω, y, one gets that $\delta_\Omega(\omega) = \sin(d_\Sigma(\omega))$.

Let Δ_S be the Laplace–Beltrami operator on the unit sphere $S := \mathbb{S}^{n-1}$. Then in spherical coordinates, the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2}$$

has the following skew-product form:

$$P_\mu u(r,\omega) = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(-\Delta_S u - \mu \frac{u}{\delta_\Omega^2(\omega)}\right), \quad r > 0,\ \omega \in \Sigma. \tag{2.6}$$

For any Lipschitz cone the Hardy inequality holds true (as in the case of a sufficiently smooth bounded domain [27]). We have

Lemma 2.6. *Let Ω be a Lipschitz cone, and let $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$. Then*

$$0 < \mu_0 \leqslant \tfrac{1}{4}. \tag{2.7}$$

In other words, the Hardy inequality

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \mu_0 \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2}\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.8}$$

where $0 < \mu_0 \leqslant 1/4$ is the best constant, holds true.

Moreover, if Ω is a weakly mean convex domain, then $\mu_0 = 1/4$.

Proof. Using Rademacher's theorem, it follows that $\partial\Omega$ admits a tangent hyperplane almost everywhere in $\partial\Omega$. Hence, [27, theorem 5] implies that

$$\mu_0 = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \leqslant \lambda_\infty(-\Delta, \delta_\Omega^{-2}, \Omega) \leqslant \tfrac{1}{4}.$$

We claim that $\mu_0 > 0$. Indeed, denote by Ω_R the truncated cone

$$\Omega_R := \{x \in \mathbb{R}^n \mid 0 < r < R,\ \omega \in \Sigma\}; \tag{2.9}$$

then

$$0 < \lambda_{0,R} := \lambda_0(-\Delta, \delta_{\Omega_R}^{-2}, \Omega_R)$$

(see, for example, [27, 31]). By comparison,

$$\mu_0 \leqslant \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) \quad \text{and} \quad 0 < \lambda_{0,R} \leqslant \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R).$$

It is well known that if $\{\Omega_k\}$ is an exhaustion of Ω, then

$$\lim_{k\to\infty} \lambda_0(P, W, \Omega_k) = \lambda_0(P, W, \Omega).$$

Hence,

$$\lim_{R\to\infty} \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \mu_0.$$

On the other hand, since δ_Ω is homogeneous of order 1, it follows that

$$\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R)$$

is R-independent. Therefore,

$$0 < \lambda_{0,1} \leqslant \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_1) = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \lim_{R\to\infty} \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \mu_0.$$

Consequently,

$$\mu_0 = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) > 0.$$

Assume furthermore that Ω is a convex cone, or even a weakly mean convex cone. Then it is well known that $\mu_0 = 1/4$ (see, for example, [8, 27]). □

Remark 2.7. Clearly, P_μ is subcritical in Ω for all $\mu < \mu_0$, and by proposition 5.8, $P_{1/4}$ is subcritical in a weakly mean convex cone. We show in theorem 5.6 that if $\mu_0 < 1/4$ and $\Sigma \in C^2$, then the operator P_{μ_0} is critical in the cone Ω (cf. [27, theorem II]).

3. Positive multiplicative solutions

As above, let Ω be a Lipschitz cone. By lemma 2.6 the generalized principal eigenvalue $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ satisfies $0 < \mu_0 \leqslant 1/4$. We have the following theorem.

Theorem 3.1. *Let $\mu \leqslant \mu_0$. Then the equation $P_\mu u = 0$ in Ω admits positive solutions of the form*

$$u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu\left(\frac{x}{|x|}\right), \tag{3.1}$$

where ϕ_μ is a positive solution of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right)\phi_\mu = \sigma(\mu)\phi_\mu \quad \text{in } \Sigma, \tag{3.2}$$

$$-\frac{(n-2)^2}{4} \leqslant \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, 1, \Sigma\right), \tag{3.3}$$

and

$$\gamma_\pm := \frac{2 - n \pm \sqrt{(n-2)^2 + 4\sigma(\mu)}}{2}. \tag{3.4}$$

Moreover, if $\sigma(\mu) > -(n-2)^2/4$, then there are two linearly independent positive solutions of the equation $P_\mu u = 0$ in Ω of the form (3.1), and P_μ is subcritical in Ω.

In particular, for any $\mu \leqslant \mu_0$ we have $\sigma(\mu) > -\infty$.

Proof. We first note that if u is a positive solution of the form (3.1), then clearly $\phi_\mu > 0$ and ϕ_μ solves (3.2), and $\gamma_\pm$ satisfies (3.4).

Fix a reference point $x_1 \in \Omega \cap \mathbb{S}^{n-1}$, and consider an *exhaustion* $\{\Sigma_k\}_{k=1}^\infty \subset \Sigma \subset \mathbb{S}^{n-1}$ of Σ (i.e. $\{\Sigma_k\}_{k=1}^\infty$ is a sequence of smooth, relatively compact domains in Σ such that $x_1 \in \Sigma_k \Subset \Sigma_{k+1}$ for $k \geqslant 1$, and $\bigcup_{k=1}^\infty \Sigma_k = \Sigma$).

Fix $\mu \leqslant \mu_0$. For $k \geqslant 1$, define the cone

$$\mathcal{W}_k := \{x \in \mathbb{R}^n \mid r > 0,\ \omega \in \Sigma_k\}.$$

Consider the convex set $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$ of all positive solutions u of the equation $P_\mu u = 0$ in $\mathcal{W}_k$ satisfying the Dirichlet boundary condition $u = 0$ on $\partial\mathcal{W}_k \setminus \{0\}$, and the normalization condition $u(x_1) = 1$.

Clearly, for $\mu \leqslant \mu_0$ we have

$$\mu \leqslant \lambda_0(-\Delta, \delta_\Omega^{-2}, \mathcal{W}_k) = \sup\{\lambda \in \mathbb{R} \mid \mathcal{K}^0_{P_\mu}(\mathcal{W}_k) \neq \emptyset\}.$$

Moreover, P_μ is subcritical in $\mathcal{W}_k$, and has Fuchsian-type singularities at the origin and at ∞. Hence, in view of [32, theorem 7.1], it follows that $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$, which is a convex compact set in the compact-open topology, has exactly two extreme points.

Next, we characterize the two extreme points of $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$ using two different approaches.

First method We use the results of [25, § 8]. Consider the multiplicative group $\mathcal{G} := \mathbb{R}^\star$ of all positive real numbers. Then $\mathcal{G}$ acts on $\overline{\mathcal{W}_k}\setminus\{0\}$ (and also on $\bar{\Omega}\setminus\{0\}$) by homotheties $x \mapsto sx$, where $s \in \mathcal{G}$ and $x \in \overline{\mathcal{W}_k} \setminus \{0\}$. This is a compactly generating (cocompact) abelian group action, and P_μ is an invariant elliptic operator with respect to this action on $\mathcal{W}_k$. In spherical coordinates, a positive $\mathcal{G}$-multiplicative function on $\mathcal{W}_k$ is of the form

$$f(r, \omega) = r^\gamma \phi(\omega), \tag{3.5}$$

where $\gamma \in \mathbb{R}$. We remark that positive solutions in $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$ satisfy a uniform boundary Harnack principle on $\partial\mathcal{W}_k \setminus \{0\}$. Recall that $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$ has exactly two extreme points. Hence, by [25, theorems 8.7 and 8.8], $\lambda_0(-\Delta, \delta_\Omega^{-2}, \mathcal{W}_k) > \mu$, and the two extreme points in $\mathcal{K}^0_{P_\mu}(\mathcal{W}_k)$ are positive $\mathcal{G}$-multiplicative solutions of the equation $P_\mu u = 0$ in $\mathcal{W}_k$, and therefore they have the form

$$u_{\pm,k}(r, \omega) = r^{\gamma_{\pm,k}} \phi_{\pm,k}(\omega). \tag{3.6}$$

In particular, $\phi_{\pm,k}$ vanish on Σ_k.

Using the spherical coordinates representation (2.6) of P_μ, it follows that $\phi_{\pm,k}$ are positive in Σ, satisfy $\phi_{\pm,k}(x_1) = 1$, and solve the eigenvalue Dirichlet problem

$$\left.\begin{aligned} \left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right)\phi_{\pm,k} &= (\gamma_{\pm,k}^2 + \gamma_{\pm,k}(n-2))\phi_{\pm,k} \quad \text{in } \Sigma_k, \\ \phi_\pm &= 0 \quad \text{on } \partial\Sigma_k. \end{aligned}\right\} \tag{3.7}$$

On the other hand, since the operator $-\Delta_S - \mu\delta_\Omega^{-2}$ has up to the boundary regular coefficients in Σ_k, it admits a unique (Dirichlet) eigenvalue σ_k with a positive eigenfunction ϕ_k satisfying $\phi_k(x_1) = 1$. Moreover, σ_k is simple. In other words, σ_k and ϕ_k are respectively the *principal* eigenvalue and eigenfunction of $-\Delta_S - \mu\delta_\Omega^{-2}$ in Σ_k.

Hence, $\phi_{\pm,k}$ are equal to ϕ_k, and

$$\sigma_k := \sigma_k(\mu) = (\gamma_{\pm,k}^2 + \gamma_{\pm,k}(n-2)).$$

By the strict monotonicity with respect to bounded domains of the principal eigenvalue of second-order elliptic operators with up to the boundary regular coefficients, it follows that $\sigma_k(\mu) > \sigma_{k+1}(\mu)$.

On the other hand, since

$$u_{\pm,k}(r, \omega) = r^{\gamma_{\pm,k}} \phi_k(\omega) > 0, \tag{3.8}$$

it follows that $\gamma_{-,k} \neq \gamma_{+,k}$, and $\gamma_{\pm,k}$ are given by

$$\gamma_{\pm,k} := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma_k}}{2}.$$

In particular,

$$\gamma_{-,k} < \gamma_{-,k+1} < \frac{2-n}{2} < \gamma_{+,k+1} < \gamma_{+,k} \quad \text{and} \quad \sigma_k > -\frac{(n-2)^2}{4}.$$

Second method We only indicate briefly the second approach. We use the results of [29]. By (2.6), the subcritical elliptic operator P_μ has a skew-product form in $\mathcal{W}_k = \mathbb{R}_+ \times \Sigma_k$ and satisfies the conditions of [29, theorem 1.1]. Therefore, the equation $P_\mu u = 0$ admits two Martin functions of the form (3.6).

Now, let $k \to \infty$. Then $\sigma_k \searrow \sigma \geqslant -(n-2)^2/4$, and up to a subsequence $\phi_k \to \phi_\mu$ locally uniformly in Σ. Clearly, σ does not depend on the exhaustion of Σ. Recall also that for any non-negative second-order elliptic operator L in a domain D and any exhaustion $\{D_k\}$ of D we have

$$\lambda_0(L, W, D) = \lim_{k\to\infty} \lambda_0(L, W, D_k).$$

Hence, $\sigma = \sigma(\mu) = \lambda_0(-\Delta_S - \mu\delta_\Omega^{-2}, 1, \Sigma)$.

Consequently, $\gamma_{\pm,k} \to \gamma_\pm$, where $\gamma_- \leqslant -(n-2)/2 \leqslant \gamma_+$. Hence, we have that

$$\lim_{k\to\infty} u_{\pm,k}(r,\omega) = \lim_{k\to\infty} r^{\gamma_{\pm,k}}\phi_k(\omega) = r^{\gamma_\pm}\phi_\mu(\omega).$$

If $\gamma_- < -(n-2)/2 < \gamma_+$ (or equivalently, $\sigma(\mu) > -(n-2)^2/4$), then we obtain two linearly independent $\mathcal{G}$-multiplicative positive solutions of the equation $P_\mu u = 0$ in Ω. In particular, P_μ is subcritical in Ω. □

Remark 3.2. Note that for $n = 2$, $\Sigma = \mathbb{S}^1$, and $\mu = \mu_0 = 0$, we obtain $\sigma(0) = 0$, $\gamma_\pm = 0$, and $P_0 = -\Delta$ is critical in the cone $\mathbb{R}^2 \setminus \{0\}$.

Remark 3.3. Let Σ be a bounded domain in a smooth Riemannian manifold M, and let d_Σ be the Riemannian distance function to the boundary $\partial\Sigma$. If Σ is smooth enough, then the Hardy inequality with respect to the weight $(d_\Sigma)^{-2}$ holds in Σ with a positive constant C_H [37]. A sufficient condition for the validity of such a Hardy inequality is that Σ is *boundary distance regular*, and this condition holds true if Σ satisfies either the *uniform interior cone condition* or the *uniform exterior ball condition* (see the definitions in [37]). For other sufficient conditions for the validity of the Hardy inequality on Riemannian manifolds, see, for example, [28].

Hence, if the cone $\Omega \subsetneqq \mathbb{R}^n \setminus \{0\}$ is smooth enough, then $\Sigma \subset \mathbb{S}^{n-1}$ is boundary distance regular. So, for such $\Sigma \subset \mathbb{S}^{n-1}$, there exists $C > 0$ such that $-\Delta_S - Cd_\Sigma^{-2} \geqslant 0$ in Σ. Note that $d_\Sigma(\omega) \asymp \delta_\Omega(\omega)|_\Sigma$ in Σ, and therefore $-\Delta_S - C_1\delta_\Omega^{-2} \geqslant 0$ in Σ for some $C_1 > 0$.

In what follows we shall need the following lemma concerning the criticality of the operator $\mathcal{L}_\mu := -\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu)$ in Σ.

https://doi.org/10.1017/S0308210516000056 Published online by Cambridge University Press

Lemma 3.4. *Consider the operator* $\mathcal{L}_\mu = -\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu)$ *on* Σ.

(1) *We have*

$$\mu_0 = \lambda_0\biggl(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\biggr). \tag{3.9}$$

(2) *Assume that* $\Sigma \in C^2$, *and* $\mu_0 < 1/4$. *Then* $\sigma(\mu_0) = -(n-2)^2/4$, *and* $\mathcal{L}_{\mu_0}$ *is critical in* Σ *with ground state* $\phi_{\mu_0} \in L^2(\Sigma, \delta_\Omega^{-2}\mathrm{d}S)$.

(3) *Assume that* $\Sigma \in C^2$, *and* $\mu_0 = 1/4$. *Then* $\mathcal{L}_{1/4}$ *is critical in* Σ *with ground state* $\phi_{1/4} \in L^2\ (\Sigma, \delta_\Omega^{-2}\log(\delta_\Omega)^{-(1+\varepsilon)}\mathrm{d}S)$, *where* ε *is any positive number.*

(4) *Assume that* $\mu < \mu_0$. *Then* $\mathcal{L}_\mu$ *is positive critical in* Σ, *that is,* $\mathcal{L}_\mu$ *admits a ground state* ϕ_μ *in* Σ, *and* $\phi_\mu \in L^2(\Sigma)$.

In particular, in all the above cases, ϕ_μ *is (up to a multiplicative constant) the unique positive (super)solution of the equation* $\mathcal{L}_\mu u = 0$ *in* Σ, *and* $\phi_\mu \in L^2(\Sigma)$.

Proof. (1) To prove (3.9) we note that theorem 3.1 implies that for $\mu \leqslant \mu_0$ there exists a positive solution ϕ_μ of

$$\mathcal{L}_\mu u = \biggl(-\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)\biggr)u = 0 \quad \text{in } \Sigma,$$

and, since for any $\mu \leqslant \mu_0$ we have $\sigma(\mu) \geqslant -(n-2)^2/4$, it follows that ϕ_μ is a positive supersolution of the equation

$$\mathcal{L}_\mu u = \biggl(-\Delta_S - \frac{\mu}{\delta_\Omega^2} + \frac{(n-2)^2}{4}\biggr)u = 0 \quad \text{in } \Sigma.$$

Thus, by the AAP theorem (theorem 2.1) we obtain

$$\mu_0 \leqslant \lambda_0\biggl(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\biggr).$$

Let us now take $\mu > \mu_0$, and assume by contradiction that $-\Delta_S + (n-2)^2/4 - \mu\delta_\Omega^{-2} \geqslant 0$ in Σ. Then, by definition, there is a positive solution ϕ_μ of the equation

$$\biggl(-\Delta_S - \frac{\mu}{\delta_\Omega^2} + \frac{(n-2)^2}{4}\biggr)u = 0 \quad \text{in } \Sigma.$$

If one defines

$$\psi(x) = |x|^{(2-n)/2}\phi_\mu\biggl(\frac{x}{|x|}\biggr),$$

then it is immediate to check that ψ is a positive solution of

$$\biggl(-\Delta - \frac{\mu}{\delta_\Omega^2}\biggr)u = 0 \quad \text{in } \Omega.$$

This implies that

$$\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \geqslant \mu > \mu_0,$$

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which is a contradiction. Thus, the operator $-\Delta_S + (n-2)^2/4 - \mu\delta_\Omega^{-2}$ cannot be non-negative in Σ for $\mu > \mu_0$, and this implies that

$$\mu_0 \geqslant \lambda_0\biggl(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\biggr).$$

Hence, (3.9) is proved.

(2) Since

$$d_\Sigma(x) \sim \delta_\Omega(x) \quad \text{as } x \in \Sigma,\ d_\Sigma(x) \to 0,$$

and in light of the proof of [27, theorem 5], our assumption that Σ is C^2 implies that

$$\lambda_\infty(-\Delta_S, \delta_\Omega^{-2}, \Sigma) = \tfrac{1}{4},$$

which in turn implies that

$$\lambda_\infty\biggl(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\biggr) = \tfrac{1}{4}.$$

On the other hand, by (1) we have

$$\lambda_0\biggl(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\biggr) = \mu_0.$$

Hence, our assumption that $\mu_0 < 1/4$ implies that there is a spectral gap between the bottom of the $L^2(\Sigma, \delta_\Omega^{-2}\mathrm{d}S)$-spectrum and the bottom of the essential spectrum of the operator $-\Delta_S + (n-2)^2/4$ in Σ. Consequently, the operator $-\Delta_S + (n-2)^2/4 - \mu_0\delta_\Omega^{-2}$ is critical in Σ, with ground state $\phi_{\mu_0} \in L^2(\Sigma, \delta_\Omega^{-2}\mathrm{d}S)$. Clearly, the criticality of $-\Delta_S + (n-2)^2/4 - \mu_0\delta_\Omega^{-2}$ in Σ implies that

$$\sigma(\mu_0) = -\frac{(n-2)^2}{4},$$

and the second part of the lemma is proved.

Before proving (3), we prove (4).

(4) The assumption that $\mu < \mu_0$ clearly implies that $\lambda_\infty(-\Delta_S - \mu\delta_\Omega^{-2}, 1, \Sigma) = \infty$. Hence,

$$-\frac{(n-2)^2}{4} \leqslant \sigma(\mu) = \lambda_0\biggl(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, 1, \Sigma\biggr) < \lambda_\infty(-\Delta_S - \mu\delta_\Omega^{-2}, 1, \Sigma) = \infty.$$

Since λ_0 (respectively, λ_∞) is the bottom of the L^2-spectrum (respectively, essential L^2-spectrum) of the operator $-\Delta_S - \mu\delta_\Omega^{-2}$ in Σ, it follows that the operator $\mathcal{L}_\mu$ is critical in Σ, and $\sigma(\mu)$ is the principal eigenvalue of the operator $-\Delta_S - \mu\delta_\Omega^{-2}$ with principal eigenfunction $\phi_\mu \in L^2(\Sigma)$. Hence, the operator $\mathcal{L}_\mu$ is positive critical in Σ.

(3) The proof uses a modification of Agmon's trick (see [3, theorem 2.7] and also [27, lemma 7]). In order to prove that $\lambda_\infty(-\Delta_S - 1/(4\delta_\Omega^2), 1, \Sigma) = \infty$, we will show

that for suitable positive constants c, ε, the function $\delta_\Omega^{1/2} - \delta_\Omega/2$ is a positive supersolution of the equation

$$\left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right)u = 0 \tag{3.10}$$

in a sufficiently small neighbourhood of the boundary of Σ.

We start by denoting a tubular neighbourhood of $\partial\Sigma$ having width $\beta > 0$ by

$$\Sigma_\beta := \{\omega \in \Sigma \mid d_\Sigma(\omega) < \beta\}.$$

Recall that since Σ is C^2, there exists $\beta_* > 0$ such that $d_\Sigma \in C^2$ in Σ_{β_*}. In particular, $-\Delta_S d_\Sigma$ is bounded on Σ_{β_*}. Also $|\nabla_S d_\Sigma| = 1$ and $\delta_\Omega = \sin(d_\Sigma)$ (by (2.5)), both on Σ_{β_*}. We may thus compute

$$-\Delta_S\delta_\Omega = -\cos(d_\Sigma)\Delta_S d_\Sigma + \sin(d_\Sigma) \quad \text{on } \Sigma_{\beta_*},$$

which implies that $\Delta_S\delta_\Omega$ is also bounded on Σ_{β_*}. In particular, we have

$$-\Delta_S\delta_\Omega(\omega) \geqslant -h \quad \text{for all } \omega \in \Sigma_{\beta_*}, \tag{3.11}$$

for some $h > 0$. Now let $c, \varepsilon > 0$ and compute on Σ_{β_*},

$$\begin{aligned}
&\left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right)\left(\delta_\Omega^{1/2} - \frac{\delta_\Omega}{2}\right) \\
&\quad = -\frac{1}{4\delta_\Omega^{3/2}}(1 - |\nabla_S\delta_\Omega|^2) - \frac{1}{2\delta_\Omega^{1/2}}(1 - \delta_\Omega^{1/2})\Delta_S\delta_\Omega + \frac{1}{8\delta_\Omega} - c\delta_\Omega^{1/2-\varepsilon} + \frac{c\delta_\Omega^{1-\varepsilon}}{2} \\
&\quad \geqslant -\frac{\delta_\Omega^{1/2}}{4} - \frac{h}{2\delta_\Omega^{1/2}}(1 - \delta_\Omega^{1/2}) + \frac{1}{8\delta_\Omega} - c\delta_\Omega^{1/2-\varepsilon} + \frac{c\delta_\Omega^{1-\varepsilon}}{2},
\end{aligned}$$

where we have used the fact that $1 - |\nabla_S\delta_\Omega|^2 = \sin^2(d_\Sigma) = \delta_\Omega^2$ on Σ_{β_*} and also (3.11). Clearly, by fixing ε in $(0, 3/2)$ we obtain that this estimate blows up to $+\infty$ as $\omega \in \Sigma_{\beta_*}$ approaches the boundary of Σ. Thus, for a smaller $\beta_* > 0$ if necessary, we proved that $\delta_\Omega^{1/2} - \delta_\Omega/2$ is a positive supersolution of (3.10) in Σ_{β_*}. The APP theorem (theorem 2.1) implies that

$$\int_{\Sigma_{\beta_*}} \left(|\nabla u|^2 - \frac{1}{4\delta_\Omega^2}\right)\varphi^2\,\mathrm{d}S \geqslant c\int_{\Sigma_{\beta_*}} \frac{\varphi^2}{\delta_\Omega^\varepsilon}\,\mathrm{d}S \quad \forall\varphi \in C_0^\infty(\Sigma_{\beta_*}), \tag{3.12}$$

which together with $\lim_{d_\Sigma(\omega)\to 0}\delta_\Omega^{-\varepsilon}(\omega) = \infty$ implies that

$$\lambda_\infty\left(\Delta_S - \frac{1}{4\delta_\Omega^2}, 1, \Sigma\right) = \infty.$$

As in the proof of (2), one concludes that $\mathcal{L} = \Delta_S - 1/(4\delta_\Omega^2) - \sigma(\mu)$ is critical, with ground state $\phi_{1/4} \in L^2(\Sigma)$.

It remains to show that in fact, $\phi_{1/4} \in L^2(\Sigma, \delta_\Omega^{-2}\log^{-(1+\varepsilon)}(\delta_\Omega)\,\mathrm{d}S)$. In fact, the arguments used in the proof of [27, lemma 9] show that, as $\omega \in \Sigma$ and $\delta_\Omega(\omega) \to 0$,

$$\phi_{1/4}(\omega) \asymp \delta_\Omega^{1/2}(\omega).$$

This implies that $\phi_{1/4} \in L^2(\Sigma, \delta_\Omega^{-2}\log^{-(1+\varepsilon)}(\delta_\Omega)\,\mathrm{d}S)$ for any $\varepsilon > 0$. □

PROPOSITION 3.5. *Let* $\sigma(\mu) = \lambda_0(-\Delta_S - \mu\delta_\Omega^{-2}, 1, \Sigma)$. *Then:*

(1) $\sigma(\mu) \geqslant -(n-2)^2/4$ *for any* $\mu \leqslant \mu_0$, *and if* $\Sigma \in C^2$ *and* $\mu_0 < 1/4$, *then* $\sigma(\mu_0) = -(n-2)^2/4$;

(2) $\sigma(\mu) = -\infty$ *for any* $\mu > 1/4$;

(3) *if* $\Sigma \in C^2$, *then* $\sigma(\mu) > -\infty$ *for all* $\mu \leqslant 1/4$.

Proof. (1) Recall that by lemma 2.6 we have that $0 < \mu_0 \leqslant 1/4$, and by theorem 3.1 $\sigma(\mu) \geqslant -(n-2)^2/4$ for all $\mu \leqslant \mu_0$. Moreover, by lemma 3.4, if $\Sigma \in C^2$ and $\mu_0 < 1/4$, then $\sigma(\mu_0) = -(n-2)^2/4$. In particular, for such a μ we have that $\sigma(\mu)$ is finite.

(2) Let $\mu > 1/4$, and suppose that $\sigma(\mu)$ is finite. Then one can find a positive function ϕ satisfying

$$(-\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu))\phi = 0 \quad \text{in } \Sigma.$$

Take $\varepsilon > 0$ such that $\mu - \varepsilon > 1/4$. Clearly,

$$\lim_{\omega \to \partial\Sigma} \delta_\Omega^{-2}(\omega) = \infty \quad \text{and} \quad \lim_{\omega \to \partial\Sigma} \frac{\delta_\Omega(\omega)}{d_\Sigma(x)} = 1,$$

where d_Σ is the Riemannian distance to the boundary of Σ. Hence, ϕ is a positive supersolution of the equation

$$(-\Delta_S - (\mu - \varepsilon)d_\Sigma^{-2})u = 0$$

in a neighbourhood of ∞ in Σ.

On the other hand, as in [27], if Σ is a Lipschitz domain, then $\lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) \leqslant 1/4$. Consequently, for such ε, one gets a contradiction to $\lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) \leqslant 1/4$.

(3) Suppose first that $\mu < 1/4$. Recall that since $\Sigma \in C^2$, we have

$$\lambda_\infty(-\Delta_S, \delta_\Omega^{-2}, \Sigma) = \lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) = 1/4.$$

Take $\varepsilon > 0$ such that $\mu + \varepsilon < 1/4$. Let ϕ be a positive solution of the equation

$$(-\Delta_S - (\mu + \varepsilon)\delta_\Omega^{-2})u = 0$$

in a neighbourhood of ∞ in Σ, and let $\tilde{\phi}$ be a nice positive function in Σ such that $\tilde{\phi} = \phi$ in a neighbourhood of $\partial\Sigma$. Then, for σ large enough, $\tilde{\phi}$ is a positive supersolution of the equation $(-\Delta_S - \mu\delta_\Omega^{-2} + \sigma)u = 0$ in Σ. Hence, $\sigma(\mu) > -\infty$ for all $\mu < 1/4$.

Suppose now that $\mu = 1/4$. By (3.10), $\psi := \delta_\Omega^{1/2} - \delta_\Omega/2$ is a positive supersolution of

$$\left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right)u = 0$$

outside a compact set $K_\varepsilon \Subset \Sigma$. Let $\tilde{\psi}$ be a nice positive function in Σ such that $\tilde{\psi} = \psi$ in a neighbourhood of $\partial\Sigma$. Hence, for σ large enough, $\tilde{\psi}$ is a positive supersolution of the equation $(-\Delta_S - 1/4\delta_\Omega^2 + \sigma)u = 0$ in Σ. Hence, $\sigma(1/4) > -\infty$. □

Remark 3.6. In lemma 3.4 and proposition 3.5 it is assumed that $\Sigma \in C^2$. The extension of the proposition to the class of Lipschitz domains remains open. We recall that by the recent result of Barbatis and Lamberti [7, proposition 1], the Hardy constant of a bounded domain is Lipschitz continuous as a function of bi-Lipschitz maps that approximate the domain. It seems that finding for a given Lipschitz domain a uniform bi-Lipschitz smooth approximation is a non-trivial problem: we note that in [12, theorem 1] Daneri and Pratelli proved that bi-Lipschitz homeomorphisms can be approximated by smooth ones in the $W^{1,p}$ topology for $p < \infty$. However, to apply the results in [7], we would need $W^{1,\infty}$-approximations.

We conclude this section with the following general result that provides us with a sufficient condition for the criticality of a Schrödinger operator on a precompact domain. For a general sufficient condition, see [33].

Lemma 3.7. *Let $P = -\Delta + V$ be a non-negative Schrödinger operator on a compact Riemannian manifold with boundary M, endowed with its Riemannian measure $\mathrm{d}x$. Denote by $\delta = \delta_M$ the distance function to the boundary of M. Assume that $M \in C^2$, V is smooth in the interior of M, and that the equation $Pu = 0$ in M admits a positive solution $\phi \in L^2(M, \delta^{-2}\log^{-2}(\delta)\,\mathrm{d}x)$. Then P is critical in M with ground state ϕ and, furthermore, there exists a null-sequence $\{\phi_k\}_{k=0}^{\infty}$ for P, which converges locally uniformly and in L^2 to φ.*

Proof. If q denotes the quadratic form of P, then using the ground state transform (see, for example, [14]) we have for every $\varphi \in C_0^\infty(M)$,

$$q(\phi\varphi) = \int_M \phi^2 |\nabla\varphi|^2 \,\mathrm{d}x.$$

This formula extends easily to every Lipschitz continuous function φ that is compactly supported in M. For $k \geqslant 2$, let us define $v_k\colon \mathbb{R}_+ \to [0,1]$ by

$$v_k(t) = \begin{cases} 0 & 0 \leqslant t \leqslant 1/k^2, \\ 1 + \dfrac{\log(kt)}{\log k} & 1/k^2 < t < 1/k, \\ 1 & t \geqslant 1/k. \end{cases}$$

Note that $0 \leqslant v_k(\delta) \leqslant 1$, and $\{v_k(\delta)\}_{k\geqslant 2}$ converges pointwise to the constant function 1 in M. Define

$$\phi_k := v_k(\delta)\phi.$$

Then, using that $\phi \in L^2_{\mathrm{loc}}$, one sees that $\{\phi_k\}_{k=0}^{\infty}$ converges locally uniformly, and hence in L^2_{loc} to ϕ. We now prove that $\{\phi_k\}_{k=2}^{\infty}$ is a null-sequence for P, which implies that P is critical with ground state ϕ. If $K \Subset M$ is a fixed precompact open set, then, clearly, there is a positive constant C such that, for k big enough,

$$\int_K \phi_k^2 \,\mathrm{d}x \asymp 1.$$

Thus, in order to prove that $\{\phi_k\}_{k=2}^{\infty}$ is a null-sequence for P, it is enough to prove that

$$\lim_{k\to\infty} \int_M \phi^2 |\nabla v_k(\delta)|^2 \,\mathrm{d}x = 0. \tag{3.13}$$

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Since $|\nabla\delta(x)| \leqslant 1$ almost everywhere in M, it is enough to show that

$$\lim_{k\to\infty}\int_M \phi^2|v_k'(\delta)|^2\,\mathrm{d}x = 0.$$

We compute

$$\int_M \phi^2|v_k'(\delta)|^2\,\mathrm{d}x = \int_{\{1/k^2<\delta<1/k\}}\left(\frac{\phi}{\delta\log(k)}\right)^2\mathrm{d}x \leqslant 4\int_{\{\delta<1/k\}}\left(\frac{\phi}{\delta\log(\delta)}\right)^2\mathrm{d}x.$$

By our hypothesis, the function $\phi^2\delta^{-2}\log^{-2}(\delta)$ is integrable on $\{\delta < 1/2\}$, and hence

$$\lim_{k\to\infty}\int_{\{\delta<1/k\}}\left(\frac{\phi}{\delta\log(\delta)}\right)^2\mathrm{d}x = 0,$$

which shows (3.13). Thus, $\{\phi_k\}_{k\geqslant 2}$ is a null-sequence for P. □

4. The structure of $\mathcal{K}^0_{P_\mu}(\Omega)$

As above, let Ω be a Lipschitz cone. By lemma 2.6, the generalized principal eigenvalue $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ satisfies $0 < \mu_0 \leqslant 1/4$.

For $\mu \leqslant \mu_0$, denote by $\mathcal{K}^0_{P_\mu}(\Omega)$ the convex set of all positive solutions u of the equation $P_\mu u = 0$ in Ω satisfying the normalization condition $u(x_1) = 1$, and the Dirichlet boundary condition $u = 0$ on $\partial\Omega\setminus\{0\}$ in the sense of the Martin boundary. That is, any $u \in \mathcal{K}^0_{P_\mu}(\Omega)$ has minimal growth on $\partial\Omega\setminus\{0\}$. For the definition of minimal growth on a portion Γ of $\partial\Omega$, see [32].

If $\mu_0 < 1/4$ and Σ is C^2, then in theorem 5.6 (to be proved in what follows) we show that the operator P_{μ_0} is critical in Ω, and therefore the equation $P_{\mu_0}u = 0$ in Ω admits (up to a multiplicative constant) a unique positive supersolution. Moreover, by theorem 3.1, the unique positive solution is a multiplicative solution of the form (3.1).

The following theorem characterizes the structure of $u \in \mathcal{K}^0_{P_\mu}(\Omega)$ for any $\mu < \mu_0$.

Theorem 4.1. *Let $\mu < \mu_0 \leqslant 1/4$. Then $\mathcal{K}^0_{P_\mu}(\Omega)$ is the convex hull of two linearly independent positive solutions of the equation $P_\mu u = 0$ in Ω of the form*

$$u_\pm(x) = |x|^{\gamma_\pm}\phi_\mu\left(\frac{x}{|x|}\right), \tag{4.1}$$

where ϕ_μ is the unique positive solution of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right)\phi_\mu = \sigma(\mu)\phi_\mu \quad \text{in } \Sigma, \tag{4.2}$$

$$-\frac{(n-2)^2}{4} < \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, 1, \Sigma\right), \tag{4.3}$$

and

$$\gamma_\pm := \frac{2-n\pm\sqrt{(2-n)^2+4\sigma(\mu)}}{2}. \tag{4.4}$$

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Proof. The assumption that $\mu < \mu_0$ implies that the operator P_μ is subcritical in Ω. In particular, $\mu < 1/4$, and therefore there exists $\varepsilon > 0$ such that the operator $P_{\mu+\varepsilon}$ is subcritical in a small neighbourhood of a given portion of $\partial\Omega \setminus \{0\}$. Since the operator P_μ and the cone Ω are invariant under scaling, it follows from the local Harnack inequality, and from the boundary Harnack principle of Ancona for the operator P_μ in Ω [4] (see also [6]), that the following uniform boundary Harnack principle holds true in the annulus $A_R \subset \Omega$. There exists $C > 0$ (independent of R) such that

$$C^{-1}\frac{v(x)}{v(y)} \leqslant C^{-1}\frac{u(x)}{u(y)} \leqslant C\frac{v(x)}{v(y)} \quad \forall x, y \in A_R, \tag{4.5}$$

for any $u, v \in \mathcal{K}^0_{P_\mu}(\Omega)$ and $R > 0$.

Hence, we can use directly the arguments in [32] to obtain that in the subcritical case the convex set $\mathcal{K}^0_{P_\mu}(\Omega)$ has exactly two extreme points. Moreover, we can use directly the method of [25, § 8], to obtain that u is an extreme point of $\mathcal{K}^0_{P_\mu}(\Omega)$ if and only if it is a positive multiplicative solution in $\mathcal{K}^0_{P_\mu}(\Omega)$. Thus, the two extreme points of $\mathcal{K}^0_{P_\mu}(\Omega)$ are of the form

$$u_\pm(x) = |x|^{\gamma_\pm}\phi_\pm\left(\frac{x}{|x|}\right),$$

where $\phi_\pm > 0$ in Σ, and solves the equation

$$\left(-\Delta_S - \frac{\mu}{\delta^2_\Omega(\omega)}\right)\phi_\pm = \sigma_\pm\phi_\pm \quad \text{in } \Sigma, \tag{4.6}$$

$$-\frac{(n-2)^2}{4} \leqslant \sigma_\pm \leqslant \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta^2_\Omega}, 1, \Sigma\right) \tag{4.7}$$

and

$$\gamma_\pm := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma_\pm}}{2}. \tag{4.8}$$

If $\gamma_+ = \gamma_-$, then (4.5) implies that $u_+ \asymp u_-$. Since $u_\pm(x)$ are two extreme points, and $\mathcal{K}^0_{P_\mu}(\Omega)$ has exactly two extreme points, it follows that $\gamma_+ \neq \gamma_-$. Therefore, $\sigma_\pm = \sigma$, where $-(n-2)^2/4 < \sigma \leqslant \sigma(\mu)$ and $\gamma_\pm$ satisfy

$$\gamma_\pm := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma}}{2}. \tag{4.9}$$

Moreover, since $\phi_\pm$ solve the same equation in Σ, and $\mathcal{K}^0_{P_\mu}(\Omega)$ has exactly two extreme points, it follows that $\phi_\pm = \phi$.

Note that by lemma 3.4, ϕ is a positive solution of minimal growth near $\partial\Sigma$ if and only if $\sigma = \sigma(\mu)$. On the other hand, $u_\pm$ have minimal growth near $\partial\Omega \setminus \{0\}$. Therefore, $\phi = \phi_\mu$ and $\sigma = \sigma(\mu)$, where ϕ_μ is a ground state satisfying (4.2), and $\sigma(\mu)$ and $\gamma_\pm$ satisfy (4.3) and (4.4), respectively. □

5. The main result

This section is devoted to our main result concerning the existence of an optimal Hardy weight for the operator P_μ, which is defined in a cone Ω. In theorem 5.4 we

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prove the case in which $\mu < \mu_0$ and Ω is a Lipschitz cone, while in theorem 5.6 we prove the $\mu = \mu_0$ case under the assumption that $\Sigma \in C^2$.

Let us recall that by theorem 3.1, if $\mu \leqslant \mu_0$, then

$$\sigma(\mu) := \lambda_0\biggl(-\Delta - \frac{\mu}{\delta_\Omega^2}, 1, \Sigma\biggr) \geqslant -\frac{(n-2)^2}{4},$$

and there exists a positive solution ϕ_μ of the equation

$$\biggl(-\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)\biggr)u = 0 \quad \text{in } \Sigma.$$

Furthermore, by lemma 3.4, the operator

$$\mathcal{L} := \mathcal{L}_\mu = -\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)$$

is critical (for any $\mu < \mu_0$, and also for $\mu = \mu_0$ if in addition $\Sigma \in C^2$), and ϕ_μ is the ground state of $\mathcal{L}$.

We first prove the following proposition.

PROPOSITION 5.1. *Let Ω be a Lipschitz cone. Let $\mu \leqslant \mu_0$, and let*

$$\lambda(\mu) := \frac{(2-n)^2 + 4\sigma(\mu)}{4}. \tag{5.1}$$

Then $\lambda(\mu) \geqslant 0$, and the following Hardy inequality holds true in Ω:

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \mu\int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2}\,\mathrm{d}x \geqslant \lambda(\mu)\int_\Omega \frac{|\varphi|^2}{|x|^2}\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega). \tag{5.2}$$

Proof. The fact that $\lambda(\mu) \geqslant 0$ follows from $\sigma(\mu) \geqslant -(n-2)^2/4$, which has been proved in theorem 3.1. Define

$$\psi(x) = |x|^{(2-n)/2}\phi_\mu\biggl(\frac{x}{|x|}\biggr).$$

Then, taking into account that

$$\biggl(-\Delta_S - \sigma(\mu) - \frac{\mu}{\delta_\Omega^2}\biggr)\phi_\mu = 0 \quad \text{in } \Sigma,$$

and writing P_μ in spherical coordinates (2.6), it follows that ψ is a positive solution of the equation

$$(P_\mu - \lambda(\mu)|x|^{-2})u = 0 \quad \text{in } \Omega.$$

Thus, the operator $P_\mu - \lambda(\mu)|x|^{-2}$ is non-negative in Ω, and so (5.2) holds by the AAP theorem (theorem 2.1). □

REMARK 5.2. In the $\mu < \mu_0$ case, the Hardy inequality (5.2) can be obtained using the *supersolution construction* of [14]: indeed, by theorem 4.1, the equation $P_\mu u = 0$ has two linearly independent, positive solutions in Ω, of the form

$$u_\pm(x) = |x|^{\gamma_\pm}\phi_\mu\biggl(\frac{x}{|x|}\biggr).$$

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By the *supersolution construction* (see [14, lemma 5.1]), the positive function

$$u_{1/2} := (u_+u_-)^{1/2} = |x|^{(2-n)/2}\phi_\mu\left(\frac{x}{|x|}\right)$$

is a solution of

$$\left(P_\mu - \frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2}\right)u = 0 \quad \text{in } \Omega.$$

It is easy to check that

$$\frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2} = \frac{\lambda(\mu)}{|x|^2},$$

and by the AAP theorem, the Hardy inequality (5.2) holds.

Remark 5.3. In the $\mu \leqslant \mu_0$ case, the Hardy inequality (5.2) can also be obtained using spherical coordinates, Fubini's theorem, and the well-known one-dimensional Hardy inequality

$$\int_0^\infty (v')^2 t^{n-1}\,\mathrm{d}t \geqslant \left(\frac{n-2}{2}\right)^2 \int_0^\infty v^2 t^{n-3}\,\mathrm{d}t, \tag{5.3}$$

valid for all functions $v \in H^1(\mathbb{R}_+)$ that vanish at ∞.

Indeed, suppose that $\varphi \in C_c^\infty(\Omega)$. Then we have that φ_{Σ_r}, the restriction of φ on Σ_r, is in $C_c^\infty(\Sigma)$. Consequently, by the definition of $\sigma(\mu)$, it follows that for all $\varphi \in C_c^\infty(\Omega)$ and each $r > 0$ we have

$$\int_{\Sigma_r} |\nabla_\omega \varphi|^2\,\mathrm{d}S_r - \mu \int_{\Sigma_r} \frac{\varphi^2}{\delta_\Omega^2(\omega)}\,\mathrm{d}S_r \geqslant \sigma(\mu) \int_{\Sigma_r} \varphi^2\,\mathrm{d}S_r.$$

Multiplying this by r^{-2} and integrating in $\mathbb{R}_+$ with respect to r, we arrive at

$$\int_0^\infty \int_{\Sigma_r} \frac{|\nabla_\omega \varphi|^2}{r^2}\,\mathrm{d}S_r\,\mathrm{d}r - \mu \int_0^\infty \int_{\Sigma_r} \frac{\varphi^2}{r^2\delta_\Omega^2(\omega)}\,\mathrm{d}S_r\,\mathrm{d}r \geqslant \sigma(\mu) \int_0^\infty \int_{\Sigma_r} \frac{\varphi^2}{r^2}\,\mathrm{d}S_r\,\mathrm{d}r.$$

Recall that in spherical coordinates we have

$$|\nabla\varphi|^2 = \frac{|\nabla_\omega \varphi|^2}{r^2} + \varphi_r^2,$$

and taking into account (2.3), the last inequality is written as

$$\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \mu \int_\Omega \frac{\varphi^2}{\delta_\Omega^2(x)}\,\mathrm{d}x \geqslant \sigma(\mu) \int_\Omega \frac{\varphi^2}{|x|^2}\,\mathrm{d}x + \int_\Sigma \int_0^\infty \varphi_r^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}S,$$

where we have used Fubini's theorem on the last term. Applying (5.3) in the inner integral of the last term and using Fubini's theorem again, we obtain (5.2).

We now investigate the optimality of the Hardy inequality (5.2) when $\mu < \mu_0$.

Theorem 5.4. *Let Ω be a Lipschitz cone, and let $\mu < \mu_0$. Then $\lambda(\mu) > 0$. Furthermore, the weight $W := \lambda(\mu)|x|^{-2}$ is an optimal Hardy weight for the operator P_μ in Ω in the following sense.*

(1) *The operator* $P_\mu - \lambda(\mu)|x|^{-2}$ *is critical in* Ω*, i.e. the Hardy inequality*

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \,\mathrm{d}x \geqslant \int_\Omega V(x)|\varphi|^2 \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega)$$

holds true for $V \geqslant W$ *if and only if* $V = W$*. In particular,*

$$\lambda_0\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

(2) *The constant* $\lambda(\mu)$ *is also the best constant for (5.2) with test functions supported either in* Ω_R *or in* $\Omega \setminus \overline{\Omega_R}$*, where* Ω_R *is a fixed truncated cone of the form (2.9). In particular,*

$$\lambda_\infty\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

(3) *The operator* $P_\mu - \lambda(\mu)|x|^{-2}$ *is null critical at* 0 *and at* ∞ *in the following sense: for any* $R > 0$ *the (Agmon) ground state of the operator* $P_\mu - \lambda(\mu)|x|^{-2}$ *given by*

$$v(x) := |x|^{(2-n)/2}\phi_\mu\left(\frac{x}{|x|}\right)$$

satisfies

$$\int_{\Omega_R}\left(|\nabla v|^2 - \mu\frac{|v|^2}{\delta_\Omega^2}\right)\mathrm{d}x = \int_{\Omega\setminus\overline{\Omega_R}}\left(|\nabla v|^2 - \mu\frac{|v|^2}{\delta_\Omega^2}\right)\mathrm{d}x = \infty.$$

In particular, the variational problem

$$\inf_{\varphi\in\mathcal{D}^{1,2}_{P_\mu}(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \mu\int_\Omega(|\varphi|^2/\delta_\Omega^2)\,\mathrm{d}x}{\int_\Omega(|\varphi|^2/|x|^2\,\mathrm{d}x)}$$

does not admit a minimizer.

(4) *The spectrum and the essential spectrum of the Friedrichs extension of the operator* $W^{-1}P_\mu = \lambda(\mu)^{-1}|x|^2 P_\mu$ *on* $L^2(\Omega, W\,\mathrm{d}x)$ *are both equal to* $[1,\infty)$*.*

REMARK 5.5. As was pointed out in remark 5.2, if $\mu < \mu_0$, then the Hardy inequality (5.2) can be obtained by applying the *supersolution construction* from [14]. Thus, theorem 5.4 extends theorem 1.1 to the particular singular case, where Ω is a cone and P_μ is the Hardy operator (which is singular on $\partial\Omega$).

Proof of theorem 5.4. In light of our assumption that $\mu < \mu_0 \leqslant 1/4$, it follows that the operator P_μ is subcritical in Ω. Moreover, by theorem 4.1, $\sigma(\mu) > -(n-2)^2/4$, so $\lambda(\mu) > 0$. For such a μ, consider the operator $\mathcal{L} = \mathcal{L}_\mu$ on $\Sigma \subset \mathbb{S}^{n-1}$ defined by

$$\mathcal{L} = -\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu),$$

with the corresponding non-negative quadratic form

$$q_{\mathcal{L}}(\psi) = \int_\Sigma\left(|\nabla_\omega\psi|^2 - \mu\frac{|\psi|^2}{\delta_\Omega^2} - \sigma(\mu)|\psi|^2\right)\mathrm{d}S, \quad \text{where } \psi \in C_0^\infty(\Sigma).$$

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Notice that by lemma 3.4, $\mathcal{L}$ is critical in Σ with the ground state $\phi_\mu \in L^2(\Sigma)$. We normalize ϕ_μ so that $\int_\Sigma \phi_\mu^2 \, \mathrm{d}S = 1$.

On the other hand, it is well known that the operator

$$\mathcal{R} := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2}$$

is critical on $\mathbb{R}_+$, and $r^{(2-n)/2}$ is its ground state. Indeed, the corresponding quadratic form $q_{\mathcal{R}}$ of $\mathcal{R}$ (endowed with the measure $r^{n-1}\,\mathrm{d}r$) is given by

$$q_{\mathcal{R}}(u) = \int_0^\infty \left[(u')^2 - \frac{(n-2)^2}{4}\frac{u^2}{r^2}\right] r^{n-1}\,\mathrm{d}r, \quad u \in C_0^\infty(\mathbb{R}_+),$$

and gives rise to the critical operator $\mathcal{R}$ on $\mathbb{R}_+$.

Recall that in spherical coordinates $P_\mu - W$ has the skew-product form

$$P_\mu - W = \mathcal{R} \otimes \mathcal{I}_\Sigma - \frac{\mathcal{I}_{\mathbb{R}_+}}{r^2} \otimes \mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2} + \frac{1}{r^2}\mathcal{L},$$

where $\mathcal{I}_A$ is the identity operator on A. Consequently, it is natural to construct a null-sequence for $P_\mu - W$ of the product form

$$\{\varphi_k(r,\omega)\}_{k=1}^\infty = \{u_k(r)\phi_k(\omega)\}_{k=1}^\infty$$

that converges locally uniformly to $r^{(2-n)/2}\phi_\mu(\omega)$, and by theorem 2.4, this implies that the operator $P_\mu - W$ is critical and $r^{(2-n)/2}\phi_\mu(\omega)$ is its ground state.

Let $\{u_k(r)\}_{k=1}^\infty$ be a null sequence for the critical operator $\mathcal{R}$ on $\mathbb{R}_+$, converging locally uniformly to $r^{(2-n)/2}$. So,

$$q_{\mathcal{R}}(u_k) \to 0, \qquad \int_1^2 (u_k)^2 r^{n-1}\,\mathrm{d}r = 1.$$

On the other hand, let $\{\phi_k(\omega)\}_{k=1}^\infty$ be (up to the normalization constants) the sequence of ground states defined by (3.7) on Σ_k, so that

$$\int_\Sigma \phi_k^2\,\mathrm{d}S = 1 \quad \text{and} \quad q_{\mathcal{L}}(\phi_k) = (\sigma_k(\mu) - \sigma(\mu))\int_\Sigma \phi_k^2\,\mathrm{d}S \to 0.$$

Note that the normalization of ϕ_k is different from the one used in the proof of theorem 3.1. Recall that the operator $\mathcal{L}_{\mu_0} = -\Delta_S - \mu_0\delta_\Omega^{-2} - \sigma(\mu_0)$ is non-negative on Σ. Therefore,

$$\frac{\mu\sigma(\mu_0)}{\mu_0}\int_\Sigma \phi_k^2\,\mathrm{d}S + \mu\int_\Sigma \frac{|\phi_k|^2}{\delta_\Omega^2}\,\mathrm{d}S \leqslant \frac{\mu}{\mu_0}\int_\Sigma |\nabla_\omega \phi_k|^2\,\mathrm{d}S. \tag{5.4}$$

On the other hand,

$$\int_\Sigma |\nabla_\omega\phi_k|^2\,\mathrm{d}S = \sigma_k\int_\Sigma \phi_k^2\,\mathrm{d}S + \mu\int_\Sigma \frac{\phi_k^2}{\delta_\Omega^2}\,\mathrm{d}S. \tag{5.5}$$

By (5.4) and (5.5) we get

$$\left(1 - \frac{\mu}{\mu_0}\right)\int_\Sigma |\nabla_\omega\phi_k|^2\,\mathrm{d}S \leqslant \left(\sigma_k - \frac{\mu\sigma(\mu_0)}{\mu_0}\right)\int_\Sigma \phi_k^2\,\mathrm{d}S \leqslant \sigma_1 - \frac{\mu\sigma(\mu_0)}{\mu_0}. \tag{5.6}$$

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Since $\mu < \mu_0$, one gets that $\{\phi_k\}$ is bounded in $W_0^{1,2}(\Sigma)$, and therefore (up to a subsequence) $\{\phi_k\}$ converges, in L^2 and locally uniformly, to ϕ, a positive solution of $\mathcal{L}u = 0$ in Σ with $\int_\Sigma \phi^2 \, \mathrm{d}S = 1$. Since $\mathcal{L}$ is critical in Σ, $\phi = \phi_\mu$. Hence, by the Harnack inequality,

$$\int_{\Sigma_1} \phi_k^2 \, \mathrm{d}S \asymp 1,$$

and therefore $\{\phi_k\}$ is a null-sequence.

We claim that there exists a subsequence $\{k_l\} \subset \mathbb{N}$ such that $\{u_l(r)\phi_{k_l}(\omega)\}$ is a null-sequence for the operator $P_\mu - W$ in Ω that converges locally uniformly to $r^{(2-n)/2}\phi_\mu(\omega)$.

Indeed, fix the pre-compact open set $B := \{(r,\omega) \mid r \in (1,2),\ \omega \in \Sigma_1\}$. Note that for the quadratic form Q of $P_\mu - W$ in Ω, if $u = u(r)$ is compactly supported in $\mathbb{R}_+$ and $\psi = \psi(\omega)$ is compactly supported in Σ, we have

$$Q(u(r)\psi(\omega)) = q_{\mathcal{R}}(u)\|\psi\|_2^2 + \left(\int_0^\infty u^2(r) r^{n-3} \, \mathrm{d}r\right) q_{\mathcal{L}}(\psi).$$

For each k, notice that by the definition of a null-sequence, u_k is compactly supported in $\mathbb{R}_+$. So, for $l \geqslant 1$, let $\{k_l\}_{l=1}^\infty$ be a subsequence such that

$$q_{\mathcal{R}}(u_l)\|\phi_{k_l}\|_2^2 = q_{\mathcal{R}}(u_l) < \frac{1}{l}$$

and

$$\left(\int_0^\infty u_l^2(r) r^{n-3} \, \mathrm{d}r\right) q_{\mathcal{L}}(\phi_{k_l}) < \frac{1}{l}.$$

Thus, $\lim_{l\to\infty} Q(u_l(r)\phi_{k_l}(\omega)) = 0$.

On the other hand, $\{u_l(r)\phi_{k_l}\}$ converges uniformly in B to $r^{(2-n)/2}\phi_\mu(\omega)$, and hence $\int_B (u_l(r)\phi_{k_l}(\omega))^2 \, \mathrm{d}x \asymp 1$.

Therefore, $\{u_l(r)\phi_{k_l}(\omega)\}_{l=1}^\infty$ is indeed a null-sequence for $P_\mu - W$. It follows that $P_\mu - W$ is critical in Ω with the ground state $r^{(2-n)/2}\phi_\mu(\omega)$. Moreover, since $\mathcal{R}$ is null critical around 0 and ∞ it follows that $P_\mu - W$ is in fact null-critical around 0 and ∞.

Next we prove that the spectrum of $W^{-1}P_\mu$ is $[1,\infty)$. Let us keep our assumption that ϕ_μ is normalized so that $\|\phi_\mu\|_2 = 1$. If $\xi \in \mathbb{R}$, then it easily checked (cf. [14]) that

$$\left(\mathcal{R} - \frac{(n-2)^2\xi^2}{|x|^2}\right)(r^{n-2})^{\mathrm{i}\xi - 1/2} = 0,$$

and therefore

$$\left(P_\mu - \left(1 + \frac{(n-2)^2}{\lambda(\mu)}\xi^2\right)W\right)((r^{n-2})^{\mathrm{i}\xi-1/2}\phi_\mu(\omega)) = 0. \tag{5.7}$$

Define the subspace $\mathcal{E}$ of $L^2(\Omega, W\,\mathrm{d}x)$ that consists of all functions of the form $u(r)\phi_\mu(\omega)$, where $u \in L^2(\mathbb{R}_+, r^{n-1}\lambda(\mu)/r^2\,\mathrm{d}r)$. We are going to define a spectral representation of $W^{-1}P_\mu$ restricted to the subspace $\mathcal{E}$. Notice that the measure on $\mathcal{E}$ is $r^{n-1}\lambda(\mu)/(r^2)\,\mathrm{d}r \otimes \mathrm{d}S$, so that

$$\mathcal{E} = L^2\left(\mathbb{R}_+, r^{n-1}\frac{\lambda(\mu)}{r^2}\,\mathrm{d}r\right) \otimes \operatorname{span}\{\phi_\mu\}.$$

Recall that the classical Mellin transform is the unitary operator $\mathcal{M}\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ defined by

$$\mathcal{M}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(r) r^{\mathrm{i}\xi - 1/2}\,\mathrm{d}r.$$

Consider the composition $\mathcal{C}$ of the unitary operator

$$\mathcal{U}\colon L^2\biggl(\mathbb{R}_+, r^{n-1}\frac{\lambda(\mu)}{r^2}\,\mathrm{d}r\biggr) \to L^2(\mathbb{R}_+)$$

given by

$$f(r) \mapsto \sqrt{\frac{\lambda(\mu)}{n-2}} f(r^{1/(n-2)}),$$

with the Mellin transform $\mathcal{M}$. Define

$$\mathcal{T}\colon \mathcal{E} \mapsto L^2(\mathbb{R}); \quad \mathcal{T}(u(r)\phi_\mu(\omega)) = (\mathcal{C}u)(\xi) = (\mathcal{M}(\mathcal{U}(u)))(\xi).$$

So, $\mathcal{T}$ is a unitary operator. By (5.7), the operator $\mathcal{T}(W^{-1}P_\mu)\mathcal{T}^{-1}$ is the multiplication by the real function $(1 + (n-2)^2\xi^2/\lambda(\mu))$ on $L^2(\mathbb{R})$, with values in $[1, \infty)$. Therefore, the spectrum of $W^{-1}P_\mu$, restricted to $\mathcal{E}$, is $[1, \infty)$. So, the spectrum of $W^{-1}P_\mu$ on $L^2(\Omega, W\,\mathrm{d}x)$ contains $[1, \infty)$. But the Hardy inequality (5.2) implies that the spectrum of $W^{-1}P_\mu$ must be included in $[1, \infty)$. Hence, the spectrum of $W^{-1}P_\mu$ on $L^2(\Omega, W\,\mathrm{d}x)$ is $[1, \infty)$.

For $k \geqslant 2$, define the subspace $\mathcal{E}_k$ (respectively, $\mathcal{E}_{1/k}$) of $L^2(\Omega, W\,\mathrm{d}x)$ consisting of functions of the form $u(r)\phi(\omega)$, where $u \in L^2((k,\infty), r^{n-1}\lambda(\mu)/r^2\,\mathrm{d}r)$ (respectively, $u \in L^2((0,1/k), r^{n-1}\lambda(\mu)/r^2\,\mathrm{d}r)$). Denote by $\mathcal{P}_k$ (respectively, $\mathcal{P}_{1/k}$) the restriction of P_μ to $\mathcal{E}_k$ (respectively, $\mathcal{E}_{1/k}$), with Dirichlet boundary conditions at $\{k\} \times \Sigma$ (respectively, at $\{1/k\} \times \Sigma$). Notice that by symmetry considerations (under $x \mapsto x^{-1}$), the spectrum of $W^{-1}\mathcal{P}_k$ and the spectrum of $W^{-1}\mathcal{P}_{1/k}$ are equal. Moreover, by the fact that the essential spectrum is stable under compactly supported perturbations, and since the discrete spectrum of $W^{-1}P_\mu$ is empty, the spectrum of $W^{-1}P_\mu$ is equal to the union of the spectrum of $W^{-1}\mathcal{P}_k$, and of the spectrum of $W^{-1}\mathcal{P}_{1/k}$. Thus, the spectra of $W^{-1}\mathcal{P}_k$ and $W^{-1}\mathcal{P}_{1/k}$ are both equal to $[1, \infty)$.

Also, the best constant C_0 for the validity of the Hardy inequality

$$\int_{\mathcal{V}_0} \biggl(|\nabla\varphi|^2 - \frac{\mu}{\delta_\Omega^2}\varphi^2\biggr)\,\mathrm{d}x \geqslant C_0 \int_{\mathcal{V}_0} W\varphi^2\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\mathcal{V}_0),$$

in $\mathcal{V}_0$, an arbitrarily small neighbourhood of zero, is equal to the bottom of the essential spectrum of $W^{-1}\mathcal{P}_{1/k}$ (for any $k \geqslant 2$). Thus, it is equal to 1. Similarly, using $W^{-1}\mathcal{P}_k$ instead, one concludes that the best constant C_∞ for the validity of the Hardy inequality

$$\int_{\mathcal{V}_\infty} \biggl(|\nabla\varphi|^2 - \frac{\mu}{\delta_\Omega^2}\varphi^2\biggr)\,\mathrm{d}x \geqslant C_\infty \int_{\mathcal{V}_\infty} W\varphi^2\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\mathcal{V}_\infty),$$

in $\mathcal{V}_\infty$, an arbitrarily small neighbourhood at ∞, is equal to 1. This finishes the proof of theorem 5.4. □

We now turn to the $\mu = \mu_0$ case, for which we need to assume more regularity on Σ.

THEOREM 5.6. *Assume that* $\Sigma \in C^2$.

(1) *If* $\mu_0 < 1/4$, *then* $\lambda(\mu_0) = 0$, *and the operator* P_{μ_0} *is critical in* Ω, *and null-critical around* 0 *and* ∞. *In particular, the Hardy inequality*

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \mu_0 \int_\Omega \frac{\varphi^2}{\delta_\Omega^2}\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega)$$

cannot be improved.

(2) *If* $\mu_0 = 1/4$ *and* $\lambda(1/4) = 0$, *then the operator* $P_{1/4}$ *is critical in* Ω, *and null-critical around* 0 *and* ∞. *In particular, the Hardy inequality*

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \frac{1}{4} \int_\Omega \frac{\varphi^2}{\delta_\Omega^2}\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega)$$

cannot be improved.

(3) *If* $\mu_0 = 1/4$ *and* $\lambda(1/4) > 0$, *then the weight* $W_{1/4} := \lambda(1/4)|x|^{-2}$ *is optimal in the sense of theorem 5.4. In particular, the Hardy inequality (5.2) cannot be improved. Moreover, the spectrum and the essential spectrum of the Friedrichs extension of the operator* $(W_{1/4})^{-1}P_{1/4}$ *on* $L^2(\Omega, W_{1/4}\mathrm{d}x)$ *are both equal to* $[1,\infty)$.

Proof. Define $W(x) := \lambda(\mu_0)|x|^{-2}$. Let us start by proving that in all cases, $P_{\mu_0} - W$ is critical. Recall that in spherical coordinates $P_{\mu_0} - W$ has the following skew-product form:

$$P_{\mu_0} - W = \mathcal{R} \otimes \mathcal{I}_\Sigma - \frac{\mathcal{I}_{\mathbb{R}_+}}{r^2} \otimes \mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2} + \frac{1}{r^2}\mathcal{L}_{\mu_0}.$$

So, as in the first part of the proof of theorem 5.4, it is natural to construct a null-sequence for $P_{\mu_0} - W$ of the product form

$$\{\varphi_k(r,\omega)\}_{k=1}^\infty = \{u_k(r)\phi_k(\omega)\}_{k=1}^\infty$$

that converges locally uniformly to $r^{(2-n)/2}\phi_{\mu_0}(\omega)$.

As in the proof of theorem 5.4, let $\{u_k(r)\}_{k=1}^\infty$ be a null sequence for the critical operator $\mathcal{R}$ on $\mathbb{R}_+$, converging locally uniformly to $r^{(2-n)/2}$. So,

$$q_{\mathcal{R}}(u_k) \to 0, \qquad \int_1^2 (u_k)^2 r^{n-1}\,\mathrm{d}r = 1.$$

However, the definition of $\{\phi_k\}$ differs from the one of theorem 5.4. Let us normalize ϕ_{μ_0} so that $\int_\Sigma \phi_{\mu_0}^2\,\mathrm{d}S = 1$ (by lemma 3.4, $\phi_\mu \in L^2(\Sigma)$). By lemmas 3.4 and 3.7, there exists a null sequence $\{\phi_k\}$ for $\mathcal{L}_{\mu_0}$, converging locally uniformly and in $L^2(\Sigma)$ to ϕ_{μ_0}. Thus, normalizing ϕ_k so that

$$\int_\Sigma \phi_k^2\,\mathrm{d}S = 1,$$

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one has for k large enough, by the Harnack inequality,

$$\int_{\Sigma_1} \phi_k^2 \, \mathrm{d}S \asymp 1.$$

Let $B = \{(r, \omega) \mid r \in (1, 2),\ \omega \in \Sigma_1\}$. We now choose the subsequence $\{k_l\} \subset \mathbb{N}$ as in the proof of theorem 5.4: let $\{k_l\}_{l=1}^{\infty}$ be a subsequence such that

$$q_{\mathcal{R}}(u_l)\|\phi_{k_l}\|_2^2 = q_{\mathcal{R}}(u_l) < \frac{1}{l}$$

and

$$\left(\int_0^{\infty} u_l^2(r) r^{n-3} \, \mathrm{d}r\right) q_{\mathcal{L}}(\phi_{k_l}) < \frac{1}{l}.$$

The same computation made in the proof of theorem 5.4 shows that

$$\lim_{l \to \infty} Q(u_l(r)\phi_{k_l}(\omega)) = 0 \quad \text{and} \quad \int_B (u_l(r)\phi_{k_l}(\omega))^2 \, \mathrm{d}x \asymp 1,$$

so that $\{u_l(r)\phi_{k_l}(\omega)\}_{l=1}^{\infty}$ is indeed a null sequence for $P_\mu - W$. It follows that $P_\mu - W$ is critical in Ω with a ground state $r^{(2-n)/2}\phi_\mu(\omega)$. Moreover, since $\mathcal{R}$ is null critical around 0 and ∞ it follows that $P_\mu - W$ is in fact null-critical around 0 and ∞.

(1) Assume now that $\mu_0 < 1/4$. By the first part of the proof, the operator $P_{\mu_0} - \lambda(\mu_0)|x|^{-2}$ is critical, and null-critical around 0 and ∞. By lemma 3.4, $\sigma(\mu_0) = -(n-2)^2/4$, so $\lambda(\mu_0) = 0$. It follows that P_{μ_0} is critical, and null-critical around 0 and ∞.

(2) Suppose that $\mu_0 = 1/4$, and $\lambda(1/4) = 0$. Then by the first part of the proof, the operator $P_{1/4} = P_{1/4} - \lambda(1/4)|x|^{-2}$ is critical, and null-critical around 0 and ∞.

(3) Assume that $\mu_0 = 1/4$, and $\lambda(1/4) > 0$. Then, following the proof of theorem 5.4, one concludes that W is an optimal weight for $P_{1/4}$. □

In the particular case of the half-space, we can compute the constants appearing in theorems 5.4 and 5.6.

Example 5.7 (see [14, example 11.9] and [19]). Let $\Omega = \mathbb{R}_+^n$, let $\mu \leqslant \mu_0 = 1/4$ and consider the subcritical operator $P_\mu := -\Delta - \mu x_1^{-2}$ in Ω. Let α_+ be the largest root of the equation $\alpha(1-\alpha) = \mu$, and let

$$\eta(\mu) := n - 1 + \sqrt{1 - 4\mu} = n - 2 + 2\alpha_+.$$

Then

$$v_0(x) := x_1^{\alpha_+}, \qquad v_1(x) := x_1^{\alpha_+}|x|^{-\eta(\mu)}$$

are two positive solutions of the equation $P_\mu u = 0$ in Ω that vanish on $\partial\Omega \setminus \{0\}$.

Therefore, $\lambda(\mu) = \eta^2(\mu)/4$, and for $\mu \leqslant \mu_0 = 1/4$ we have the following optimal Hardy inequality

$$\int_{\mathbb{R}_+^n} |\nabla\varphi|^2 \, \mathrm{d}x - \mu \int_{\mathbb{R}_+^n} \frac{\varphi^2}{x_1^2} \, \mathrm{d}x \geqslant \frac{\eta^2(\mu)}{4} \int_{\mathbb{R}_+^n} \frac{\varphi^2}{|x|^2} \, \mathrm{d}x \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}_+^n).$$

In particular, the operator $-\Delta - \mu x_1^{-2} - \lambda(\mu)|x|^{-2}$ is critical in $\mathbb{R}^n_+$ with the ground state $\psi(x) := x_1^{\alpha_+}|x|^{-\eta(\mu)/2}$. Note that for $\mu = 0$ we obtain the well-known (optimal) Hardy inequality (see [30])

$$\int_{\mathbb{R}^n_+} |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \frac{n^2}{4}\int_{\mathbb{R}^n_+} \frac{\varphi^2}{|x|^2}\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n_+),$$

while for $\mu = \mu_0 = 1/4$ we obtain the optimal double Hardy inequality (see [19])

$$\int_{\mathbb{R}^n_+} |\nabla\varphi|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{1}{x_1^2}\varphi^2\,\mathrm{d}x \geqslant \frac{(n-1)^2}{4}\int_{\mathbb{R}^n_+} \frac{\varphi^2}{|x|^2}\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n_+). \tag{5.8}$$

It turns out that in the weakly mean convex case, $\lambda(1/4)$ is always positive.

Proposition 5.8. *Assume that $\Sigma \in C^2$ and Ω is weakly mean convex. Then $\lambda(1/4) > 0$.*

Proof. Since Ω is weakly mean convex (i.e. $-\Delta\delta_\Omega \geqslant 0$ in Ω), it follows that $\delta_\Omega^{1/2}$ is a positive supersolution of $P_{1/4}u = 0$ in Ω. We proceed by contradiction: assume that $\lambda(1/4) = 0$. Then by theorem 5.6 the operator $P_{1/4}$ is critical, and therefore $\delta_\Omega^{1/2}$ is a positive solution of $P_{1/4}u = 0$ in Ω. Thus, necessarily $-\Delta\delta_\Omega = 0$ in the sense of distributions. Since $\delta_\Omega \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (or directly by Weyl's lemma) we have that δ_Ω is harmonic and in particular $\delta_\Omega \in C^\infty(\Omega)$. This means that the singular set of δ_Ω,

$$\begin{aligned}\mathrm{Sing}(\delta_\Omega) &:= \{x \in \Omega \mid \delta_\Omega(x) \text{ is achieved by more than one boundary point}\}\\ &= \{x \in \Omega \mid \delta_\Omega \text{ is not differentiable}\}\end{aligned}$$

(see, for example, [16, theorem 3.3]), is empty. In light of Motzkin's theorem [38, theorem 1.2.4], $\mathbb{R}^n \setminus \Omega$ is convex. We may without loss of generality assume that $0 \in \partial\Omega$. By considering a supporting hyperplane of $\mathbb{R}^n \setminus \Omega$ at 0, we find that necessarily $\mathbb{R}^n \setminus \Omega$ is included in a half-space. This implies that Σ contains a half-sphere. If this half-sphere is strictly contained in Σ, then $K := \mathbb{R}^n \setminus \Omega$ is a closed convex cone not containing a line (i.e. K is *pointed*). Hence, its dual cone K^*, and thus its polar cone $K^o = -K^* \subset \Omega$, has non-empty interior (see, for instance, [9, p. 53]). Clearly, $\delta_\Omega(x) = |x|$ whenever $x \in K^o$, but this contradicts the harmonicity of δ_Ω in Ω.

Hence, Σ is precisely a half-sphere, and thus Ω is a half-space. But by example 5.7, in the half-space $\{x_1 > 0\}$ we have $\lambda(1/4) = (n-1)^2/4 > 0$, and we have arrived at a contradiction. □

Assume that Ω is a domain admitting a supporting hyperplane H at zero. Without loss of generality, we may assume that $H = \partial\mathbb{R}^n_+$. Recall that in this case $\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \leqslant 1/4$ [27, theorem 5]. Also, $\delta_\Omega \leqslant \delta_H$ in Ω. Consequently, for appropriate test functions φ_ε supported in a relative small neighbourhood of the origin in Ω, we have that for $0 \leqslant \mu \leqslant 1/4$ the corresponding Rayleigh–Ritz quo-

https://doi.org/10.1017/S0308210516000056 Published online by Cambridge University Press

tients satisfy the inequality

$$\frac{\int_\Omega (|\nabla\varphi_\varepsilon|^2 - \mu(|\varphi_\varepsilon|^2/\delta_\Omega^2))\,\mathrm{d}x}{\int_\Omega (|\varphi_\varepsilon|^2/|x|^2)\,\mathrm{d}x} \leqslant \frac{\int_H (|\nabla\varphi_\varepsilon|^2 - \mu(|\varphi_\varepsilon|^2/\delta_H^2))\,\mathrm{d}x}{\int_H (|\varphi_\varepsilon|^2/|x|^2)\,\mathrm{d}x} = \frac{(n-1+\sqrt{1-4\mu})^2}{4} + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Thus, example 5.7 implies the following corollary.

COROLLARY 5.9. *Suppose that a domain Ω admits a supporting hyperplane at zero, and let $P_\mu = -\Delta - \mu\delta_\Omega^{-2}$, where $0 \leqslant \mu \leqslant 1/4$. Then*

$$\lambda_0(P_\mu, |x|^{-2}, \Omega) \leqslant \frac{(n-1+\sqrt{1-4\mu})^2}{4}.$$

6. On the optimality of an inequality by Filippas *et al.*

In this section we generalize examples 1.4 and 5.7 concerning the half-space $\mathbb{R}^n_+$. We consider the following family of Hardy inequalities in $\mathbb{R}^n_+$, obtained by Filippas *et al*. [19]:

$$\int_{\mathbb{R}^n_+} |\nabla\varphi|^2\,\mathrm{d}x \geqslant \int_{\mathbb{R}^n_+} \left(\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_1^2+x_2^2} + \cdots + \frac{\beta_n}{x_1^2+\cdots+x_n^2}\right)\varphi^2\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n_+). \tag{6.1}$$

According to [19, theorem A], the Hardy inequality (6.1) holds if and only if the β_is are of the form

$$\beta_1 = -\alpha_1^2 + \tfrac{1}{4}, \quad \beta_i = -\alpha_i^2 + (\alpha_{i-1} - \tfrac{1}{2})^2, \quad i = 2,\dots,n, \tag{6.2}$$

where the α_i, $i = 1,\dots,n$, are arbitrary real numbers. Without loss of generality, we can – and will – assume that the α_i, $i = 1,\dots,n$, in (6.2) are non-positive. Define

$$V(\beta_1,\dots,\beta_j) = \left(\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_1^2+x_2^2} + \cdots + \frac{\beta_j}{x_1^2+\cdots+x_j^2}\right), \quad j = 1,\dots,n.$$

Let $2^* = 2n/(n-2)$ be the Sobolev exponent. In [19, theorem B], it was shown that (6.1) can be improved by adding to the right-hand side a Sobolev term of the form $C(\int_{\mathbb{R}^n_+} |\varphi|^{2^*}\,\mathrm{d}x)^{2/2^*}$ if and only if $\alpha_n < 0$. Notice that, $\beta_1,\dots,\beta_{n-1}$ being fixed, taking $\alpha_n = 0$ corresponds to taking the greatest β_n possible in (6.2).

Our aim in this section is to show that when $\alpha_n = 0$ (and under an extra assumption on the α_i, $i = 1,\dots,n-1$), not only can one not add a Sobolev term, but in fact one cannot even add any term of the form $\int_{\mathbb{R}^n_+} W\varphi^2\,\mathrm{d}x$, $W \gneqq 0$, to the right-hand side of (6.1). In other words, if $\alpha_n = 0$, the operator $-\Delta - V(\beta_1,\dots,\beta_n)$ is *critical* in $\mathbb{R}^n_+$. This implies in particular (see [35]) that (6.1) cannot be improved by adding to the right-hand side any weighted Sobolev term of the form $C(\int_{\mathbb{R}^n_+} \rho|\varphi|^{2^*}\,\mathrm{d}x)^{2/2^*}$, where $\rho \gneqq 0$; an improvement of the result obtained in [19].

Theorem 6.1. *Consider the Hardy inequality (6.1), where the β_i, $i = 1, \dots, n$, are defined in terms of non-positive α_i, $i = 1, \dots, n$, by (6.2). Assume that $\alpha_n = 0$ and that $\alpha_1, \dots, \alpha_{n-1}$ are either all distinct or all negative. Then the operator $P := -\Delta - V(\beta_1, \dots, \beta_n)$ is critical in $\mathbb{R}^n_+$, i.e. the Hardy inequality (6.1) cannot be improved. Furthermore, the weight $\beta_n|x|^{-2}$ is an optimal weight for the subcritical operator $-\Delta - V(\beta_1, \dots, \beta_{n-1})$ in $\mathbb{R}^n_+$.*

Proof. Define $X_k(x) := (x_1, \dots, x_k, 0, \dots, 0)$. Let $(\beta_i)_{i=1}^n$ satisfy (6.2), and define

$$\psi(x) := |X_1|^{-\gamma_1}|X_2|^{-\gamma_2} \cdots |X_n|^{-\gamma_n},$$

where γ_i are defined by

$$\gamma_1 = \alpha_1 - \tfrac{1}{2}, \quad \gamma_i = \alpha_i - \alpha_{i-1} + \tfrac{1}{2}, \quad i = 2, \dots, n.$$

Then

$$\beta_1 = -\gamma_1(1 + \gamma_1), \quad \beta_i = -\gamma_i\biggl(2 - i + \gamma_i + 2\sum_{k=1}^{i-1}\gamma_k\biggr), \quad i = 2, \dots, n,$$

and, according to [19, (2.3)],

$$-\frac{\Delta\psi}{\psi} = V(\beta_1, \dots, \beta_n).$$

Hence, ψ is a positive solution of the equation $Pu = 0$ in $\mathbb{R}^n_+$. By the AAP theorem, this implies the validity of (6.1).

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ \setminus \{0\}$, define

$$r = |x|, \quad \omega = \frac{x}{|x|}, \quad \omega_i = \frac{x_i}{r}, \quad 1 \leqslant i \leqslant n.$$

Notice that $\omega \in \mathbb{S}_+ := \mathbb{S}^{n-1} \cap \{x_1 > 0\}$. Since $\alpha_n = 0$ we have

$$\psi(x) = \phi(\omega)r^{-\sum_{i=1}^n \gamma_i} = \phi(\omega)r^{(2-n)/2},$$

where

$$\phi(\omega) := \psi|_{\mathbb{S}_+} = \omega_1^{-\gamma_1}(\omega_1^2 + \omega_2^2)^{-\gamma_2/2} \cdots (\omega_1^2 + \dots + \omega_n^2)^{-\gamma_n/2}.$$

Define

$$W(\omega) := \frac{\beta_1}{\omega_1^2} + \dots + \frac{\beta_{n-1}}{\omega_1^2 + \dots + \omega_{n-1}^2},$$

and let

$$\mathcal{L} := -\Delta_{\mathbb{S}^{n-1}} - W(\omega) - \beta_n + \frac{(n-2)^2}{4}$$

and

$$\mathcal{R} := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2}.$$

Then, in spherical coordinates, P has the skew-product form

$$P = \mathcal{R} + \frac{1}{r^2}\mathcal{L}.$$

Recall that $\mathcal{R}$ is critical on $(0, \infty)$, and its ground state is $r^{(2-n)/2}$.

Lemma 6.2. *The operator $\mathcal{L}$ is critical on $\mathbb{S}_+$, with ground state $\phi \in L^2(\mathbb{S}_+)$.*

Once lemma 6.2 is proved, the rest of the proof of theorem 6.1 follows along the lines of the proof of theorem 5.4. □

Proof of lemma 6.2. We have

$$P\psi = 0 = \phi \mathcal{R} r^{(2-n)/2} + r^{-(n+2)/2} \mathcal{L}\phi.$$

Since

$$\mathcal{R} r^{-(n-2)/2} = 0 \quad \text{in } \mathbb{R}_+,$$

one concludes that

$$\mathcal{L}\phi = 0 \quad \text{in } \mathbb{S}_+.$$

For $x \in \mathbb{S}_+$, let ρ be the spherical distance function to $\partial\mathbb{S}_+ = \{\omega \in \mathbb{S}_+ \mid \omega_1 = 0\}$, the boundary of $\mathbb{S}_+$. Let $\mathrm{d}S$ be the Riemannian measure on $\mathbb{S}_+$. We claim that

$$\int_{\mathbb{S}_+ \cap \{\rho \leqslant 1/2\}} \left(\frac{\phi(\omega)}{\rho \log(\rho)} \right)^2 \mathrm{d}S < \infty. \tag{6.3}$$

Clearly, (6.3) implies that $\phi \in L^2(\mathbb{S}_+)$ and, moreover, by lemma 3.7, (6.3) implies that $\mathcal{L}$ is critical, with ground state ϕ. In fact, since ϕ is smooth in the interior of $\mathbb{S}_+$ and

$$\rho(\omega) \sim \omega_1(\omega) \quad \text{as } \omega \in \mathbb{S}_+ \text{ and } \rho(\omega) \to 0,$$

(6.3) is equivalent to

$$\int_{\mathbb{S}_+ \cap \{\omega_1 \leqslant 1/2\}} \left(\frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 \mathrm{d}S < \infty. \tag{6.4}$$

For $i = 1, \dots, n-1$, define

$$\mathcal{E}_i = \{\omega \in \mathbb{S}_+ \mid \omega_1 \leqslant \varepsilon,\ \ \dots,\ \omega_i^2 \leqslant \varepsilon,\ \omega_{i+1}^2 > \varepsilon\}.$$

Then all the $\mathcal{E}_i$ are disjoint, and if $\varepsilon < 1/n$, one can write the ε-neighbourhood $\mathbb{S}_+ \cap \{\omega_1 \leqslant \varepsilon\}$ of $\partial\mathbb{S}_+$ as the disjoint union

$$\mathbb{S}_+ \cap \{\omega_1 \leqslant \varepsilon\} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_{n-1}.$$

Notice that on $\mathcal{E}_i$,

$$\phi(\omega) \leqslant C_\varepsilon \omega_1^{-\gamma_1} (\omega_1^2 + \omega_2^2)^{-\gamma_2/2} \cdots (\omega_1^2 + \dots + \omega_i^2)^{-\gamma_i/2}.$$

Hence,

$$\int_{\mathcal{E}_i} \left(\frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 \mathrm{d}S \leqslant C_\varepsilon \int_{\mathcal{E}_i} \log^{-2}(\omega_1) \omega_1^{-2} \omega_1^{-2\gamma_1} \cdots (\omega_1^2 + \dots + \omega_i^2)^{-\gamma_i} \, \mathrm{d}S.$$

If ε is small enough, then on $\mathcal{E}_i$,

$$\mathrm{d}S \simeq \mathrm{d}\omega_1 \otimes \dots \otimes \mathrm{d}\omega_i \otimes \mathrm{d}\nu(\omega_1, \dots, \omega_i),$$

where $\mathrm{d}\nu(\omega_1, \ldots, \omega_i)$ is the standard Hausdorff measure on the $(n-i-1)$-sphere $\omega_{i+1}^2 + \cdots + \omega_n^2 = \sigma^2$, with $\sigma^2 = 1 - (\omega_1^2 + \cdots + \omega_i^2)$. Thus,

$$\int_{\mathcal{E}_i} \left(\frac{\phi(\omega)}{\omega_1 \log(\omega_1)}\right)^2 \mathrm{d}S \leqslant \tilde{C}_\varepsilon \int_{[0,\varepsilon]^i} \log^{-2}(\omega_1)\omega_1^{-2}\omega_1^{-2\gamma_1} \cdots (\omega_1^2 + \cdots + \omega_i^2)^{-\gamma_i} \,\mathrm{d}\omega_1 \cdots \mathrm{d}\omega_i. \tag{6.5}$$

For $\lambda_1, \ldots, \lambda_i$ real numbers and k integer, define

$$I_i(\lambda_1, \ldots, \lambda_i, k) := \int_{[0,\varepsilon]^i} \log^{-2}(\omega_1)\omega_1^{-2}\omega_1^{-2\lambda_1} \cdots (\omega_1^2 + \cdots + \omega_i^2)^{-\lambda_i} \times |\log^k(\omega_1^2 + \cdots + \omega_i^2)| \,\mathrm{d}\omega_1 \cdots \mathrm{d}\omega_i.$$

One has the elementary fact

$$I_i(\lambda_1, \ldots, \lambda_i, k) \leqslant C_\varepsilon \begin{cases} I_{i-1}(\lambda_1, \ldots, \lambda_{i-2}, \lambda_{i-1} + \lambda_i - 1/2, k), & \lambda_i > 1/2, \\ I_{i-1}(\lambda_1, \ldots, \lambda_{i-1}, k), & \lambda_i < 1/2, \\ I_{i-1}(\lambda_1, \ldots, \lambda_{i-1}, k+1), & \lambda_i = 1/2. \end{cases} \tag{6.6}$$

Case 1 (assume the α_k, $k = 1, \ldots, n-1$, are all distinct). Then, for every $2 \leqslant j \leqslant k \leqslant i$,

$$\gamma_j + \sum_{l=j+1}^{k} (\gamma_l - \tfrac{1}{2}) = \alpha_k - \alpha_{j-1} + \tfrac{1}{2} \neq \tfrac{1}{2}.$$

Moreover,

$$-2 - 2\gamma_1 - 2\sum_{j=2}^{k} (\gamma_j - \tfrac{1}{2}) = -2 - 2\alpha_k - (k-2) + (k-1) = -2\alpha_k - 1. \tag{6.7}$$

Thus, by using (6.6) i times in (6.5), and (6.7), one gets

$$\begin{aligned}\int_{\mathcal{E}_i} \left(\frac{\phi(\omega)}{\omega_1 \log(\omega_1)}\right)^2 \mathrm{d}S &\leqslant C \sum_{k=1}^{i} \int_0^\varepsilon \log(\omega_1)^{-2} \omega_1^{-2-2\gamma_1-2\sum_{j=2}^{k}(\gamma_j - 1/2)} \,\mathrm{d}\omega_1 \\ &\leqslant C \sum_{k=1}^{i} \int_0^\varepsilon \log(\omega_1)^{-2} \omega_1^{-2\alpha_k - 1} \,\mathrm{d}\omega_1,\end{aligned}$$

where by convention the sum

$$\sum_{j=2}^{k} (\gamma_j - 1/2)$$

is zero when $k = 1$. By hypothesis, $\alpha_k \leqslant 0$, and therefore $\log(\omega_1)^{-2}\omega_1^{-2\alpha_k - 1}$ is integrable at zero, and thus one concludes the validity of (6.3).

Case 2 (assume $\alpha_k < 0$, for all $k = 1, \dots, n-1$). Then, by using (6.6) i times in (6.5), and (6.7), one gets

$$\int_{\mathcal{E}_i} \left(\frac{\phi(\omega)}{\omega_1 \log(\omega_1)}\right)^2 \mathrm{d}S \leqslant C \sum_{k=1}^{i} \int_0^{\varepsilon} |\log^{n(k)}(\omega_1)| \omega_1^{-2-2\gamma_1-2\sum_{j=2}^{k}(\gamma_j-1/2)} \,\mathrm{d}\omega_1$$
$$\leqslant C \sum_{k=1}^{i} \int_0^{\varepsilon} |\log^{n(k)}(\omega_1)| \omega_1^{-2\alpha_k-1} \,\mathrm{d}\omega_1,$$

where $n(k)$ is an integer. Since $\alpha_k < 0$, the function $|\log^{n(k)}(\omega_1)|\omega_1^{-2\alpha_k-1}$ is integrable at zero, and therefore (6.3) holds. □

Remark 6.3. We believe that theorem 6.1 should hold in the general case, without any extra assumption on $\alpha_1, \dots, \alpha_{n-1}$. We leave this for future investigation.

7. A differential inequality

Throughout this section, Ω denotes a domain in $\mathbb{R}^n$ such that $0 \in \partial\Omega$, and $P_\mu = -\Delta - \mu\delta_\Omega^{-2}$. Our aim is to obtain a Hardy-type inequality with the best constant for the (non-negative) operator P_μ in Ω, assuming that δ_Ω satisfies the linear differential inequality

$$-\Delta\delta_\Omega + \frac{n-1+\sqrt{1-4\mu}}{|x|^2}(x \cdot \nabla\delta_\Omega - \delta_\Omega) \geqslant 0 \quad \text{in } \Omega. \tag{7.1}$$

The above differential inequality certainly holds true for any $\mu \leqslant 1/4$ if Ω is a weakly mean convex cone (see definition 2.5); it also holds for $\mu = 1/4$ if Ω is a ball touching the origin (see remark 7.2).

For $\mu = 1/4$, (7.1) is equivalent to the differential inequality

$$-|x|^{n-1} \operatorname{div}(|x|^{1-n}\nabla\delta_\Omega) - \frac{n-1}{|x|^2}\delta_\Omega \geqslant 0 \quad \text{in } \Omega.$$

It is worth mentioning here that Filippas *et al.* [18, theorem 3.2] obtained an improved Hardy inequality under the assumption that Ω is a *bounded* domain such that $0 \in \Omega$, and δ_Ω satisfies the differential inequality

$$-\operatorname{div}(|x|^{2-n}\nabla\delta_\Omega) \geqslant 0 \quad \text{in } \Omega,$$

while Gkikas [22] proved the Hardy inequality in an *exterior* domain Ω such that $0 \in \mathbb{R}^n \setminus \bar{\Omega}$, and δ_Ω satisfies the differential inequality

$$-\operatorname{div}(|x|^{1-n}\nabla\delta_\Omega) \geqslant 0 \quad \text{in } \Omega.$$

Let

$$\eta(\mu) := n-1+\sqrt{1-4\mu}. \tag{7.2}$$

Recall that for $\Omega = \mathbb{R}^n_+$, we obtained in example 5.7 that $\lambda_0(P_\mu, |x|^{-2}, \Omega) = \eta^2(\mu)/4$. The following theorem shows that if Ω is a domain such that δ_Ω is a positive supersolution of a certain second-order linear elliptic equation, then $\lambda_0(P_\mu, |x|^{-2}, \Omega) \geqslant \eta^2(\mu)/4$.

THEOREM 7.1. *Let Ω be a domain in $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Fix $\mu \leqslant 1/4$, and let $\eta(\mu)$ be as in (7.2). Suppose that δ_Ω satisfies the differential inequality*

$$-\Delta\delta_\Omega + \frac{\eta(\mu)}{|x|^2}(x \cdot \nabla\delta_\Omega - \delta_\Omega) \geqslant 0 \quad \text{in } \Omega \tag{7.3}$$

in the sense of distributions. Then the following improved Hardy inequality holds:

$$\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} \, \mathrm{d}x \geqslant \frac{\eta^2(\mu)}{4} \int_\Omega \frac{|\varphi|^2}{|x|^2} \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega). \tag{7.4}$$

Assume, furthermore, that Ω admits a supporting hyperplane at zero and $\mu \geqslant 0$; then

$$\lambda_0(P_\mu, |x|^{-2}, \Omega) = \frac{\eta^2(\mu)}{4}.$$

Proof. As in example 5.7, we write α_+ for the largest root of the equation $\alpha(1-\alpha) = \mu$, and $\psi := \delta_\Omega^{\alpha_+}|x|^{-\eta(\mu)/2}$. We will show that ψ is a supersolution of the equation

$$(P_\mu - (\eta(\mu)/2)^2|x|^{-2})u = 0 \quad \text{in } \Omega,$$

and then (7.4) follows from the AAP theorem (theorem 2.1). By direct computation we obtain

$$\begin{aligned}\left(P_\mu - \frac{\eta^2(\mu)}{4|x|^2}\right)\psi &= \alpha_+\left(-\Delta\delta_\Omega + \frac{\eta(\mu)}{|x|^2} x \cdot \nabla\delta_\Omega\right)\delta_\Omega^{\alpha_+ - 1}|x|^{-\eta(\mu)/2} \\ &\quad + \frac{\eta(\mu)}{2}(n - 2 - \eta(\mu))\delta_\Omega^{\alpha_+}|x|^{-\eta(\mu)/2-2} \\ &= \alpha_+\left(-\Delta\delta_\Omega + \frac{\eta(\mu)}{|x|^2}(x \cdot \nabla\delta_\Omega - \delta_\Omega)\right)\delta_\Omega^{\alpha_+ - 1}|x|^{-\eta(\mu)/2} \\ &\geqslant 0,\end{aligned}$$

where for the second equality we have used the fact that $n - 2 - \eta(\mu) = -2\alpha_+$, which follows from our choice of α_+.

Assume that Ω is a domain admitting a supporting hyperplane H at zero. Without loss of generality, we may assume that $H = \partial\mathbb{R}^n_+$. Then by corollary 5.9 we have that $\lambda_0(P_\mu, |x|^{-2}, \Omega) \leqslant \eta^2(\mu)/4$. Thus, $\lambda_0(P_\mu, |x|^{-2}, \Omega) = \eta^2(\mu)/4$. □

REMARK 7.2. (1) By (2.4), inequality (7.3) holds true for any $\mu \leqslant 1/4$ if Ω is a weakly mean convex cone.

We claim that (7.3) holds true also for $\mu = 1/4$ in any ball B with $0 \in \partial B$, and consequently, the Hardy inequality (7.4) is valid in this case.

Indeed, let $B = B_R(x_0)$ be an open ball in $\mathbb{R}^n$ centred at x_0 such that $|x_0| = R$. Then for $x \in B$ we have $\delta_B(x) = R - |x_0 - x|$, and simple computations show that for any $x \in B \setminus \{x_0\}$,

$$\nabla\delta_B(x) = \frac{x_0 - x}{|x_0 - x|} \quad \text{and} \quad -\Delta\delta_B(x) = \frac{n-1}{|x_0 - x|}.$$

Thus, for (7.3) to be true it is enough that for any $x \in B \setminus \{x_0\}$ we have

$$-\Delta\delta_B + \frac{\eta(\mu)}{|x|^2}(x \cdot \nabla\delta_B - \delta_B) = \frac{n-1}{|x_0 - x|} + \frac{n-1}{|x|^2}\left(x \cdot \frac{(x_0 - x)}{|x_0 - x|} - R + |x_0 - x|\right) \geqslant 0.$$

https://doi.org/10.1017/S0308210516000056 Published online by Cambridge University Press

After some cancellations this is equivalent to

$$|x|^2 \geqslant (R|x_0 - x| - x_0 \cdot (x_0 - x)) \quad \forall x \in B. \tag{7.5}$$

Some further simple computations imply that (7.5) is equivalent to

$$(x_0 - x) \cdot x \leqslant R^2 - R|x_0 - x| \quad \forall x \in B.$$

This is true since

$$2(x_0 - x) \cdot x = R^2 - |x|^2 - |x_0 - x|^2 \leqslant R^2 - |x_0 - x|^2 \leqslant 2(R^2 - R|x_0 - x|),$$

where in the last inequality we have used $\alpha^2 - \beta^2 \leqslant 2(\alpha^2 - \alpha\beta)$ for all $\alpha, \beta \in \mathbb{R}$.

(2) If the origin is an isolated point of $\partial\Omega$, then the classical Hardy inequality near 0 and theorem 2.1 imply that inequality (7.3) cannot hold.

(3) It would be interesting to characterize the domains for which (7.3) holds true.

Acknowledgements

The authors thank Professor Alano Ancona for valuable discussions, and Professor Achilles Tertikas for having pointed out the results of [19] to them. The authors are also grateful to the anonymous referees for their careful reading and valuable comments. The authors acknowledge the support of the Israel Science Foundation (Grant no. 963/11) founded by the Israel Academy of Sciences and Humanities. B.D. was supported in part by a Technion fellowship and by the Natural Sciences and Engineering Research council of Canada (NSERC). G.P. was supported in part at the Technion by a Fine Fellowship. Finally, Y.P. thanks Alessia Kogoj and Sergio Polidoro, and the Bruno Pini Mathematical Analysis Seminar for the kind hospitality during his visit at Università di Bologna and Università degli Studi di Modena e Reggio Emilia.

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