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A DECOMPOSITION OF RINGS GENERATED BY FAITHFUL CYCLIC MODULES

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ABSTRACT. A ring R is said to be *generated by faithful right cyclics* (right finitely pseudo-Frobenius), denoted by $GFC(FPF)$, if every faithful cyclic (finitely generated) right R-module generates the category of right R-modules. The class of right GFC rings includes right FPF rings, commutative rings (thus every ring has a GFC subring – its center), strongly regular rings, and continuous regular rings of bounded index. Our main results are: (1) a decomposition of a semi-prime quasi-Baer right GFC ring (e.g., a semiprime right FPF ring) is achieved by considering the set of nilpotent elements and the centrality of idempotnents; (2) a generalization of S. Page's decomposition theorem for a right FPF ring.

Introduction. All rings are associative with unity, unless specifically stated otherwise. $R, Z_r(R), P(R)$, and $N(R)$ denote a ring, the right singular ideal of R, the prime radical of R, and the set of nilpotent elements of R, respectively. Let X and Y be right ideals such that $X \subseteq Y \subseteq R$; we say X is *ideal essential* in Y if every nonzero ideal of R which is contained in Y has nonzero intersection with X. If V is a set, then $r_R(V)$ and $l_R(V)(r(V)$ and $l(V)$ when unambiguous) will denote the right and left annihilator of V in R, respectively. A right ideal X of R is *densely nil*, DN, if either $X = 0$ or for every nonzero $x \in X$, there exists $s \in R$ such that $xs \neq 0$ but $(xs)^2 = 0$. From [1] the minimal direct summand containing the nilpotent elements, $MDSN$, is a completely semiprime ideal (i.e., $x^n \in MDSN \Rightarrow x \in MDSN$) which equals the intersection of all direct summands which contain the set of nilpotent elements of the ring. The subring (without unity) of R generated by $\bigcup_{e\in E} eR(1-e)$ where E is the set of idempotent elements of R will be denoted by $\langle N_E(R)\rangle$. A right ideal X is *reduced* if it contains no nonzero nilpotent elements. From [9] and [16] a ring R is (*quasi-*) *Baer* if the right annihilator of every (ideal) non-empty subset of R is a direct summand. Semi-prime right FPF rings are quasi-Baer [10, p. 168]. A Baer ring is abelian (i.e., every idempotent is central) if and only if it is reduced. From [7] R is right CS if every right ideal is essential in a direct summand. From [11], R is *strongly right bounded* if every nonzero right ideal contains a nonzero ideal. R satisfies the *ideal intersection left (right) annihilator sum property*, $IILAS(IIRAS)$, if whenever X and Y are ideals of R such that $X \cap Y = 0$, then $R = l(X) + l(Y)$ $(R = r(X) + r(Y))$. Right FPF rings

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[10, p. 168], right selfinjective rings [18, p. 275], dual rings [15], and uniform rings have the *IILAS* property. Other terminology can be found in [13].

1. **Semiprime Quasi-Baer Right GFC Rings.**

LEMMA 1.1. *(i) $\langle N_E(R)\rangle$ is an ideal of R. (ii) $\langle N_E(R)\rangle = 0$ if and only if every idempotent is central. (iii) $\langle N_E(R)\rangle \subseteq \langle E(R)\rangle$ and $\langle N_E(R)\rangle \subseteq \langle U(R)\rangle$ where $\langle E(R)\rangle$ and $\langle U(R)\rangle$ are the subrings generated by the sets of idempotents and units of R, respectively. (iv) If $ReR = R$ where $e = e^2$, then $R(1-e)R \subseteq \langle N_E(R)\rangle$. (v) If R is a right p.p. ring (i.e., principal right ideals are projective), then $\langle N_E(R)\rangle$ equals the ideal generated by $N(R)$.*

PROOF. (i) Let $s, t \in R$ and $e = e^2$. Then $et(1-e)s = [et(1-e)][(1-e)se] + et(1-e)s(1-e) \in \langle N_E(R)\rangle$. It follows that $\langle N_E(R)\rangle$ is a right ideal and similarly it can be shown that $\langle N_E(R)\rangle$ is a left ideal.

(ii) Let $e = e^2$ and $t \in R$. Then $et = ete + et(1-e)$. Hence $et = ete$ for all $t \in R$ if and only if $eR(1-e) = 0$. It follows that $\langle N_E(R)\rangle = 0$ if and only if every idempotent is central.

(iii) Let $x \in eR(1-e)$ where $e = e^2$. Then $e + x \in E(R)$. Hence $x = e + x - e \in \langle E(R)\rangle$. Thus $\langle N_E(R)\rangle \subseteq \langle E(R)\rangle$. Also $x = (x-1) + 1 \in \langle U(R)\rangle$. Hence $\langle N_E(R)\rangle \subseteq \langle U(R)\rangle$.

(iv) There exist $t_i, v_i \in R$ such that $1 - e = \Sigma t_i e v_i = \Sigma t_i e v_i(1-e) \in \langle N_E(R)\rangle$. Thus $R(1-e)R \subseteq \langle N_E(R)\rangle$.

(v) Let N_i denote the set of nilpotent elements of index i. Let $y \in N_2$; then there exists $e = e^2$ such that $y \in eR = r(y)$. Thus $y = eye + ey(1-e) = ey(1-e) \in N_E$. Hence, $N_2 \subseteq \langle N_E(R)\rangle$. Now assume $N_j \subseteq \langle N_E(R)\rangle$ for all $2 < j \leqq k$. Let $s \in N_{k+1}$. Then there exists $c = c^2$ such that $s \in cR = r(s^k)$. Now $s = sc + cs(1-c)$. But $(sc)^k = s^k c = 0$. Thus $s \in \langle N_E(R)\rangle$. By induction $N(R) \subseteq \langle N_E(R)\rangle$. Consequently, $\langle N_E(R)\rangle$ equals the ideal generated by $N(R)$. □

We note that Baer rings and regular rings are p.p., hence Lemma 1.2(v) generalizes a result of Stephenson in [14, Proposition 3.3]. Also Baer right *GFC* rings are semiprime [3].

PROPOSITION 1.2. *(i) R is an abelian Baer right GFC ring if and only if R is a quasi-Baer strongly right bounded ring.*

(ii) Let R be a right GFC ring. Then R is a Baer ring if and only if R is a right nonsingular right CS ring.

(iii) Let R be a Baer right GFC ring and X a right ideal which contains no nonzero ideals. Then $X \subseteq \langle N_E(R)\rangle$.

PROOF. (i) If R is an abelian Baer (hence reduced) right *GFC* ring, the implication follows from [3, Corollary 1.3]. Conversely, let $0 \neq y \in Z_r(R)$. Then there exists an essential right ideal L such that $yL = 0$. By [11, Note 1.3D], there is an ideal $K \subseteq L$ which is an essential right ideal. Hence there exists $e = e^2 \neq 0$ such that

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$l(K) = Re \subseteq Z_r(R)$. Contradiction! Thus $Z_r(R) = 0$. By [3, Proposition 1.4], R is semiprime so it is reduced. Since a reduced quasi-Baer ring is an abelian Baer ring, the implication follows from [3, Corollary 1.3].

(ii) This part follows from [3, Proposition 2.5 and Corollary 2.7] and [8, Theorem 2.1].

(iii) By part (ii) there exists $e = e^2$ such that X is essential in eR. From [3, Lemma 1.1 and Corollary 2.7], $R(1-e)R = R$. By Lemma 1.1, $ReR \subseteq \langle N_E(R) \rangle$. □

From [3] we see that commutative domains and continuous regular rings of bounded index are Baer GFC rings.

PROPOSITION 1.3. *Let R be a semiprime quasi-Baer right GFC ring and $0 \neq e = e^2 \in R$. Then there exist idempotents $b, c, d \in R$ such that:*

(i) $eR = bR \oplus cR$ where b is central and cR contains no nonzero ideals of R.

(ii) $RcR = dR$ where d is central and $dR = d\langle N_E(R) \rangle$.

PROOF. Let J be the sum of the ideals of R which are contained in eR. By [3, Proposition 2.6 (i)], there is a central idempotent b such that J is essential in bR. Hence $be = b$. Therefore $eR = bR \oplus cR$ where cR contains no nonzero ideals of R. From [3, Corollary 2.7 (v)], there is a central idempotent d such that $RcR = dR$. By [3, Lemma 1.1], $R(1-c)R = R$. Lemma 1.1 shows that $RcR = \langle N_E(dR) \rangle = dR \cap \langle N_E(R) \rangle = d\langle N_E(R) \rangle$. □

THEOREM 1.4. *Let R be a semiprime quasi-Baer right GFC ring. Then $R = A \oplus B \oplus C$ where:*

(i) A is an abelian Baer ring.

(ii) B is a ring in which every idempotent is central.

(iii) C is a ring which is an essential extension of $\langle N_E(R) \rangle$.

(iv) $B \oplus C$ is the densely nil MDSN of R.

(v) If R contains no infinite set of orthogonal central idempotents, then R is a finite direct product of prime right Goldie rings and $C = \langle N_E(R) \rangle$.

PROOF. Parts (i), (ii) and (iii) follow from Lemma 1.1, [3, Lemma 2.2], and [2, Corollary 5]. Let X be a nonzero reduced right ideal of $B \oplus C$. By [3, Proposition 1.4], X is an essential extension of an ideal. A contradiction follows from [2, Lemma 2]. Hence $B \oplus C$ is DN. Part (v) follows from [3, Theorem 3.11] and Proposition 1.3. □

Note that if R is a Baer right GFC ring, then $B = 0$ in Theorem 1.4.

2. **Decompositions.**

LEMMA 2.1. *Let R be a ring with right ideals X and T such that T is an essential extension of X and $Z_2 = \{x \in R | x + Z_r(R) \in Z_r(R/Z_r(R))\}$. Then:*

(i) $Z_2 = \{x \in R | xL \subseteq Z_r(R)$ where L is some essential right ideal of $R\}$ is a densely nil ideal.

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(ii) $Z_r(R)$ *is essential in* Z_2.

(iii) Z_2 *is a closed right ideal of* R *and* $Z_r(R/Z_2) = 0$ *as a ring and as an* R*-module.*

(iv) If $X \subseteq Z_2$, *then* $T \subseteq Z_2$.

(v) If R *is right GFC, then* $P(R) \subseteq Z_r(R)$.

PROOF. From [1, Lemma 3.3], Z_2 is DN. The remainder of parts (i), (ii), and (iii) can be found in [13, pp. 36–48].

(iv) Let $0 \neq y + t \in Z_2 + T$ where $y \in Z_2$ and $t \in T$ and $t^{-1}X = \{r \in R | tr \in X\}$. Now $t^{-1}X$ is an essential right ideal of R. If $(y+t)t^{-1}X = 0$, then $y + t \in Z_r(R) \subseteq Z_2$. Otherwise, there exists $s \in t^{-1}X$ such that $0 \neq (y+t)s = ys + ts \in Z_2$. Consequently, $Z_2 + T$ is an essential extension of Z_2. From part (iii), $T \subseteq Z_2$.

(v) The proof is similar to [12, Lemma 1.2]. □

PROPOSITION 2.2. *Let* R *be a right GFC ring such that* $P(R)$ *is ideal essential in* Z_2. *If* X *is a right ideal of* R *such that* $X \cap P(R) = 0$ *and* $X \subseteq Z_2$, *then* R/X *generates* mod-R *and* $X \subseteq Z_r(R)$. *Hence, if* $t \in Z_2$ *such that* $t \notin Z_r(R)$, *then* $tR \cap P(R) \neq 0$.

PROOF. By [3, Lemma 1.1], R/X is a generator. Let Y be a relative complement of $P(R)$ in Z_2 such that $X \subseteq Y$. Hence R/Y generates Y. Thus $Y = \Sigma y_i R$ where $y_i \in Y \cap l(Y)$. Let K be a relative complement of Z_2 in R. Then $Z_2 K \subseteq P$. Consequently, $y_i(k \oplus Y \oplus P(R)) = y_i K \subseteq P(R) \cap Y = 0$. Hence $y_i \in Z_r(R)$. Therefore, $X \subseteq Y \subseteq Z_r(R)$. □

LEMMA 2.3. *If* R *is a right GFC ring and* B *is a relative complement of* $Z_r(R)$, *then* B *is an essential extension of an ideal of* R.

PROOF. Assume $B \neq 0$. Let $x \in l(B) \cap B$. Then $x(Z_r(R) \oplus B) = 0$. Hence $x \in Z_r(R) \cap B = 0$. By [3, Proposition 1.4], B contains a nonzero ideal of R. Let J be the sum of all ideals of R contained in B. Assume there is a right ideal $K \neq 0$ such that $J \oplus K$ is essential in B. Let $k \in l(K) \cap K$. Then $k(Z_r(R) \oplus J \oplus K) = 0$. Hence $k \in Z_r(R) \cap B = 0$. Again by [3, Proposition 1.4], K contains a nonzero ideal of R. Contradiction! Thus J is essential in B. □

The following result generalizes S. Page's [17] decomposition for right FPF rings.

THEOREM 2.4. *Let* R *be a right GFC ring which satisfies at least one of the following conditions:*

(i) IILAS (e.g., if R *is right FPF);*

(ii) If X *is a closed right ideal of* R, *then* $r(X)$ *is essential in a direct summand of* R;

(iii) every ideal which is closed as a right ideal is a direct summand of R.

Then $R = S \oplus Z_2$ *(right ideal decomposition) where* S *is a semiprime quasi-Baer right GFC ring. Also the prime radical* $P(R)$ *is ideal essential (for condition (ii)* $P(R)$ *is essential) in* Z_2.

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PROOF. For each condition (i), (ii), and (iii) we will show that $R = S \oplus Z_2$ and that $P(R)$ is ideal essential in xR where $x^2 = x$. We will conclude by proving $Z_2 \subseteq xR$.

(i) Suppose R is *IILAS*. Let B be a relative complement of $Z_R(R)$. If $Z_r(R)$ has no nonzero relative complement, then $Z_2 = R$. So assume $B \neq 0$. From Lemma 2.3, B is an essential extension of an ideal J. Thus $R = l(Z_r(R)) + l(J) = l(Z_2) + l(J)$. But $l(J) = Z_2$. Hence, $R = l(Z_2) + Z_2$. Therefore, there exists $e = e^2$ such that $Z_2 = eR$. Let $S = (1-e)R$. Since eR is an ideal, S is a ring with unity. From [3, Proposition 1.4], S is a semiprime ring. We claim that S is an *IILAS* ring. Let X, Y be ideals of S such that $X \cap Y = 0$. By Lemma 2.3, S is an essential extension of an ideal H of R. Then $X_1 = X \cap H$ and $Y_1 = Y \cap H$ are ideals of R. Hence $R = l(X_1) + l(Y_1)$. Thus $S = l_S(X_1) + l_S(Y_1)$. But X_1 and Y_1 are essential in X and Y, respectively; and $Z_r(S) = 0$. Hence $l_S(X) = l_S(X_1)$ and $l_S(Y) = l_S(Y_1)$. Consequently, S is *IILAS*. By [3, Lemma 2.2], S is quasi-Baer. Now let K be an ideal of R which is maximal among ideals having zero intersection with $P(R)$. Hence $l(K) \cap K = 0$. Then $R = l(l(K)) + l(K)$. Hence there exists $x = x^2$ such that $P(R) \subseteq l(K) = xR$.

(ii) Assume R satisfies condition (ii). Let B be a relative complement of Z_2. Hence there exists $e = e^2$ such that $r(B)$ is essential in eR. But $(B \cap r(B)) \oplus Z_2$ is essential in eR. By Lemma 2.1, $Z_2 = eR$. Let $S = (1-e)R$. Since S is a ring with unity such that any right ideal of S is a right ideal of R, then S inherits condition (ii) from R. Since S is semiprime, the above argument could be used to show that any closed ideal of S is a direct summand. By [3, Lemma 2.2], S is a quasi-Baer ring. Now let K be an ideal of R which is maximal among ideals having zero intersection with $P(R)$. It follows that $r(K) = l(K)$ and that $l(K)$ is a closed right ideal. Hence there exists $x = x^2$ such that $P(R) \subseteq l(K) = xR$.

(iii) Assume R satisfies condition (iii). From Lemma 2.1 and [3, Lemma 2.2], $R = S \oplus Z_2$ where S is a quasi-Baer ring. For K defined as in part (ii), the same argument shows that there exists $x = x^2$ such that $P(R) \subseteq l(K) = xR$.

Now $R = (1-x)R \oplus xR$ where xR is an ideal and $P(R)$ is ideal essential in xR. Thus, by the above argument, $(1-x)R = (1-x)R(1-x)$ is also a semiprime quasi-Baer right *GFC* ring. From [3, Proposition 2.5], $(1-x)R(1-x)$ is right nonsingular. Hence $(1-x)R \cap Z_2 = 0$. Therefore, $P(R) \subseteq Z_2 = xZ_2$. Consequently, $P(R)$ is ideal essential in Z_2. Furthermore if R satisfies condition (ii), by an argument used in part (ii) of this proof, $P(R)$ is essential in a direct summand W of R. By Lemma 2.1, $Z_2 = tR \oplus W$ where $t = t^2$. From Proposition 2.2, $t \in Z_r(R)$. Hence $P(R)$ is essential in Z_2. □

COROLLARY 2.5. *Let R be a right GFC ring which satisfies at least one of the conditions (i), (ii), (iii), and its left-sided version (indicated in brackets):*

(i) IILAS[IIRAS] (e.g., if R is FPF);

(ii) If X is a closed right [left] ideal of R, then $r(X)[l(X)]$ is essential in an idempotent generated right [left] ideal (e.g., if R is quasi-Baer);

(iii) Every ideal which is a closed right [left] ideal is an idempotent generated right [left] ideal.

Then $R = S \oplus Z_2$ is a ring decomposition.

PROOF. From Theorem 2.4, $Z_2 = eR$ where $e = e^2$, and $r(Z_2) \subseteq (1-e)R = (1-e)R(1-e)$. Hence $Z_2 \subseteq r((1-e)R) \subseteq r(r(Z_2))$. Let $B = r(Z_2) \cap r(r(Z_2))$. Then $B^2 = 0$. But $B \cap Z_2 = 0$. By Lemma 2.1, $r(Z_2) \cap r(r(Z_2)) = 0$. Next we will show that for each condition (i), (ii), or (iii) there exists a central idempotent c such that $r(r(Z_2)) = cR$. Finally, we will prove that $Z_2 = cR$.

(i) Assume R satisfies condition (i). Since $r(Z_2) \cap Z_2 = 0$, then $R = r(Z_2) + r(r(Z_2))$. Thus $r(r(Z_2)) = cR$ where c is a central idempotent.

(ii) Assume R satisfies condition (ii). Since $r_R(r_R(Z_2))$ is closed as a left ideal which contains Z_2, the left sided version of the argument used in part (ii) of the proof of Theorem 2.4 yields $r_R(r_R(Z_2)) = Rc$ where $c^2 = c$. Assume $Rc \neq Z_2$. Then there exists a right ideal $X \neq 0$ such that $Rc = Z_2 \oplus X$. Let $y \in l(X) \cap X$ and W be a relative complement (on the right) for Rc. Then $y(W \oplus Rc) = yW \subseteq P(R)$. By Lemma 2.1, $y \in X \cap Z_2 = 0$. From [3, Proposition 1.4], X contains an ideal $J \neq 0$. But $J \subseteq r(Z_2) \cap r(r(Z_2)) = 0$. Hence $Rc = Z_2 = eR$. Thus $Z_2 = eR$ where e is a central idempotent.

(iii) Assume R satisfies condition (iii). Now $r(Z_2)$ is a relative complement of Z_2 as a left ideal. Thus there exists an idempotent b such that $r(Z_2) = Rb$. Since Rb is an ideal, $Rb = (1-b)Rb \oplus bR$ (right ideal decomposition). But $r(Z_2) \cap Z_2 = 0$. By Lemma 2.1 $(1-b)Rb = 0$. Hence b is a central idempotent. Thus $1-b$ is a central idempotent. Let $c = 1-b$. Then $r(r(Z_2)) = cR$ where c is a central idempotent.

Now in all cases, Z_2 is faithful in cR. Hence $Z_2 = eR$ is a generator for cR. Thus trace $(Z_2) = \text{trace}(eR) = eR = cR = Z_2$. Consequently, $R = S \oplus Z_2$ is a ring decomposition. □

EXAMPLE 2.6. Let T be the semigroup ring over I_2 (integers modulo 2) where A is the semigroup on the set $\{a, b\}$ satisfying the relation $xy = y$ for $x, y \in A$. Thus $T = \{0, a, b, a+b\}$. Let T_1 denote the ring with unity formed by extending (i.e., Dorroh extension) T to $T \times I$ (I denotes the integers). T_1 is neither right CS nor is the right annihilator of a closed right ideal necessarily a direct summand (e.g., $r(T, 0) = (0, 2I)$), hence T_1 is not quasi-Baer. However, T_1 satisfies the following conditions: (i) strongly right bounded (hence right GFC [3, Proposition 1.2]; (ii) $IILAS$; (iii) every ideal is essential in a direct summand, thus the right annihilator of a closed right ideal is essential in a direct summand; (iv) every ideal which is closed as a right ideal is a direct summand. Further details can be found in [4, Example 2.3] and [6]. Therefore T_1 provides a nontrivial example for Theorem 2.4 where $S = (a, 1)T_1$ and $Z_2 = (T, 0)$.

EXAMPLE 2.7. Let $S = I_2[x]/\langle x^3 \rangle$, where I_2 denotes the ring of integers modulo 2, x is an indeterminate, $\langle x^3 \rangle$ is the ideal generated by x^3, and $\bar{x} = x + \langle x^3 \rangle$. Let R be the subring of $\left(\begin{smallmatrix} S & S \\ 0 & S \end{smallmatrix}\right)$ generated by $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \bar{x} \end{smallmatrix}\right)$, and $t = \left(\begin{smallmatrix} 0 & \bar{x} \\ 0 & 0 \end{smallmatrix}\right)$. R has the following characteristics: (i) by observing that if $y \in R$, then $y = k_1e + k_2b + k_3t + k_4b^2 + k_5tb$, where $k_i \in I_2$, it follows that R is a local ring with nilpotent Jacobson radical of index

3, and R is an algebra of dimension five (hence $|R| = 32$) over I_2. (ii) A principal right ideal yR is a proper ideal if and only if $k_1 = k_2 = 0$. However, the minimal right ideals (i.e., b^2R, tbR, and $(b^2+tb)R$) are ideals. Therefore, R is strongly right bounded (hence right GFC). Note b^2 and tb are in the center of R. Let $soc(R)$ denote the right socle of R. If X is a nonzero right ideal, then $soc(R) \subseteq r(X)$. Hence $r(X)$ is essential in R. Thus condition (ii) of Theorem 2.4 is satisfied. However, R is not quasi-Baer; and R is not *IILAS* (hence not right *FPF*), since $b^2R \cap tbR = 0$, but $R \neq l(b^2R) + l(tbR)$. Note $R = Z_2$. Further details on this example can be found in [5].

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