Some Remarks on Uniform Convergence

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Introduction.

A useful test for uniform convergence is that first established by Buchanan and Hildebrandt [4] which is as follows.

(A) "If a sequence $f_n(x)$ of monotonic functions converges to a continuous function $f(x)$ in $[a, b]$ then this convergence is uniform."

In §1 of this paper it is shown that this test is included in a sequence of theorems, each of which establishes a type of uniform convergence. The first is a well-known topological theorem on limit sets, the second is a result on the limits of rectifiable arcs, the third is a generalisation of (A) due to Behrend [3], the fourth is (A) itself, the fifth is a one-sided version of Bendixson's test and the sixth is Bendixson's test.

In §2 these theorems are partially extended to theorems of more than one variable.

A similar result to (A) in which the range $[a, b]$ is replaced by $(-\infty, \infty)$ has been established by Pólya [8] and extended by Conti [5]. There are corresponding extensions of the theorems proved here. The proof of Theorem 2 is related to the ideas used in papers by Radó and Reichelderfer [9], Ayer [1], Tsuji [11], and Ayer and Radó [2]. The paper by Goodstein [6] is also concerned with a theorem similar to (A).

(A) may be stated in the alternative form

"If a sequence of additive non-negative functions $F_n(x)$ defined for all measurable subsets of $[a, b]$ converges to an absolulely continuous limit function $F(x)$, then the convergence is uniform."

An example of the use of the alternative form of (A) is given in Scheffé [10].

§1.

Let S be a compact metric space and ρ denote the distance function in S. For any subset X of S let $\mathfrak{U}(X, \epsilon)$ denote the set of all points whose distance from X is less than ϵ, ϵ being a positive number.

THEOREM 1. *Let X_n be any sequence of sets in S; then for every $\epsilon > 0$ there exists an integer N such that for $n \geqq N$*

$$\mathfrak{U}(\overline{\lim}\, X_n, \epsilon) \supset X_n$$

where $\overline{\lim}\, X_n$ is the upper topological limit of X_n.

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For suppose that the contrary were the case. Then for some positive number ϵ_0 there is a sequence of integers n_i such that each X_{n_i} contains a point x_{n_i} and

$$(1) \qquad \lim_{i \to \infty} n_i = \infty, \qquad \rho(x_{n_i}, \overline{\lim}\, X_{n_i}) \geqq \epsilon_0.$$

As S is compact, a subsequence of the x_{n_i} tends to a point p. By (1)

$$(2) \qquad \rho(p, \overline{\lim}\, X_n) \geqq \epsilon_0.$$

But by the definition of $\overline{\lim}\, X_n$, $p \subset \overline{\lim}\, X_n$, a contradiction of (2)

THEOREM 2. *Let S be a bounded portion of Euclidean space of any (finite) number of dimensions and X_n be a sequence of arcs in S. Suppose that the length of X_n is l_n. If $\lim_{n \to \infty} l_n = l$ and there is an arc X of length l contained in $\underline{\lim}\, X_n$, then for every $\epsilon > 0$ there exists an integer N such that for $n \geqq N$ and x_n belonging to X_n,*

$$\rho(x_n, X) < \epsilon.$$

By Theorem 1 it is sufficient to show that

$$X = \overline{\lim}\, X_n.$$

If this is not the case then there is a point y contained in $\overline{\lim}\, X_n - X$. As X is a closed set $\rho(y, X) = \eta > 0$. There is a positive number δ such that $\eta > 8\delta$ and such that every polygonal line joining the two end-points of X, whose vertices belong to X and whose segments are of length less than or equal to δ, has total length greater than

$$l - \eta/4.$$

Such a polygonal line is constructed as follows. One end-point of the arc X, say p_0, is selected and called the first point, the other end-point is called the last point p', and the other points are ordered in accordance with this nomenclature.

p_0 is the first point of the polygonal line. p_1 is p' if $\rho(p_0, p') \leqq \delta$, otherwise it is the last point after p_0 whose distance from p_0 is δ.

Generally, p_{i+1} is p' if $\rho(p_i, p') \leqq \delta$, otherwise it is the last point after p_i for which $\rho(p_i, p_{i+1}) = \delta$. Let $p_0 p_1 \ldots p_{t-1} p'$ be the polygonal line so formed and write $p' = p_t$.

The circles $C(p_i, \delta/2)$, $i = 0, 1, \ldots, t$, of centre p_i and radius $\delta/2$, are non-overlapping and

$$(3) \qquad (t+1)\,\delta > l - \eta/4.$$

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As $p_0, p_1, \ldots, p_t$ belong to $\underline{\lim}\ X_n$, there is an integer N_0 such that

$$\rho(p_i, X_n) \leqq \eta/8(t+1), \qquad i = 0, \ldots, t, \qquad n \geqq N_0.$$

If the circle $C(p_i, \delta/2)$ does not contain an end-point of X_n, the part of X_n contained in it has length at least $\delta - \eta/4(t+1)$. If it does contain an end-point the length contained is at least $\delta/2 - \eta/8(t+1)$.

X_n has two end-points; thus the set of circles $C(p_i, \delta/2)$, $i=0, \ldots, t$ contain a part of X_n of length at least

$$(4) \qquad t(\delta - \eta/4(t+1)) > l - \eta/2 - \delta > l - 5\eta/8.$$

Now $y \subset \overline{\lim}\ X_n$, and thus for some integer n arbitrarily large, say $n = m \geqq N_0$,

$$\rho(y, X_m) < \eta/4.$$

The circle $C(y, \eta - \delta/2)$ does not overlap any of the $C(p_i, \delta/2)$ and contains a part of X_n of length greater than or equal to

$$(5) \qquad \eta - \delta/2 - \eta/4 > 11\eta/16.$$

By (4), (5)

$$l_m \geqq l + \eta/16.$$

Since m is arbitrarily large this is in contradiction with the fact that $\lim_{n\to\infty} l_n = l$.

COROLLARY. *Let $f_n(x)$ be a set of continuous functions defined over $a \leqq x \leqq b$ such that*

(i) $\lim_{n\to\infty} f_n(x) = f(x), \qquad a \leqq x \leqq b,$

(ii) $$\lim_{n\to\infty}\left\{\lim_{h\to 0}\int_a^b\left[1+\left(\frac{f_n(x+h)-f_n(x)}{h}\right)^2\right]^{\frac{1}{2}}dx\right\} = \lim_{h\to 0}\int_a^b\left[1+\left(\frac{f(x+h)-f(x)}{h}\right)^2\right]^{\frac{1}{2}}dx,$$

(iii) $f(x)$ *is continuous.*

Then the convergence of the sequence $f_n(x)$ to $f(x)$ is uniform in x, $a \leqq x \leqq b$.

Denote by X_n the set of points in the (x, y) Euclidean plane with coordinates $(x, f_n(x))$, $a \leqq x \leqq b$, and by X the set of points $(x, f(x))$, $a \leqq x \leqq b$. X_n and X are arcs with lengths, say l_n and l.

Condition (i) implies that $X \subset \underline{\lim}\ X_n$ and condition (ii) that $\lim_{n\to\infty} l_n = l$. Thus by Theorem 2 for a given $\epsilon > 0$,

$$(6) \qquad X_n \subset \mathfrak{U}(X, \epsilon) \qquad n \geqq N = N(\epsilon).$$

For each point p of X form the segment whose midpoint is p and which is parallel to the y-axis and is of length δ. Let the set of points formed from all such segments be called $Y(X, \delta)$. Because $f(x)$ is continuous, for a given $\delta > 0$ there is an $\epsilon > 0$ such that

$$\mathcal{U}_1(X, \epsilon) \subset Y(X, \delta) \tag{7}$$

where $\mathcal{U}_1(X, \epsilon)$ denotes that part of $\mathcal{U}(X, \epsilon)$ lying in the closed strip $a \leqq x \leqq b$.

The conclusion now follows from (6) and (7).

Remark. The continuity condition on the $f_n(x)$, $f(x)$ can be considerably relaxed: see e.g. Saks, *Theory of the integral*, p. 184.

THEOREM 3. *If $f_n(x)$, $f(x)$ are monotonic in $[a, b]$ then $f_n(x) \to f(x)$ uniformily in $[a, b]$ if and only if* (i) *$f_n(x) \to f(x)$ for all x of an everywhere dense set E in $[a, b]$ containing the set D of all discontinuities of $f(x)$, and*

(ii) $f_n(x-0) \to f(x-0)$, $\qquad f_n(x+0) \to f(x+0)$ *for all x in D.*

The necessity of the conditions is trivial. To prove their sufficiency the given conditions are used to reduce this theorem to the case of continuous $f(x)$ which is discussed in Theorem 4.

Write the discontinuities of $f(x)$ as $c_1, c_2, \ldots$. Define the functions

$$\phi(x) = \sum_{c_k < x} (f(c_k) - f(c_k - 0)), \quad \phi_n(x) = \sum_{c_k < x} (f_n(c_k) - f_n(c_k - 0)),$$

$$\psi(x) = \sum_{c_k < x} (f(c_k + 0) - f(c_k)), \quad \psi_n(x) = \sum_{c_k < x} (f_n(c_k + 0) - f_n(c_k)),$$

$$g(x) = f(x) - \phi(x) - \psi(x), \qquad g_n(x) = f_n(x) - \phi_n(x) - \psi_n(x).$$

Then $g(x)$, $g_n(x)$ are monotone and $g(x)$ is continuous. Conditions (i) and (ii) imply that

$$\phi_n(x) \to \phi(x), \qquad \psi_n(x) \to \psi(x)$$

uniformly in $[a, b]$. Hence

$$g_n(x) \to g(x).$$

By the next theorem this convergence is uniform in x.

It follows that $f_n(x) \to f(x)$ uniformly in x, $a \leqq x \leqq b$.

THEOREM 4. (*Buchanan and Hildebrandt*) *If a sequence $f_n(x)$ of monotonic functions converges to a continuous function in $a \leqq x \leqq b$, say $f(x)$, then the convergence is uniform.*

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An auxiliary sequence of functions $F_i(x)$ is defined as follows.

$F_i(x_k) = f(x_k)$ where $x_k = a + (b-a)\,k/i$, $k = 0, 1, 2, \dots i$,

and for other x of $a \leqq x \leqq b$, $F_i(x)$ is defined by linear interpolation.

Then $F_i(x) \to f(x)$, uniformly by Theorem 2. Also

$$\sup_{a < x < b} |f_n(x) - F_i(x)| \leqq \max_{k=0,\dots,i-1} \{ |f_n(x_k) - F_i(x_{k+1})| + |f_n(x_{k+1}) - F_i(x_k)| \}.$$

Thus $f_n(x) \to f(x)$ uniformly.

THEOREM 5. *Let $f_n(x)$ be a sequence of continuous functions for each of which the upper right-hand derivative $f_n^+(x)$ satisfies* (i) $f_n^+(x) > -\infty$ *at every point x, $a \leqq x \leqq b$, except at most at those of an enumerable set,* (ii) $f_n^+(x) \geqq g(x)$ *for almost all x, where $g(x)$ is Perron-integrable over $a \leqq x \leqq b$.*

Suppose also that $f_n(x)$ tends to a continuous limit function $f(x)$. Then

$$f_n(x) \to f(x) \text{ uniformly in } x,\ a \leqq x \leqq b.$$

Write $\phi_n(x) = f_n(x) - \int_a^x g(x)\,dx$ where the integral sign denotes the Perron integral. Conditions (i) and (ii) imply that,

$$\phi_n(x+h) - \phi_n(x) = f_n(x+h) - f_n(x) - \int_x^{x+h} g(x)\,dx \geqq 0$$

for $a \leqq x < x + h \leqq b$.

By Theorem 4, $\phi_n(x) \to f(x) - \int_a^x g(x)\,dx$ uniformly in x. Thus $f_n(x) \to f(x)$ uniformly in x, $a \leqq x \leqq b$.

THEOREM 6. (*Bendixson*). *If $f_n(x)$ is a sequence of differentiable functions in $a \leqq x \leqq b$ and $|f_n'(x)| < k$ for all n and x, then if $f_n(x) \to f(x)$ as $n \to \infty$ it does so uniformly in x, $a \leqq x \leqq b$.*

By Theorem 5 all that needs to be proved is that the limit function $f(x)$ is continuous. If this were not so there would be a point x_0 and a sequence of points $x_1, x_2, \dots$ such that

$$\lim_{i\to\infty} x_i = x_0, \quad \lim_{i\to\infty} f(x_i) \neq f(x_0).$$

Thus there are two points x_0, x_k such that

$$|(f(x_0) - f(x_k))/(x_0 - x_k)| > K.$$

For n sufficiently large

$$|(f_n(x_0) - f_n(x_k))/(x_0 - x_k)| > K.$$

This is not possible because of the condition $|f_n'(x)| < K$.

Thus the theorem is established.

§2.

Of the preceding theorems, 2 and 3 do not extend directly to the case of functions of more than one variable. This is related to the fact that the behaviour of the areas of a sequence of surfaces is more complicated than that of the lengths of a sequence of arcs. The analogues of Theorems 4 and 6 are true but now require direct proofs. Similar direct proofs hold for Theorems 4 and 6.

THEOREM 7. *Let $f_n(x_1, x_2, \ldots, x_r)$ be a sequence of functions of r variables $x_1, x_2, \ldots, x_r$, defined over the ranges $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \ldots, r$, such that each function is monotone in each of the $x_2, x_3, \ldots, x_r$ variables separately when all the other variables are fixed. Suppose also that the sequence converges to a continuous limit function $f(x_1, x_2, \ldots, x_r)$, and that this convergence is uniform in x_1 when $x_2, x_3, \ldots, x_r$ are kept fixed. Then the convergence is uniform in all the variables $x_1, x_2, \ldots, x_r$ simultaneously.*

Since $f(x_1, x_2, \ldots, x_r)$ is continuous, for each $\epsilon > 0$ there is an integer $N = N(\epsilon)$ such that

$$|f(x_1', x_2', \ldots, x_r') - f(x_1, x_2, \ldots, x_r)| < \epsilon \tag{8}$$

provided $|x_i - x_i'| < 1/N$, $a_i \leqq x_i \leqq b_i$, $a_i \leqq x_i' \leqq b_i$, $i = 1, 2, \ldots, r$.

For each $\epsilon > 0$, there is an integer $n_0(\epsilon)$ such that

$$\begin{aligned} |f_n(x_1, a_2 + (b_2 - a_2) j_2/N, \ldots, a_r + (b_r - a_r) j_r/N) \\ - f(x_1, a_2 + (b_2 - a_2) j_2/N, \ldots, a_r + (b_r - a_r) j_r/N)| < \epsilon \end{aligned} \tag{9}$$

where $j_2, j_3, \ldots, j_r$ range over the integers $0, 1, \ldots, N$ independently and $a_1 \leqq x_1 \leqq b_1$, $n \geqq n_0(\epsilon)$.

Consider now any point $(x_1, x_2, \ldots, x_r)$ where $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \ldots, r$. There are two sets of integers $j_2, j_3, \ldots, j_r$, and $j_2', j_3', \ldots, j_r'$, such that

$$\left.\begin{aligned} a_i + (b_i - a_i) j_i/N \leqq x_i < a_i + (b_i - a_i) j_i'/N \\ j_i' = j_i + 1, \ 0 \leqq j_i < N - 1 \end{aligned}\right\} i = 2, 3, \ldots, r.$$

Since the function is monotone

$$f_n(x_1, y_2, \ldots, y_r) \geqq f_n(x_1, x_2, \ldots, x_r) \geqq f_n(x_1', y_2', \ldots, y_r'),$$

where $y_i = a_i + (b_i - a_i) j_i/N$ or $y_i = a_i + (b_i - a_i) j_i'/N$ and similarly for y_i'. (The particular value depends on whether the function is increasing or decreasing in the i^{th} variable at the particular values of the other variables concerned and it may vary with n.)

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By (8) and (9), for $n \geqq n_0(\epsilon)$,

$$| f_n(x_1, x_2, \ldots, x_r) - f(x_1, x_2, \ldots, x_r) | < 2\epsilon.$$

This proves the theorem.

Remarks.

(i) This theorem contains the direct analogue of Theorem 4 because if the functions are monotonic in x_1 then the convergence with respect to x_1 is necessarily uniform.

(ii) This theorem also contains a more general result in which the condition on $f_n(x_1, x_2, \ldots, x_r)$ as a function of x_1 is that of the corollary to Theorem 2.

(iii) A similar conclusion to that of the theorem is true if the conditions on the sequence are "monotone in $r - s$ variables and uniformly convergent simultaneously in the other s variables." This is not however true if the convergence is given to be uniform with respect to the s variables separately.

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