Alice's adventures in inverse tan land – mathematical argument, language and proof

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1. *Introduction*

Andrew Palfreyman's article [1] reminds us of the result

$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi, \tag{1}$$

having been set the challenge of finding the value of the left-hand side by his head of department at the start of a departmental meeting. The article then went on to explore an interesting relationship between the inverse tangent of three unit fractions, namely

$$\tan^{-1} \frac{1}{a} = \tan^{-1} \frac{1}{a + 1} + \tan^{-1} \frac{1}{a^2 + a + 1}, \tag{2}$$

where $a > 0$, as a special case of

$$\tan^{-1} \left(\frac{p + q}{1 - pq}\right) = \tan^{-1} p + \tan^{-1} q. \qquad -1 < p, q < 1 \tag{3}$$

with $p = \dfrac{1}{a + 1}$ and $q = \dfrac{1}{a^2 + a + 1}$, where $-1 < p, q < 1$. One natural generalisation of (2) is

$$\tan^{-1} \frac{1}{a} = \tan^{-1} \frac{1}{a + b} + \tan^{-1} \frac{b}{a^2 + ab + 1}$$

for $a, b > 0$, which we leave readers to check.

The result in (3) was known to Alice of 'wonderland' and 'looking glass' fame, as is evident from the article by her creator the mathematician Charles Dodgson, better known by his pseudonym Lewis Carroll, cited in [2] in the form

$$\tan^{-1} \frac{1}{a} = \tan^{-1} \frac{1}{a + x} + \tan^{-1} \frac{1}{b + y}, \text{ where } xy = a^2 + 1,$$

which we also leave as an exercise.

One well-known exchange in Alice in Wonderland goes as follows:

"Would you tell me, please, which way I ought to go from here?"

"That depends a good deal on where you want to get to," said the Cat.

"I don't much care where –" said Alice.

"Then it doesn't matter which way you go," said the Cat.

"– so long as I get SOMEWHERE," Alice added as an explanation.

"Oh, you're sure to do that," said the Cat, "if you only walk long enough."

In this article we consider two tangents (or should I say inverse tangents, but not *in verse*!) that Alice could now go off on from these

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starting points if she was studying for the reformed A level in Mathematics [3] with its *Overarching Theme (OT1): Mathematical argument, language and proof* given in Table 1.

	Knowledge/Skill
OT1.1	[Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable]
OT1.2	[Understand and use mathematical language and syntax as set out in the content]
OT1.3	[Understand and use language and symbols associated with set theory, as set out in the content] [Apply to solutions of inequalities] and probability
OT1.4	Understand and use the definition of a function; domain and range of functions
OT1.5	[Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics]

TABLE 1

We show some opportunities open to Alice to address some of these aspects by exploring generalisations of (1). Alice also realises that, as well as her mathematical prowess, it would help to have use of the latest technology – at least a spreadsheet, and potentially basic computer algebra software or online app – to continue her journey at certain stages. This will help her address one of the other important themes in the reformed A levels:

> The use of technology, in particular mathematical and statistical graphing tools and spreadsheets, must permeate the study of AS and A level mathematics.

This suggests another tangent Alice might head off along by investigating an infinite series utilising (2), for which a spreadsheet is pretty essential.

2. *Tangent* 1

Starting with the well-known addition formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{4}$$

we have

$$A + B = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) + m\pi, \tag{5}$$

where m is an integer to be determined. Since $\tan^{-1} x$ has principal values in the interval $-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$ the value of m in (5) depends on the size of the angles A and B. If $A = \tan^{-1} a$ and $B = \tan^{-1} b$ for $1 \leqslant a < b$, then

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$\frac{1}{4}\pi \leqslant A < B < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < A + B < \pi$, so for the right-hand side of (5) to lie in this interval we must have $m = 1$. In this case (5) becomes

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a + b}{1 - ab}\right) + \pi \text{ for } 1 \leqslant a < b. \tag{6}$$

If we now replace B by $B + C$ in (4) we have

$$\begin{aligned}\tan(A + B + C) &= \tan\big((A + B) + C\big) \\ &= \frac{\tan(A + B) + \tan C}{1 - \tan(A + B)\tan C} \\ &= \frac{\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) + \tan C}{1 - \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)\tan C} \\ &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}, \end{aligned} \tag{7}$$

where we have used the addition formula to expand $\tan(A + B)$, and thus

$$A + B + C = \tan^{-1}\left(\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}\right) + m\pi. \tag{8}$$

where m is to be determined as before. If $A = \tan^{-1} a$, $B = \tan^{-1} b$ and $C = \tan^{-1} c$ for $1 \leqslant a < b < c$, then $\frac{1}{4}\pi \leqslant A < B < C < \frac{1}{2}\pi$ and $\frac{3}{4}\pi \leqslant A + B + C < \frac{3}{2}\pi$. Thus, given that $-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$, for the right-hand side in (8) to lie in this interval we must again have $m = 1$. In this case (8) becomes

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \tan^{-1}\left(\frac{a + b + c - abc}{1 - ab - bc - ca}\right) + \pi, \text{ for } 1 \leqslant a < b < c. \tag{9}$$

We can now obtain the result in (1) from (9) by setting $a = 1$, $b = 2$, $c = 3$ since $a + b + c - abc = 1 + 2 + 3 - 1 \times 2 \times 3 = 0$ and $1 - ab - bc - ca = 1 - (1 \times 2) - (2 \times 3) - (3 \times 1) = -10 \neq 0$, so that $\tan^{-1}\left(\frac{a + b + c - abc}{1 - ab - bc - ca}\right) = \tan^{-1} 0 = 0$.

As an alternative approach to using the addition formula again in (7) (and one which makes life easier for Alice on her journey later on) we could use (3) and (6) to establish (9), as follows.

Consider adding $\tan^{-1} c$ to both sides of (6) to give

$$\begin{aligned}\tan^{-1} a + \tan^{-1} b + \tan^{-1} c &= \tan^{-1}\left(\frac{a + b}{1 - ab}\right) + \tan^{-1} c + \pi \\ &= \tan^{-1}\left(\frac{\left(\frac{a + b}{1 - ab}\right) + c}{1 - \left(\frac{a + b}{1 - ab}\right)c}\right) + m\pi \end{aligned} \tag{10}$$

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$$= \tan^{-1}\left(\frac{a + b + c - abc}{1 - ab - bc - ca}\right) + m\pi$$

where $m = 1$ or 2, and where we have used a combination of (3) and (6) in the second line. The reason that we cannot use just (3) is because that is valid for arguments $-1 < p, q < 1$, whereas $c > 1$; equally we cannot use just (6) because that is for arguments $1 \leqslant a < b$, whereas the argument $\frac{a + b}{1 - ab} < 0$. Whether or not this is less than -1 determines whether we might need to add an additional π on the right hand side, and hence why we could have $m = 1$ or 2, a consideration that is also important later on. However in this case (as discussed previously for angles A, B, C and $A + B + C$) we have $\frac{3}{4}\pi \leqslant \tan^{-1}a < \tan^{-1}b < \tan^{-1}c < \frac{1}{2}\pi$ and so $\frac{1}{4}\pi < \tan^{-1}a + \tan^{-1}b + \tan^{-1}c < \frac{3}{2}\pi$. Thus, given that $-\frac{1}{2}\pi < \tan^{-1}x < \frac{1}{2}\pi$, for the right-hand side of (10) to lie in this interval we must again have $m = 1$, so that (10) becomes the same expression as in (9).

2.1 *Three angle case*

Because the result in (1) and the expression in (9) relate to the sum of three angles being equal to π (or an integer multiple thereof) we refer to this as the *three angle case*. So for the first part of our diversion we now ask in this three angle case for what other integers $1 \leqslant a < b < c$ is

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi? \qquad (11)$$

From (9) it is clear that we are seeking (distinct) positive integer solutions of $a + b + c - abc = 0$ for which $1 - ac - ac - bc \neq 0$. We leave readers to experiment with this, say using a spreadsheet or computer program, computing many possible combinations of $a + b + c - abc$ for positive integers a, b, c and seeking a combination where the value is zero. But it turns out there are no other solutions. Thus the smallest conceivable solution is a solution, and this is the *only* solution. This is relatively easy to prove, as follows.

Suppose there is another, 'larger' solution $a = 1 + \alpha$, $b = 2 + \beta$, $c = 3 + \gamma$ for integers $\alpha, \beta, \gamma \geqslant 0$ where at least one of α, β, γ is positive, i.e. where at least one of the values is different. This means that

$$a + b + c - abc = 1 + \alpha + 2 + \beta + 3 + \gamma - (1 + \alpha)(2 + \beta)(3 + \gamma)$$

$$= -5\alpha - 2\beta - \gamma - 3\alpha\beta - \beta\gamma - 2\gamma\alpha - \alpha\beta\gamma < 0$$

using these properties of α, β, γ. Thus apart from the solution $a = 1$, $b = 2$, $c = 3$ there can be no other solutions of $a + b + c - abc = 0$, and hence of equation (11).

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2.2 *Four angle case*

The fun begins as we ask for integer values of a, b, c, d where $1 \leqslant a < b < c < d$ for which

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d = k\pi \tag{12}$$

for some positive integer k. First we note that since $\frac{1}{4}\pi \leqslant \tan^{-1} a < \frac{1}{2}\pi$, and $\frac{1}{4}\pi \leqslant \tan^{-1} b < \frac{1}{2}\pi$ and similarly for c, d, then the left-hand side of (12) must lie in the interval $(\pi,\ 2\pi)$. This tells us therefore that (12) has no solutions!

Alternatively, if we proceed as for the three angle case above and add in one more angle, $D = \tan^{-1} d$, then we arrive at the same result but through a different route, which I believe we should encourage students to do to show them how powerful proof is in mathematics – a universal truth is just that, regardless of how it is arrived at. (Remember what the cat said to Alice?). We now take this alternative route, not only for this purpose but because we will shortly move to the five angle case when we will need an expression similar to (9) for the left-hand side of (12); moreover, the simple argument based on the range of the angles doesn't work later on when we consider, for example, the six angle case.

Proceeding as above, and using (9) together with a combination of (3) and (6), we have:

$$\begin{aligned} &\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d \\ &= \tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right) + \tan^{-1} d + \pi \\ &= \tan^{-1}\left(\frac{\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)+d}{1-\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)d}\right) + m\pi \qquad (13) \\ &= \tan^{-1}\left(\frac{a+b+c+d-abc-bcd-cda-dab}{1-ab-ac-ad-bc-bd-cd+abcd}\right) + m\pi \end{aligned}$$

where $m = 1$ or 2 as before. Thus the solution of (12) is found by determining (distinct) positive integer solutions of

$$a + b + c + d - abc - bcd - cda - dab = 0$$

for which

$$1 - ab - ac - ad - bc - bd - cd + abcd \neq 0.$$

We can again experiment using a spreadsheet or computer program to seek solutions by computing many possible combinations of the numerator in (13) for positive integers a, b, c seeking a combination where the value is zero.

The first possible set of integers is $a = 1$, $b = 2$, $c = 3$, $d = 4$, for which $a + b + c + d - abc - bcd - cda - dab = -40$, so this clearly isn't a

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solution. If we look for other solutions these will be of the form $a = 1 + \alpha$, $b = 2 + \beta$, $c = 3 + \gamma$, $d = 4 + \delta$ for integers $\alpha, \beta, \gamma, \delta \geqslant 0$ where at least one of $\alpha, \beta, \gamma, \delta$ is positive, i.e. where at least one of the values is different. We leave readers to check that substituting this solution into $a + b + c + d - abc - bcd - cda - dab$ gives a value which is strictly negative, and so (12) has no solutions, as expected. (For completeness this expression is

$$-25\alpha - 18\beta - 13\gamma - 10\delta - 7\alpha\beta - 6\alpha\gamma - 5\beta\gamma - 4\beta\delta - 3\gamma\delta - 5\alpha\delta$$
$$- \alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta - \beta\gamma\delta - 40,$$

which is clearly negative. The algebra required to determine this expression is fairly tedious so a computer algebra package is easier, and more reliable, provided of course that the expressions are entered accurately.)

This might be a good time to try to clear out of the way all similar cases, i.e. all even numbered angle cases, by using the first argument about the range on the left-hand side. Unfortunately, for the six angle case, for example, when we have

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e + \tan^{-1} f = k\pi, \quad (14)$$

the left-hand side of (14) will lie in the interval $\left(\frac{3}{2}\pi, 3\pi\right)$ which doesn't rule out ther being a solution with $k = 2$. The same will be true for other even-numbered angle cases. Thus to find solutions of the six angle case (if indeed there are any) we would need to determine the right-hand side of the corresponding expression to the one in (13), first for the five angle case and only then for the six angle case. We leave this as an exercise, but it is not difficult to do if we follow the same arguments as we did to establish (13). Readers will then spot, if they haven't already, a pattern emerging from (9) and (13), which can be written as two general cases: even-numbered angle cases and odd-numbered angle cases.

2.3 *Five angle case*

Here we seek solutions of:

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e = k\pi. \quad (15)$$

What you will need to show first is that

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e$$
$$= \tan^{-1}\left(\frac{\begin{array}{c}a + b + c + d + e - abc - bcd - cda - dab \\ -abe - ace - ade - bce - bde - cde + abcde\end{array}}{\begin{array}{c}1 - ab - ac - ad - bc - bd - cd - ae - be \\ -ce - de + abcd + abce + bcde + cdae + dabe\end{array}}\right) + m\pi \quad (16)$$

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where $m = 1$ or 2 as before. (Note that the term on the right-hand side of (16) can be written as

$$\tan^{-1}\left(\frac{\sum a - \sum abc + \sum abcde}{1 - \sum ab + \sum abcd}\right)$$

where, for example, the notation $\sum abc$ means the sum of all possible combinations of the three values chosen from the five values a, b, c, d, e, with the special cases $\sum a = a + b + c + d + e$ and $\sum abcde = abcde$. (We also note that the elementary symmetric functions of a, b, c, d, e on the numerator and denominator of (16) are the coefficients of the corresponding quintic polynomial

$$(x - a)(x - b)(x - c)(x - d)(x - e)$$

$$\equiv x^5 - \left(\sum a\right)x^4 + \left(\sum ab\right)x^3 - \left(\sum abc\right)x^2 + \left(\sum abcd\right)x - \left(\sum abcde\right).$$

Hence solutions of (15) are roots of quintic equations for which

$$\sum a - \sum abc + \sum abcde = 0,$$

i.e. where

$$\text{coefficient of } x^4 - \text{coefficient of } x^2 + \text{constant term} = 0.$$

Similar remarks apply to other angle cases, for example equations (10) and (13) in the three and four angle case, respectively.

Using (16) we see that solutions of (15) must satisfy

$$a + b + c + d + e - abc - bcd - cda - dab - abe$$

$$-ace - ade - bce - bde - cde + abcde = 0$$

for which

$$1 - ab - ac - ad - bc - bd - cd - ae - be - ce - de$$

$$+abcd + abce + bcde + cdae + dabe \neq 0.$$

At last we are in luck again, and this time we have a solution, and in fact more than one. (Note that we can always establish whether there are definitely no solutions by substituting $a = 1 + \alpha$ etc into the relevant expression to see if it is one-signed or could, theoretically, vanish, which is the case here and we leave this for readers to check.)

The solutions where all values are less than or equal to 30 are shown in Table 2:

a	b	c	d	e
1	2	4	23	30
1	2	5	13	21
1	2	7	8	18
1	3	4	7	13
1	3	5	7	8

TABLE 2

and the value of k in (8) is 2, so for the last solution in Table 2 we have

$$\tan^{-1} 1 + \tan^{-1} 3 + \tan^{-1} 5 + \tan^{-1} 7 + \tan^{-1} 8 = 2\pi.$$

We also see from the Table 2 that some solutions have values in common, so for example from the first two rows we have that

$$\tan^{-1} 4 + \tan^{-1} 23 + \tan^{-1} 30 = \tan^{-1} 5 + \tan^{-1} 13 + \tan^{-1} 21.$$

Notice that all the solutions in Table 2 start with 1. Thus, the first question that arises is whether there are solutions without 1 as a member. For example, is there one starting with 2? Is the last one as given in Table 2 the 'smallest' one in the sense that the sum of the values is least? Are there infinitely many solutions and, if not, what is the 'largest' one, and which one has the largest first member? Are there solutions where the values are consecutive, or just even, or just odd, or members of well-known sequences, such as squares, Fibonacci numbers, and so on?

2.4 *Six angle case*

Likewise for the six angle case we seek solutions of

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e + \tan^{-1} f = k\pi \quad (17)$$

for which you will need

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e + \tan^{-1} f$$

$$= \left(\frac{\begin{gathered} a+b+c+d+e+f-abc-bcd-cda-dab-abe-ace-ade \\ -bce-bde-cde-abf-acf-adf-bcf-bdf-cdf-aef \\ -bef-cef-def+abcde+abcdf+abcef+bcdef+cdaef+dabef \end{gathered}}{\begin{gathered} 1-ab-ac-ad-bc-bd-cd-ae-be-ce-de-af-bf-cf \\ -df-ef+abcd+abce+bcde+cdae+dabe+abcf+bcdf \\ +cdaf+dabf+abef+acef+adef+bcef+bdef+cdef-abcdef \end{gathered}} \right) + m\pi \quad (18)$$

where now $m = 2$ or 3. (Note that the term on the right-hand side of (15)

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could be more succinctly expressed using the notation above as

$$\tan^{-1}\left(\frac{\sum a - \sum abc + \sum abcde}{1 - \sum ab + \sum abcd - \sum abcdef}\right).)$$

Having established the expression in (16) for the five angle case and (18) for the six angle case as typical even and odd-numbered cases, and spotted a pattern, we could now generate all the expressions we need to explore solutions of generalisations of (9), (13), (16) and (18).

So for the six angle case we ask what, if any, solutions of (17) there are, for which we need the numerator in the fraction on the right-hand side of (18) to be zero and the denominator non-zero. It turns out that this case also has no solutions, although as remarked earlier this cannot be inferred from (17) using the range of the left-hand side, namely $\left(\frac{3}{2}\pi,\ 3\pi\right)$, and the right-hand side which lies inside this interval with $k = 2$. We therefore have to proceed as for the four angle case to prove this by noting that the first possible set of integers is $a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$, for which the numerator on the right-hand side in (18) is 1050 , so this clearly isn't a solution. If we then look for other solutions these will be of the form $a = 1 + \alpha, b = 2 + \beta, c = 3 + \gamma, d = 4 + \delta, e = 5 + \varepsilon, f = 6 + \zeta$ for integers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \geqslant 0$ where at least one of $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ is positive, i.e. where at least one of the values is different. We leave readers to check that substituting this solution into the numerator in the right-hand side of (18) gives a value which is strictly positive, and so (17) has no solutions. (Note that this expression comprises some 62 terms, together with the constant term 1050.)

This begs the questions as to whether any of the even-numbered angle cases has a solution, and also whether all odd-numbered angle cases always have at least one solution.

2.5 *Seven angle case*

For

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c + \tan^{-1}d + \tan^{-1}e + \tan^{-1}f + \tan^{-1}g = k\pi \quad (19)$$

there are solutions, and those for which the values are less than or equal to 30 are shown in Table 3:

a	b	c	d	e	f	g
1	2	7	18	21	23	30
1	2	12	13	17	18	21
1	2	5	7	21	23	30
1	3	5	12	13	17	21
1	3	7	8	12	17	18

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1	4	5	7	8	23	30
1	4	5	8	12	13	17
2	3	4	5	7	8	13

TABLE 3

and the value of k in (19) is 3. The expression you need to explore this can be written in summary form as

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e + \tan^{-1} f + \tan^{-1} g$$

$$= \tan^{-1}\left(\frac{\sum a - \sum abc + \sum abcde - \sum abcdefg}{1 - \sum ab + \sum abcd - \sum abcdef}\right) + m\pi \qquad (20)$$

where $m = 2$ or 3 based on the range of both the left-hand and right-hand sides of (20).

We now also see that there is a solution (the last one in the table) where 1 is not a member, giving

$$\tan^{-1} 2 + \tan^{-1} 3 + \tan^{-1} 4 + \tan^{-1} 5 + \tan^{-1} 7 + \tan^{-1} 8 + \tan^{-1} 13 = 3\pi.$$

As an aside we also see from the solutions in rows 1 and 3 that the following must be true:

$$\tan^{-1} 2 + \tan^{-1} 18 = \tan^{-1} 3 + \tan^{-1} 5$$

and from rows 6 and 7 that:

$$\tan^{-1} 7 + \tan^{-1} 23 + \tan^{-1} 30 = \tan^{-1} 12 + \tan^{-1} 13 + \tan^{-1} 17.$$

2.6 *Even-numbered cases from other cases*

Readers may now have realised that solutions of even-numbered cases could be formed if there are two solutions of the corresponding case with half as many angles which have no values in common. For example, if we can find two such solutions of the five angle case (15) we can add these to form a solution of the ten angle case where the sum will be 4. Unfortunately none of those in Table 2 have this property. Similar remarks apply to Table 3 in respect of the seven angle case and fourteen angle case, unfortunately.

This latter remark provokes one final question in this Tangent 1: Are there instances of two solutions in any case which have no values in common? Maybe we'll leave the rabbit to see if he can pull that one out of his hat.

3. *Tangent* 2

We conclude with a completely different area that Alice, readers, teachers and their students, might wish to explore. It includes an understanding of the addition formula (3), sums of series, convergence, and the use of technology to investigate the following problem. What is the sum

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of the infinite series

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \tan^{-1}\frac{1}{31} + \tan^{-1}\frac{1}{43} + \tan^{-1}\frac{1}{57} + \tan^{-1}\frac{1}{73} + \tan^{-1}\frac{1}{91} + \ldots ? \tag{21}$$

Various stages in the investigation of this problem could be suitable starting points for students, with students either left to their own devices, or with suggestions or hints made to them at suitable junctures.

The denominators of the unit fractions in (21) increase by 2, 4, 6, 8, … , and so ultimately we can show that these form the quadratic sequence: $1 + n + n^2$, starting with $n = 1$, and thus the series in (21) can be written as

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \ldots$$
$$= \tan^{-1}\left(\frac{1}{1+1+1^2}\right) + \tan^{-1}\left(\frac{1}{1+2+2^3}\right) + \tan^{-1}\left(\frac{1}{1+3+3^2}\right) + \ldots$$
$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{1+n+n^2}\right).$$

Using a spreadsheet we find the partial sums $S_N = \sum_{n=1}^{N} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$ in Table 4 for the values of N indicated

N	10	50	100	500	1000	5000	10000
S_N	0.685730	0.765401	0.775398	0.783398	0.784398	0.785198	0.785298

TABLE 4

(To do this using an *Excel* spreadsheet we populate column A (cells A1,A2,A3, …) with the positive integers $n = 1, 2, 3, \ldots$. This can be done by entering 1 in cell A1, then =A1+1 in cell A2, and then replicate using the drag and drop facility to populate further entries in column A from cells A3 onwards down to cell A10000. Similarly enter $=\text{ATAN}\left(\frac{1}{1 + \text{A1}+(\text{A1})\wedge 2}\right)$ in cell B1 (which will compute $\tan^{-1}\left(\frac{1}{1+1+1^2}\right)$ and then replicate using the drag and drop facility again to populate further entries in column B from cells B2 down to cell B10000. Now enter =SUM(B1:B1) in cell C1, and then replicate this down to cell C10000. This will compute in cell C10, for example, the value of $S_{10} = \sum_{n=1}^{10} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$, and similarly further partial sums S_N, down to

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$$S_{10000} = \sum_{n=1}^{10000} \tan^{-1}\left(\frac{1}{1 + n + n^2}\right)$$

in cell C10000.)

The values in Table 4 which would appear to indicate convergence, albeit rather slowly, although not what the sum might be. If we multiply the values of the partial sums in Table 4 by 4, however, we obtain the results in Table 5.

N	10	50	100	500	1000	5000	10000
$4S_N$	2.742918	3.061603	3.101594	3.133593	3.137593	3.140793	3.141193

TABLE 5

This is looking distinctly more promising, with it appearing that the values might be approaching π, so that the sum in (21) *might* therefore be $\frac{1}{4}\pi$. Figure 1 shows the partial sums (multiplied by 4) up to $N = 50$ which is more convincing that the series might be convergent.

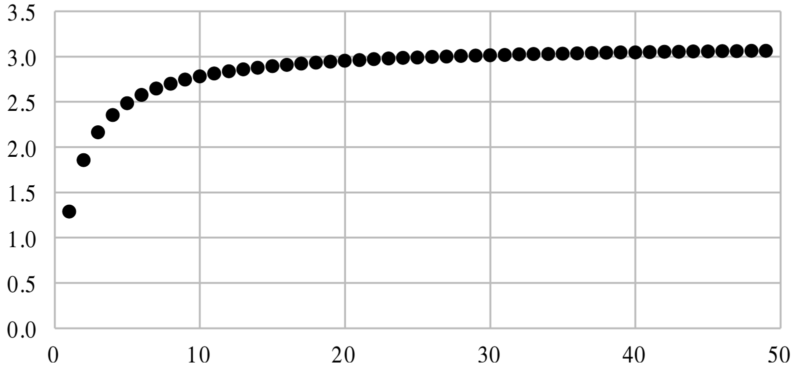

FIGURE 1: N th partial sum versus N

The next stage is to prove this, which ultimately requires knowing, or stumbling upon (2) in the form:

$$\tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{1 + n} = \tan^{-1}\frac{1}{1 + n + n^2}. \tag{22}$$

How one achieves this is not so clear, however. Knowledge of (3) or the equivalent expression for the difference:

$$\tan^{-1}p - \tan^{-1}q = \tan^{-1}\frac{p - q}{1 + pq},$$

say for $0 < q \leqslant p \leqslant 1$, and setting $p = \frac{1}{n}$ and $q = \frac{1}{1 + n}$ from which we can prove (2) (or, equivalently, (22)) would help, or maybe considering adding $\tan^{-1}\frac{1}{1 + n}$ to $\tan^{-1}\frac{1}{1 + n + n^2}$, which will simplify to $\tan^{-1}\frac{1}{n}$ as in (2) or (22). Ultimately one probably needs to spot that with the sum of the

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series (22) appearing to be $\frac{1}{4}\pi$ which is $\tan^{-1} 1$, then the series of partial sums $\sum_{n=1}^{N} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$ must somehow 'collapse' to give a term which is $\frac{1}{4}\pi$ and another which tends to 0 as $N \to \infty$. For this to happen we would need to rewrite $\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$ as the difference $a_n - a_{n+1}$ between two terms in a sequence a_n, i.e. the result in (22). However Alice reaches this point, she will finish her journey with

$$\sum_{n=1}^{N} \tan^{-1}\left(\frac{1}{1+n+n^2}\right) = \sum_{n=1}^{N}\left(\tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{1+n}\right)$$

$$= \tan^{-1}\frac{1}{1} - \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3} - \ldots - \tan^{-1}\frac{1}{1+N}$$

$$= \tan^{-1}1 - \tan^{-1}\frac{1}{1+N} \to \tan^{-1}1 = \tfrac{1}{4}\pi$$

as $N \to \infty$, as expected.

I wonder what the Queen would have made of all this.

References

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The answers to the *Nemo* page from July on paradoxes were:

1.	Oscar Wilde	A Portrait of Dorian Gray	Chapter 17
2.	Charlotte Brontë	Villette	Chapter 42
3.	Shakespeare	Hamlet	Act 3 Scene 1
4	GB Shaw	Back to Methuselah	Part IV Act 1
5.	Hermann Melville	Mardi: and a Voyage Thither	Chapter 30
6.	James Joyce	Ulysses	Scylla and Charybdis

Congratulations to Martin Lukarevski on tracking all of these down.

Quotations are on page 408.