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Pointwise-recurrent graph maps

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Abstract. In this paper we show that a continuous map f from a connected graph G to itself is pointwise-recurrent if and only if one of the following two statements holds: (1) G is a circle, and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle S^1; (2) f is a periodic homeomorphism.

1. *Introduction*

A continuous map f from a topological space X to itself is said to have some homogeneous property if all points in X under f have a common property, for example, all points in X are periodic, almost-periodic, recurrent, or chain recurrent. It is interesting to describe maps which have some homogeneous properties. Montgomery [**15**] showed that a connected topological manifold M has the property that every pairwise periodic homeomorphism of M must be periodic; see also [**16**]. Weaver [**19**] showed that a continuum C embedded in an orientable 2-manifold has this property with respect to homeomorphisms which are C^1 and orientation-preserving. In [**12**] we proved that if X is a compact locally connected subset of a closed 2-manifold M which has no cut-points, then every orientation-preserving (or orientation-reversing, or orientation-relatively-preserving) pointwise-periodic homeomorphism $f : X \to X$ is periodic, and f can be extended to a periodic homeomorphism of a compact submanifold of M. Brechner [**8**] showed that almost-periodic homeomorphisms of the plane are periodic. Oversteegen and Tymchatyn [**18**] showed that recurrent homeomorphisms of the plane are also periodic. Kolev and Pérouème [**10**] proved that recurrent homeomorphisms of a compact surface with negative Euler characteristic are still periodic. However, on the annulus, there exists a recurrent area- and orientation-preserving diffeomorphism without periodic points which is not conjugate to a rotation; see [**9**]. In [**14**], Mai and Ye proved that a continuous map of a metric space is pointwise-recurrent and has the pseudo-orbit-tracing property if and only if the map is uniformly conjugate to an adding-machine-like map restricted to some invariant subset.

It is easy to show that every pointwise non-wandering circle map without periodic points is topologically conjugate to an irrational rotation; see [**1**]. Block and Coven [**3**] studied pointwise chain-recurrent interval maps. In this paper we will study pointwise-recurrent graph maps. Our main results are the following theorems.

THEOREM 3.4. *Let G be a graph, $f \in C^0(G)$, and let W be a minimal set of f. If W contains interior points, then:*

(1) *W is the union of finitely many pairwise disjoint circles;*

(2) *there exists $\varepsilon > 0$ such that $(B(W, \varepsilon) - W) \cap \Omega(f) = \emptyset$.*

THEOREM 4.4. *Let G be a connected graph, and $f : G \to G$ be a continuous map. Then f is pointwise-recurrent if and only if one of the following two statements holds:*

(1) *G is a circle and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle S^1;*

(2) *f is a periodic homeomorphism.*

2. *Minimal sets of maps*

Let $\mathbb{N}$ denote the set of positive integers. For any topological space X, denote by $C^0(X)$ the set of all continuous maps from X to X. Suppose $f \in C^0(X)$, and $n \in \mathbb{N}$. A point $x \in X$ is called a *periodic point of f with period n* (or an *n-periodic point of f*) if $f^n(x) = x$ and $f^k(x) \neq x$ for each $k \in (0, n) \cap \mathbb{N}$. A point x is called a *fixed point* of f if $f(x) = x$. A point x is called a *recurrent* (respectively *non-wandering*) *point* of f if for any neighborhood U of x there exists $i \in \mathbb{N}$ (respectively $y \in U$ and $i \in \mathbb{N}$) such that $f^i(x) \in U$ (respectively $f^i(y) \in U$). Denote by $\mathrm{Fix}(f)$ (respectively $P_n(f)$, $R(f)$ and $\Omega(f)$) the set of all fixed (respectively n-periodic, recurrent and non-wandering) points of f. Write $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$. A map f is said to be *pointwise-recurrent* (respectively *pointwise-periodic*) if all points in X are recurrent (respectively periodic). A map f is said to be *periodic* if there exists $m \in \mathbb{N}$ such that f^m is the identity map of X. Note that every pointwise-periodic continuous map of a compact metric space is a homeomorphism.

Write $O(x, f) = (x, f(x), f^2(x), \dots)$, and call such a sequence an *orbit* of f. Let $S_n = (x_0, x_1, \dots, x_n)$ and $S = (y_0, y_1, y_2, \dots)$ be sequences of points in X. If $f(x_i) = x_{i-1}$ for each $i \in \{1, \dots, n\}$, then S_n is called a *finite inverse orbit* of f with length n, and x_0 and x_n are respectively called the *starting point* and *terminal point* of this inverse orbit. If $f(y_i) = y_{i-1}$ for all $i \in \mathbb{N}$, then S is called an *infinite inverse orbit* of f. Sometimes we regard the orbit $O(x, f)$ and the inverse orbits S and S_n as sets. The following lemma clearly holds.

LEMMA 2.1. *Let X be a topological space, $Y \subset X$ and $f \in C^0(X)$. If $f(Y) \supset Y$, then every finite inverse orbit $(x_0, x_1, \dots, x_n)$ in Y can be extended to an infinite inverse orbit $(x_0, x_1, \dots, x_n, x_{n+1}, \dots)$ in Y.*

A subset W of X is said to be *invariant* or *f-invariant* if $f(W) \subset W$. A set W is called an *f-minimal set* or a *minimal set* of f if it is non-empty, closed and f-invariant, and it contains no proper subset having these three properties. $f : X \to X$ is called

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a *minimal map* if X itself is a (unique) minimal set of f. The following lemma can be directly derived from [**2**, Proposition V.5 and Lemma V.7].

LEMMA 2.2. *Let X be a compact metric space, $f \in C^0(X)$, $y \in X$ and $p \in \mathbb{N}$. Then $\overline{O(y, f)}$ is a minimal set of f if and only if $\overline{O(y, f^p)}$ is a minimal set of f^p.*

THEOREM 2.3. *Let X be a metric space, W be a non-empty compact subset of X, and $f \in C^0(X)$. Then the following four properties are equivalent.*

(i) *W is a minimal set of f.*
(ii) *For every $y \in W$, $\overline{O(y, f)} = W$.*
(iii) *$f(W) \subset W$, and every orbit in W is dense in W.*
(iv) *$f(W) \supset W$, and every infinite inverse orbit in W is dense in W.*

Furthermore, if the non-empty compact set W is connected, then the above four properties and the following two properties are also equivalent.

(v) *$f(W) \subset W$, and there exists a $p \in \mathbb{N}$ such that W is a minimal set of f^p.*
(vi) *For all $p \in \mathbb{N}$, W is a minimal set of f^p.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (vi) $\Rightarrow$ (v) are trivial.

(i) $\Rightarrow$ (iv) Assume that (i) is true. Then, since W is compact, we have $f(W) = W$. Let $S = (x_0, x_1, x_2, \dots)$ be an infinite inverse orbit of f contained in W. Then there exist $y \in \overline{S}$ and positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\lim_{i\to\infty} x_{k_i} = y$. Noting that $f^n(y) = \lim_{i\to\infty} x_{k_i - n} \in \overline{S}$, for all $n \in \mathbb{N}$, we have $W = \overline{O(y, f)} \subset \overline{S}$. Thus, S is dense in W.

(iv) $\Rightarrow$ (i) Assume that (iv) is true. Then, by Lemma 2.1, there is an infinite inverse orbit $S = (x_0, x_1, x_2, \dots)$ of f contained in W. For every $n \in \mathbb{N} \cup \{0\}$, write $S_n = (x_n, x_{n+1}, x_{n+2}, \dots)$. Then S_n is also an infinite inverse orbit in W. Let $\alpha(S) = \bigcap_{n=0}^{\infty} \overline{S_n}$, and call it the *$\alpha$-limit set* of S. It is easy to see that $\alpha(S)$ is an f-invariant closed subset of W. Noting that all S_n are dense in W, we have $\alpha(S) = W$. If W is not a minimal set of f, then W should contain a non-empty, closed and f-invariant proper subset W_0, and W_0 should contain an infinite inverse orbit of f which is not dense in W. This leads to a contradiction. Thus W must be a minimal set of f.

(v) $\Rightarrow$ (i) Assume that (v) is true. Then we have $W \supset \overline{O(y, f)} \supset \overline{O(y, f^p)} = W$, for all $y \in W$. Thus W is a minimal set of f.

(i) $\Rightarrow$ (vi) Assume that (i) is true, and W is connected. In order to show that (vi) is true, it suffices to consider the case that p is a prime number. Take a point $y \in W$. For every $i \in \mathbb{N} \cup \{0\}$, write $y_i = f^i(y_0)$, and $W_i = \overline{O(y_i, f^p)}$. Then $f(W_i) = W_{i+1}$, and $W = \overline{O(y, f)} = \bigcup_{n=0}^{p-1} W_n$. By Lemma 2.2, W_i is a minimal set of f^p and $W_{i+p} = W_i$. Since W is connected, there exists $k \in \{1, \dots, p-1\}$ such that $W_k \cap W_0 \neq \emptyset$, which implies

$$W_{ik+k} \cap W_{ik} = f^k(W_{ik}) \cap f^k(W_{ik-k}) \supset f^k(W_{ik} \cap W_{ik-k}) \neq \emptyset, \quad \text{for } i = 1, 2, 3, \dots .$$

Noting that any two intersecting minimal sets of f^p are the same, we have $W_{ik+k} = W_{ik} = W_{ik-k} = \cdots = W_0$. For each $n \in \{0, 1, \dots, p-1\}$, since k and p are relatively prime, there exists $j = j(n) \in \mathbb{N}$ such that $jk \equiv n \pmod p$. Thus $W_n = W_{jk} = W_0$, and hence $W = \bigcup_{n=0}^{p-1} W_n = W_0$ is a minimal set of f^p. Theorem 2.3 is proven. □

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From Theorem 2.3 we obtain the following.

COROLLARY 2.4. *Let X be a compact metric space and $f \in C^0(X)$. If there exists a non-empty closed proper subset Y of X such that $f(Y) \subset Y$ or $f(Y) \supset Y$, then f is not a minimal map.*

Proof. If $f(Y) \subset Y$, then X is not a minimal set since it has a non-empty closed f-invariant proper subset Y. If $f(Y) \supset Y$, then f has an infinite inverse orbit S in Y. Since S is not dense in X, by Theorem 2.3, X is not a minimal set of f. Thus, f is not a minimal map. □

3. *Minimal sets of graph maps*

We now consider graph maps. A (finite) *graph* G is a topological space without isolated points which is homeomorphic to the polyhedron $|K|$ of a finite one-dimensional simplicial complex K in the three-dimensional Euclidean space $\mathbb{R}^3$. A continuous map from G to G is called a *graph map*. For convenience, we may assume $G = |K|$. Then every zero-dimensional (respectively one-dimensional) simplex of K is called a *vertex* (respectively *edge*) of G. Let $V(G)$ be the set of vertices of G. For $v \in V(G)$, the number of edges with v being an endpoint is called the *valence* of v and is written $\text{val}(v)$. A vertex v is called an *endpoint* (respectively *branching point*) of G if $\text{val}(v) = 1$ (respectively $\text{val}(v) \geq 3$). For $x \in G - V(G)$, we put $\text{val}(x) = 2$. Define the *standard metric* d_K on $G = |K|$ as in [**13**]. Then the length of each edge of G is 1, and for any two points x and y lying on the same connected component of G, $d_K(x, y)$ is the minimal length of arcs in G whose endpoints are x and y. For any non-empty subset Y of G and any $r > 0$, we write $B(Y, r) = \{x \in G \mid d_K(x, Y) \leq r\}$.

For any $x, y \in G$ with $x \neq y$, let $[x, y] = [x, y]_G$ be the arc in G whose endpoints are x and y and whose length is $d_K(x, y)$, if there exists a unique such arc, and let $[x, y) = (y, x] = [x, y] - \{y\}$, $(x, y) = [x, y) - \{x\}$, $[x, x] = \{x\}$, $[x, x) = (x, x] = (x, x) = \emptyset$. Generally, for any arc A in G and any $\{x, y\} \subset A$, we denote by ∂A the two endpoints of A, by $\mathring{A} = A - \partial A$ the interior of A, and by $[x, y]_A$ the subarc of A whose endpoints are x and y.

LEMMA 3.1. *Let G be a graph, $f \in C^0(G)$, and let A be an arc in G with $\partial A = \{a, b\}$. If $\mathring{A} \cap V(G) = \emptyset$ and $f(a) = f(b) \in A$, then f is not a minimal map.*

Proof. Write $c = f(a)$. It suffices to consider only the case that $c \in \mathring{A}$. Let $X_0 = \{x \in A : f(x) \in [x, b]\}$, $X_1 = \{y \in A : f(y) \in [a, y]\}$. Suppose that $[a, v]$ is the connected component of X_0 containing a, and $[w, b]$ is the connected component of X_1 containing b. Then $\{v, w\} \subset \mathring{A}$. If $f(v) = v$ or $f(w) = w$, then f has a fixed point and hence is not a minimal map. If $f(v) \neq v$ and $f(w) \neq w$, then $f(v) = b$, $f(w) = a$ and $v \in (a, w)$. Let $Y = [a, v] \cup [w, b]$. Noting that $[c, b] \subset f([a, v]) \subset A$ and $[a, c] \subset f([w, b]) \subset A$, we have $f(Y) = A \supset Y$ and $G - Y \supset A - Y = (v, w) \neq \emptyset$. Thus, by Corollary 2.4, f is not a minimal map. Lemma 3.1 is proven. □

THEOREM 3.2. *Let $G = |K|$ be a connected graph but not a circle. Then there exists no minimal map from G to itself.*

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Proof. If Theorem 3.2 is not true, then there exists a minimal map $f : G \to G$. By Theorem 2.3, $f^p : G \to G$ is also minimal, for all $p \in \mathbb{N}$.

CLAIM 1. *For any edge A of G, the restriction $f|A$ of f on A is injective.*

In fact, if Claim 1 does not hold then there are two points $a \neq b$ in A such that $f(a) = f(b) = c$, and by the minimality of f there exists a $p \in \mathbb{N}$ such that $f^p(a) = f^p(b) = f^{p-1}(c) \in [a, b]$. Therefore, by Lemma 3.1, f^p is not minimal. This yields a contradiction. Hence, Claim 1 holds.

Let $V_b(G)$ denote the set of branching points of G. Since there is no minimal map from an interval to itself, $G = |K|$ is not an arc. Thus, $V_b(G)$ is a non-empty finite set, and there exists $v \in V_b(G)$ such that $f(v) \notin V_b(G)$. Choose $\varepsilon \in (0, 1/2]$ such that $f(B(v, \varepsilon)) \cap V_b(G) = \emptyset$. Write $v_1 = f(v)$. Since $\mathrm{val}(v) \geq 3$ and $\mathrm{val}(v_1) \leq 2$, it follows from Claim 1 that there exist $\{x_0, x_1\} \subset B(v, \varepsilon) - \{v\}$ and $z \in B(v_1, \varepsilon) - \{v_1\}$ such that $f(x_0) = f(x_1) = z$, $[x_0, v] \cap [x_1, v] = \{v\}$ and

$$f([x_0, v]) = f([x_1, v]) = [z, v_1]. \tag{3.1}$$

Let $Y = G - (x_1, v)$. Then Y is a non-empty closed proper subset of G, and $Y \supset [x_0, v]$. Noting that f is surjective, from (3.1) we get $f(Y) = f(G - (x_1, v)) \cup f([x_0, v]) \supset (f(G) - f([x_1, v])) \cup f([x_0, v]) = f(G) = G \supset Y$. Hence, by Corollary 2.4, f is not a minimal map. This leads to a contradiction. Thus, Theorem 3.2 must be true. □

Remark 3.3. In [**4**] Blokh proved that transitive maps of connected graphs have the so-called specification property. Theorem 3.2 can also be derived from this property. In addition, Kolyada *et al* recently proved that every minimal map of a compact metric space is almost one-to-one (see [**11**, Theorem 2.7]). From this result and (3.1) we can also derive Theorem 3.2.

It is easy to show that a finite minimal set on a metric space is a periodic orbit, and an infinite minimal set of a graph map containing no interior points is homeomorphic to the Cantor set. To give a description of the topological structure of minimal sets of graph maps containing interior points, we present the following theorem.

THEOREM 3.4. *Let G be a graph, $f \in C^0(G)$, and let W be a minimal set of f. If W contains interior points, then:*

(1) *W is the union of finitely many pairwise disjoint circles;*

(2) *there exists $\varepsilon > 0$ such that $(B(W, \varepsilon) - W) \cap \Omega(f) = \emptyset$.*

Proof. (1) Let w be an interior point of W and let U be the connected component of W containing w. Then U is a subgraph of G. Let n be the smallest positive integer which satisfies $f^n(U) \cap U \neq \emptyset$. Then $f^n(U) \cup U$ is connected. Thus, $f^n(U) \subset U$. Furthermore, since $f^i(U) \cap U = \emptyset$ for $1 \leq i < n$, and W is a minimal set of f, we have $f^n(U) = U$, and U is a minimal set of f^n. Therefore, for $1 \leq i < n$, $f^i(U)$ is also a connected minimal set of f^n containing interior points. By Theorem 3.2, for every $i \in \{0, 1, \ldots, n-1\}$, $f^i(U)$ is a circle. Evidently, $U, f(U), \ldots, f^{n-1}(U)$ are pairwise disjoint, and $W = \bigcup_{i=0}^{n-1} f^i(U)$.

(2) Let Y be the set of branching points of G lying on W. Then Y is a finite set. If $Y = \emptyset$, then all of $U, f(U), \ldots, f^{n-1}(U)$ are connected components of G, and by

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the definition of metric d_K given in [**13**] we have $B(W, 1) = W$. If $Y \neq \emptyset$, then there exists $p \in \mathbb{N}$ such that $f^p(Y) \cap Y = \emptyset$. Take $\varepsilon > 0$ such that $f^p(B(Y, \varepsilon)) \cap Y = \emptyset$. Then we have $f^p(B(Y, \varepsilon)) \subset W$, and hence $f^k(B(Y, \varepsilon)) \subset W$ for all $k \geq p$. Thus, $(B(W, \varepsilon) - W) \cap \Omega(f) = (B(Y, \varepsilon) - W) \cap \Omega(f) = \emptyset$. Theorem 3.4 is proven. □

COROLLARY 3.5. *Let G be a graph and $f \in C^0(G)$.*

(1) *If G is a forest, i.e. G is a graph containing no circles, then every minimal set of f contains no interior points.*

(2) *If G is connected and contains branching points, and f has a minimal set W which contains interior points, then f has wandering points.*

Proof. (1) is clear, by Theorem 3.4.

(2) Let Y be the set of branching points of G lying on W. Then, by (1) of Theorem 3.4, we have $Y \neq \emptyset$, and for any $\varepsilon > 0$, $B(Y, \varepsilon) - W \neq \emptyset$. By (2) of Theorem 3.4, for sufficiently small $\varepsilon > 0$, all points in $B(Y, \varepsilon) - W$ are wandering points of f. □

4. *Pointwise-recurrent graph maps*

In this section we study the structure of pointwise-recurrent graph maps.

LEMMA 4.1. *Let G be a connected graph, $f : G \to G$ be a pointwise-recurrent continuous map, and let W be a minimal set of f. If $W \neq G$, then W is a periodic orbit of f.*

Proof. Since $W \neq G$, $G - W$ is a non-empty open set. Let U be a connected component of $G - W$. Then $\overline{U} - U = \overline{U} \cap W \neq \emptyset$. It follows from $U \subset R(f)$ that there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. Obviously, $f^n(U) \cup U$ is connected, and $f^n(U) \cap W = \emptyset$ (if a point of U gets mapped into W then this point is not recurrent). Thus, $f^n(U) \subset U$, and hence $f^n(\overline{U}) \subset \overline{U}$. This with $f^n(W) \subset W$ implies $f^n(\overline{U} \cap W) \subset \overline{U} \cap W$. Noting that $\overline{U} \cap W$ is a finite set, we have $W \cap P(f) = W \cap P(f^n) \supset (\overline{U} \cap W) \cap P(f^n) \neq \emptyset$. Thus, the minimal set W is a periodic orbit of f. □

LEMMA 4.2. *Let G be a graph, $f : G \to G$ be a pointwise-recurrent continuous map, and let A be an arc contained in an edge of G with $\partial A = \{a, b\}$. If $\{a, b\} \subset \mathrm{Fix}(f)$ and there exists $c \in \mathring{A}$ such that $f(c) \in A$, then $A \subset \mathrm{Fix}(f)$.*

Proof. If $f(\mathring{A}) \not\subset \mathring{A}$ then there will be a point $x \in A$ such that $f(x) \in \{a, b\}$, and x will not be recurrent. This contradicts the condition of the lemma. Thus, we have $f(\mathring{A}) \subset \mathring{A}$.

If $A \not\subset \mathrm{Fix}(f)$, then there will be a subarc $A_1 = [a_1, b_1]$ of A such that $A_1 \cap \mathrm{Fix}(f) = \partial A_1$ and $f(\mathring{A}_1) \subset \mathring{A}_1$. This will lead to $\lim_{n\to\infty} f^n(x) = a_1$ or b_1 for all $x \in \mathring{A}_1$, which also yields a contradiction. Thus, we have $A \subset \mathrm{Fix}(f)$. □

LEMMA 4.3. *Let G be a connected graph, and $f : G \to G$ be a pointwise-recurrent continuous map. If G is not a circle, or G is a circle but f is not a minimal map, then every point $x \in G$ is periodic.*

Proof. For any given $x \in G$, there exists $v \in \overline{O(x, f)}$ such that $\overline{O(v, f)}$ is a minimal set of f. By Theorem 3.2 and Lemma 4.1 (if G is not a circle), or by the assumption and Lemma 4.1 (if G is a circle), $\overline{O(v, f)} = O(v, f)$ is a periodic orbit. Let n be the period of v under f. If $v \in O(x, f)$, then $x \in O(v, f)$ is periodic since x is recurrent. If $v \notin O(x, f)$, then there exist positive integers $k_1 < k_2 < k_3 < \cdots$ and $m \in \{0, 1, \ldots, n-1\}$ such that $\lim_{i\to\infty} f^{k_i}(x) = v$ and $k_i \equiv m \pmod{n}$ for all $i \in \mathbb{N}$. Therefore, since $\mathrm{val}(v)$ is finite, there exist $\lambda, \mu \in \{k_1, k_2, k_3, \ldots\}$ with $\lambda > \mu$ such that $f^{\lambda n+m}(x)$, $f^{\mu n+m}(x)$ and v lie on the same edge of G, and $f^{\lambda n+m}(x) \in (v, f^{\mu n+m}(x))$. Write $w = f^{\mu n+m}(x)$, $w_1 = f^{\lambda n+m}(x)$ and $g = f^{\lambda n-\mu n}$. Then, $w_1 = g(w)$ and $v \in \mathrm{Fix}(g)$. By [**2**, Lemma IV.25], we have $R(g) = R(f) = G$. Thus, g is still pointwise-recurrent, which implies $g^{-1}(v) = \{v\}$, and hence $v \notin g((v, w])$. Let $Y = \{y \in (v, w] : g(y) \in (v, y)\}$, and let L be the connected component of Y containing w. If $L = (v, w]$, then $\lim_{i\to\infty} g^i(w) = v$, which contradicts $w \in R(g)$. If $L \neq (v, w]$, then there exists $v_1 \in (v, w)$ such that $L = (v_1, w]$ and $g(v_1) = v_1$. Note that $g((v_1, w]) \cap \{v, v_1\} = \emptyset$, since $(v_1, w] \subset R(g)$. If $w_1 = g(w) \in (v_1, w]$, then $\lim_{i\to\infty} g^i(w) = v_1$, which still contradicts $w \in R(g)$. If $w_1 \in (v, v_1)$, then, since $w_1 \in R(g)$, there exists $k \in \mathbb{N}$ such that $g^k(w_1) \in (v, v_1)$. By Lemma 4.2, we have $w_1 \in [v, v_1] \subset \mathrm{Fix}(g^k) \subset P(f)$. Noting that $w_1 \in O(x, f)$ and $x \in R(f)$, we have $x \in O(w_1, f) \subset P(f)$. Lemma 4.3 is proven. □

THEOREM 4.4. *Let G be a connected graph, and $f : G \to G$ be a continuous map. Then f is pointwise-recurrent if and only if one of the following two statements holds:*

(1) *G is a circle and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle S^1;*

(2) *f is a periodic homeomorphism.*

Proof. The sufficiency is clear. We now show the necessity. Assume that f is pointwise-recurrent.

(1) If f has no periodic points, then, by Lemma 4.1, f is a minimal map, and by Theorem 3.2, G is a circle. From Lemma 3.1 (or from [**1**, Corollary 1]) we see that f is a homeomorphism. It is well-known (for example, see [**17**]) that every transitive homeomorphism of a circle is topologically conjugate to an irrational rotation of S^1.

(2) If f has periodic points, then f is not a minimal map. By Lemma 4.3, all points in G are periodic. Let $\{E_1, E_2, \ldots, E_n\}$ be the set of all edges of G. For $i = 1, \ldots, n$, suppose $\partial E_i = \{v_{i1}, v_{i2}\}$, and let v_{i3} be a point in $\mathring{E}_i$. Let k_{ij} be the period of v_{ij} under f, k_i be the least common multiple of k_{i1}, k_{i2} and k_{i3}, and k be the least common multiple of $k_1, k_2, \ldots, k_n$. Then it follows from Lemma 4.2 that $E_i \subset \mathrm{Fix}(f^{k_i})$, which implies $G = \bigcup_{i=1}^n E_i \subset \mathrm{Fix}(f^k)$. Thus, f is a periodic homeomorphism. □

Remark 4.5. In [**5–7**] Blokh studied the spectral decomposition for graph maps. From results of [**5–7**] we can also deduce Theorem 4.4.

Remark 4.6. Let G be a connected graph but not a circle. Noting that $\mathrm{val}(f(x)) = \mathrm{val}(x)$ for any homeomorphism $f : G \to G$ and any $x \in G$, by Theorem 4.4 we can easily prove that there are only finitely many topological conjugacy equivalence classes in the set of pointwise-recurrent continuous maps of G.

Note that there exist infinitely many topological conjugacy equivalence classes in the set of pointwise-recurrent continuous maps of a circle.

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Remark 4.7. Suppose that G is a disconnected graph and $f : G \to G$ is a pointwise-recurrent continuous map. Then the connected components of G can be numbered to be $\{G_{ij} : i = 1, \ldots, m;\ j = 1, \ldots, n_i\}$ such that $f(G_{in_i}) = G_{i1}$ and $f(G_{ij}) = G_{i,j+1}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i - 1$. By [**2**, Lemma IV.25], $f^{n_i}|G_{ij} : G_{ij} \to G_{ij}$ is pointwise-recurrent, for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$. Therefore, every pointwise-recurrent continuous map f of a disconnected graph G is still a homeomorphism, and if G has no connected component being a circle then f must be a periodic homeomorphism.

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REFERENCES

[1] J. Auslander and Y. Katznelson. Continuous maps of the circle without periodic points. *Israel. J. Math.* **32** (1979), 375–381.

[2] L. Block and W. A. Coppel. *Dynamics in One Dimension (Lecture Notes in Mathematics, 1513)*. Springer, New York, 1992.

[3] L. Block and E. M. Coven. Maps of the interval with every point chain recurrent. *Proc. Amer. Math. Soc.* **98** (1986), 513–515.

[4] A. Blokh. On transitive maps of one-dimensional branched manifolds. *Differential–Difference Equations and Problems of Mathematical Physics (Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984)*, pp. 3–9 (in Russian).

[5] A. Blokh. Dynamical systems on one-dimensional branched manifolds, 1. *Theory of Functions, Functional Analysis and Applications (Kharkov)* **46** (1986), 8–18 (in Russian). Translation in *J. Soviet Math.* **48**(5) (1990), 500–508.

[6] A. Blokh. Dynamical systems on one-dimensional branched manifolds, 2. *Theory of Functions, Functional Analysis and Applications (Kharkov)* **47** (1987), 67–77 (in Russian). Translation in *J. Soviet Math.* **48**(6) (1990), 668–674.

[7] A. Blokh. Dynamical systems on one-dimensional branched manifolds, 3. *Theory of Functions, Functional Analysis and Applications (Kharkov)* **48** (1987), 32–46 (in Russian). Translation in *J. Soviet Math.* **49**(2) (1990), 875–883.

[8] B. L. Brechner. Almost periodic homeomorphisms of E^2 are periodic. *Pacific J. Math.* **59** (1975), 367–374.

[9] R. J. Fokkink and L. G. Oversteegen. A recurrent nonrotational homeomorphism on the annulus. *Trans. Amer. Math. Soc.* **333** (1992), 865–875.

[10] B. Kolev and M.-C. Pérouème. Recurrent surface homeomorphisms. *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 161–168.

[11] S. Kolyada, L. Snoha and S. Trofimchuk. Noninvertible minimal maps. *Fund. Math.* **168** (2001), 141–163.

[12] J.-H. Mai. Pointwise periodic self-maps of subspaces of 2-dimensional manifolds. *Sci. China Ser. A*, **33** (1990), 145–155.

[13] J.-H. Mai. Scrambled sets of continuous maps of 1-dimensional polyhedra. *Trans. Amer. Math. Soc.* **351** (1999), 353–362.

[14] J.-H. Mai and X. Ye. The structure of pointwise recurrent maps having the pseudo orbit tracing property. *Nagoya Math. J.* **166** (2002), 83–92.

[15] D. Montgomery. Pointwise periodic homeomorphisms. *Amer. J. Math.* **59** (1937), 118–120.

[16] D. Montgomery and L. Zippin. *Topological Transformation Groups*. Interscience Publishers, New York, 1955.

[17] Z. Nitecki. *Differentiable Dynamics*. MIT Press, Cambridge, MA, 1971.
[18] L. G. Oversteegen and E. D. Tymchatyn. Recurrent homeomorphisms on $\mathbb{R}^2$ are periodic. *Proc. Amer. Math. Soc.* **110** (1990), 1083–1088.
[19] N. Weaver. Pointwise periodic homeomorphisms of continua. *Ann. Math.* **95** (1972), 83–85.