*Probability in the Engineering and Informational Sciences*, **24**, 2010, 289–301.
doi:10.1017/S0269964809990283

# SUBEXPONENTIAL INTERVAL GRAPHS GENERATED BY IMMIGRATION–DEATH PROCESSES

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We propose a simple model of random interval graphs generated by immigration–death processes (also known as $M/G/\infty$ queuing processes), where the length of each interval follows a subexponential distribution, and provide a condition under which the stationary degree distribution is also subexponential. Furthermore, we consider the conditional expectation of the cluster coefficient of a vertex given the degree and show that it vanishes in the limit as the degree goes to infinity under the same condition as that for obtaining the tail asymptotics of the stationary degree distribution.

## 1. INTRODUCTION

Scale-free graphs have recently attracted much attention since so-called scale-free phenomena have really appeared in various physical and social networks, where we say that a graph is scale-free if the distribution of degrees (the numbers of edges incident to respective vertices) has a power-law tail. To throw light on such phenomena in the real world, many models of random graphs realizing the scale-free property have so far been proposed and investigated since the early works by Watts and Strogatz [17] and Barabási and Albert [3]. Among them, the authors' previous work [13] proposed a simple model of random interval graphs generated by immigration–death processes

 0269-9648/10 $25.00 

(also known as $M/G/\infty$ queuing processes; see, e.g., Cox and Isham [6, Sect. 5.6]) and showed that when the interval lengths follow a power-law distribution, the generated interval graph is scale-free. Here, a graph $G = (V, E)$ is said to be an interval graph when $G$ has an interval representation $\mathcal{I}$, the set of intervals on the real line, such that each vertex $v \in V$ corresponds to an interval $I_v \in \mathcal{I}$ and there is an edge $(u, v) \in E$ connecting two vertices $u,v \in V$ if and only if $I_u \cap I_u \neq \emptyset$. In [13], each interval is then given as the period of a customer's stay in the $M/G/\infty$ queue; that is, the interval lengths correspond to the service (sojourn) times of customers. Interval graphs form one of the most important classes of graphs since they have several nice features, so they have been studied thoroughly in graph theory (see, e.g., Golumbic [10, Chap. 8] and McKee and McMorris [12]).

In the current paper, we generalize the result in [13] to the model where the distribution of interval lengths is subexponential (see, e.g., Embrechts, Klüppelberg, and Mikosch [8, Sect. 1.3 and A3] or Rolski, Schmidli, Schmidt, and Teugels [15, Sect. 2.5] for subexponential distributions)—namely we consider random interval graphs generated by immigration–death processes with subexponential lifetime distributions, which we call *subexponential interval graphs*. We provide a condition on the lifetime (service time, interval length) distribution $F$ under which the stationary degree distribution of the generated interval graph has a tail equivalent to that of $F(x/\lambda)$; that is, the stationary degree distribution is also subexponential, where $\lambda$ denotes the arrival rate of intervals. This derivation is based on the recent results on sampling of a stochastic process at random times according to subexponential distributions (see Asmussen, Klüppelberg, and Sigman [2], Foss and Korshunov [9], and Jelenković, Momčilović, and Zwart [11]).

Furthermore, we consider the conditional expectation of the cluster coefficient of a vertex given its degree. In a given graph, the cluster coefficient of a vertex represents the fraction of couples of its neighbors such that the couple is also connected by an edge, and it is observed that many scale-free graphs have high cluster coefficients (see, e.g., Newman [14]). In fact, the previous work [13] also demonstrated by a combinatorial argument that the random interval graphs similar to ones in this article have high cluster coefficients, on average, over vertices. In this article, however, we show that the conditionally expected cluster coefficient given the degree vanishes in the limit as the degree goes to infinity under the same condition as that for obtaining the tail asymptotics of the stationary degree distribution. This result states that the vertices with high degrees, which correspond to very long intervals, have extremely small cluster coefficients and does not contradict the result in [13], in which relatively short intervals play an essential role.

The rest of the article is organized as follows. In the next section, we describe the immigration–death process and exhibit an algorithm constructing random interval graphs based on that process. In Section 3 we analyze the subexponential interval graph generated by the immigration–death process in the steady state, in which we discuss the tail asymptotics of the stationary degree distribution and the conditionally expected cluster coefficient of a vertex given its degree in the limit as the degree goes to infinity. Finally, Section 4 is a concluding remark.

## 2. INTERVAL GRAPHS GENERATED BY IMMIGRATION–DEATH PROCESSES

In this section, we describe an immigration–death process (also known as an $M/G/\infty$ queuing process; see, e.g., [6, Sect. 5.6]) and construct a random interval graph based on that process. Let $\{T_n\}_{n\in\mathbb{Z}_+}$ denote a random sequence on $\mathbb{R}_+$ satisfying $0 = T_0 < T_1 < T_2 < \cdots$, at each of which an individual arrives and enters a system. We refer to the individual arriving at $T_n$ as individual $n$ ($\in \mathbb{Z}_+$). The lifetime (service time) of individual $n$ in the system is denoted by $L_n$ ($\geq 0$), so that individual $n$ departs from the system at $T_n + L_n$. We assume that $\{T_n\}_{n\in\mathbb{N}}$ follows a homogeneous Poisson process with intensity $\lambda \in (0,\infty)$ and $\{L_n\}_{n\in\mathbb{Z}_+}$ is a sequence of mutually independent nonnegative random variables according to a common distribution $F$, where $\{T_n\}_{n\in\mathbb{N}}$ and $\{L_n\}_{n\in\mathbb{Z}_+}$ are also independent of each other. The distribution $F$ is assumed to have its mean $\mu^{-1} = \int_0^\infty \overline{F}(x)\,\mathrm{d}x < \infty$, where $\overline{F}(x) = 1 - F(x)$, $x \geq 0$. Let $I_n = [T_n, T_n + L_n]$, $n \in \mathbb{Z}_+$, and $Z(t) = \sum_{n\in\mathbb{Z}_+} 1_{I_n}(t)$, $t \geq 0$, where $1_A$ denotes the indicator function for set $A$. Note that $Z(t)$ represents the number of individuals in the system at time $t \geq 0$ (the reason for the choice of $I_n = [T_n, T_n + L_n]$ rather than $[T_n, T_n + L_n)$ will be clarified in Remark 1). It is well known that $\{Z(t)\}_{t\geq 0}$ has a stationary regime when both $\lambda$ and $\mu$ are nonzero and finite (see, e.g., [6, Sect. 5.6] or Takács [16, Sect. 3.2]).

Based on this immigration–death process, we consider a random interval graph $G_0 = (V_0, E_0)$ with interval representation $\mathcal{I}_0 = \{I_n\}_{n\in V_0}$, where $V_0 = \{0, 1, \ldots, n_0 - 1\}$ and $n_0$ is a predetermined positive integer—namely each individual in $V_0$ corresponds to a vertex of the graph and two vertices $n$ and $m \in V_0$ are connected by an edge if and only if $I_n \cap I_m \neq \emptyset$. Note that such a graph has no multiedges or self-loops. Given $n_0$, $\lambda$, and distribution $F$, a simple algorithm constructing such random interval graphs is as follows, where *Sample*$(F)$ denotes the sampled value extracted according to $F$ and $\mathrm{Exp}(\lambda)$ denotes the exponential distribution with parameter $\lambda$.

**procedure** *generate_graph*$(n_0, \lambda, F)$
  $T = 0, V = \{0\}, E = \emptyset, Q = \{0\}, U_0 = \mathit{Sample}(F), n = 1;$ {$Q$: Set of individuals in the system; $U_n$: departure time of individual $n$}
  **while** $n < n_0$ **do**
    $T \leftarrow T + \mathit{Sample}(\mathrm{Exp}(\lambda));\ V \leftarrow V \cup \{n\};$ {Individual $n$ arrives $\Rightarrow$ Add vertex $n$}
    **for** $i$ such that $i \in Q$ **do**
      **if** $U_i < T$ **then**
        $Q \leftarrow Q \setminus \{i\};$
      **else**
        $E \leftarrow E \cup \{(i,n)\};$ {Individual $i$ is still in the system at individual $n$'s arrival $\Rightarrow$ Add edge $(i,n)$}
      **end if**
    **end for**

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$U_n = T + Sample(F)$; $Q \leftarrow Q \cup \{n\}$;
$n \leftarrow n + 1$;
**end while**

*Remark 1*: When $n_0$ is large, the random interval graph constructed by the above algorithm ends up having many connected components with random but finite sizes and the size of any connected component does not tend to infinity even as $n_0 \to \infty$. Against such a feature, one might want to have one large connected graph. In such a case, it can be realized by adding extra intervals $J_n = [A_n, B_n]$, $n \in \mathbb{N}$, where $A_n = \inf\{t > B_{n-1} \mid Z(t) = 0\}$ and $B_n = \inf_{k\in\mathbb{N}}\{T_k > A_n\}$ with $B_0 = 0$; that is, $J_n$, $n \in \mathbb{N}$, represent the idle periods of the corresponding $M/G/\infty$ queue. Two connected components in the original graph $G_0$ are then connected through a vertex with two edges in the modified graph $\widetilde{G}_0$, which has the interval representation $\{I_n\}_{n\in V_0} \cup \{J_m\}_{m\in W_0}$ with $W_0 = \{m \in \mathbb{Z}_+ : B_m < T_{n_0}\}$ (note that for any $m \in W_0$, there exists an $n \in V_0$ such that $B_m = T_n$).

## 3. STATIONARY ANALYSIS OF SUBEXPONENTIAL INTERVAL GRAPHS

In this section, we analyze the *subexponential interval graph*—that is, the random interval graph proposed in the preceding section such that the lifetime (interval length) distribution is subexponential. In the analysis, we extend the time range of the immigration–death process to the whole real line $\mathbb{R}$ and consider it to be stationary; namely a sequence $\{T_n\}_{n\in\mathbb{Z}}$ follows a homogeneous Poisson process with intensity $\lambda \in (0,\infty)$ satisfying $\cdots < T_0 \le 0 < T_1 < \cdots$ and $\{L_n\}_{n\in\mathbb{Z}}$ denotes a sequence of mutually independent nonnegative random variables according to the identical distribution $F$ with mean $\mu^{-1} \in (0,\infty)$, where $\{L_n\}_{n\in\mathbb{Z}}$ is also independent of $\{T_n\}_{n\in\mathbb{Z}}$. Let $Q(t)$, $t \in \mathbb{R}$, denote the set of individuals in the system at time $t$; that is, $Q(t) = \{n \in \mathbb{Z} : t \in I_n\}$ for $I_n = [T_n, T_n + L_n]$. Then, clearly, $|Q(t)| = Z(t) = \sum_{n\in\mathbb{Z}} 1_{I_n}(t)$, $t \in \mathbb{R}$, where $|A|$ denotes the cardinality of set $A$. When $Z(t) > 0$, let $n_i(t)$, $i = 1,\dots,Z(t)$, denote the $i$th element of $Q(t)$ satisfying $n_i(t) < n_j(t)$ when $i < j$. Also let $R_{(i)}(t) = T_{n_i(t)} + L_{n_i(t)} - t$ $(\ge 0)$, $i = 1,\dots,Z(t)$; that is, the residual lifetime of individual $n_i(t)$ at time $t \in \mathbb{R}$. It is then known that (see, e.g., [16, Sect. 3.2]) when both $\lambda$ and $\mu$ are positive and finite, the stationary distribution of $\{Z(t), R_{(i)}(t), i = 1,\dots,Z(t)\}_{t\in\mathbb{R}}$ is given by

$$\mathrm{P}\big(Z(0) = l, R_{(1)}(0) \le x_1, \dots, R_{(l)}(0) \le x_l\big) = \frac{(\lambda/\mu)^l}{l!}\, e^{-\lambda/\mu} \prod_{i=1}^{l} F_e(x_i),$$
$$l \in \mathbb{Z}_+,\ x_1,\dots,x_l \in \mathbb{R}_+, \qquad \mathbf{(1)}$$

where $F_e$ denotes the equilibrium residual lifetime distribution of $F$ defined by $F_e(x) = \mu\int_0^x \overline{F}(y)\,dy$, $x \ge 0$, and when $l = 0$, the left-hand side is just reduced to $\mathrm{P}(Z(0) = 0)$ and, conventionally, $\prod_{i=1}^0 \cdot = 1$ on the right-hand side. Formula (1) states

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that, in the steady state, the number of individuals in the system follows the Poisson distribution with mean $\lambda/\mu$, and the residual lifetimes of the individuals in the system are mutually independent and identically distributed according to $F_e$. By the PASTA (Poisson arrivals see time averages) property (see Wolff [18]), the right-hand side of (1) also gives the distribution of $\{Z(T_n-), R_{(i)}(T_n-), i = 1, \ldots, Z(T_n-)\}_{n\in\mathbb{Z}}$ just before the arrivals of individuals.

In the following two subsections, we consider the infinite size of random interval graph $G = (V, E)$, $V = \mathbb{Z}$, with interval representation $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ and the subexponential interval length (lifetime) distribution $F$. We discuss the tail asymptotics of the stationary degree distribution and the conditionally expected cluster coefficient of a vertex given its degree in the limit as the degree goes to infinity. In the analysis, we use the standard notation that for any two real functions $f(x)$ and $g(x)$ on $\mathbb{R}$, $f(x) \sim g(x)$ as $x \to a$ stands for $\lim_{x\to a} f(x)/g(x) = 1$, where $a$ is possibly infinity.

### 3.1. Degree Distribution

A random graph $G = (V, E)$ is said to be scale-free if its degree distribution has a power-law tail; that is, for some constants $C > 0$ and $\gamma > 0$,

$$\mathrm{P}(D_0 = k) \sim \frac{C}{k^\gamma} \quad \text{as } k \to \infty, \tag{2}$$

where $D_n = \sum_{i\in V} 1_E(n, i)$ denotes the degree of vertex $n \in V$. Note that $D_0$ satisfying (2) has the $m$th moment if $\gamma > m + 1$. Previous work [13] showed that, in a discrete-time model setting, the random interval graph $G = (V, E)$ with interval representation $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ is scale-free in the steady state when the interval length distribution $F$ has a power-law tail. Here we extend this by applying the recent results on sampling of a stochastic process at random times according to subexponential distributions (see [2,9,11]) and provide a more general condition on $F$ under which the stationary degree distribution satisfies

$$\mathrm{P}(D_0 > k) \sim \overline{F}\left(\frac{k}{\lambda}\right) \quad \text{as } k \to \infty. \tag{3}$$

We will see that the power-law distribution $F$ such that $\overline{F}(x) \sim c/x^\alpha$ as $x \to \infty$ with $c > 0$ and $\alpha > 1$ fulfills the provided condition, so that (3) leads to $\mathrm{P}(D_0 > k) \sim c\,(\lambda/k)^\alpha$ as $k \to \infty$, which implies (2) with $C = c\alpha\lambda^\alpha$ and $\gamma = \alpha + 1$.

To provide the condition on the lifetime distribution under which (3) holds, we first give the definition of subexponential distributions. A distribution $F$ and corresponding random variables are said to be *subexponential* (see, e.g., Chistyakov [4] or [8, Sects. 1.3 and A3], [15, Sect. 2.5]) if $\overline{F}(x) > 0$ for all $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2, \tag{4}$$

where $F^{*n}$ denotes the $n$th-fold convolution of $F$ with itself. Note that if $F$ is subexponential, then $\overline{F}(x + a) \sim \overline{F}(x)$ as $x \to \infty$ for any $a \in \mathbb{R}$; that is, subexponential

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distributions are long-tailed. The following is a well-known and basic property of the subexponential distributions.

LEMMA 1 (see, e.g., Cline [5]): *Let F denote a subexponential distribution and let $G_i$, $i = 1, 2$, denote distributions on* $[0, \infty)$ *such that* $\lim_{x\to\infty} \overline{G_i}(x)/\overline{F}(x) = c_i \in [0, \infty)$. *Then* $\lim_{x\to\infty} \overline{G_1 * G_2}(x)/\overline{F}(x) = c_1 + c_2$, *where* $G_1 * G_2$ *denotes the convolution of* $G_1$ *and* $G_2$.

Another important class of heavy-tailed distributions is recently introduced in [11] in problems of random time sampling and reduced load equivalence (see also [2, 9]). A distribution $F$ and corresponding random variables are said to be *square-root insensitive* if $\overline{F}(x) > 0$ for all $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\overline{F}(x - \sqrt{x})}{\overline{F}(x)} = 1. \tag{5}$$

Note that if $F$ is square-root insensitive, then $\overline{F}(x - a\sqrt{x}) \sim \overline{F}(x)$ as $x \to \infty$ for any $a \in \mathbb{R}$. Additionally, a random variable $X$ is square-root insensitive if and only if $\sqrt{X^+}$ is long-tailed; that is, $\mathrm{P}(\sqrt{X^+} > x + a) \sim \mathrm{P}(\sqrt{X^+} > x)$ as $x \to \infty$ for any $a \in \mathbb{R}$ (see [11]), where $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$. It is known that distribution $F$ is square-root insensitive when its tail is heavier than $\exp(-x^\beta)$ with $\beta < 1/2$, whereas any distribution with a tail lighter than $e^{-\sqrt{x}}$ is not square-root insensitive (see [2]).

LEMMA 2 (see [2,9,11]): *Let N denote a (delayed or nondelayed) renewal process with interrenewal sequence* $\{\tau_i\}_{i\in\mathbb{Z}_+}$ *satisfying* $\mathrm{E}(\tau_1{}^2) < \infty$ *and let L denote a nonnegative random variable independent of N. If L follows a square-root insensitive distribution F, then* $\mathrm{P}(N((0, L]) > k) \sim \mathrm{P}(\lambda L > k) = \overline{F}(k/\lambda)$ *as* $k \to \infty$, *where* $\lambda = 1/\mathrm{E}\tau_1$.

Asmussen et al. [2] and Jelenković et al. [11] considered a more general case including that $N$ in Lemma 2 is replaced with a regenerative process. Foss and Korshunov [9] also considered another general case in which $\mathrm{E}(\tau_1{}^\beta) < \infty$ for $\beta \in [1, 2)$. In this article, however, the above form of the lemma is sufficient to show the following.

THEOREM 1: *If the lifetime distribution F is subexponential and square-root insensitive—that is, F fulfills* (4) *and* (5)*—then the stationary degree distribution of the random interval graph* $G = (V, E)$ *satisfies* (3).

Theorem 1 states that if the lifetime distribution $F$ is subexponential and square-root insensitive, then so is the stationary degree distribution of the obtained random interval graph. The power-law distributions are subexponential and square-root insensitive, so that Theorem 1 covers the previous result in [13]. In the proof below and thereafter, $N$ denotes the counting measure corresponding to $\{T_n\}_{n\in\mathbb{Z}}$; that is, $N(A)$

represents the number of points of $\{T_n\}_{n\in\mathbb{Z}}$ in $A \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma$-field on $\mathbb{R}$.

PROOF: Here we consider the Palm version satisfying $T_0 = 0$; that is, an arrival occurs at the origin. It is then known that $\{T_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is also the Poisson process with the same intensity $\lambda$ (see, e.g., Daley and Vere-Jones [7, Example 13.1(c)]). We can observe that the degree of vertex 0 consists of the number of individuals in the system just before the arrival of individual 0 and the number of new arrivals during the lifetime of individual 0; that is,

$$D_0 = \sum_{n<0} 1_E(0,n) + \sum_{n>0} 1_E(0,n) = Z(0-) + N(I_0) \quad \text{a.s.} \tag{6}$$

Since a Poisson process has independent increments and the lifetimes of individuals are mutually independent, $Z(0-)$ and $N(I_0)$ are also independent of each other, so that the distribution of $D_0$ is given as the convolution of those of $Z(0-)$ and $N(I_0)$. Since $F$ is square-root insensitive, Lemma 2 implies that $\mathrm{P}(N(I_0) > k) \sim \overline{F}(k/\lambda)$ as $k \to \infty$. By (1), on the other hand, $Z(0-)$ follows the Poisson distribution with mean $\lambda/\mu$, so that $\mathrm{P}(Z(0-) > k)/\overline{F}(k/\lambda) \to 0$ as $k \to \infty$ since $F$ is subexponential. Hence, we have by Lemma 1 that

$$\mathrm{P}(D_0 > k) = \mathrm{P}(Z(0-) + N(I_0) > k) \sim \overline{F}\left(\frac{k}{\lambda}\right) \quad \text{as } k \to \infty.$$

■

### 3.2. Conditionally Expected Cluster Coefficient

In a given graph, the cluster coefficient of a vertex represents the fraction of couples of its neighbors such that the couple is also connected by an edge. The cluster coefficient of vertex 0 of graph $G = (V, E)$, $V = \mathbb{Z}$, is then given by

$$C_0 = \frac{\displaystyle\sum_{n\in\mathbb{Z}} \sum_{m>n} 1_E(0,n)\, 1_E(0,m)\, 1_E(n,m)}{\binom{D_0}{2}}. \tag{7}$$

The previous work [13] demonstrated by a combinatorial argument that the random interval graph in the discrete-time model setting has the high cluster coefficient, on average, over vertices. Here, however, we provide a contrastive result that the conditional expectation $\mathrm{E}(C_0 \mid D_0 > k)$ converges to zero as $k$ goes to infinity under the same condition as in Theorem 1.

THEOREM 2: *If the lifetime distribution $F$ is subexponential and square-root insensitive—that is, $F$ fulfills (4) and (5)—then* $\lim_{k\to\infty} \mathrm{E}(C_0 \mid D_0 > k) = 0$.

This result states that the vertices with high degrees, which correspond to very long intervals, have extremely small cluster coefficients and does not contradict the result in [13], in which relatively short intervals play an essential role. To prove Theorem 2, we need the following lemma.

LEMMA 3: *Let $N$ and $L$ be the same as in Lemma 2. Then, for any constant $c > 0$,*

$$\mathrm{E}\left[\left\{\left(1-\frac{c}{L}\right)^{+}\right\}^{2} 1_{\{N((0,L])>k\}}\right] \sim \mathrm{P}(N((0,L])>k) \sim \mathrm{P}(\lambda L > k) = \overline{F}(k/\lambda)$$

*as $k \to \infty$.*

The proof of Lemma 3 follows that of Theorem 3.6 in [2] (see also the proof of Theorem 3 in [11]) and is given in the Appendix. With this lemma, we now provide the proof of Theorem 2.

PROOF OF THEOREM 2: We consider the Palm version satisfying $T_0 = 0$ as in the proof of Theorem 1 and demonstrate that $\mathrm{E}(C_0\, 1_{\{D_0>k\}})/\mathrm{P}(D_0 > k) \to 0$ as $k \to \infty$. For simplicity of notation, we write the event $A(0,n,m) = \{(0,n) \in E,\ (0,m) \in E,\ (n,m) \in E\}$, $n, m \in \mathbb{Z}$ $(n < m)$. Recall that $D_0 = Z(0-) + N(I_0)$ a.s., as seen in (6). We then have from (7) that

$$\begin{aligned}\mathrm{E}(C_0\, 1_{\{D_0>k\}}) &= \sum_{l=k+1}^{\infty} \frac{2}{l\,(l-1)}\, \mathrm{E}\left[\sum_{n\in\mathbb{Z}}\sum_{m>n} 1_E(0,n)\, 1_E(0,m)\, 1_E(n,m)\, 1_{\{D_0=l\}}\right] \\ &= \sum_{l=k+1}^{\infty} \frac{2}{l\,(l-1)} \sum_{j=0}^{l}\sum_{n\in\mathbb{Z}}\sum_{m>n} \mathrm{P}\big(A(0,n,m), Z(0-)=j, N(I_0)=l-j\big) \\ &= S\{n<m<0\} + S\{n<0<m\} + S\{0<n<m\}, \end{aligned} \tag{8}$$

where the sum over $-\infty < n < m < +\infty$ is separated into three cases: (i) $n < m < 0$, (ii) $n < 0 < m$, and (iii) $0 < n < m$, and each case is denoted by $S\{\cdot\}$ in the last equality. We show below that each case leads to the term of $o(\mathrm{P}(D_0 > k))$ as $k \to \infty$.

(i) Case of $n < m < 0$. Whenever $(0,n) \in E$ and $(0,m) \in E$ for $n$, $m < 0$, it is necessary that $(n,m) \in E$ since individuals $n$ and $m$ are in the system when individual 0 arrives, so that

$$\begin{aligned}&\sum_{n=-\infty}^{-2}\sum_{m=n+1}^{-1} \mathrm{P}\big(A(0,n,m), Z(0-)=j, N(I_0)=l-j\big) \\ &\quad = \binom{j}{2}\, \mathrm{P}\big(Z(0-)=j, N(I_0)=l-j\big).\end{aligned}$$

Substituting this into (8), we have

$$S\{n < m < 0\} = \sum_{l=k+1}^{\infty} \sum_{j=2}^{l} \frac{j\,(j-1)}{l\,(l-1)} \mathrm{P}\big(Z(0-) = j, N(I_0) = l-j\big)$$

$$= \left(\frac{\lambda}{\mu}\right)^2 \sum_{l=k+1}^{\infty} \frac{1}{l\,(l-1)} \sum_{j=2}^{l} \mathrm{P}(Z(0-) = j-2)$$

$$\times\, \mathrm{P}(N(I_0) = l-j),$$

where we use the fact that $Z(0-)$ follows the Poisson distribution with mean $\lambda/\mu$, so that $j\,(j-1)\,\mathrm{P}(Z(0-) = j) = (\lambda/\mu)^2\,\mathrm{P}(Z(0-) = j-2)$, $j = 2, 3, \ldots$. Additionally, for any $\epsilon > 0$, there exists a $k_\epsilon > 0$ such that $1/[l\,(l-1)] < \epsilon$ for $l > k_\epsilon$. Thus, we have $S\{n < m < 0\} \le \epsilon\,(\lambda/\mu)^2\,\mathrm{P}(D_0 > k-2)$ for $k \ge k_\epsilon$. Since $D_0$ is long-tailed and $\epsilon$ is arbitrarily small, this implies that $S\{n < m < 0\} = o(\mathrm{P}(D_0 > k))$ as $k \to \infty$.

(ii) Case of $n < 0 < m$. Given that $Z(0-) = j$, we have by (1) that the residual lifetimes of these $j$ individuals at time 0 are independent and identically distributed according to $F_e$. Additionally, given that $L_0 = x\ (> 0)$ and $N(I_0) = l-j$, the property of Poisson processes implies that the arrival times of these $l-j$ individuals are independent and uniformly distributed on $[0, x]$ (see, e.g., [7, Sect. 2.1]). Note that interval $I_n$, $n < 0$, has an overlap with interval $I_m$, $m > 0$, which has its left end point at $y \in [0, x]$, when the residual lifetime of individual $n$ at time 0 is longer than $y$. Therefore,

$$\sum_{n=-\infty}^{-1} \sum_{m=1}^{+\infty} \mathrm{P}\big(A(0,n,m), Z(0-) = j, N(I_0) = l-j\big)$$

$$= \sum_{n=-\infty}^{-1} \sum_{m=1}^{+\infty} \int_0^{\infty} \mathrm{P}\big(A(0,n,m), Z(0-) = j, N(I_0) = l-j \mid L_0 = x\big)\, dF(x)$$

$$= j\,(l-j) \int_0^{\infty} \frac{1}{x} \int_0^{x} \overline{F_e}(y)\, dy\, \mathrm{P}\big(Z(0-) = j, N((0,x]) = l-j\big)\, dF(x)$$

$$\le j\,(l-j)\,\mathrm{P}\big(Z(0-) = j, N(I_0) = l-j\big),$$

where $(1/x) \int_0^x \overline{F_e}(y)\, dy \le 1$, $x > 0$, is used in the last inequality. Applying this in (8), we have

$$S\{n < 0 < m\} \le \sum_{l=k+1}^{\infty} \sum_{j=1}^{l-1} \frac{2j\,(l-j)}{l\,(l-1)} \mathrm{P}(Z(0-) = j, N(I_0) = l-j)$$

$$\le \frac{2\lambda}{\mu} \sum_{l=k+1}^{\infty} \sum_{j=1}^{l-1} \frac{l-j}{l\,(l-1)} \mathrm{P}(Z(0-) = j-1)\,\mathrm{P}(N(I_0) = l-j),$$

where we use $j\,\mathrm{P}(Z(0-)=j) = (\lambda/\mu)\,\mathrm{P}(Z(0-)=j-1), j=1,2,\ldots$. Here, for any $\epsilon > 0$, there exists a $k_\epsilon > 0$ such that $(l-j)/[l\,(l-1)] \le \epsilon$ for $l > k_\epsilon$ and $1 \le j < l$, so that we have $S\{n < 0 < m\} \le 2\,\epsilon\,(\lambda/\mu)\,\mathrm{P}(D_0 > k-1)$ for $k \ge k_\epsilon$. Since $D_0$ is long-tailed and $\epsilon$ is arbitrarily small, this leads to $S\{n < 0 < m\} = o(\mathrm{P}(D_0 > k))$ as $k \to \infty$.

(iii) Case of $0 < n < m$. Given that $L_0 = x$ $(> 0)$ and $N(I_0) = l-j$, the arrival times of these $l-j$ individuals are independent and uniformly distributed on $[0,x]$. The event that interval $I_n$ whose left end point is at $y \in [0,x]$ has an overlap with interval $I_m$ whose left end point is at $z \in [y,x]$ is realized when $L_n > z-y$, so that

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\sum_{m=n+1}^{+\infty} \mathrm{P}\big(A(0,n,m), Z(0-)=j, N(I_0)=l-j\big)\\
&\quad= \sum_{n=1}^{+\infty}\sum_{m=n+1}^{+\infty}\int_0^\infty \mathrm{P}\big(A(0,n,m), Z(0-)=j, N(I_0)=l-j \mid L_0 = x\big)\,dF(x)\\
&\quad= \binom{l-j}{2}\int_0^\infty \frac{2}{x^2}\int_0^x\int_y^x \overline{F}(z-y)\,dz\,dy\\
&\quad\times \mathrm{P}\big(Z(0-)=j, N((0,x])=l-j\big)\,dF(x). \qquad \textbf{(9)}
\end{aligned}$$

Here, an easy calculation yields

$$\frac{2}{x^2}\int_0^x\int_y^x \overline{F}(z-y)\,dz\,dy = 1 - \mathrm{E}\left[\left\{\left(1-\frac{L_1}{x}\right)^+\right\}^2\right],$$

where $L_1$ denotes a random variable according to distribution $F$ and independent of $L_0$ and $N$. Thus, taking this into account and substituting (9) into (8), we have

$$\begin{aligned}
S\{0 < n < m\} \le \sum_{l=k+1}^{\infty}\sum_{j=0}^{l-2} &\mathrm{P}(Z(0-)=j)\\
&\times \mathrm{E}\left(\left[1-\left\{\left(1-\frac{L_1}{L_0}\right)^+\right\}^2\right] 1_{\{N(I_0)=l-j\}}\right), \qquad \textbf{(10)}
\end{aligned}$$

where $(l-j)\,(l-j-1)/[l\,(l-1)] \le 1$ for $l > 1$ and $0 \le j < l$ is used. We now consider a random variable $X_H$, which is independent of $Z(0-)$ and has the following proper distribution:

$$\mathrm{P}(X_H \le k) = \frac{\mathrm{E}\big(\big[1 - \big\{(1 - L_1/L_0)^+\big\}^2\big]\, 1_{\{N(I_0) \le k\}}\big)}{\mathrm{E}\big(1 - \big\{(1 - L_1/L_0)^+\big\}^2\big)}, \qquad k \in \mathbb{Z}_+.$$

Then (10) yields

$$S\{0 < n < m\} \le \mathrm{E}\left(1 - \left\{\left(1 - \frac{L_1}{L_0}\right)^+\right\}^2\right) \mathrm{P}\big(Z(0-) + X_H > k\big). \quad \textbf{(11)}$$

By Lemma 3, on the other hand, since $L_1$ is independent of $L_0$ and $N$, we have

$$\frac{\mathrm{P}(X_H > k)}{\mathrm{P}(N(I_0) > k)} = \frac{1}{\mathrm{E}\big(1 - \big\{(1 - L_1/L_0)^+\big\}^2\big)} \times \left(1 - \frac{\mathrm{E}\big(\big\{(1 - L_1/L_0)^+\big\}^2\, 1_{\{N(I_0) > k\}}\big)}{\mathrm{P}(N(I_0) > k)}\right) \to 0 \quad \text{as } k \to \infty.$$

Hence, applying Lemma 1 in (11) leads to $S\{0 < n < m\} = o(\mathrm{P}(N(I_0) > k)) = o(\mathrm{P}(D_0 > k))$ as $k \to \infty$.

■

*Remark 2*: In considering the connected interval graph $\widetilde{G}$ in Remark 1, we have to modify slightly the result on the stationary degree distribution in Theorem 1. The fact that $\mathrm{P}(Z(0-) = 0) = e^{-\lambda/\mu}$ from (1) states that $\{A_n\}_{n\in\mathbb{Z}}$ is a stationary point process with intensity $\lambda\, e^{-\lambda/\mu}$. Thus, since the superposed point process $\{T_n\}_{n\in\mathbb{Z}} \cup \{A_n\}_{n\in\mathbb{Z}}$ has intensity $\lambda\,(1 + e^{-\lambda/\mu})$, the probability that an arbitrary chosen vertex is not the one that is extraneously added in Remark 1 is given by $(1 + e^{-\lambda/\mu})^{-1}$, so that the tail asymptotics of the stationary degree distribution in the modified graph $\widetilde{G}$ becomes $\mathrm{P}(\widetilde{D}_0 > k) \sim (1 + e^{-\lambda/\mu})^{-1}\, \overline{F}(k/\lambda)$ as $k \to \infty$. The limit of the conditionally expected cluster coefficient, on the other hand, remains the same as that in the original $G$.

## 4. CONCLUDING REMARK

In this article, we have analyzed the stationary subexponential interval graphs generated by immigration–death processes—namely we have derived the tail asymptotics of the stationary degree distribution when the lifetime distribution of the immigration–death process is subexponential and square-root insensitive. Furthermore, we have shown that the conditionally expected cluster coefficient of a vertex given its degree vanishes in the limit as the degree goes to infinity under the same condition as that for obtaining the tail asymptotics of the stationary degree distribution. In future works, we can consider problems like evaluating the stationary distribution of the sizes of connected components and the diameter of a connected component, which represents the length of the shortest path connecting any pair of vertices in the connected component.

*Acknowledgments*

The authors wish to thank Osamu Watanabe for valuable discussions. The third author's work was supported in part by JSPS (Japan Society for the Promotion of Science) Global COE (Centers of Excellence) Program "Computation as a Foundation for the Sciences."

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## APPENDIX PROOF OF LEMMA 3

Here we provide the proof of Lemma 3, which mainly follows those of Theorem 3.6 in [2] and Theorem 3 in [11]. In the following, for any two real functions $f(x)$ and $g(x)$ on $\mathbb{R}$, $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$ as $x \to a$ stand for respectively $\limsup_{x\to a} f(x)/g(x) \le 1$ and $\liminf_{x\to a} f(x)/g(x) \ge 1$, where $a$ is possibly infinity.

For the asymptotic upper bound, it is clear that

$$\mathrm{E}\left[\left\{\left(1-\frac{c}{L}\right)^{+}\right\}^{2} 1_{\{N((0,L])>k\}}\right] \lesssim \mathrm{P}(N((0,L])>k) \quad \text{as } k\to\infty,$$

since $\{(1-c/L)^{+}\}^{2} \le 1$ a.s. We now consider the asymptotic lower bound. We have for $a>0$,

$$\begin{aligned}
&\mathrm{E}\left[\left\{\left(1-\frac{c}{L}\right)^{+}\right\}^{2} 1_{\{N((0,L])>k\}}\right] \\
&\quad\ge \mathrm{E}\left[\left\{\left(1-\frac{c}{L}\right)^{+}\right\}^{2} 1_{\{N((0,L])>k,\, L>k/\lambda+a\sqrt{k/\lambda}\}}\right] \\
&\quad\ge \left\{\left(1-\frac{\lambda c}{k}\right)^{+}\right\}^{2} \mathrm{P}\left(N\left(\left(0,\frac{k}{\lambda}+a\sqrt{\frac{k}{\lambda}}\right]\right)>k\right)\overline{F}\left(\frac{k}{\lambda}+a\sqrt{\frac{k}{\lambda}}\right),
\end{aligned}$$

where we use $\{(1-c/L)^{+}\}^{2} \ge \{(1-\lambda c/k)^{+}\}^{2}$ a.s. and $N((0,L]) \ge N\big((0,k/\lambda+a\sqrt{k/\lambda}]\big)$ a.s. when $L>k/\lambda+a\sqrt{k/\lambda}$ in the second inequality. Here, one obtains for $k\ge a^{2}\lambda$,

$$\begin{aligned}
&\mathrm{P}\left(N\left(\left(0,\frac{k}{\lambda}+a\sqrt{\frac{k}{\lambda}}\right]\right)>k\right) \\
&\quad= \mathrm{P}\left(\frac{N((0,k/\lambda+a\sqrt{k/\lambda}])-(k+a\sqrt{\lambda k})}{\sqrt{k/\lambda+a\sqrt{k/\lambda}}} > -\frac{a\lambda}{\sqrt{1+a\sqrt{\lambda/k}}}\right) \\
&\quad\ge \mathrm{P}\left(\frac{N((0,k/\lambda+a\sqrt{k/\lambda}])-(k+a\sqrt{\lambda k})}{\sqrt{k/\lambda+a\sqrt{k/\lambda}}} > -\frac{a\lambda}{\sqrt{2}}\right).
\end{aligned}$$

Additionally, for any $\epsilon>0$, there exists a $k_\epsilon>0$ such that $\{(1-\lambda c/k)^{+}\}^{2}>1-\epsilon$ for $k\ge k_\epsilon$. Therefore, the square-root insensitivity of $F$ and the central limit theorem for renewal processes (see, e.g., Asmussen [1, Chap. V, Thm. 6.3]) result in, for an appropriate $\sigma>0$,

$$\mathrm{E}\left[\left\{\left(1-\frac{c}{L}\right)^{+}\right\}^{2} 1_{\{N((0,L])>k\}}\right] \gtrsim (1-\epsilon)\,\Phi\left(\frac{a\lambda}{\sigma\sqrt{2}}\right)\overline{F}\left(\frac{k}{\lambda}\right) \quad \text{as } k\to\infty,$$

where $\Phi$ denotes the standard normal distribution. Finally, letting $\epsilon\to 0$ and $a\to\infty$ completes the proof.