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COMMUTATOR EQUATIONS IN FINITE GROUPS

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Abstract. The problem of finding the number of ordered commuting tuples of elements in a finite group is equivalent to finding the size of the solution set of the system of equations determined by the commutator relations that impose commutativity among any pair of elements from an ordered tuple. We consider this type of systems for the case of ordered triples and express the size of the solution set in terms of the irreducible characters of the group. The obtained formulas are natural extensions of Frobenius' character formula that calculates the number of ways a group element is a commutator of an ordered pair of elements in a finite group. We discuss how our formulas can be used to study the probability distributions afforded by these systems of equations, and we show explicit calculations for dihedral groups.

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1. Introduction. Systems of equations in finite groups have been studied using a variety of tools such as probability and representation theory. One of the simplest systems of equations one can consider in a finite group G is the one determined by the commutator equation:

$$[x, y] = g,$$

where g is a fixed element in G and $[x, y] = x^{-1}y^{-1}xy$. According to Ore's Conjecture (proved in [9]), this latter equation always has a solution in finite non-abelian simple groups. If one considers the probability $P_2(g)$ that a randomly chosen ordered pair of elements in a finite group G has commutator equal to g, then Ore's Conjecture is equivalent to saying that $P_2(g)$ is always positive for finite non-abelian simple groups. One of the key tools in proving Ore's Conjecture is Frobenius's character formula which allows us to express the number of solutions to the equation $[x, y] = g$ in terms of the set $\mathrm{Irr}(G)$ of irreducible characters of G. More precisely, we have:

THEOREM 1.1. *[Frobenius] Suppose $g \in G$. Then*

$$|\{(x, y) \in G \times G : [x, y] = g\}| = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)} \chi(g).$$

A complete proof of Frobenius' formula can be found in [3]. If $k(G)$ is the number of conjugacy classes of G, then Frobenius' formula yields $P_2(1) = k(G)/|G|$, which is the probability of randomly selecting a commuting ordered pair.

This latter probability can be generalized in a number of ways, for instance we can consider the probability $P_n(1)$ that a randomly chosen n-tuple of elements in G commutes:

$$P_n(1) = \frac{|\{(x_1, \ldots, x_n) \in G^n : [x_i, x_j] = 1 \text{ for } i < j\}|}{|G|^n}.$$

The probability $P_n(1)$ was also studied in [8] under the name of commutativity degree. Note that the set of commuting n-tuples $\{(x_1, \ldots, x_n) \in G^n : [x_i, x_j] = 1 \text{ for } i < j\}$ can be identified with the set of group homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$. Since $\mathrm{Hom}(\mathbb{Z}^{n+1}, G) \subseteq \mathrm{Hom}(\mathbb{Z}^n, G) \times G \subseteq G^{n+1}$, it follows that

$$1 \geq P_2(1) \geq \cdots \geq P_n(1) \geq P_{n+1}(1) \geq \cdots$$

with equality between any two (and hence between all) if and only if G is abelian. Moreover, since $\mathrm{Hom}(\mathbb{Z}^n, G) \times 1 \subset \mathrm{Hom}(\mathbb{Z}^{n+1}, G)$, we see that $P_n(1)/|G| \leq P_{n+1}(1)$, and thus $P_2(1) \leq P_{2+n}(1)|G|^n$. The sets $\mathrm{Hom}(\mathbb{Z}^n, G)$ reflect a number of structural properties of G as well as topological properties. These sets have been used in [2] and [1] to construct the classifying space $B_{com}G$ for commutative G-bundles and in [13] to study further properties of $P_n(1)$ related to $B_{com}G$.

To extend the definition of $P_n(1)$ to other elements $g \in G$, we will consider the class function

$$f_n(g) = \left| \left\{ (x_1, \ldots, x_n) \in G^n : \left[x_i, x_j\right] = g \text{ for } i < j \right\} \right|,$$

and we will set $P_n(g) = f_n(g)/|G|^n$. This latter is the probability that a randomly chosen n-tuple in G^n is a solution to the system of commutator equations $[x_i, x_j] = g$, for all $i < j$.

The calculation of the value of $f_2(g)$ is given by Frobenius' formula, which is in terms of the irreducible characters of G and their degree. In contrast, calculating the value of $f_n(g)$ for different elements $g \in G$ when $n > 2$ requires a careful analysis of the lattice of centralizers in G and their cosets as we will show with the case $n = 3$. Recall that if f is a class function, then we can write it as $f = \sum_{\chi \in \mathrm{Irr}(G)} \alpha_\chi \chi$, where

$$\alpha_\chi = \langle f, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\chi(g)}.$$

THEOREM 1.2. *Suppose $\chi \in \mathrm{Irr}(G)$. If we let*

$$\theta_\chi(a) = \sum_{b \in G} |C_G(ab)b \cap C_G(a)| \chi([a, b]),$$

and

$$m_\chi = \sum_{a \in G} \theta_\chi(a),$$

then θ_χ is a class function and

$$f_3(g) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} m_\chi \chi(g).$$

The calculation of f_3 could be pretty involved, but it is possible to get an upper bound for the value of $f_3(g)$ by considering the class function:

$$t_3(g) = \left| \left\{ (x, y, z) \in G^3 : [x, y] = g = [x, z] \right\} \right|$$

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Of course, $f_3(g) \leq t_3(g)$, and the calculation of t_3 in terms of the set $\mathrm{Irr}(G)$ is much simpler.

THEOREM 1.3. *Suppose that $x_1, \ldots, x_{k(G)}$ is a full set of representatives of the conjugacy classes of G. If $\chi \in \mathrm{Irr}(G)$, then*

$$\langle t_3, \chi \rangle = \sum_{i=1}^{k(G)} \frac{|G|}{\chi(1)} |\chi(x_i)|^2.$$

As an application of f_3, we show that:

THEOREM 1.4. *If $n \geq 3$ and g is an arbitrary element in the alternating group A_n, then the system of commutator equations:*

$$[x_1, x_2] = g$$
$$[x_1, x_3] = g$$
$$[x_2, x_3] = g$$

is always consistent in the symmetric group Σ_n.

REMARK 1.5. Based on a number of calculations in GAP [12], we have conjectured that the system in Theorem 1.4 is always consistent in A_n when $n \geq 5$. Further calculations also seem to indicate that this may be very well the case for any finite non-abelian simple group. This would be a natural extension of Ore's conjecture (see Remark 4.4).

The organization of this paper is as follows: in Section 2, we prove the character formulas for f_3 and t_3; in Section 3, we calculate the coefficients of f_3 and t_3 for dihedral groups; and in Section 4, we revisit some upper bounds for $P_n(1)$, obtain estimates for $P_3(g)$, and discuss some properties of the probability distribution afforded by f_3.

2. Character formulas.

In order to write t_3 in terms of the irreducible characters of G, we will use the following formula (see Problem 3.12 of [7]):

$$\chi(g)\chi(h) = \frac{\chi(1)}{|G|} \sum_{z \in G} \chi(gh^z), \tag{2.1}$$

where $\chi \in \mathrm{Irr}(G)$ and $h^z = z^{-1}hz$. Our first result in this section is a slight generalization of Theorem 1.3.

THEOREM 2.1. *Let $t_n(g) = |\{(x_1, \ldots, x_n) \in G^n : [x_1, x_i] = g \text{ for } i > 1\}|$, and suppose that ϑ is the character of the conjugation action of G on itself. If χ is an irreducible character of G, then*

$$\langle t_n, \chi \rangle = \frac{|G|}{\chi(1)} \left\langle \vartheta^{n-2} \cdot \chi, \chi \right\rangle$$

and hence t_n is a character of G.

Proof. Note that $[x_1, x_2] = [x_1, x_i]$ if and only if $x_2 x_i^{-1} \in C_G(x_1)$ for all $i \geq 2$. This implies that the set that affords the value of $t_n(g)$ is determined by g and pairs (x_1, x_2) with $[x_1, x_2] = g$. Then using (2.1), we have

$$\langle t_n, \overline{\chi}\rangle = \frac{1}{|G|}\sum_{g\in G} t_n(g)\chi(g) = \frac{1}{|G|}\sum_{x_1\in G}\sum_{x_2\in G}|C_G(x_1)|^{n-2}\chi([x_1, x_2])$$
$$= \frac{1}{|G|}\sum_{x_1\in G}|C_G(x_1)|^{n-2}\frac{|G|}{\chi(1)}\chi\left(x_1^{-1}\right)\chi(x_1)$$
$$= \frac{|G|}{\chi(1)}\left(\frac{1}{|G|}\sum_{x_1\in G}\left(\vartheta^{n-2}(x_1)\chi(x_1)\right)\overline{\chi(x_1)}\right).$$

So, $\langle t_n, \overline{\chi}\rangle$ is $|G|/\chi(1)$ times the multiplicity of χ in $\vartheta^{n-2}\cdot\chi$, which is the product of two non-negative integers. Thus, $\langle t_n, \overline{\chi}\rangle = \overline{\langle t_n, \overline{\chi}\rangle} = \langle t_n, \chi\rangle$. This completes the proof. □

REMARK 2.2. If we fix $g \in G$, then there is a bijection between the sets

$$\left\{(x, y, z) \in G^3 : [x, y] = [x, z] = [y, z] = g\right\}$$

and

$$\left\{(x, y, z) \in G^3 : [x, y] = [x, z] = [z, y] = g\right\}$$

given $(x, y, z) \mapsto (x, z, y)$.

Now, we proceed to prove our formula for f_3.

Proof of Theorem 1.2: Let T_g be the set $\{(x, y, z) \in G^3 : [x, y] = [x, z] = [z, y] = g\}$. Then according to the previous remark $f_3(g) = |T_g|$. Suppose that $(x, y, z) \in \sqcup_{g\in G} T_g$. Then, the condition $[x, y] = [x, z]$ holds if and only if $y = cz$ for some $c \in C_G(x)$, or equivalently $y = cz$ for some c such that $x \in C_G(c)$. In addition, $[x, y] = [z, y]$ if and only if $[y, x] = [y, z]$, if and only if $x \in C_G(y)z$. Then in order to form a triple (x, y, z) in a set T_g, we can first pick any two elements $z, c \in G$, then set $y = cz$, and find $x \in G$ such that $x \in C_G(cz)z \cap C_G(c)$. Then, we have

$$\sum_{g\in G} f_3(g)\chi(g) = \sum_{c\in G}\sum_{z\in G}|C_G(cz)z \cap C_G(c)|\chi([z, cz]).$$

Since $[z, cz] = [z, z][z, c]^z$ and $\chi([z, c]) = \overline{\chi([c, z])}$, it follows that we can write this latter as:

$$|G|\langle f_3, \chi\rangle = \sum_{g\in G} f_3(g)\overline{\chi(g)} = \sum_{a\in G}\sum_{b\in G}|C_G(ab)b \cap C_G(a)|\chi([a, b]).$$

Hence

$$\langle f_3, \chi\rangle = \frac{1}{|G|}\sum_{a\in G}\theta_\chi(a) = \frac{m_\chi}{|G|},$$

as wanted. To show that θ_χ is a class function fix an element w in G and note that:

$$\theta(a^w) = \sum_{b\in G}|C_G(a^w b)b \cap C_G(a^w)|\chi([a^w, b])$$
$$= \sum_{b^w\in G}|C_G(a^w b^w)b^w \cap C_G(a^w)|\chi([a^w, b^w])$$
$$= \theta(a).$$

□

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When studying systems of commutator equations in a group G, there will be times when we will consider solutions consisting of tuples of group elements in specific subgroups of G. This motivates the following class function: let H be a subgroup of G and let $f_{n,H\leq G}\colon G\to\mathbb{N}$ given by

$$f_{n,H\leq G}(g)=|\left\{(x_1,\ldots,x_n)\in H^n:\left[x_i,x_j\right]=g \text{ for } i<j\right\}|.$$

Note that $f_{n,G\leq G}=f_n$. When $H=G$, we will write $f_{n,G}$ or simply f_n. Now we can prove multiple properties of the functions f_n and t_n.

Proposition 2.3. *Suppose that $g\in G$ and that H and K are subgroups of G. We have the following:*

(1) *If $H\leq K$, then $f_{n,H\leq G}(g)\leq f_{n,K\leq G}(g)$.*
(2) *If $\chi\in\operatorname{Irr}(G)$ and we let*

$$\tau_\chi(b)=\sum_{a\in G}|C_G(ab)b\cap C_G(a)|\chi([a,b]),$$

then τ_χ is a class function and $\tau_\chi(b^{-1})=\tau_{\overline{\chi}}(b)=\overline{\tau_\chi(b)}$.
(3) *$m_\chi=\sum_{b\in G}\tau_\chi(b)$ and $m_\chi\in\mathbb{R}$.*
(4) *Both f_n and t_n are invariant under isoclinism between groups of the same order. In particular, P_n is invariant under isoclinism.*
(5) *$t_n(g)\leq t_n(1)$.*
(6) *$f_3(g)\leq t_3(g)$, and if $g\neq 1$ then $f_3(g)\leq t_3(g)-f_2(g)$.*
(7) *$f_n(g^{-1})=f_n(g)$.*
(8) *$t_n(g^{-1})=t_n(g)$.*

Proof. (1) This statement is straightforward.

(2) To prove that τ_χ is a class function, we can proceed as we did for θ_χ in Theorem 1.2, so we leave this to the interested reader. Note that for any two non-empty subsets A, B of G and any element v in G, we have $|Av\cap B|=|A\cap Bv^{-1}|$. Then

$$\begin{aligned}\tau_\chi\left(b^{-1}\right)&=\sum_{a\in G}|C_G\left(ab^{-1}\right)b^{-1}\cap C_G\left(a\right)|\chi\left(\left[a,b^{-1}\right]\right)\\&=\sum_{a\in G}|C_G\left(ab^{-1}\right)\cap C_G\left(a\right)b|\chi\left(\left[a,b^{-1}\right]\right)\\&=\sum_{a^{-1}\in G}|C_G\left(a^{-1}b^{-1}\right)\cap C_G\left(a^{-1}\right)b|\chi\left(\left[a^{-1},b^{-1}\right]\right).\end{aligned}$$

So if we set $u=a^{-1}b^{-1}$ and use the identities $[ub,b^{-1}]=[u,b^{-1}]^b$ and $[u,b^{-1}]=[b,u]^{b^{-1}}$, we get:

$$\begin{aligned}\tau_\chi\left(b^{-1}\right)&=\sum_{u\in G}|C_G(u)\cap C_G(ub)b|\chi\left(\left[ub,b^{-1}\right]\right)\\&=\sum_{u\in G}|C_G(u)\cap C_G(ub)b|\chi\left(\left[u,b^{-1}\right]\right)\\&=\sum_{u\in G}|C_G(u)\cap C_G(ub)b|\chi([b,u])\\&=\sum_{u\in G}|C_G(u)\cap C_G(ub)b|\overline{\chi([u,b])}\\&=\overline{\tau_\chi(b)}=\tau_{\overline{\chi}}(b).\end{aligned}$$

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(3) That $m_\chi = \sum_{b\in G} \tau_\chi(b)$ follows from the definition of m_χ. Moreover, we have:

$$\overline{m_\chi} = \sum_{b\in G} \overline{\tau_\chi(b)} = \sum_{b^{-1}\in G} \overline{\tau_\chi(b^{-1})} = \sum_{b^{-1}\in G} \tau_\chi(b) = m_\chi.$$

(4) To prove that f_n is invariant under isoclinism between groups of the same order, we refer the reader to the proof in [8] of the invariance of the value of $P_n(1)$. The argument for t_n is the same.

(5) If we write $t_n = \sum_{\chi\in \mathrm{Irr}(G)} a_\chi \chi$, then according to Theorem 2.1, the coefficients a_χ are non-negative integers. Thus,

$$t_n(g) \leq \sum_{\chi\in \mathrm{Irr}(G)} a_\chi \chi(1) = t_n(1),$$

as wanted.

(6) The first inequality is straightforward, while for the second note that if $g \neq 1$, then a triple (x, y, z) satisfying $[x, y] = [x, z] = [y, z] = g$ cannot have $y = z$. The triples that do satisfy $y = z$ are of the form (x, y, y) and can be counted by the function $f_2(g)$. These latter triples can be knocked off from the set of triples that are counted by t_3 yielding a set containing the set of triples that are counted by f_3.

(7) The map given by $(a_1, \ldots, a_n) \mapsto (a_n, \ldots, a_1)$ defines a bijection between the set $\{(x_1, \ldots, x_n) \colon [x_i, x_j] = g \text{ for } i < j\}$ and the set $\{(x_1, \ldots, x_n) : [x_i, x_j] = g^{-1} \text{ for } i < j\}$. Their size is precisely $f_n(g)$ and $f_n(g^{-1})$.

(8) If we write $t_n = \sum_{\chi\in \mathrm{Irr}(G)} a_\chi \chi$, then

$$t_n\left(g^{-1}\right) = \sum_{\chi\in \mathrm{Irr}(G)} a_\chi \overline{\chi(g)} = \overline{\sum_{\chi\in \mathrm{Irr}(G)} a_\chi \chi(g)} = \overline{t_n(g)} = t_n(g). \qquad \square$$

EXAMPLE 2.4. Using Proposition 2.3, one can simplify the calculations yielding the coefficients of f_3 and t_3. For the alternating group A_5, we have:

(1) $f_2 = 60\chi_1 + 20\chi_2 + 20\chi_3 + 15\chi_4 + 12\chi_5$.
(2) $f_3 = 40\chi_1 + 64\chi_2 + 64\chi_3 + 84\chi_4 + 112\chi_5$.
(3) $t_3 = 300\chi_1 + 260\chi_2 + 260\chi_3 + 285\chi_4 + 324\chi_5$.
(4) $P_2(1) = 1/12$ and $P_3(1) = 11/1800$.

Here, χ_1 is the character of the trivial representation, χ_2 and χ_3 have degree 3, whereas χ_4 and χ_5 have degree 4 and 5, respectively.

REMARK 2.5. We have done numerous calculations in GAP, and we have conjectured that f_3 is always a character.

3. Calculations for Dihedral groups. In this section, we calculate the coefficients of f_3 and t_3 for the dihedral group: $D_{2n} = \langle a, b \colon a^n = b^2 = 1, a^b = a^{-1}\rangle$. For convenience of the reader, we include the character table(s) of D_{2n}. For more details, we refer the reader to [10].

When n is odd, D_{2n} has two linear characters χ_1, χ_2, and $(n-1)/2$ degree-two irreducible characters $\psi_1, \ldots, \psi_{(n-1)/2}$. The conjugacy classes in this case are: $\{1\}$; $\{a^r, a^{-r}\}$

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for $1 \leq r \leq (n-1)/2$; and $\{a^s b : 0 \leq s \leq n-1\}$. If we set $\omega = e^{2\pi i/n}$, we have the following table.

Class	1	a^r	b
size	1	2	n
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\omega^{jr} + \omega^{-jr}$	0

When n is even, D_{2n} has four linear characters $\chi_1, \ldots, \chi_4$ and $(n-2)/2$ degree two irreducible characters $\psi_1, \ldots, \psi_{(n-2)/2}$. In this case, if we write $n = 2l$, then the conjugacy classes are as follows: $\{1\}$; $\{a^l\}$; $\{a^r, a^{-r}\}$ for $1 \leq r \leq l-1$; $\{a^r b\colon r \text{ even}\}$; and $\{a^s b : s \text{ odd}\}$. We have the following table.

Class	1	a^l	a^r	b	ab
size	1	1	2	l	l
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^l$	$(-1)^r$	1	-1
χ_4	1	$(-1)^l$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\omega^{jr} + \omega^{-jr}$	0	0

LEMMA 3.1. *If n is a positive integer, then*

$$\sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1;$$

and if n is an odd integer greater than 1, then we also have

$$\sum_{k=1}^{\frac{n-1}{2}} 2\cos\left(\frac{4k\pi}{n}\right) = -1.$$

Proof. Recall that $\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then we have $\omega^n - 1 = (\omega - 1)\sum_{k=0}^{n-1}\omega^k = 0$. So, it follows that

$$\sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1.$$

If n is a positive odd integer greater than 1, then

$$\sum_{k=1}^{\frac{n-1}{2}} 2\cos\left(\frac{4k\pi}{n}\right) = \sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1. \qquad \square$$

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THEOREM 3.2. *The coefficients of f_3 for D_{2n} are given as follows:*

(1) When $n \equiv 1, 3 \mod 4$,

$$\langle f_3, \chi_i\rangle = \frac{1}{2}(n^2+2n+5) \text{ and } \langle f_3, \psi_j\rangle = n^2+5.$$

(2) When $n \equiv 0 \mod 4$,

$$\langle f_3, \chi_i\rangle = \frac{1}{2}(n^2+4n+24) \text{ and } \langle f_3, \psi_j\rangle = \begin{cases} n^2+16 \text{ if } j \text{ is odd} \\ n^2+24 \text{ if } j \text{ is even.} \end{cases}$$

(3) When $n \equiv 2 \mod 4$,

$$\langle f_3, \chi_i\rangle = \frac{1}{2}(n^2+4n+20) \text{ and } \langle f_3, \psi_j\rangle = n^2+20.$$

Proof. Let χ be an irreducible character of D_{2n}. To simplify notation, we will write only $C(y)$ to denote the centralizer of y in D_{2n}. According to Theorem 1.2, we have:

$$\langle f_3, \chi\rangle = \frac{1}{|D_{2n}|}\sum_{x\in D_{2n}} \theta_\chi(x), \text{ where } \theta_\chi(x) = \sum_{y\in D_{2n}} |C(xy)y \cap C(x)|\chi([x,y]).$$

We will make use of Lemma 3.1 and of the fact that θ_χ is a class function (see Theorem 1.2).

Suppose $n \equiv 1, 3 \mod 4$. A careful analysis shows that we have the following calculations:

$$\begin{aligned}
\theta_\chi(1) &= \sum_{y\in G} |C(y)|\chi(1) \\
&= [2n+(n-1)n+2n]\chi(1) \\
&= \left(n^2+3n\right)\chi(1). \\
\theta_\chi(a^r) &= \sum_{y\in G} |C(a^r y)y \cap C(a^r)|\chi([a^r, y]) \\
&= n^2\chi(1) + n\chi\left(a^{2r}\right). \\
\theta_\chi(a^r b) &= \sum_{y\in G} |C(a^r by)y \cap C(a^r b)|\chi([a^r b, y]) \\
&= 4\chi(1) + \sum_{i=1}^{n-1}\chi\left(a^i\right).
\end{aligned}$$

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Thus,

$$\begin{aligned}\langle f_3, \chi\rangle &= \frac{1}{2n}\left[\left(n^2+3n\right)\chi(1)+\sum_{i=1}^{n-1}\left[n^2\chi(1)+n\chi\left(a^{2i}\right)\right]+n\left[4\chi(1)+\sum_{i=1}^{n-1}\chi\left(a^i\right)\right]\right]\\ &= \frac{1}{2n}\left[\left(n^3+7n\right)\chi(1)+\sum_{i=1}^{\frac{n-1}{2}}2n\chi\left(a^{2i}\right)+n\sum_{i=1}^{n-1}\chi\left(a^i\right)\right]\\ &= \frac{1}{2}\left[\left(n^2+7\right)\chi(1)+\sum_{i=1}^{\frac{n-1}{2}}2\chi\left(a^{2i}\right)+\sum_{i=1}^{n-1}\chi\left(a^i\right)\right].\end{aligned}$$

Hence, if $\chi = \chi_i$, then

$$\begin{aligned}\langle f_3, \chi_i\rangle &= \frac{1}{2}\left[\left(n^2+7\right)+(n-1)+(n-1)\right]\\ &= \frac{1}{2}\left(n^2+2n+5\right),\end{aligned}$$

and if $\chi = \psi_i$, then

$$\begin{aligned}\langle f_3, \psi_i\rangle &= \frac{1}{2}\left[2\left(n^2+7\right)+\sum_{i=1}^{\frac{n-1}{2}}4\cos\left(\frac{4i\pi}{n}\right)+\sum_{i=1}^{n-1}2\cos\left(\frac{2i\pi}{n}\right)\right]\\ &= n^2+7-1-1\\ &= n^2+5.\end{aligned}$$

Likewise, when $n \equiv 0 \mod 4$, we have the following calculations:

$$\begin{aligned}\theta_\chi(1) &= \sum_{y\in G}|C(y)|\chi(1)\\ &= [2\cdot 2n+(n-2)n+2n]\chi(1)\\ &= (n^2+6n)\chi(1).\\ \theta_\chi(a^{\frac{n}{2}}) &= \sum_{y\in G}|C(y)|\chi(1)\\ &= \left(n^2+6n\right)\chi(1).\end{aligned}$$

$$\text{When } r \neq \frac{n}{2}: \quad \begin{aligned}\theta_\chi(a^r) &= \sum_{y\in G}|C(a^r y)y\cap C(a^r)|\chi([a^r, y])\\ &= n^2\chi(1)+2n\chi\left(a^{-2r}\right).\end{aligned}$$

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When r is even: $$\theta\,(a^r b) = \sum_{y\in G} |C\,(a^r by)\,y \cap C\,(a^r b)\,|\chi([a^r b, y])$$

$$= 2\cdot 4\chi\,(1) + 2\cdot 4\chi\,(1) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 4\chi\left(a^{2i}\right)$$

$$= 16\chi\,(e) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 4\chi\left(a^{2i}\right).$$

When r is odd: $$\theta_\chi(a^r b) = 16\chi\,(1) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 4\chi\left(a^{2i}\right).$$

Thus,

$$\langle f_3, \chi\rangle = \frac{1}{2n}\left[2\left(n^2+6n\right)\chi\,(1) + (n-2)n^2\chi\,(e) + \sum_{i=1}^{\frac{n}{2}-1} 4n\chi\left(a^{2i}\right)\right.$$
$$\left. + 2\cdot\frac{n}{2}\left(16\chi\,(1) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 4\chi\left(a^{2i}\right)\right)\right]$$
$$= \frac{1}{2}\left[\left(n^2+28\right)\chi\,(1) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 8\chi\left(a^{2i}\right)\right].$$

Hence, when $\chi = \chi_i$, we have

$$\langle f_3, \chi_i\rangle = \frac{1}{2}\left[\left(n^2+28\right) + 4 + \sum_{i=1}^{\frac{n}{2}-1} 8\right]$$
$$= \frac{1}{2}\left(n^2+4n+24\right),$$

and when $\chi = \psi_j$, we get

$$\langle f_3, \psi_j\rangle = \frac{1}{2}\left[2(n^2+28) + 8\cos(j\pi) + \sum_{i=1}^{\frac{n}{2}-1} 16\cos\left(\frac{2i\pi j}{n}\right)\right]$$
$$= \begin{cases} n^2+16 \text{ if } j \text{ is odd} \\ n^2+24 \text{ if } j \text{ is even.}\end{cases}$$

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Finally, when $n \equiv 2 \mod 4$, a similar analysis yields:

$$\langle f_3, \chi \rangle = \frac{1}{2n}\left[2\left(n^2+6n\right)\chi(e) + (n-2)n^2\chi(1) + 2\cdot\frac{n}{2}\left(16\chi(1) + 4\chi\left(a^{\frac{n}{2}}\right) + \sum_{i=1}^{\frac{n}{2}-1} 4\chi\left(a^{2i}\right)\right)\right]$$
$$= \frac{1}{2}\left[\left(n^2+28\right)\chi(1) + \sum_{i=1}^{\frac{n}{2}-1} 8\chi\left(a^{2i}\right)\right].$$

Thus, if $\chi = \chi_i$, then

$$\langle f_3, \chi \rangle = \frac{1}{2}\left(n^2+4n+20\right).$$

And if $\chi = \psi_j$, then

$$\langle f_3, \chi \rangle = n^2 + 20. \qquad \square$$

THEOREM 3.3. *Suppose that $n \geq 3$. If g is in the commutator subgroup of D_{2n}, then $f_3(g) > 0$. More precisely, we have the following:*

(1)

$$f_3(1) = \begin{cases} n^3 + 7n & \text{if } n \text{ is odd,} \\ n^3 + 28n & \text{if } n \text{ is even.} \end{cases}$$

(2) *For any integer s with $0 < s < \frac{n}{2}$,*

$$f_3(a^{2s}) = \begin{cases} 2n & \text{if } n \text{ is odd,} \\ 8n & \text{if } n \text{ is even and } s \neq \frac{n}{4}, \\ 12n & \text{if } n \text{ is even and } s = \frac{n}{4}. \end{cases}$$

Proof. Recall that $[D_{2n}, D_{2n}] = \langle a^2 \rangle$. We will go over the case when n is odd and will leave the cases $n \equiv 0, 2 \mod 4$ to the reader.

When n is odd, it suffices to calculate $f_3(a^{2s})$ where $0 \leq 2s \leq (n-1)/2$. According to Theorem 3.2, when $n \equiv 1, 3 \mod 4$, we have:

$$f_3(a^{2s}) = \frac{1}{2}\left(n^2+2n+5\right)(1+1) + \left(n^2+5\right)\sum_{k=1}^{\frac{n-1}{2}} 2\cos\left(\frac{4\pi sk}{n}\right) = 2n,$$

and

$$f_3(1) = \frac{1}{2}\left(n^2+2n+5\right)(1+1) + \left(n^2+5\right)\left(\frac{n-1}{2}\right)2 = n^3 + 7. \qquad \square$$

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Recall that the coefficients of t_3 are given by

$$\langle t_3, \chi\rangle = |G| \sum_{x\in K} \frac{|\chi(x)|^2}{\chi(1)},$$

where K is a system of representatives of the conjugacy classes of G, and $\chi \in \mathrm{Irr}(G)$. Using this formula, we easily obtain the following result, whose proof we omit.

THEOREM 3.4. *The coefficients of t_3 for the dihedral group D_{2n} are given as follows:*

(1) *When n is odd,*

$$\langle t_3, \chi_i\rangle = n^2 + 3n, \ \textit{and} \ \langle t_3, \psi_i\rangle = n^2 + 2n,$$

(2) *and when n is even,*

$$\langle t_3, \chi_i\rangle = n^2 + 6n, \ \textit{and} \ \langle t_3, \psi_i\rangle = n^2 + 4n.$$

4. Probability distributions. Calculating the exact value of $f_3(g)$, or equivalently that of $P_3(g)$, could be a pretty challenging task even in the case when $g = 1$. Nevertheless, it is possible to obtain some estimates as we will show in this section. Some results estimating $P_2(1)$ and $P_2(g)$ can be found for instance in [5] and [11].

A key observation that has been used to estimate $P_n(1)$ is to write $f_n(1)$ recursively as follows:

$$f_n(1) = |\{(x_1, \ldots, x_n) \in (G \setminus Z(G)) \times G^{n-1} : [x_i, x_j] = 1 \text{ for } i < j\}| + |Z(G)| f_{n-1}(1).$$

A similar recursive formula can be obtained if we consider the function f_n restricted to tuples formed with elements in the centralizers of elements of the group G. More precisely,

$$f_n(1) = \sum_{g\in G} f_{n-1, C_G(g)}(1).$$

EXAMPLE 4.1. Recall that a group G is called TC (for transitively commutative) if commutativity is a transitive relation on the set of non-central elements of G. This latter condition is equivalent to requiring non-central elements to have abelian centralizers. For TC groups (also known as CA groups), both recursive formulas simplify to:

$$f_n(1) = |G| \sum_{x_i \notin Z(G)} |C_G(x_i)|^{n-2} + |Z(G)| f_{n-1}(1).$$

where $x_1, \ldots, x_k$ is a full set of representatives of the conjugacy classes of G.

It is also possible to approach the calculation of $P_n(1)$ by considering the poset of abelian subgroups of G as was shown in [13]. The structure of this poset simplifies when G is a TC group. For instance, we showed in [13] that for the alternating group A_5 (which is a TC group), we have:

$$P_n(1) = \frac{6}{12^n} + \frac{5}{15^n} + \frac{10}{20^n} - \frac{20}{60^n}.$$

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It is well-known that $P_2(1) \leq 5/8$ for any non-abelian group G (see for instance [6]) and that this upper bound is attained by groups that satisfy $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This upper bound was extended in [8] to:

$$P_n(1) \leq \frac{3 \cdot 2^{n-1} - 1}{2^{2n-1}}.$$

This inequality can be slightly improved if we take into account the index of the center of G. We will show this in the following result only when $n = 3$, although it is not hard to extend it to higher values of n.

PROPOSITION 4.2. *Suppose that G is a finite non-abelian group. Then for all $g \in G$, we have:*

(1) $P_3(1) \leq \frac{1}{2}(P_2(1) - \alpha) + \alpha P_2(1) \leq \frac{11}{32}$, *where* $\alpha^{-1} = |G : Z(G)|$.
(2) $\frac{1}{|G||G'|} \leq P_3(g) \leq \frac{P_2(1)}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)|\chi(g)| \leq P_2(1)$.
(3) $P_3(g) \leq P_2(1)\sqrt{\frac{|C_G(g)|}{|G|}}$.

Proof. (1) If we write the class equation of G as $|G| = |Z(G)| + s_1 + \cdots + s_m$, then each s_i is at least two and so $m \leq (|G| - |Z(G)|)/2$. Since G is not abelian, it follows that $\alpha \leq 1/4$, and hence

$$P_2(1) = \frac{k(G)}{|G|} = \frac{|Z(G)| + m}{|G|} \leq \frac{1+\alpha}{2}.$$

Now, we apply this idea again:

$$\begin{aligned} P_3(1) &= \sum_{i=i}^{m} \frac{|G|}{|C_G(x_i)|} \frac{f_{2,C_G(x_i)}(1)}{|G|^3} + \frac{|Z(G)|}{|G|} P_2(1) \\ &= \sum_{i=i}^{m} \frac{|G||C_G(x_i)|}{|G|^3} \frac{f_{2,C_G(x_i)}(1)}{|C_G(x_i)|^2} + \alpha P_2(1) \\ &\leq (k(G) - |Z(G)|) \frac{1}{|G|} \cdot \frac{1}{2} \cdot 1 + \alpha P_2(1) \\ &= \frac{1}{2}(P_2(1) - \alpha) + \alpha P_2(1) \leq \frac{1}{2}\left(\frac{1-\alpha}{2}\right) + \alpha\left(\frac{1+\alpha}{2}\right) \\ &= \frac{2\alpha^2 + \alpha + 1}{4} \leq \frac{11}{32}. \end{aligned}$$

(2) Note that

$$\frac{|\theta_\chi(x)|}{|C_G(x)|} \leq \sum_{y \in G} \chi(1) = |G|\chi(1).$$

Thus

$$|m_\chi| \leq \sum_{x \in G} |\theta_\chi(x)| = \sum_{x^G} \frac{|G|}{|C_G(x)|} |\theta_\chi(x)| \leq \sum_{x^G} |G||G|\chi(1).$$

and so

$$\frac{|m_\chi|}{|G|^3} \leq P_2(1)\chi(1).$$

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It follows that

$$P_3(g) \leq \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{|m_\chi|}{|G|^3} \chi(g) \leq \frac{1}{|G|} P_2(1) \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)|\chi(g)| \leq P_2(1).$$

The lower bound follows from $P_2(1)/|G| \leq P_3(1)$ and $1/|G'| \leq P_2(1)$ according to [5].

(3) Suppose that K is a full system of representatives of the conjugacy classes of G. Then by Theorem 1.3 and applying the Cauchy–Schwarz inequality, we have:

$$\langle t_3, \chi \rangle = |G| \sum_{x \in K} \frac{|\chi(x)|^2}{\chi(1)} \leq |G| \sum_{x \in K} \chi(1) \leq |G| k(G)^{1/2} |G|^{1/2}.$$

Thus,

$$\begin{aligned} t_3(g) &= \sum_{\chi \in \mathrm{Irr}(G)} \langle t_3, \chi \rangle \chi(g) \\ &\leq |G|^{3/2} k(G)^{1/2} \sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)| \\ &\leq |G|^{3/2} k(G)^{1/2} k(G)^{1/2} |C_G(g)|^{1/2}, \end{aligned}$$

where the last inequality is an application of the Cauchy–Schwarz inequality. The desired inequality follows from the inequality $f_3(g) \leq t_3(g)$ (see Proposition 2.3). □

Unlike the values of $P_2(g)$, the set of probability values $\{P_3(g) : g \in G\}$ does not constitute a probability distribution on G. Nevertheless, we can normalize f_3 to define a probability distribution on a group G by setting:

$$Q_3(g) = \frac{f_3(g)}{\sum_{x \in G} f_3(x)}.$$

Using the coefficients of f_3, we can also write Q_3 in terms of $\mathrm{Irr}(G)$. For instance, for the alternating group A_5, we have (see Example 2.4):

$$Q_3(g) = \frac{1}{60} \left(\chi_1 + \frac{8}{5}\chi_2 + \frac{8}{5}\chi_3 + \frac{21}{10}\chi_4 + \frac{14}{5}\chi_5 \right).$$

It has been shown in [4] that the distribution $P_2(g) = \frac{f_2(g)}{|G|^2}$ converges in the L_1-norm to the uniform distribution $U(g) = \frac{1}{|G|}$ for finite non-abelian simple groups as $|G| \to \infty$. Several computations seem to indicate that this latter is not the case for the distribution Q_3. For instance, we can see in the chart below that the distribution of Q_3 on A_5 is heavily skewed at the identity (all the percentages are approximations).

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Class	1	(12)(34)	(123)	(12345)	(12354)
size	1	15	20	12	12
f_2	300	32	63	65	65
P_2	8.3%	0.8%	1.75%	1.8%	1.8%
f_3	1320	24	12	20	20
P_3	0.6%	0.01%	0.005%	0.009%	0.009%
Q_3	55%	1%	0.5%	0.8%	0.8%

We will close this section showing that the distribution Q_3 over the symmetric group Σ_n is always positive for even permutations when $n \geq 3$. The following Lemma is straightforward, and we will omit its proof.

LEMMA 4.3. *Suppose that* (x_1, x_2, x_3) *and* (y_1, y_2, y_3) *are 3-tuples of elements from G that satisfy* $[x_i, x_j] = g$ *and* $[y_i, y_j] = h$ *for* $i < j$. *If* $[x_i, y_j] = 1$ *for all i and j, then the 3-tuple* (x_1y_1, x_2y_2, x_3y_3) *satisfies* $[x_iy_i, x_jy_j] = gh$ *for* $i < j$.

Proof of Theorem 1.4: We want to prove that if $n \geq 3$ and $g \in A_n$, then $f_{3,\Sigma_n}(g) > 0$. We will proceed by induction on n. Using Theorem 1.3, one can check that for Σ_3 and Σ_4, we have the following:

(1) $f_{3,\Sigma_3}(1) = 48$,
(2) $f_{3,\Sigma_3}((123)) = 6$,
(3) $f_{3,\Sigma_4}((12)(34)) = 72$, and
(4) $f_{3,\Sigma_4}((123)) = 12$.

Now assume the result is true for integers greater than 4 and less than n. We write g as a product of disjoint cycles: $g = \sigma_1 \cdots \sigma_u \tau_1 \cdots \tau_v$, so that each σ_i is a r_i-cycle of even length and each τ_i is a t_i-cycle of odd length (including 1-cycles). Thus, $n = r_1 + \cdots + r_u + t_1 + \cdots + t_v$, and u must be an even number equal to 0, or greater than or equal to 4. By relabeling if necessary, we can assume that $g \in A_r \times A_t$, where $r = r_1 + \cdots + r_u$ and $t = t_1 + \cdots + t_v$.

Then, we have to consider two cases:

(1) If each of the cycles σ_i and τ_j has length less than n, then both u and v are less than n and u is equal to 0, or greater than or equal to 4. Then by inductive hypothesis, there exist triples (x_1, x_2, x_3) in Σ_u and (y_1, y_2, y_3) in Σ_v such that $[x_i, x_j] = \sigma_1 \cdots \sigma_u$ and $[y_i, y_j] = \tau_1 \cdots \tau_v$ for all $i < j$. Hence, by Lemma 4.3, it follows that $g = [x_iy_i, x_jy_j]$ for all $i < j$, as wanted.
(2) If one of the cycles of g has length equal to n, then this implies that g must be an n-cycle. Since n-cycles are conjugate in Σ_n, it suffices to prove that $f_{3,\Sigma_n}((12 \cdots n)) > 0$. According to Theorem 3.3 and Proposition 2.3, we have:

$$f_{3,\Sigma_n}((12 \cdots n)) \geq f_{3,D_{2n}}((12 \cdots n)) > 0,$$

as wanted. □

REMARK 4.4. We have conjectured that $f_{3,A_n}(g) > 0$ for all g in A_n when $n \geq 5$. This conjecture is in a way an extension of Ore's conjecture. More precisely, if we set

$$\mathcal{O}_{k,G} = \{g \in G : f_k(g) > 0\},$$

then Ore's conjecture states that $\mathcal{O}_{2,G} = G$ for any non-abelian simple group G. With this notation, Theorem 1.4 states that $\mathcal{O}_{3,\Sigma_n} = A_n$ for $n \geq 3$. We conjecture that $\mathcal{O}_{3,A_n} = A_n$ for $n \geq 5$.

On the other hand, we know that $f_{3,A_5}((123)) = 12$, so from Proposition 2.3 and the fact that 3-cycles are conjugate in A_n when $n \geq 5$, it follows that $f_{3,A_n}(g) > 0$ whenever g is the identity or a 3-cycle. Moreover, since any permutation in A_n can be written as a product of at most $n/2$ 3-cycles, it follows that the support of the k-iterated convolution product Q_3^{*k} is equal to A_n for $k \geq n/2$. Therefore, the random walk driven by Q_3 is ergodic and Q_3^{*k} converges in the L_1-norm to the uniform distribution. This latter in turn implies that if we fix both $\epsilon > 0$ and $n \geq 5$, then we can find k large enough (see Corollary 1.2 of [4]) so that

$$|A_n| \geq |\mathcal{O}_{A_n}^{*k}| \geq (1-\epsilon)|A_n|,$$

where $\mathcal{O}_{A_n}^{*k} = \{g \in A_n : Q_3^{*k}(g) > 0\}$. This is remarkable to some extent because it is telling that almost every element in A_n can be written as a product of k factors each of which is in $\mathcal{O}_{3,A_n}$, for some k large enough.

As for $\mathcal{O}_{k,G}$ when $k > 3$, we have computational evidence to conjecture that $\mathcal{O}_{k,\Sigma_n} = \{1\}$ when $k > 3$, $n > 2$. Note that it suffices to show the case when $k = 4$ (since $f_n(g) \neq 0$ implies $f_{n-1}(g) \neq 0$).

REFERENCES

1. A. Adem, F. Cohen and E. Torres-Giese, Commuting elements, simplicial spaces and filtrations of classifying spaces, *Math. Proc. Cambridge Philos. Soc.* **152**(1) (2012), 91–114.

2. A. Adem and J. M. Gómez, A classifying space for commutativity in Lie groups, *Algebr. Geom. Topol.* **15**(1) (2015), 493–535.

3. M. Deaconescu and G. Walls, Remarks on commutators in finite groups, *J. Reine Angew. Math.* **732** (2017), 247–253.

4. S. Garion and A. Shalev, Commutator maps, measure preservation, and T-systems, *Trans. Amer. Math. Soc.* **361**(9) (2009), 4631–4651.

5. R. M. Guralnick and G. R. Robinson, On the commuting probability in finite groups, *J. Algebra* **300**(2) (2006), 509–528.

6. W. H. Gustafson, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973), 1031–1034.

7. I. M. Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics, vol. 69 (Academic Press, New York-London, 1976).

8. P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177**(3) (1995), 847–869.

9. M. Liebeck, E. A. O'Brien, A. Shalev and P. H. Tiep, The Ore conjecture, *J. Eur. Math. Soc.* **12**(4) (2010), 939–1008.

10. G. James and M. Liebeck, *Representations and characters of groups*, 2nd edition (Cambridge University Press, New York, 2001).

11. M. R. Pournaki and R. Sobhani, Probability that the commutator of two group elements is equal to a given element, *J. Pure Appl. Algebra* **212**(4) (2008), 727–734.

12. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*; 2020, https://www.gap-system.org.

13. E. Torres-Giese, Higher commutativity and nilpotency in finite groups, *Bull. Lond. Math. Soc.* **44**(6) (2012), 1259–1273.