Monotonic series for fractions near π and their convergents

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We describe various methods to derive monotonic infinite series for fractions near π and obtain a variety of series for the special case of its convergents. These series immediately show that π is clearly different from these fractions, replicating with series the results in Dalzell [1, 2] and Lucas [3] that used integrals with non-negative integrands to represent the gaps between π and fractions.

1. *Introduction*

Of all mathematical constants, π is the most interesting and best-known transcendental number. Derived from a geometrical setting (ratio of the circumference of a circle to its diameter), it appears in many contexts in mathematics. Most early civilisations took 3 (a very crude approximation) as the value of π. Archimedes (287-212 BC) demonstrated geometrically that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The fraction $\frac{22}{7}$ has become almost synonymous with π in the minds of laymen. The next famous fraction $\frac{355}{113}$, good enough for most practical purposes, was derived c. 470 by the Chinese mathematician Zu Chongzhi (429-500) [4, pp. 82-84]. It was rediscovered in Europe by the German mathematician Valentinus Otho in 1573 and by the Dutchman Adriaen Anthoniszoon in 1585. Anthoniszoon first showed that $\frac{333}{106} < \pi < \frac{377}{120}$ and then averaged the numerators and denominators to obtain the fraction for π [5, p.310, fn.7]. In 1767, the Swiss-German mathematician J. H. Lambert proved by means of a continued fraction of the tangent function and an argument of infinite descent that if x is a non-zero rational number then $\tan x$ cannot be rational; since $\tan\frac{\pi}{4} = 1$, $\frac{\pi}{4}$ cannot be rational. It implies the irrationality of π.

Since the rational numbers are dense in the continuum of real numbers, there are rational numbers as close as we wish to any irrational number. A consequence of Dirichlet's approximation theorem [6, p. 34] states that for any irrational number α, there are infinitely many reduced fractions $\frac{p}{q} (q > 0)$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$. It was improved by Hurwitz to $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$. The source of best possible approximations of an irrational number is its continued fraction [7]. The 'best' approximants are known as convergents.

The aim of this paper is to show how to represent the gap between π and a fraction by means of a monotonic series, extending the work of Dalzell and Lucas. The first four convergents of π are: 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$. Beukers [8] deals with the explicit construction of good rational approximations to π.

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The following series contain three well known convergents [9, eqs(3.25)–(3.28)]:

$$\sum_{n=0}^{\infty}\frac{n^2 2^n}{\binom{2n}{n}} = \frac{1}{2}(22+7\pi)\quad \sum_{n=0}^{\infty}\frac{n^3 2^n}{\binom{2n}{n}} = \frac{5}{2}(22+7\pi)\quad \sum_{n=0}^{\infty}\frac{n^4 2^n}{\binom{2n}{n}} = 355+113\pi,$$

but the sign before the terms involving π is + and not − for measuring the gap $\frac{22}{7} - \pi$ and $\frac{355}{113} - \pi$. Mathematical software (like *WolframAlpha* or *Mathematica*) can evaluate such sums. For integer $m \geqslant -1$ we have:

$$\sum_{n=0}^{\infty}\frac{n^m 2^n}{\binom{2n}{n}} = r_1 + r_2\pi.$$

with $r_1, r_2 \in \mathbb{Q}$. But no value of m yields $333 + 106\pi$.

The serendipitous discovery by one of the present authors of (4) (see later) led to the quest for such series. An inspection of existing formulae for π revealed that such series do not occur naturally. We had to manipulate certain series to obtain 'tweaked' series that measure the gap between and its convergents.

2. *A few alternating series for* $\frac{22}{7} - \pi$

2.1 *Euler's series*. In a paper [10] presented to the St Petersburg Academy on 25 April, 1737, Euler obtained a series made up of odd powers of odd numbers which are not powers, the powers with odd exponents of the form $4m - 1$ being increased by one, and those with exponents of the form $4m + 1$ decreased by one, so that the denominators are mulitples of 4:

$$\frac{\pi}{4} - \frac{3}{4} = \left\{\begin{array}{l} \frac{1}{3^3+1} + \frac{1}{3^5+1} + \frac{1}{3^7+1} + \ldots \\ -\frac{1}{5^3-1} - \frac{1}{5^5-1} - \frac{1}{5^7-1} - \ldots \\ \frac{1}{7^3+1} + \frac{1}{7^5+1} + \frac{1}{7^7+1} + \ldots \\ \frac{1}{11^3+1} + \frac{1}{11^5+1} + \frac{1}{11^7+1} + \ldots \\ -\frac{1}{13^3-1} - \frac{1}{13^5-1} - \frac{1}{13^7-1} - \ldots \\ +\frac{1}{15^3+1} + \frac{1}{15^5+1} + \frac{1}{15^7+1} + \ldots \\ \qquad\ldots \end{array}\right\}$$

As all the denominators are divisible by 4, he multiplied both sides by 4 and transposed 3 to the right and rewrote the series as

$$\pi = 3 + \frac{1}{7} - \frac{1}{31} + \frac{1}{61} + \frac{1}{86} + \frac{1}{333} + \frac{1}{547} - \frac{1}{549} - \frac{1}{781} + \frac{1}{844} - \ldots .$$

He commented: *Quae series ideo notari meretur, quod eius duo primi termini iam dent Archimedis proportionem peripharae circuli ad diametrum*. It translates as: "It merits to be noted that the two first terms of

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that series give Archimedes' ratio of the periphery of a circle to the diameter."

Note that the terms of above-noted series are irregular in sign and no formula exists to represent its general term. Hence it is practically useless. Further, since its terms alternate irregularly, it does not show unambiguously that π is strictly less than $\frac{22}{7}$.

2.2 *Dalzell's integral and rapid series*

We refer here to the classic integral of Dalzell [1]:

$$\pi = \frac{22}{7} - \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx,$$

whereby he established the famous inequality due to Archimedes. In a later paper [2, eq(4)], he gave a rapid alternating series:

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1}\left[\frac{3(4n)!^2}{(8n+1)!} + \frac{(4n+1)!^2}{(8n+3)!} - \frac{1}{2}\frac{(4n+2)!^2}{(8n+5)!} - \frac{(4n+3)!^2}{(8n+7)!}\right].$$

We obtained an equivalent form with rate of convergence $\frac{1}{1024}$:

$$\pi = \frac{1}{4^5}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \frac{1}{\binom{8n}{4n}}\left(\frac{3183}{8n+1} + \frac{117}{8n+3} - \frac{15}{8n+5} - \frac{5}{8n+7}\right)$$

or

$$\frac{\pi}{2} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \frac{820n^3 + 1553n^2 + 902n + 165}{\binom{8n}{4n}(8n+1)(8n+3)(8n+5)(8n+7)}.$$

In addition, Dazell refined the earlier estimates by showing that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}; \qquad \frac{355}{113} - \frac{33}{10^8} < \pi < \frac{22}{7} - \frac{24}{10^8}.$$

2.3 *A similar series*

We will now obtain a series of similar type (see [11]).

$$\sum_{n=0}^{\infty}\frac{1}{b^n(kn+c)} = b^{c/k}\sum_{n=0}^{\infty}\frac{1}{kn+c}\left(\frac{1}{b^{1/k}}\right)^{kn+c} = b^{c/k}\sum_{n=0}^{\infty}\left[\frac{x^{kn+c}}{kn+c}\right]_0^{1/(b^{1/k})}$$

$$= b^{c/k}\sum_{n=0}^{\infty}\int_0^{1/(b^{1/k})} x^{kn+c-1}dx = b^{c/k}\int_0^{1/(b^{1/k})}\sum_{n=0}^{\infty}(x^k)^n x^{c-1}dx$$

$$= b^{c/k}\sum_{n=0}^{\infty}\int_0^{1/(b^{1/k})}\frac{x^{c-1}}{1-x^k}dx.$$

And on putting $b^{1/k}x = y$ in the integral on the right-hand side, we obtain

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$$\sum_{n=0}^{\infty} \frac{1}{b^n (kn + c)} = b \int_0^1 \frac{y^{c-1}}{b - y^k} dy.$$

With $b = -4$, $k = 4$ the last formula becomes becomes

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n + c} = 4 \int_0^1 \frac{y^{c-1}}{4 + y^4} dy.$$

The corresponding indefinite integrals for $c = 1, 2, 3$ occur in [12, p. 86, 17.15.1, 2, 3], using which we get

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left[\frac{2}{4n + 1} + \frac{2}{4n + 2} + \frac{1}{4n + 3}\right] = \pi.$$

We observe that

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n + 6} = -4 \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n + 2}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n + 7} = -4 \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n + 3},$$

so that

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left[\frac{2}{4n + 1} - \frac{6}{4n + 2} + \frac{5}{4n + 3} - \frac{2}{4n + 6} + \frac{1}{4n + 7}\right]$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \left[\frac{2}{4n + 1} + \frac{2}{4n + 2} + \frac{1}{4n + 3}\right] = \pi - \frac{8}{3}.$$

This leads to

$$\sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \left[\frac{2}{4n + 1} - \frac{6}{4n + 2} + \frac{5}{4n + 3} - \frac{2}{4n + 6} + \frac{1}{4n + 7}\right] = \pi - \frac{22}{7}.$$

Fortuitously, collecting the terms leaves only a constant in the numerator resulting in a neat expression:

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \frac{30}{(4n + 1)(4n + 2)(4n + 3)(4n + 6)(4n + 7)}, \qquad (1)$$

which can be transformed into a series that consists of only positive terms:

$$\pi = \frac{22}{7} - \sum_{n=1}^{\infty} \frac{30(768n^3 + 1984n^2 + 836n + 87)}{16^n(16n^2 - 1)(64n^2 - 1)(64n^2 - 9)(4n + 3)(8n + 7)}, \qquad (2)$$

clearly showing that $\pi < \frac{22}{7}$. Its rate of convergence equals $\frac{1}{16}$.

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2.4 *A slower series*

The next series is comparatively slower:

$$\pi = \frac{22}{7} - \sum_{n=2}^{\infty} (-1)^n \frac{24}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}. \tag{3}$$

It comes from

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)2n(2n+1)(2n+2)(2n+3)} = \frac{10 - 3\pi}{72},$$

whose derivation is explained in [13, eq(3)] where it was converted into this continued fraction:

$$\frac{6}{10 - 3\pi} = 10 + \cfrac{1 \times 5}{10 + \cfrac{3 \times 7}{10 + \cfrac{5 \times 9}{10 + \cfrac{7 \times 11}{10 + \ldots}}}}.$$

Equation (3) can be converted into a series with positive terms only:

$$\pi = \frac{22}{7} - \sum_{k=1}^{\infty} \frac{240}{(4k+1)(4k+2)(4k+3)(4k+5)(4k+6)(4k+7)}.$$

3. *Using cotangent series to derive series for* $\pm\pi \mp \frac{p}{q}$

The series (3) was a sequel to the following series discovered by using the cotangent series, but was published in [14] without derivation:

$$\pi = \frac{22}{7} - \sum_{n=2}^{\infty} \frac{60}{(4n^2 - 1)(16n^2 - 1)(16n^2 - 9)}. \tag{4}$$

Note that the sum begins with $n = 2$. The first term was extracted to get the convergent $\frac{22}{7}$. Though the series converges slowly, it is simple and elegant.

We now explain a method to derive (4) and many more such series. We begin with the Fourier series [15, p. 62] for $\cos(\mu x)$

$$\cos(\mu x) = \frac{2\mu \sin(\mu x)}{\pi}\left[\frac{1}{2\mu^2} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(nx)}{\mu^2 - n^2}\right], \qquad -\pi \leqslant x \leqslant \pi,$$

where μ is a fraction.

Setting $x = \pi$ in the preceding Fourier series results in the series expansion for the cotangent function:

$$\pi \cot(\mu\pi) = \frac{1}{\mu} - \sum_{n=1}^{\infty} \frac{2\mu}{n^2 - \mu^2}.$$

Special cases with $\mu = \frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$ are

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}; \quad \sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} = \frac{1}{2} - \frac{\pi}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{16n^2 - 9} = \frac{1}{18} + \frac{\pi}{24}.$$

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We find after trial that no combination of these sums lead to (4). So let us extract the first terms for each thereby getting

$$\sum_{n=2}^{\infty}\frac{1}{4n^2-1}=\frac{1}{6};\ \sum_{n=2}^{\infty}\frac{1}{16n^2-1}=\frac{13}{30}-\frac{\pi}{8}\text{ and}\sum_{n=2}^{\infty}\frac{1}{16n^2-9}=\frac{\pi}{24}-\frac{11}{126}.$$

Combining them gives

$$\sum_{n=2}^{\infty}\frac{a}{4n^2-1}+\frac{b}{16n^2-1}+\frac{c}{16n^2-9}=\left(\frac{c}{24}-\frac{b}{8}\right)\pi+\left(\frac{a}{6}+\frac{13b}{30}-\frac{11c}{126}\right).$$

Choosing $c = \dfrac{210k - 35a - 728}{12}$ and $b = \dfrac{210k - 35a - 440}{36}$ gives

$$\pi = k-\sum_{n=2}^{\infty}\frac{2\begin{bmatrix}32(a+210k-656)n^4-10(23a+294k-832)n^2\\ -3(4a-105k+256)\end{bmatrix}}{9(4n^2-1)(16n^2-1)(16n^2-9)}. \tag{5}$$

We may now assign any fractional value to k and choose suitable parameters to get a series for the fraction. With $k = 22/7$, the equation (5) becomes

$$\pi = \frac{22}{7}-\sum_{n=2}^{\infty}\frac{4\left[16(a+4)n^4-115(a+4)n^2-3(2a-37)\right]}{9(4n^2-1)(16n^2-1)(16n^2-9)}.$$

Setting $a = -4$ gives (4). The choice $a = -7/2$ also yields another series of (4) type but it is not as simple-looking.

3.1 *A series and integral for* $\pi - \frac{333}{106}$

The next convergent in the continued fraction of π is 333/106, which is so close to π that we need to pull one more term off our original cotangent series, leading to

$$\pi = \frac{333}{106}+\sum_{n=3}^{\infty}\frac{3(1136n^2-7349)}{371(4n^2-1)(16n^2-1)(16n^2-9)}. \tag{6}$$

As the sum starts with $n = 3$, all terms are positive, $\pi > 333/106$. We have this corresponding integral formula from Lucas [3]

$$\pi = \frac{333}{106}+\int_0^1\frac{x^5(1-x)^6(197+462x^2)}{530(1+x^2)}dx.$$

3.2 *A series and integral for* $\frac{355}{113} - \pi$

We found that for the next convergent we need to extract the first eight terms and the sum formula, unavoidably cumbersome, begins with $n = 9$.

$$\pi = \frac{355}{113}-\sum_{n=9}^{\infty}\frac{2(39925136n^2+259704581471)}{113\times 76608285(4n^2-1)(16n^2-1)(16n^2-9)}. \tag{7}$$

It may be compared with the following result from Lucas [3]:

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$$\pi = \frac{355}{113} - \int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}dx.$$

4. *Using logarithmic series*

Integrating the Maclaurin series for $\ln(1-x^d)$ with $d > 0$ leads to

$$\sum_{n=1}^{\infty} \frac{1}{n(nd+1)} = -\int_0^1 \ln(1-x^d)\,dx.$$

Since for natural numbers p and q we have

$$\sum_{n=1}^{\infty} \frac{1}{n(pd+q)} = \sum_{n=1}^{\infty} \frac{\frac{1}{q}}{n\{(p/q)n+1\}} = -\frac{1}{q}\int_0^1 \ln(1-x^{p/q})\,dx,$$

we can evaluate the integrals to form sums. Using the notation

$$S_{(p,q)} \equiv \sum_{n=1}^{\infty} \frac{1}{n(pn+q)},$$

a few of the simplest results with coprime p, q are

$$S_{(1,1)} = 1, \qquad S_{(2,1)} = 2 - 2\ln(2),$$
$$S_{(3,1)} = 3 - \tfrac{3}{2}\ln(3) - \tfrac{\sqrt{3}}{6}\pi, \qquad S_{(3,2)} = \tfrac{3}{4} - \tfrac{3}{4}\ln(3) + \tfrac{\sqrt{3}}{12}\pi,$$
$$S_{(4,1)} = 4 - 3\ln(2) - \tfrac{1}{2}\pi, \qquad S_{(4,3)} = \tfrac{4}{9} - \ln(2) + \tfrac{1}{6}\pi,$$
$$S_{(6,1)} = 6 - 2\ln(2) - \tfrac{3}{2}\ln(3) - \tfrac{\sqrt{3}}{2}\pi, \qquad S_{(6,5)} = \tfrac{3}{8} - \tfrac{3}{8}\ln(3) + \tfrac{\sqrt{3}}{24}\pi.$$

Some of these can be combined as

$$\sum_{n=1}^{\infty} \frac{a}{n(n+1)} + \frac{b}{n(2n+1)} - \frac{3+b}{3n(4n+1)} + \frac{3-b}{n(4n+3)}$$
$$= \frac{9a+2b-24}{9} + \pi,$$

or

$$\pi = \frac{24-9a-2b}{9} + \sum_{n=1}^{\infty} \frac{\left[\begin{matrix}(96a+16b+48)n^3 + (144a+36b+72)n^2 + \\ (66a+23b+24)n + 9a+3b\end{matrix}\right]}{3n(n+1)(2n+1)(4n+1)(4n+3)}.$$

With $b = -\frac{3}{2}(1+3a)$,

$$\pi = 3 + \sum_{n=1}^{\infty} \frac{(16+16a)n^3 + (12-12a)n^2 - (7+25a)n - 3a - 3}{2n(n+1)(2n+1)(4n+1)(4n+3)}.$$

Good choices of a eliminate the n^3 term, or form a numerator with a common factor with the denominator that can be eliminated. We then get a

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number of series including

$$\pi = 3 + \sum_{n=1}^{\infty} \frac{3}{(n+1)(2n+1)(4n+1)},$$

the one described in [14].

Extracting terms and cancelling the cubic term in the numerator, we get

$$\pi = \frac{22}{7} - \frac{3}{71} \sum_{n=1}^{\infty} \frac{8n^2 + 292n + 143}{4n(n+1)(2n+1)(4n+1)(4n+3)}. \tag{8}$$

5. *Series via binomial expansions*

5.1 *Binomial coefficient in numerator*

Euler got a series (with rate of convergence $\frac{1}{4}$) by using the expansion of the arcsin function:

$$\pi = 3 + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n+1)2^{4n}}.$$

We obtained a series for $\frac{22}{7}$ comprising more factors in the denominator:

$$\pi = \frac{22}{7} - \sum_{n=2}^{\infty} \frac{8(6n^3 + 23n^2 - 6n - 47)\binom{2n}{n}}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)2^{4n}}. \tag{9}$$

Recall the binomial series $(-1 < x \leqslant 1)$

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \times 3}{2 \times 4}x^2 - \frac{1 \times 3 \times 5}{2 \times 4 \times 6}x^3 + \dots .$$

Replacing x by $-y^m$, and multiplying by y^r leads to

$$\frac{y^r}{\sqrt{1 - y^m}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n} y^{mn+r}}{4^n}. \tag{10}$$

Integrating (10) with respect to y on $[0, t]$, the sum replaces x^{mn+r} by $\frac{t^{mn+r+1}}{mn+r+1}$. With $m = 2$ and $t = \frac{1}{2}$, we get different sums involving π and/or $\sqrt{3}$ corresponding to various values of r which may be suitably be combined to get desired result.

Now to establish the sum noted above, take the partial fraction decomposition of

$$\frac{6n^3 + 23n^2 - 6n - 47}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}$$

which is

$$\frac{2}{n+1} - \frac{3}{4(n+2)} - \frac{13}{8(2n+1)} - \frac{13}{8(2n+3)} + \frac{3}{4(2n+5)}.$$

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On integrating the two sides of (10) for different values of r, we obtain:

$$r = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)2^{4n}} = \frac{\pi}{3},$$

$$r = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)2^{4n}} = 8 - 4\sqrt{3},$$

$$r = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+3)2^{4n}} = \frac{2\pi}{3} - \sqrt{3},$$

$$r = 3 \Rightarrow \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+2)2^{4n}} = \frac{64}{3} - 12\sqrt{3},$$

$$r = 4 \Rightarrow \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+5)2^{4n}} = 2\pi - \frac{7\sqrt{3}}{2}.$$

Their combination followed by extraction of the two terms for $n = 0$ and $n = 1$ yields (9).

5.2 *Binomial coefficient in denominator*

Starting with the Maclaurin series for $\dfrac{2x \arcsin(x)}{\sqrt{1 - x^2}}$, putting $x = \frac{1}{\sqrt{2}}$ and shifting index, lead to series such as

$$\pi = \frac{22}{7} - \frac{1}{120} \sum_{n=6}^{\infty} \frac{2^n (n + 242)}{(2n+1)(2n+3)\binom{2n}{n}}. \tag{11}$$

Concluding remarks

We have derived a number of series for $\pi - 3$, $\frac{22}{7} - \pi$, $\pi - \frac{333}{106}$ and $\frac{355}{113} - \pi$. These are just a few examples from some classes of formulas. There are also other classes and each class has numerous formulas, admittedly of increasing complexity. These include those derived by using an infinite series for cosecant, and by means of the digamma function. It is an open question whether formulas more neat and elegant than those presented here exist, and if so how to find them.

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