Proceedings of the Edinburgh Mathematical Society (2016) **59**, 473–481
DOI:10.1017/S0013091515000140

SOME HOMOLOGICAL CRITERIA FOR REGULAR, COMPLETE INTERSECTION AND GORENSTEIN RINGS

JAVIER MAJADAS

Departamento de Álgebra, Facultad de Matemáticas,
Universidad de Santiago de Compostela,
15782 Santiago de Compostela,
Spain (j.majadas@usc.es)

(Received 25 February 2013)

Abstract Regularity, complete intersection and Gorenstein properties of a local ring can be characterized by homological conditions on the canonical homomorphism into its residue field. In positive characteristic, the Frobenius endomorphism (and, more generally, any contracting endomorphism) can also be used for these characterizations. We introduce here a class of local homomorphisms, in some sense larger than all above, for which these characterizations still hold, providing an unified treatment for this class of homomorphisms.

Keywords: regular local ring; finite flat dimension; complete intersection

2010 *Mathematics subject classification:* Primary 13H05; 13H10
Secondary 13D03; 13D05

For a module of finite type M over a commutative Noetherian local ring $(A, \mathfrak{m}, k)$ we consider the projective dimension $\mathrm{pd}_A(M)$, the complete intersection dimension $\text{CI-dim}_A(M)$ (see [**10**] for the definition) and the Gorenstein dimension $\text{G-dim}_A(M)$ (see [**3**]). We have the following theorem.

Theorem 1 (Serre [24], Avramov *et al.* [10], Auslander and Bridger [3]).

(i) *A is regular* $\iff \mathrm{pd}_A(k) < \infty$.

(ii) *A is complete intersection* $\iff \text{CI-dim}_A(k) < \infty$.

(iii) *A is Gorenstein* $\iff \text{G-dim}_A(k) < \infty$.

Subsequently, some results were obtained showing that if A contains a field of positive characteristic, the above results are also valid if we replace the map $A \to k$ by the Frobenius endomorphism $\phi\colon A \to A$. We summarize these results in the following theorem.

Theorem 2 (Kunz [18], Rodicio [22], Blanco and Majadas [13], Herzog [15], Takahashi and Yoshino [26], Iyengar and Sather-Wagstaff [16]). *Let A be a*

Noetherian local ring containing a field of characteristic $p > 0$, let $\phi\colon A \to A$, $\phi(a) = a^p$, be the Frobenius homomorphism and let ${}^{\phi}A$ be the ring A considered as A-module via ϕ. Then

(i) *A is regular* $\iff \operatorname{fd}_A({}^{\phi}A) < \infty$,

(ii) *A is complete intersection* $\iff \text{CI-dim}_A({}^{\phi}A) < \infty$,

(iii) *A is Gorenstein* $\iff \text{G-dim}_A({}^{\phi}A) < \infty$.

In this theorem, fd denotes flat dimension; for the definitions of CI-dim and G-dim when ${}^{\phi}A$ is not of finite type over A, see, for example, [**23**].

Some of these results were extended from the Frobenius endomorphism to the larger class of contracting endomorphisms (that is, endomorphisms $f\colon A \to A$ such that $f^i(\mathfrak{m}) \subset \mathfrak{m}^2$ for some $i > 0$) in [**11**, **12**, **16**, **17**, **21**, **26**].

In this paper we introduce a class of local homomorphisms (called h_2-vanishing) and for this class we prove similar criteria for regularity, complete intersection and Gorenstein-ness. The interest in working with this class is twofold. First, our class contains both the canonical epimorphism into the residue field and (powers of) all contracting endomorphisms. In fact, our class is much larger. It not only contains the canonical epimorphism $A \to k$ of a local ring into its residue field, but more generally any local homomorphism $A \to B$ that factors through a regular ring (see Example 6 (i)), and even in the case of an endomorphism, the proof given in Example 6 (ii) gives us an idea of what else is needed for an h_2-vanishing endomorphism to be contracting. Note, for example, that if A has a contracting endomorphism, then necessarily A contains a field (the subring fixed by f), while this is not needed for h_2-vanishing homomorphisms.

Second, our treatment for this class of homomorphisms is uniform, giving a single proof that is valid, in particular, at once for the residue field and for the Frobenius endomorphism. This fact is a step in allowing us to understand better why some (and what) classes of local homomorphisms are 'test' for regularity, complete intersection and Gorenstein-ness. We obtain these results as easy consequences of strong theorems in the André–Quillen homology of commutative algebras. In particular, the following theorem by Avramov [**5**] will play a key role.

Theorem 3. *Let $f\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ be a local homomorphism of Noetherian local rings. Let $a_1, \dots, a_r$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of A, and let $f(a_1), \dots, f(a_r), b_1, \dots, b_s$ be a set of generators of the ideal $\mathfrak{n}$. Assume that $\operatorname{fd}_A(B) < \infty$. Then the induced homomorphism between the first Koszul homology modules*

$$H_1(a_1, \dots a_r; A) \otimes_k l \to H_1(f(a_1), \dots, f(a_r), b_1, \dots, b_s; B)$$

is injective.

For the convenience of the reader, we start by recalling some facts of André–Quillen homology that we use throughout the paper. Associated with a homomorphism of (always commutative) rings $f\colon A \to B$ and with a B-module M we have André–Quillen homology B-modules $H_n(A, B, M)$ for all integers $n \geqslant 0$, which are functorial in all three variables.

(1) If $B = A/I$, then $H_0(A, B, M) = 0$ and $H_1(A, B, M) = I/I^2 \otimes_B M$ [**1**, Lemme 4.60, Proposition 6.1].

(2) *Base change*: let $A \to B$ and $A \to C$ be ring homomorphisms such that B or C is flat as A-module, and let M be a $B \otimes_A C$-module. Then, $H_n(A, B, M) = H_n(C, B \otimes_A C, M)$ for all n [**1**, Proposition 4.54].

(3) Let B be an A-algebra, let C be a B-algebra and let M be a flat C-module. Then $H_n(A, B, M) = H_n(A, B, C) \otimes_C M$ for all n [**1**, Lemme 3.20].

(4) *Jacobi–Zariski exact sequence*: if $A \to B \to C$ are ring homomorphisms and M is a C-module, we have a natural exact sequence [**1**, Théorème 5.1]

$$\cdots \to H_{n+1}(B, C, M) \to H_n(A, B, M) \to H_n(A, C, M) \to \\ H_n(B, C, M) \to H_{n-1}(A, B, M) \to \cdots \to H_0(B, C, M) \to 0.$$

(5) If $K \to L$ is a field extension and M is an L-module, we have $H_n(K, L, M) = 0$ for all $n \geqslant 2$ [**1**, Proposition 7.4]. So if $A \to K \to L$ are ring homomorphisms with K and L fields, from (4) we obtain $H_n(A, K, L) = H_n(A, L, L)$ for all $n \geqslant 2$, which, using (3), gives $H_n(A, K, K) \otimes_K L = H_n(A, L, L)$ for all $n \geqslant 2$.

(6) If I is an ideal of a Noetherian local ring $(A, \mathfrak{m}, k)$, then the following are equivalent:

   (i) I is generated by a regular sequence;

   (ii) $H_2(A, A/I, k) = 0$;

   (iii) $H_n(A, A/I, M) = 0$ for any A/I-module M for all $n \geqslant 2$ [**1**, Théorème 6.25]; in particular, a Noetherian local ring $(A, \mathfrak{m}, k)$ is regular if and only if $H_2(A, k, k) = 0$.

(7) If $(A, \mathfrak{m}, k)$ is a Noetherian local ring and $\hat{A}$ is its $\mathfrak{m}$-completion, then $H_n(A, k, k) = H_n(\hat{A}, k, k)$ for all $n \geqslant 0$ [**1**, Proposition 10.18].

(8) If $(A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ is a local homomorphism of Noetherian local rings with $\mathrm{fd}_A(B) < \infty$, then the theorem of Avramov cited above says that the homomorphism $H_2(A, l, l) \to H_2(B, l, l)$ is injective [**1**, Proposition 15.12] (see details in the proof of [**20**, Corollary 4.2.2]). In fact, Avramov shows much more. In Remark 12 we will use that $H_4(A, l, l) \to H_4(B, l, l)$ is also injective.

Definition 4. Let $f\colon (A,\mathfrak{m},k) \to (B,\mathfrak{n},l)$ be a local homomorphism of Noetherian local rings. We say that f has the *h_2-vanishing property* if the homomorphism induced by f,

$$H_2(A,l,l) \to H_2(B,l,l),$$

vanishes. If f has the h_2-vanishing property, so does gf for any local homomorphism $g\colon (B,\mathfrak{n},l) \to (B',\mathfrak{n}',l')$.

Proposition 5. *If $f\colon (A,\mathfrak{m},k) \to (B,\mathfrak{n},l)$ has the h_2-vanishing property and there exists a local homomorphism of Noetherian local rings $B \to C$ such that $\mathrm{fd}_A(C) < \infty$ (that is, $\mathrm{Tor}_n^A(C,-) = 0$ for all $n \gg 0$), then A is a regular local ring.*

Proof. Since the composition $A \to B \to C$ has the h_2-vanishing property, we can assume that $B = C$. Therefore, the zero map $H_2(A,l,l) \to H_2(B,l,l)$ is injective by Avramov's theorem, so $H_2(A,l,l) = 0$ and then A is regular. □

Examples 6. (i) If the homomorphism $f\colon A \to B$ factors through a local homomorphism $f\colon A \to R \to B$, where R is a regular local ring, then f has the h_2-vanishing property. This includes as a particular case the canonical epimorphism $A \to k$ of a local ring $(A,\mathfrak{m},k)$ into its residue field.

(ii) Let $f\colon (A,\mathfrak{m},k) \to (A,\mathfrak{m},k)$ be a contracting homomorphism [**11**, § 12], that is, there exists some $i > 0$ such that $f^i(\mathfrak{m}) \subset \mathfrak{m}^2$. This means that for any $t > 0$, $f^j(\mathfrak{m}) \subset \mathfrak{m}^t$ for some $j > 0$. We will see that some power of f has the h_2-vanishing property and how to apply Proposition 5 to the contracting endomorphism f.

Assume first that A is complete. Since f is contracting, the subring $F \subset A$ of elements fixed by f is clearly a field [**12**, Remark 5.9]. Let F_0 be a perfect subfield of F. Then $F_0 \to k$ is formally smooth and so there exists a section h of the F_0-algebra homomorphism $A \to k$ [**20**, Lemma 2.2.3]. So $k_0 := \mathrm{Im}(h) \subset A$ is a coefficient field for A, and there exists a surjective ring homomorphism $\pi\colon R := k_0[[X_1,\dots,X_n]] \to A$ sending $X_1,\dots,X_n$ to a set of generators $x_1,\dots,x_n$ of the maximal ideal $\mathfrak{m}$ of A. Let I be its kernel. For each j, choose a power series $P_j(x_1,\dots,x_n)$ in $x_1,\dots,x_n$ over k_0 representing $f^i(x_j) \in \mathfrak{m}^2$. We can choose P_j of order greater than or equal to 2. Since F_0 is invariant by f, we have F_0-algebra homomorphisms

$$\begin{array}{ccc} k_0 & & R \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{f^i} & A \end{array}$$

By [**14**, Chapitre 0_{IV}, Proposition 19.3.10], there exists an F_0-algebra homomorphism $g_0\colon k_0 \to R$ making the diagram a commutative square. We define $g\colon R \to R$ by $g_{|k_0} := g_0$, $g(X_j) := P_j(X_1,\dots,X_n)$. Since g is a lifting of f^i, we have $g(I) \subset I$. Also, $g(\mathfrak{q}) \subset \mathfrak{q}^2$, where $\mathfrak{q} = (X_1,\dots,X_n)$ is the maximal ideal of R.

Therefore, for any $t > 0$ there exists some s such that $g^s(\mathfrak{q}) \subset \mathfrak{q}^t$, and then $g^s(I) \subset \mathfrak{q}^t \cap I$. By the Artin–Rees lemma, some s verifies $g^s(I) \subset \mathfrak{q}I$. Consider the Jacobi–Zariski exact sequence associated with $R \to A \to k$,

$$0 = H_2(R,k,k) \to H_2(A,k,k) \to I/\mathfrak{q}I.$$

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The endomorphism induced by f^{is} on $H_2(A,k,k)$ is the restriction of the one induced by g^s on $I/\mathfrak{q}I$, which is zero. We have proved that if $f\colon A \to A$ is contracting, then some power f^{is} of f has the h_2-vanishing property when A is complete. But since $H_2(A,k,k) = H_2(\hat{A},k,k)$, this is also valid when A is not complete.

Note that the fact that we have had to replace f by some power f^{is} does not interfere with our result. More generally, let $f\colon A \to A$ be contracting and assume that there exists a Noetherian local A-algebra $(C,\mathfrak{p},l)$ such that $\mathrm{fd}_A({}^{f^t}C) < \infty$ for *some* t. By Avramov's theorem, the homomorphism $H_2(A,k,l) \to H_2(C,l,l)$ induced by $A \xrightarrow{f^t} A \to C$ is injective, and then so is the endomorphism $H_2(A,k,k) \to H_2(A,k,k)$ induced by f^t. Thus, for any r the endomorphism $H_2(A,k,k) \to H_2(A,k,k)$ induced by f^{tr} is also injective. But choosing r so that f^{tr} has the h_2-vanishing property, we obtain $H_2(A,k,k) = 0$ and then A is regular. This case of a contracting endomorphism was proved in [**17**, Proposition 2.6] and [**26**, Remark 4.4] and includes the particular case of a finite A-algebra C (localizing C at a prime ideal contracting in A to $\mathfrak{m}$) with $\mathrm{fd}_A({}^{f^t}C) < \infty$, which was proved with different methods in [**12**] in the more general case of a finite A-*module* $C \neq 0$ with $\mathrm{fd}_A({}^{f^t}C) < \infty$.

(iii) The good properties of André–Quillen homology (for example, the ones stated above and its behaviour under tensor products [**1**, Proposition 5.21]) allow us to obtain new examples of h_2-vanishing homomorphisms from others.

Remarks 7. (i) It can be proved that if $f\colon A \to A$ is contracting, then for any $n \geqslant 0$, there exists s (depending on n) such that f^s has the h_n-vanishing property, that is, it induces the zero map $H_n(A,k,k) \to H_n(A,k,k)$ [**19**, Proposition 10]. However, our *ad hoc* proof given here for the case in which $n = 2$, besides being simpler and shorter, gives a better idea of the relationship between contracting and h_2-vanishing.

(ii) If $f\colon A \to A$ is the Frobenius endomorphism, then f^s has the h_n-vanishing property for all $s > 0$, $n \geqslant 0$ [**2**, Lemme 53].

Avramov's theorem also allows us to obtain similar criteria for complete intersection, Gorenstein, and Cohen–Macaulay rings, provided that we use the adequate definitions for homological dimensions in terms of 'deformations' (see [**7**, § 8]) and flat dimension. We start by recalling the definition of upper complete intersection dimension introduced in [**25**] (see also [**10**]).

We say that a finite module $M \neq 0$ over a Noetherian local ring A has finite upper complete intersection dimension and denote it by $\text{CI*-dim}_A(M) < \infty$ if there exists a flat local homomorphism of Noetherian local rings $(A,\mathfrak{m},k) \to (A',\mathfrak{m}',k')$ such that $A' \otimes_A k$ is a regular local ring, and a surjective homomorphism of Noetherian local rings $Q \to A'$ with kernel generated by a regular sequence, such that $\mathrm{pd}_Q(M \otimes_A A') < \infty$, where pd denotes the projective dimension.

If $f\colon (A,\mathfrak{m},k) \to (B,\mathfrak{n},l)$ is a local homomorphism of Noetherian local rings, a Cohen factorization of f is a factorization $A \xrightarrow{i} R \xrightarrow{p} B$ of f, where R is a Noetherian local ring, i is a flat local homomorphism, $R \otimes_A k$ is a regular local ring and p is surjective. If B is complete, a Cohen factorization always exists [**9**].

We say that a local homomorphism of Noetherian local rings $f\colon (A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$ is of finite upper complete intersection dimension (and denote it by CI*-dim$(f)<\infty$) if there exists a Cohen factorization $A\to R\to\hat{B}$ such that CI*-dim$_R(\hat{B})<\infty$.

The following lemma is essentially [**6**, Lemma 1.7].

Lemma 8. *Let $(A,k)\to(R,l)\to(D,E)$ be local homomorphisms of Noetherian local rings such that R is a flat A-module and $R\otimes_A k$ is regular. Then $H_n(A,D,E)=H_n(R,D,E)$ for all $n\geqslant 2$.*

Proof. By flat base change $H_n(A,R,E)=H_n(k,R\otimes_A k,E)$, and by the Jacobi–Zariski exact sequence associated with $k\to R\otimes_A k\to E$, we have $H_n(k,R\otimes_A k,E)=H_{n+1}(R\otimes_A k,E,E)=0$ for all $n\geqslant 2$. So the Jacobi–Zariski exact sequence

$$\cdots\to H_n(A,R,E)\to H_n(A,D,E)\to H_n(R,D,E)\to H_{n-1}(A,R,E)\to\cdots$$

gives isomorphisms $H_n(A,D,E)=H_n(R,D,E)$ for all $n\geqslant 3$ and an exact sequence

$$0\to H_2(A,D,E)\to H_2(R,D,E)\to H_1(A,R,E)\xrightarrow{\alpha} H_1(A,D,E)\to\cdots.$$

The injectivity of α follows from the commutative diagram with exact upper row

$$\begin{array}{ccccc}
0=H_2(R\otimes_A k,E,E) & \longrightarrow & H_1(k,R\otimes_A k,E) & \longrightarrow & H_1(k,E,E)\\
 & & \simeq\uparrow & & \uparrow\\
 & & H_1(A,R,E) & \xrightarrow{\alpha} & H_2(A,D,E)
\end{array}$$

□

Proposition 9. *If $f\colon(A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$ has the h_2-vanishing property and there exists a local homomorphism of Noetherian local rings $g\colon(B,\mathfrak{n},l)\to(C,\mathfrak{p},E)$ such that* CI*-dim$(gf)<\infty$, *then A is a complete intersection ring.*

Proof. Let $A\to R\to\hat{C}$ be a Cohen factorization, let $R\to R'$ be a flat local homomorphism with regular closed fibre, and let $Q\to R'$ be a surjective homomorphism of Noetherian local rings with kernel generated by a regular sequence such that $\mathrm{pd}_Q(\hat{C}\otimes_R R')<\infty$. We have a commutative triangle

$$\begin{array}{ccccc}
 & & H_2(R,E,E) & & \\
 & \overset{\beta}{\nearrow} & & \overset{\gamma}{\searrow} & \\
H_2(A,E,E) & & \xrightarrow{\quad\alpha\quad} & & H_2(\hat{C},E,E)
\end{array}$$

where $\alpha=0$ since f has the h_2-vanishing property, and β is surjective by Lemma 8. Then $\gamma=0$ and so the homomorphism $H_2(R,E'E')\to H_2(\hat{C},E',E')$ also vanishes, where E'

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is the residue field of R' and $\hat{C} \otimes_R R'$. We have a commutative diagram

$$\begin{array}{ccc} H_2(R, E', E') & \xrightarrow{\;0\;} & H_2(\hat{C}, E', E') \\ \Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow \\ H_2(R', E', E') & \xrightarrow{\;\mu\;} & H_2(\hat{C} \otimes_R R', E', E') \end{array}$$

where λ is an isomorphism by Lemma 8. We see that $\mu = 0$, that is, $R' \to \hat{C} \otimes_R R'$ has the h_2-vanishing property. Composing with $Q \to R'$, we deduce that $Q \to \hat{C} \otimes_R R'$ has the h_2-vanishing property. By Proposition 5, Q is a regular local ring, and then R' is a complete intersection ring. By flat descent (see [**4**] or [**5**]), A is a complete intersection ring. □

Following in part [**27**], we define the upper Gorenstein dimension as follows. If $M \neq 0$ is a finite module over a Noetherian local ring A, we say that M has finite upper Gorenstein dimension if there exists a flat local homomorphism of Noetherian local rings $A \to A'$ with regular closed fibre and a surjective homomorphism of Noetherian local rings $Q \to A'$ verifying that $\mathrm{Ext}^n_Q(A', Q) = A'$ for some n and $\mathrm{Ext}^i_Q(A', Q) = 0$ for all $i \neq n$, such that $\mathrm{pd}_Q(M \otimes_A A') < \infty$. We say that a local homomorphism of Noetherian local rings $A \to B$ has finite upper Gorenstein dimension if for some Cohen factorization $A \to R \to \hat{B}$ the R-module $\hat{B}$ has finite upper Gorenstein dimension.

Proposition 10. *If $A \to B$ has the h_2-vanishing property and there exists a local homomorphism of Noetherian local rings $g\colon B \to C$ such that gf has finite upper Gorenstein dimension, then A is a Gorenstein ring.*

Proof. As in the proof of Proposition 9, we know that there exist a Cohen factorization $A \to R \to \hat{C}$, a flat local homomorphism $R \to R'$, a regular local ring Q and a surjective homomorphism $Q \to R'$ such that $\mathrm{Ext}^n_Q(R', Q) = R'$ for some n and $\mathrm{Ext}^i_Q(R', Q) = 0$ for all $i \neq n$. Now, if E' is the residue field of R', from the change of rings spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{R'}(E', \mathrm{Ext}^q_Q(R', Q)) \Rightarrow \mathrm{Ext}^{p+q}_Q(E', Q),$$

we deduce that R' is Gorenstein. By flat descent, A is also Gorenstein. □

Remark 11. For a Noetherian local ring S let $\mathrm{cmd}(S) := \dim(S) - \mathrm{depth}(S)$ (see [**14**, Chapitre 0_{IV}, Définition 16.4.9] and [**8**]). If we say that a finite A-module $M \neq 0$ has finite Cohen–Macaulay dimension when there exists a flat local homomorphism of Noetherian local rings $A \to A'$ with regular closed fibre and a surjective homomorphism of Noetherian local rings $Q \to A'$ verifying that $\mathrm{cmd}(Q) = \mathrm{cmd}(A')$ (compare with [**8**, (3.2)]) and $\mathrm{pd}_Q(M \otimes_A A') < \infty$, then we have a similar criterion for Cohen–Macaulay rings.

Remarks 12. (i) A detailed inspection of the proofs (also using [**1**, Théorème 17.13, Proposition 6.27]) shows that these criteria for complete intersection, Gorenstein and Cohen–Macaulay are also valid if, instead of assuming the h_2-vanishing property, we assume the h_4-vanishing property (see Remark 7 (i)). For instance, any local homomorphism factorizing through a complete intersection ring is h_4-vanishing but not necessarily h_2-vanishing.

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(ii) Two particular cases of these facts are then given as the following.

- Let $f\colon A \to B$ be a local homomorphism factorizing into local homomorphisms $A \to R \to B$, where R is a regular local ring. If $\mathrm{fd}_A(B) < \infty$, then A is a regular ring.

- Let $f\colon A \to B$ be a local homomorphism factorizing into local homomorphisms $A \to R \to B$, where R is a complete intersection local ring. If CI*-dim$(f) < \infty$, then A is a complete intersection ring.

We do not know if a similar result exists for the Gorenstein property.

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